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The Truncated Induced Ailamujia Distribution: Properties, Estimation, and Applications in Medical and Engineering Data

Mohammed Elgarhy [1,2], Doaa Basalamah [3,*], Ahmed Mohamed El Gazar [4], Dina A. Ramadan [5], Ghareeb A. Marei [1], Said G. Nassr [6]

[1] *Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt*
[2] *Department of Computer Engineering, Biruni University, 34010, Istanbul, Turkey*
[3] *Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*
[4] *Department of Basic Sciences, Higher Institute for Commercial Sciences, Almahlla Alkubra, Egypt*
[5] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 33516, Egypt*
[6] *Department of Statistics and Insurance, Faculty of Commerce, Arish University, El-Arish 45511, Egypt*

Abstract Existing probability distributions often suffer from significant limitations when modeling real-world data constrained to a bounded interval. when modeling bounded lifetime data exhibiting right-skewness and increasing hazard rate patterns. To address this, we construct a new continuous probability model called the truncated induced Ailamujia (TIAL) distribution for data bounded in the interval $(0, 1)$. To deal with this problem, the mathematical and statistical features of the proposed TIAL distribution are rigorously derived. For estimating model parameters, we employ eight conventional estimation techniques. The Monte Carlo simulation study provides an efficient way to analyze the behavior of different estimation methods. The main practical value of this model is its use in reliability and survival analysis. Real datasets from the medicine and engineering industries are used to demonstrate the flexibility, resilience, and better-fitting capabilities of the TIAL model.

Keywords Induced Ailamujia distribution, Moments, Quantile function, Least square estimation method, Simulation study, Real data applications

AMS 2010 subject classifications 60E05, 62E10, 62F10, 62N05

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1. Introduction

Many significant probability distributions have been created throughout the past century for use as mathematical models in a wide range of scientific fields. The creation of new classes of univariate continuous distributions that expand the applicability and flexibility of the current models has drawn increasing attention in recent years. Modern datasets in diverse domains often exhibit phenomena such as skewness, kurtosis, heavy tails and multimodal distributions. These complexities require new more flexible distributions that can better explain the intricacies of real data, particularly in modeling extreme events or in assessing risk and forecasting performance. To achieve this flexibility, various generalization strategies are employed, among which the truncation method is particularly significant. Truncated distributions are indispensable when data is inherently bound, finite, or restricted to a specific interval of values. Examples include contaminant concentrations in environmental studies often below a detection limit, or in manufacturing, the exclusion of products not meeting specified criteria. Truncation is also vital for improving parameter estimation methodologies and constructing robust Bayesian priors. Many researchers

*Correspondence to: Doaa Basalamah (Email: dabasalamah@uqu.edu.sa). Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia.

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have proposed truncated variants of conventional statistical distributions to improve the analysis of limited and constrained data. For example, Behdani et al. [1] outlined specific characteristics of truncated distributions in the context of reliability and economic analyses; see also [2, 3, 4, 5, 6, 7, 8, 9].

In this context, the Ailamujia (AL) distribution is a very flexible, recently developed longevity model introduced in [10], where the basic mathematical properties of the distribution were discussed. From its inception, several authors have contributed to the development of the AL model. Pan et al. [11] suggested this distribution for interval estimation and hypothesis testing for limited sample sets. Further extensions include the weighted version of the AL distribution by [12], the inverse Ailamujia distribution by [13] and the size-biased AL distribution developed by [14] for engineering and medical disciplines. A Bayesian estimation of the inverse AL distribution was proposed by [15]. An area-biased AL distribution was studied in the context of bladder cancer data by [16]. Furthermore, the induced form of the AL distribution was presented in [17].

Recent contributions to this literature include the inverse power induced Bilal distribution [18], the generalized Rayleigh distribution with its properties and applications [19], and the Gompertz inverse Rayleigh distribution with classical and Bayesian estimation techniques [20]. In parallel, several new one-parameter and flexible lifetime models have recently been introduced together with classical and Bayesian inference procedures, see for instance [21, 22, 23, 24]. Moreover, considerable attention has been paid to reliability and stress–strength inference under various censoring and record schemes, with engineering and medical applications [25, 26, 27, 28, 29, 30].

Let X be a random variable which follows the IAL model; then the probability density function (PDF) and cumulative distribution function (CDF) are given as

$$f(x;\alpha) = \alpha\,(1+2\alpha x)\,e^{-2\alpha x}, \tag{1.1}$$

and

$$F(x;\alpha) = 1-(1+\alpha x)\,e^{-2\alpha x}. \tag{1.2}$$

The PDF and CDF of the IAL distribution demonstrate how versatile it is in handling various kinds of data. The PDF of the IAL distribution is strictly decreasing and right-skewed for all $\alpha > 0$. Because of its adaptability, the model can identify various data patterns found in environmental, survival, and dependability studies.

The purpose of this paper is to introduce the truncated induced Ailamujia (TIAL) distribution using the flexibility of the induced generalization and the power of truncation. To demonstrate the theoretical viability of this unique model, we test its performance on two different real-world datasets from essential disciplines. The first dataset consists of survival times of breast cancer patients, which points out the importance of the model in clinical survival analysis. The second dataset comprises tensile strength measurements for polyester fibers, demonstrating the model's effectiveness in the area of material durability assessment and reliability engineering. The TIAL model is fitted to these datasets and shows more fitting flexibility than traditional models. This work links theoretical probability advances with real-world decision making in both medical and technical fields.

The remainder of this paper is structured as follows. Section 2 presents the derivation and structural characteristics of the TIAL distribution. Section 3 explores its core statistical properties, including quantiles, moments, incomplete moments, order statistics, and entropy measures. Section 4 details the parameter estimation methodologies, derives the asymptotic normality framework, and introduces a bootstrap bias-correction procedure. Section 5 presents a comprehensive Monte Carlo simulation study evaluating estimator efficiency. Section 6 demonstrates the practical applicability of the model using two real-world datasets from the biomedical and engineering fields. Finally, Section 7 concludes the paper with key findings and potential future research directions.

2. Construction of the truncated induced Ailamujia distribution

In this section, the truncation technique is used to obtain the truncated induced Ailamujia distribution, and formulas for its PDF and CDF are given. The PDF of the TIAL distribution is

$$f(x;\alpha) = A\,\alpha\,(1+2\alpha x)\,e^{-2\alpha x}, \qquad 0<x<1,\ \alpha>0, \tag{2.1}$$

and the corresponding CDF is

$$F(x;\alpha) = A\left[1-(1+\alpha x)\,e^{-2\alpha x}\right], \tag{2.2}$$

where

$$A = \left[1-(1+\alpha)\,e^{-2\alpha}\right]^{-1}, \tag{2.3}$$

and $\alpha > 0$ is a shape parameter. It is worth noting that the PDF of the TIAL distribution in Eq. (2.1) is proportional to the un-truncated IAL density (1.1) on the domain $x \in (0,1)$, scaled by the normalizing constant A in (2.3). The primary structural purpose of this truncated formulation is not to introduce non-monotonic shape variants unattainable by the parent model, but to strictly constrain the probability support to $(0,1)$, thereby ensuring proper probabilistic normalization and exact statistical inference for bounded percentage and lifetime data.

Figure 1 depicts the PDF and CDF graphs for the TIAL model at different specified parameter values. The PDF of the TIAL distribution is strictly decreasing on $(0,1)$ for all $\alpha > 0$, with its mode at the lower boundary $x = 0$. The distribution is right-skewed, and the shape parameter α controls the rate of decay and the concentration of probability mass near the origin. This flexibility demonstrates the model's capacity to fit skewed lifetime data sets. The validity of the TIAL model as a proper probability model is confirmed by the CDF plots, which demonstrate that the CDF of the proposed distribution is monotonically increasing from 0 to 1 for all the chosen parameter values. Additionally, Figure 2 displays the 3D plots for the PDF and CDF of the TIAL model.

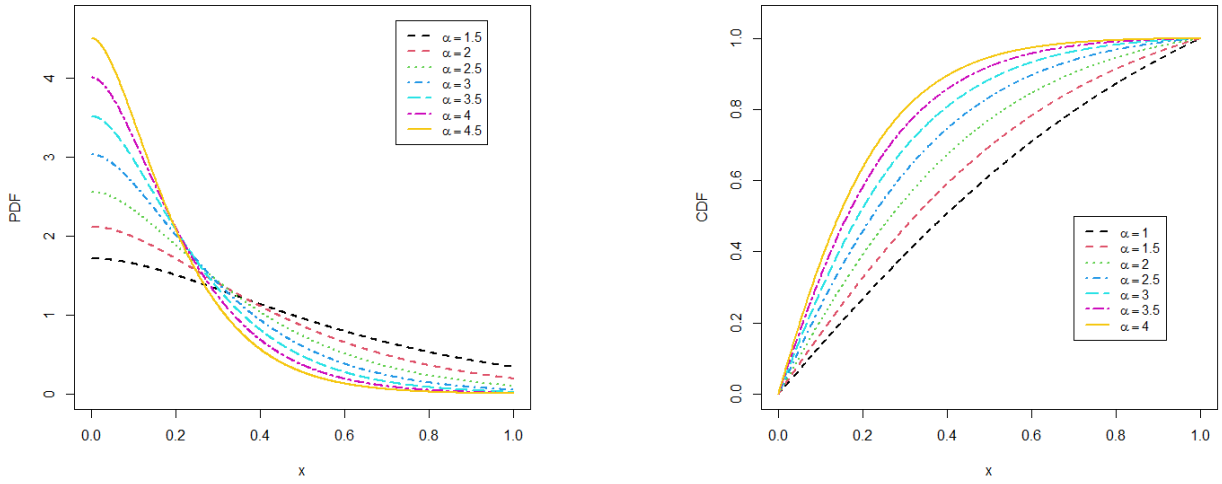


Figure 1. PDF (left) and CDF (right) plots of the TIAL model for various parameter values.

According to the PDF (2.1) and CDF (2.2), the survival function (SF) and hazard rate function (HRF) of the TIAL model are, respectively,

$$S(x) = 1 - A\left[1-(1+\alpha x)\,e^{-2\alpha x}\right], \tag{2.4}$$

and

$$h(x) = \frac{A\alpha\,(1+2\alpha x)\,e^{-2\alpha x}}{1-A\left[1-(1+\alpha x)\,e^{-2\alpha x}\right]}. \tag{2.5}$$

The SF and the HRF of the TIAL distribution for several values of the parameter are presented in Figure 3. The SF plots are monotonically decreasing for all the chosen values of the parameter, starting from 1 and approaching 0 with time, which is consistent with the physical features of survival functions. On the other hand, the HRF has a very flexible pattern. It increases gradually over time under the specified parameter settings, which indicates that the model is suitable for modeling aging systems or wear-out failure processes in reliability and survival analysis. Also, the 3D plots of the SF and HRF for the TIAL model are presented in Figure 4.

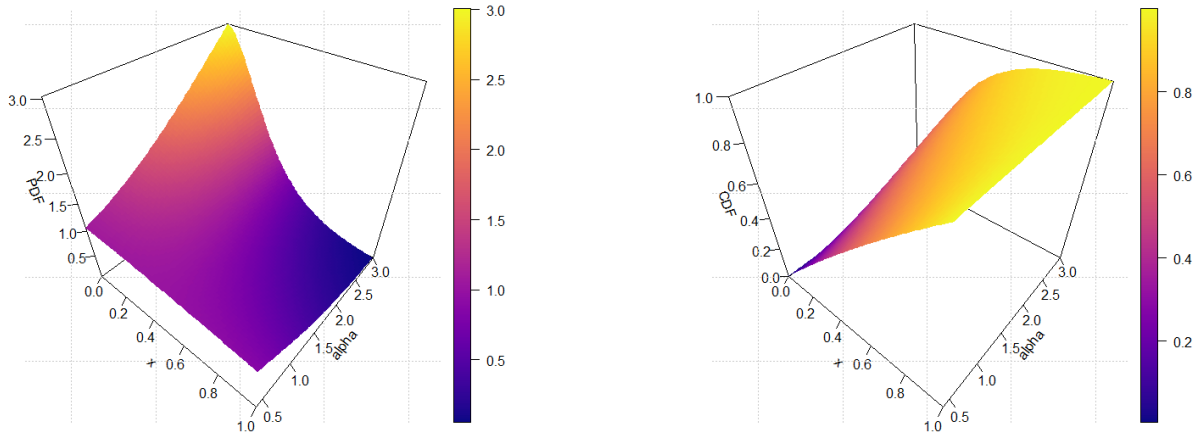


Figure 2. Three-dimensional plots of the PDF (left) and CDF (right) of the TIAL model.

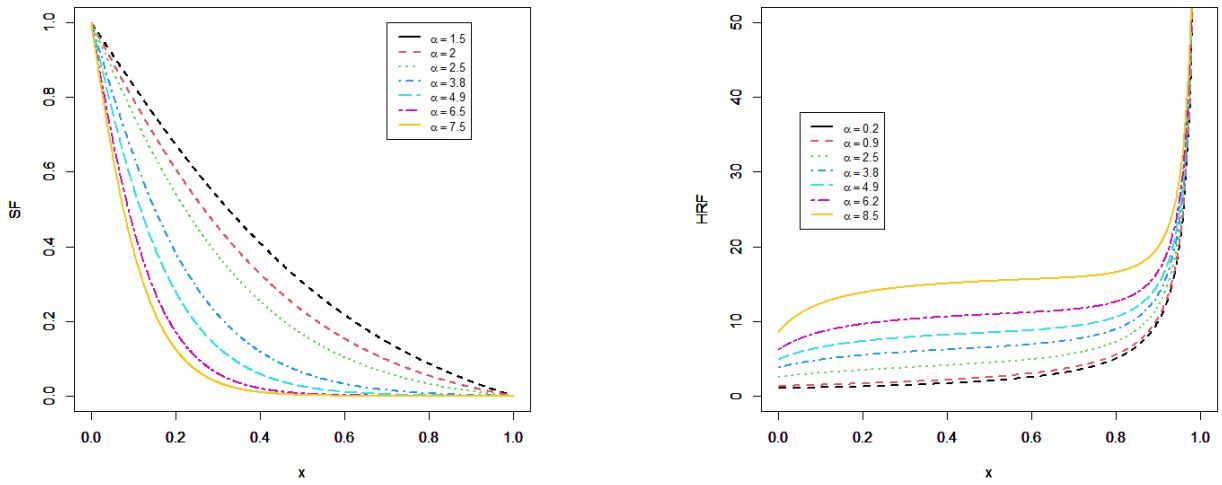


Figure 3. SF (left) and HRF (right) plots of the TIAL model for some choices of the parameter.

It is instructive to examine the behavior of the TIAL density as $\alpha \to 0^+$. Evaluating the limit of $f(x;\alpha)$ as $\alpha \to 0^+$ yields

$$\lim_{\alpha\to 0^+} f(x;\alpha) = \lim_{\alpha\to 0^+} A\,\alpha\,(1+2\alpha x)\,e^{-2\alpha x} = 1, \qquad \forall\, x \in (0,1). \tag{2.6}$$

Thus, as $\alpha \to 0^+$, the TIAL distribution continuously converges to the standard uniform distribution $U(0,1)$. Because the parameter α simultaneously governs both the normalizing denominator and the exponential decay rate, the log-likelihood gradient flattens near $\alpha \to 0$, leading to decreased identification sensitivity and higher relative upward bias in very small samples when the true parameter is close to zero.

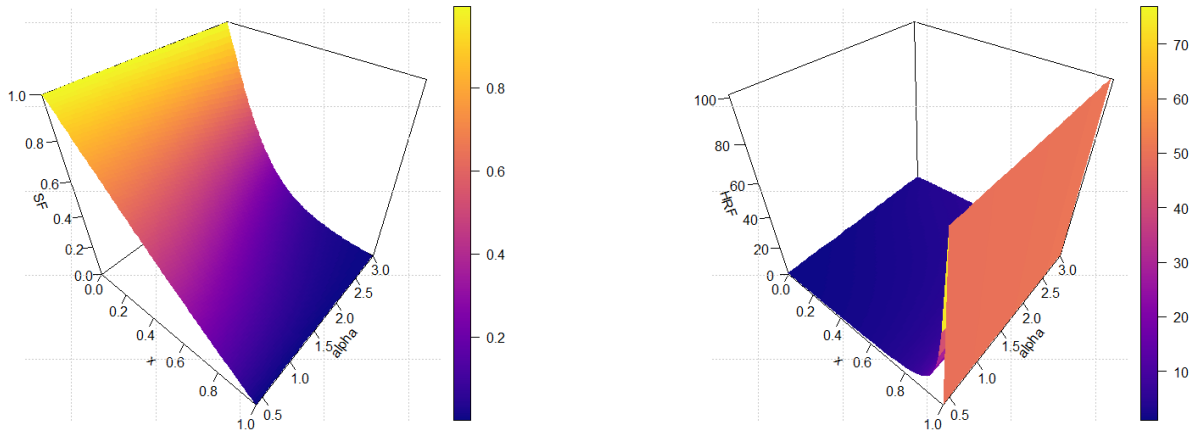


Figure 4. Three-dimensional plots of the SF (left) and HRF (right) of the TIAL model.

3. Mathematical properties of the TIAL model

This section provides and explores many important properties of the TIAL distribution.

3.1. Quantile function

The quantile function (QF) of a distribution is obtained by inverting its CDF, and it provides a full description of the statistical behavior of the distribution. Thus, a good understanding of its components is of considerable importance. Let X be a TIAL distributed random variable. The QF of the TIAL distribution is obtained by setting the CDF equal to u, where u is a uniform variate on the unit interval, as follows:

$$A\left[1-(1+\alpha x)\,e^{-2\alpha x}\right]=u. \tag{3.1}$$

By solving Eq. (3.1), the quantile function of the TIAL distribution can be obtained. Substituting $u=0.25$, 0.5 and 0.75 in Eq. (3.1) yields the first quartile (Q_1), the median (Q_2) and the third quartile (Q_3), respectively. The quantiles are also used to calculate Bowley's skewness (BSK) and Moors' kurtosis (MKUR), which are useful to explain the skewness and kurtosis properties of the TIAL distribution. These measures have the great advantage of being computationally simple, and they do not depend on the existence of ordinary moments. The mathematical expressions of BSK and MKUR are

$$\text{BSK}=\frac{Q(0.75)-2Q(0.5)+Q(0.25)}{Q(0.75)-Q(0.25)}, \tag{3.2}$$

and

$$\text{MKUR}=\frac{Q(0.875)-Q(0.625)-Q(0.375)+Q(0.125)}{Q(0.75)-Q(0.25)}. \tag{3.3}$$

Table 1 reports the quantiles Q_1, Q_2, Q_3, Bowley's skewness (3.2) and Moors' kurtosis (3.3) of the TIAL model. The quantiles fall monotonically as the parameter increases from 0.2 to 6.5. It is to be noted that all the BSK values are positive, which shows that the TIAL distribution is always right-skewed. The variation in MKUR reflects the effect of α on the concentration of probability mass near the boundaries; because the support is bounded, these measures should not be interpreted as evidence of heavy tails.

Table 1. Some evaluated values of quantiles, BSK and MKUR of the TIAL model.

α	Q_1	Q_2	Q_3	BSK	MKUR
0.2	0.2449	0.4919	0.7429	0.0081	1.0028
0.8	0.2019	0.4208	0.6757	0.0760	1.0384
1.2	0.1707	0.3640	0.6114	0.1226	1.0841
1.5	0.1499	0.3239	0.5594	0.1502	1.1231
2.0	0.1219	0.2668	0.4755	0.1805	1.1812
2.5	0.1010	0.2224	0.4030	0.1960	1.2202
3.2	0.0803	0.1775	0.3247	0.2047	1.2454
4.0	0.0646	0.1429	0.2624	0.2076	1.2543
4.5	0.0575	0.1272	0.2337	0.2081	1.2559
5.0	0.0518	0.1146	0.2104	0.2082	1.2568
5.7	0.0455	0.1005	0.1847	0.2087	1.2569
6.5	0.0399	0.0882	0.1619	0.2087	1.2572

3.2. Moments

Moments are an essential statistical tool for describing the characteristics of a probability distribution. They provide important information on the shape, dispersion and central tendency of the data. The rth moment of the TIAL distribution can be expressed as

$$\mu'_r = E\left(X^r\right) = \int_0^1 x^r \, f_{\mathrm{TIAL}}(x;\alpha)\, dx. \tag{3.4}$$

Substituting Eq. (2.1) in Eq. (3.4), we have

$$\mu'_r = A\alpha \int_0^1 \left[x^r e^{-2\alpha x} + 2\alpha\, x^{r+1} e^{-2\alpha x}\right] dx. \tag{3.5}$$

Letting $z = 2\alpha x$ and $dx = \frac{1}{2\alpha}\, dz$, and simplifying, we obtain

$$\mu'_r = \frac{A\alpha}{(2\alpha)^{r+1}} \left[\gamma(r+1, 2\alpha) + \gamma(r+2, 2\alpha)\right], \qquad r > -1, \tag{3.6}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function.

In particular, the mean and variance of the TIAL distribution follow from (3.6) as

$$E(X) = \frac{A}{4\alpha} \left[\gamma(2, 2\alpha) + \gamma(3, 2\alpha)\right], \tag{3.7}$$

and

$$\mathrm{Var}(X) = \frac{A}{8\alpha^2} \left[\gamma(3, 2\alpha) + \gamma(4, 2\alpha)\right] - \frac{A^2}{16\alpha^2} \left[\gamma(2, 2\alpha) + \gamma(3, 2\alpha)\right]^2. \tag{3.8}$$

The coefficients of skewness and kurtosis are defined using central moments as

$$\rho_3 = \frac{\mu_3}{\mu_2^{3/2}}, \qquad \rho_4 = \frac{\mu_4}{\mu_2^2}, \tag{3.9}$$

where

$$\mu_r = E\left[(X - E(X))^r\right], \qquad r = 2, 3, 4. \tag{3.10}$$

Subsequently, we calculate the coefficients of skewness (ρ_3) and kurtosis (ρ_4) in (3.9). Table 2 provides exact numerical values of the variance (σ^2), ρ_3, ρ_4 and the first four moments μ'_1, μ'_2, μ'_3 and μ'_4.

The numerical values of the first four raw moments and some descriptive characteristics of the TIAL model are reported in Table 2. As the shape parameter α increases, the first four raw moments and the variance decrease

progressively, indicating that the probability mass becomes increasingly concentrated near the origin. The index of dispersion satisfies ID < 1 for all considered values of α, so the TIAL distribution exhibits under-dispersion over this range. It should be noted, however, that under-dispersion alone is not a sufficient criterion for suitability in lifetime or reliability modelling; the adequacy of the model should be assessed through goodness-of-fit procedures and application-specific considerations. The coefficients of skewness and kurtosis in Table 2 are computed from central moments, and their positive values indicate right skewness and a leptokurtic shape. Because the support of the distribution is bounded, these measures describe the concentration and asymmetry of the probability mass within $(0, 1)$; they should not be interpreted as evidence of heavy tails.

Figure 5 presents three-dimensional plots of several statistical measures of the TIAL model for different values of α. The plots show that the moments and the coefficient of variation (CV) are sensitive to α. In particular, the CV increases monotonically with α, whereas the index of dispersion (ID) first increases and then decreases after reaching a maximum around $\alpha = 1.5$. This non-monotonic behavior of ID, together with the decline in the raw moments and variance, reflects the changing concentration of probability mass near the origin as α varies.

Table 2. Numerical results of the moments with some measures of the TIAL model.

α	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	ρ_3	ρ_4	CV	ID
0.2	0.4946	0.3277	0.2447	0.1952	0.0830	0.0225	1.8046	0.5825	0.1678
0.8	0.4459	0.2780	0.1997	0.1550	0.0792	0.2278	1.9025	0.6311	0.1776
1.2	0.4040	0.2373	0.1640	0.1239	0.0741	0.4071	2.0831	0.6738	0.1834
1.5	0.3721	0.2075	0.1385	0.1023	0.0691	0.5473	2.2905	0.7064	0.1857
2.0	0.3217	0.1628	0.1017	0.0717	0.0593	0.7771	2.7675	0.7569	0.1843
2.5	0.2776	0.1262	0.0731	0.0488	0.0491	0.9887	3.3752	0.7986	0.1770
3.2	0.2278	0.0884	0.0454	0.0277	0.0366	1.2336	4.3246	0.8394	0.1605
4.0	0.1859	0.0604	0.0268	0.0145	0.0258	1.4258	5.3210	0.8647	0.1390
4.5	0.1660	0.0486	0.0196	0.0098	0.0210	1.5018	5.8064	0.8727	0.1264
5.0	0.1497	0.0397	0.0146	0.0068	0.0173	1.5513	6.1670	0.8772	0.1152
5.7	0.1315	0.0307	0.0100	0.0042	0.0134	1.5899	6.4884	0.8801	0.1019
6.5	0.1154	0.0236	0.0068	0.0025	0.0103	1.6092	6.6727	0.8814	0.0896

3.3. Moment generating function

A fundamental tool in statistical analysis and probability theory is the moment generating function (MGF). Because of the convolution property, it makes it simpler to analyze sums of random variables and to determine the moments of a random variable. Moreover, the uniqueness property of the MGF implies that the probability distributions of two random variables are the same if they have the same MGF. The MGF of the TIAL distribution can be determined from the PDF given in Eq. (2.1) as follows:

$$M_X(t) = E\left(e^{tX}\right) = \int_0^1 e^{tx}\, f_{\mathrm{TIAL}}(x;\alpha)\, dx. \tag{3.11}$$

Next, using the exponential expansion, we get

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 x^r\, f_{\mathrm{TIAL}}(x;\alpha)\, dx = \sum_{r=0}^{\infty} \frac{t^r}{r!}\, \mu'_r. \tag{3.12}$$

Utilizing Eqs. (3.6) and (3.12), the MGF of the TIAL model is expressed as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{A\,\alpha\, t^r}{r!\,(2\alpha)^{r+1}} \left[\gamma(r+1, 2\alpha) + \gamma(r+2, 2\alpha)\right]. \tag{3.13}$$

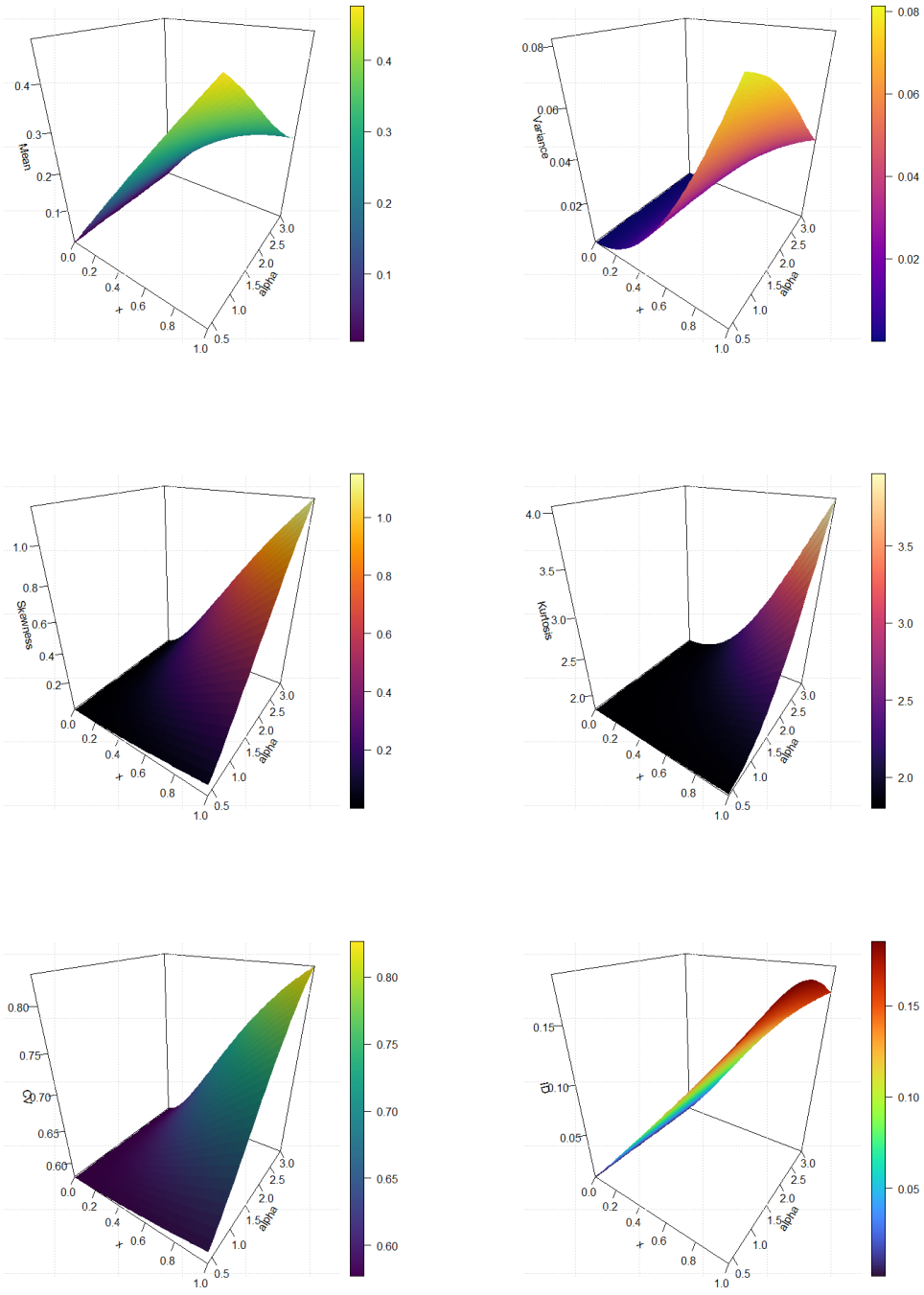


Figure 5. 3D graphs of some statistical measures of the TIAL model.

3.4. *Inverse moments*

Inverse moments are valuable tools in statistical analysis, particularly in reliability theory and survival analysis, where reciprocal measures of lifetime variables are of interest. These moments provide further insight into the lower-tail behavior of the TIAL distribution. Since $f(0^+) = A\alpha > 0$, we have $x^{-r} f(x;\alpha) \sim A\alpha\, x^{-r}$ as $x \to 0^+$. Hence, for any real $r < 1$ (in particular, for positive inverse moments, $0 < r < 1$), the rth inverse moment of X can be determined as follows:

$$\mu'_{-r} = \int_0^1 x^{-r} f(x;\alpha)\, dx = \frac{A\,\alpha}{(2\alpha)^{-r+1}} \left[\gamma(-r+1, 2\alpha) + \gamma(-r+2, 2\alpha)\right], \qquad r < 1. \tag{3.14}$$

3.5. *Incomplete moments*

The Lorenz curve, income quintiles, and the Pietra and Gini measures of inequality are just a few of the statistical applications of incomplete moments in the income distribution. The rth incomplete moment of the TIAL distribution is defined as

$$f_s(t) = \int_0^t x^r f(x)\, dx = A\alpha \left[\frac{\gamma(r+1, 2t\alpha)}{(2\alpha)^{r+1}} + \frac{\gamma(r+2, 2t\alpha)}{(2\alpha)^{r+1}}\right]. \tag{3.15}$$

Inequality measures are often computed for distributions other than expenditures. The Lorenz and Bonferroni curves are computed below:

$$L_F(z) = \frac{\int_0^z x f(x)\, dx}{E(X)} = \frac{A\alpha}{E(X)} \left[\frac{\gamma(2, 2\alpha z)}{(2\alpha)^2} + \frac{\gamma(3, 2\alpha z)}{(2\alpha)^2}\right]. \tag{3.16}$$

Therefore, using (3.7), the Lorenz curve is given as

$$L_F(z) = \frac{\gamma(2, 2\alpha z) + \gamma(3, 2\alpha z)}{\gamma(2, 2\alpha) + \gamma(3, 2\alpha)}, \tag{3.17}$$

and the Bonferroni curve is defined as

$$B_F(z) = \frac{L_F(z)}{F(z)} = \frac{\gamma(2, 2\alpha z) + \gamma(3, 2\alpha z)}{A\left[\gamma(2, 2\alpha) + \gamma(3, 2\alpha)\right]\left[1 - (1 + \alpha z)e^{-2\alpha z}\right]}. \tag{3.18}$$

3.6. *Order statistics*

Order statistics moments are beneficial for performing quality assessments and forecasting future item failures according to a restricted set of initial failures. Given independent, identically distributed random variables $X_1, X_2, \ldots, X_n$ taken from the TIAL distribution, the density of the jth order statistic is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!}\, f_X(x)\, [F_X(x)]^{j-1}\, [1 - F_X(x)]^{n-j}, \qquad j = 1, \ldots, n. \tag{3.19}$$

For the TIAL distribution, substituting (2.1) and (2.2) in (3.19), the PDF of the jth order statistic is

$$\begin{aligned} f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} &\times A\alpha\,(1 + 2\alpha x)\, e^{-2\alpha x} \left\{A\left[1 - (1 + \alpha x)e^{-2\alpha x}\right]\right\}^{j-1} \\ &\times \left\{1 - A\left[1 - (1 + \alpha x)e^{-2\alpha x}\right]\right\}^{n-j}. \end{aligned} \tag{3.20}$$

As a result, the PDF of the largest order statistic $X_{(n)}$ is

$$f_{X_{(n)}}(x) = n\, A\alpha\,(1 + 2\alpha x)\, e^{-2\alpha x} \times \left\{A\left[1 - (1 + \alpha x)e^{-2\alpha x}\right]\right\}^{n-1}, \tag{3.21}$$

and the PDF of the smallest order statistic $X_{(1)}$ is

$$f_{X_{(1)}}(x) = n\, A\alpha\,(1 + 2\alpha x)\, e^{-2\alpha x} \times \left\{1 - A\left[1 - (1 + \alpha x)e^{-2\alpha x}\right]\right\}^{n-1}. \tag{3.22}$$

3.7. *Probability weighted moments*

For parameter estimation in probability models whose inverse forms are difficult to obtain, the probability weighted moment (PWM) method is frequently used. Originally introduced in [31], this approach has become particularly popular in hydrological applications due to its effectiveness in estimation procedures. The PWM of the TIAL model can be derived as below:

$$\varphi_{\mathrm{PWM}} = E\left[X^r F^m(X;\alpha)\right] = \int_0^1 x^r\, F^m(x;\alpha)\, f(x;\alpha)\, dx. \tag{3.23}$$

By substituting Eqs. (2.1) and (2.2) in Eq. (3.23), we get

$$\varphi_{\mathrm{PWM}} = \alpha A^{m+1} \int_0^1 x^r \left[1-(1+\alpha x)e^{-2\alpha x}\right]^m (1+2\alpha x)\, e^{-2\alpha x}\, dx. \tag{3.24}$$

The generalized binomial series is defined as

$$(a+b)^{\varepsilon} = \sum_{i=0}^{\infty} \binom{\varepsilon}{i} a^{\varepsilon-i} b^i. \tag{3.25}$$

Using the series (3.25), we can write

$$\begin{aligned} \left[1-(1+\alpha x)e^{-2\alpha x}\right]^m &= \sum_{k=0}^{\infty} \binom{m}{k} (-1)^k (1+\alpha x)^k e^{-2k\alpha x} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{m}{k}\binom{k}{l} (-1)^k \alpha^l x^l e^{-2k\alpha x}. \end{aligned} \tag{3.26}$$

By substituting (3.26) in Eq. (3.24), then

$$\varphi_{\mathrm{PWM}} = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \Omega(\alpha) \int_0^1 \left(x^{r+l} + 2\alpha\, x^{r+l+1}\right) e^{-2(k+1)\alpha x}\, dx, \tag{3.27}$$

where

$$\Omega(\alpha) = \binom{m}{k}\binom{k}{l} (-1)^k \alpha^{l+1} A^{m+1}. \tag{3.28}$$

The PWM is obtained as follows using the incomplete gamma function:

$$\varphi_{\mathrm{PWM}} = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{\Omega(\alpha)}{[2(k+1)\alpha]^{r+l+1}} \left[\Gamma\left(r+l+1, 2(k+1)\alpha\right) + \frac{1}{k+1}\Gamma\left(r+l+2, 2(k+1)\alpha\right)\right]. \tag{3.29}$$

3.8. *Rényi entropy*

The variation of the uncertainty of the distribution is quantified by the Rényi entropy proposed by Rényi [32]. The Rényi entropy is given by

$$\begin{aligned} \mathrm{RE}(\delta) &= \frac{1}{1-\delta} \log\left[\int_0^1 f^{\delta}(x)\, dx\right] \\ &= \frac{1}{1-\delta} \log\left[\int_0^1 A^{\delta}\alpha^{\delta}(1+2\alpha x)^{\delta} e^{-2\alpha\delta x}\, dx\right]. \end{aligned} \tag{3.30}$$

For any real numbers $a, b > 0$ and $|\varepsilon| < 1$, the generalized binomial series is defined in (3.25). Using this series,

$$(1+2\alpha x)^{\delta} = \sum_{k=0}^{\infty} \binom{\delta}{k} (2\alpha)^k x^k. \tag{3.31}$$

Therefore,

$$I_{\delta}(x) = \frac{1}{1-\delta} \log \left[\sum_{k=0}^{\infty} \binom{\delta}{k} A^{\delta} \alpha^{\delta+k} 2^k \int_0^1 x^k e^{-2\alpha\delta x} \, dx \right]. \tag{3.32}$$

Letting $z = 2\alpha\delta x$ and $dx = \frac{1}{2\alpha\delta} \, dz$, and simplifying, the Rényi entropy is developed as

$$I_{\delta}(x) = \frac{1}{1-\delta} \log \left[\sum_{k=0}^{\infty} \binom{\delta}{k} A^{\delta} \alpha^{\delta+k} \frac{1}{2(\alpha\delta)^{k+1}} \Gamma\left(k+1, \frac{1}{2\alpha\delta}\right) \right]. \tag{3.33}$$

3.9. Tsallis entropy

Tsallis entropy, which is defined as follows, was introduced by Tsallis [33] in order to generalize standard statistical mechanics:

$$\mathrm{TE}(\delta) = \frac{1}{1-\delta} \log \left[1 - \int_0^1 f^{\delta}(x) \, dx \right]. \tag{3.34}$$

The Tsallis entropy is obtained as

$$\mathrm{TE}(\delta) = \frac{1}{1-\delta} \log \left[\sum_{k=0}^{\infty} \binom{\delta}{k} A^{\delta} \alpha^{\delta+k} \frac{1}{2(\alpha\delta)^{k+1}} \Gamma\left(k+1, \frac{1}{2\alpha\delta}\right) \right]. \tag{3.35}$$

Some numerical values of the Rényi (3.30) and Tsallis (3.34) entropies for various parameter values are shown in Table 3. It is observed that all entropy values are negative and decrease monotonically with increases in both α and δ, confirming the consistency of the numerical integration with the probabilistic behavior of the TIAL distribution.

4. Methods of estimation

4.1. Maximum likelihood estimation

Let $x_1, \ldots, x_n$ be a random sample drawn from the TIAL model. The following likelihood function is maximized in order to estimate the parameter α using the maximum likelihood (MLE) approach:

$$L(\alpha \,|\, \mathbf{x}) = \prod_{i=1}^{n} \frac{\alpha}{1-(1+\alpha)e^{-2\alpha}} (1+2\alpha x_i) \, e^{-2\alpha x_i} = \frac{\alpha^n}{[1-(1+\alpha)e^{-2\alpha}]^n} \prod_{i=1}^{n} (1+2\alpha x_i) \, e^{-2\alpha x_i}. \tag{4.1}$$

Thus, the log-likelihood function ℓ can be written as

$$\ell = \log L = n \ln \alpha - n \ln \left[1-(1+\alpha)e^{-2\alpha}\right] + \sum_{i=1}^{n} \ln(1+2\alpha x_i) - 2\alpha \sum_{i=1}^{n} x_i. \tag{4.2}$$

The MLE of the parameter is derived by maximizing (4.2). Calculating the first derivative of the function and equating the result to zero, we obtain

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + n \frac{e^{-2\alpha} + 2(1+\alpha)e^{-2\alpha}}{1-(1+\alpha)e^{-2\alpha}} + \sum_{i=1}^{n} \frac{2x_i}{1+2\alpha x_i} - 2 \sum_{i=1}^{n} x_i. \tag{4.3}$$

Table 3. Several Rényi and Tsallis entropy values of the TIAL model.

δ	α	RE(δ)	TE(δ)
0.5	0.2	−0.00009	−0.00007
	0.5	−0.00213	−0.00211
	0.8	−0.00925	−0.00923
	1.2	−0.02975	−0.02953
	1.5	−0.05412	−0.05339
	1.8	−0.08591	−0.08409
	2.4	−0.16832	−0.16143
	2.8	−0.23357	−0.22045
	3.0	−0.26826	−0.25104
0.7	0.2	−0.00013	−0.00013
	0.5	−0.00297	−0.00297
	0.8	−0.01285	−0.01283
	1.2	−0.04106	−0.04081
	1.5	−0.07420	−0.07338
	1.8	−0.11686	−0.11484
	2.4	−0.22468	−0.21727
	2.8	−0.30742	−0.29367
	3.0	−0.35051	−0.33271
1.5	0.2	−0.00027	−0.00027
	0.5	−0.00628	−0.00629
	0.8	−0.02670	−0.02688
	1.2	−0.08256	−0.08429
	1.5	−0.14464	−0.15000
	1.8	−0.21995	−0.23250
	2.4	−0.39208	−0.43315
	2.8	−0.51053	−0.58161
	3.0	−0.56855	−0.65759

By equating Eq. (4.3) to zero, the estimate $\hat{\alpha}$ of the parameter α can be obtained.

The asymptotic variance of $\hat{\alpha}$ is obtained by inverting the observed information $I(\hat{\alpha}) = -\partial^2\ell/\partial\alpha^2$, evaluated at $\hat{\alpha}$. Under standard asymptotic regularity conditions, $\hat{\alpha} \sim N\left(\alpha, I^{-1}(\hat{\alpha})\right)$. Consequently, an approximate $(1-\gamma)100\%$ asymptotic confidence interval for α is given by

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{I^{-1}(\hat{\alpha})}. \tag{4.4}$$

4.2. Maximum product of spacings estimation

The maximum product of spacings (MPS) method is used to provide the estimator $\hat{\alpha}$ of the proposed model; see Cheng and Amin [34] for additional details regarding this strategy. The MPS estimator maximizes

$$H = \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i, \qquad D_i = F\left(x_{i:n}\,|\,\alpha\right) - F\left(x_{i-1:n}\,|\,\alpha\right). \tag{4.5}$$

4.3. Least squares and weighted least squares methods

Swain et al. [35] proposed the least squares and weighted least squares estimators for estimating the parameters of Beta distributions. The parameter of the TIAL distribution is estimated in the current study using the same

methodology. The LSE of the parameter α of the TIAL model is found by minimizing

$$\sum_{j=1}^{n}\left[F\left(x_{(j)}\right)-\frac{j}{n+1}\right]^2 \tag{4.6}$$

with respect to α. Therefore, using (2.2), the LSE of α is determined by minimizing

$$\sum_{j=1}^{n}\left\{A\left[1-(1+\alpha x_{(j)})e^{-2\alpha x_{(j)}}\right]-\frac{j}{n+1}\right\}^2. \tag{4.7}$$

The WLSE of the parameter α is obtained by minimizing

$$\sum_{j=1}^{n} w_j\left[F\left(x_{(j)}\right)-\frac{j}{n+1}\right]^2, \qquad w_j=\frac{(n+1)^2(n+2)}{j(n-j+1)}. \tag{4.8}$$

Hence, the WLSE of α is found by minimizing

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left\{A\left[1-(1+\alpha x_{(j)})e^{-2\alpha x_{(j)}}\right]-\frac{j}{n+1}\right\}^2. \tag{4.9}$$

4.4. Cramér–von Mises method

The Cramér–von Mises technique is based on the difference between the estimated CDF and the empirical distribution function. Macdonald [36] showed that the bias of these estimators is smaller than that of the other minimum distance estimators. The CME $\hat{\alpha}_{\mathrm{CME}}$ of α is obtained by minimizing

$$C(\alpha)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\,|\,\alpha\right)-\frac{2i-1}{2n}\right]^2. \tag{4.10}$$

These estimates can also be calculated by solving the non-linear equation

$$\sum_{i=1}^{n}\left[F\left(x_{i:n}\,|\,\alpha\right)-\frac{2i-1}{2n}\right]\Delta_1\left(x_{i:n}\,|\,\alpha\right)=0. \tag{4.11}$$

4.5. Anderson–Darling method

Another type of minimum distance estimator is the Anderson–Darling estimator (ADE), first presented by Anderson and Darling [37]. The ADE $\hat{\alpha}_{\mathrm{ADE}}$ of the parameter α is obtained by minimizing

$$A(\alpha)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{i:n}\,|\,\alpha\right)+\log\overline{F}\left(x_{n+1-i:n}\,|\,\alpha\right)\right]. \tag{4.12}$$

Equivalently, these estimates can be obtained by solving

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1\left(x_{i:n}\,|\,\alpha\right)}{F\left(x_{i:n}\,|\,\alpha\right)}-\frac{\Delta_1\left(x_{n+1-i:n}\,|\,\alpha\right)}{\overline{F}\left(x_{n+1-i:n}\,|\,\alpha\right)}\right]=0, \tag{4.13}$$

where $\Delta_1(\cdot\,|\,\alpha)$ is the first derivative of the CDF of the TIAL model with respect to α, and $\overline{F}=1-F$.

4.6. *Right-tail Anderson–Darling method*

The RADE $\hat{\alpha}_{\mathrm{RADE}}$ is obtained by minimizing

$$\mathrm{RADE}(\alpha) = \frac{n}{2} - 2\sum_{i=1}^{n} F\left(x_{i:n} \mid \alpha\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \overline{F}\left(x_{n+1-i:n} \mid \alpha\right). \tag{4.14}$$

These estimates can also be obtained by solving

$$-2\sum_{i=1}^{n} \Delta_1\left(x_{i:n} \mid \alpha\right) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_1\left(x_{n+1-i:n} \mid \alpha\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha\right)} = 0. \tag{4.15}$$

4.7. *Left-tail Anderson–Darling method*

The LADE $\hat{\alpha}_{\mathrm{LADE}}$ is obtained by minimizing

$$\mathrm{LADE}(\alpha) = -\frac{3n}{2} + 2\sum_{i=1}^{n} F\left(x_{i:n} \mid \alpha\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F\left(x_{i:n} \mid \alpha\right). \tag{4.16}$$

By solving the following equation, we get the estimate of the unknown parameter:

$$2\sum_{i=1}^{n} \Delta_1\left(x_{i:n} \mid \alpha\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_1\left(x_{n+1-i:n} \mid \alpha\right)}{\overline{F}\left(x_{n+1-i:n} \mid \alpha\right)} = 0. \tag{4.17}$$

4.8. *Parametric bootstrap bias reduction*

To mitigate finite-sample upward bias in small sample regimes ($n \leq 50$), we adopt a parametric bootstrap bias-correction algorithm. Given a sample $\mathbf{x} = (x_1, \ldots, x_n)$ and a point estimate $\hat{\alpha}$:

I. Draw $B = 1000$ independent pseudo-random samples $\mathbf{x}^*_{(1)}, \ldots, \mathbf{x}^*_{(B)}$ of size n from $\mathrm{TIAL}(\hat{\alpha})$.
II. Compute the bootstrap estimates $\hat{\alpha}^*_b$ for each sample, $b = 1, \ldots, B$.
III. Construct the bias-corrected estimator

$$\hat{\alpha}_{\mathrm{BC}} = 2\hat{\alpha} - \frac{1}{B}\sum_{b=1}^{B}\hat{\alpha}^*_b. \tag{4.18}$$

5. Simulation study

We now assess the performance of the estimation methods described in Section 4. We generated simulated datasets from the TIAL model and estimated the unknown parameter with the estimation techniques under consideration. The performance was analyzed with two different metrics: the absolute relative bias and the mean squared error (MSE). Random samples are generated from the TIAL distribution by the inversion technique, i.e., by solving Eq. (3.1) for uniform variates u.

Random samples $X_1, X_2, \ldots, X_n$ of sizes $n = 50$, 100, 150, 200, 250 and 300 are created from the TIAL distribution. The parameter values $\alpha = 0.2$, 0.5, 0.9, 1.4, 1.9, 2.5, 3 and 3.5 are analyzed. The TIAL model estimates are evaluated based on these parameter values and sample sizes, and the biases and MSEs of the estimators are calculated. All computations were performed in R (4.6.0) under the Mersenne-Twister algorithm. Numerical optimization for all eight loss metrics was executed utilizing the bounded L-BFGS-B algorithm with lower bound $\alpha \geq 10^{-5}$, initial guess $\alpha_0 = 1.0$, and convergence control tolerances $factr = 10^7$ and $pgtol = 10^{-8}$. Iterations failing optimization convergence ($< 0.05\%$ of runs) were omitted and re-sampled until 10,000 valid estimates were obtained per scenario. Tables 4–12 report the empirical findings. These results allow us to make the following observations:

- The TIAL distribution has stability, as indicated by the small biases and MSEs of its parameter estimates.
- Increasing the sample size results in diminished bias and MSE across all methodologies.
- As the sample size increases, the MSE values of all estimators diminish, signifying improved accuracy in the estimation process.
- Each sample size demonstrates a positive bias in the estimators.

The simulation study is carried out to find the suitable estimation procedure for the proposed model. Table 12 displays the partial and overall rankings of every estimation strategy. Table 12 shows that all parameter estimation methods for the proposed model exhibit high accuracy and almost match the true values. As the sample size n increases, the calculated measures generally decrease. It is shown that the RADE is the most efficient parameter estimation method, achieving a total score of 19. Furthermore, the ADE can be considered a viable substitute for the RADE, with a cumulative score of 21.

Table 4. Simulation outcomes of the TIAL model at $\alpha = 0.2$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	0.3102	0.2885	0.3078	0.3059	0.3113	0.3061	0.2994	0.3304
	MSE	0.1331^{3}	0.1229^{1}	0.1368^{6}	0.1339^{5}	0.1385^{7}	0.1337^{4}	0.1279^{2}	0.1598^{8}
	Bias	0.5508^{6}	0.4427^{1}	0.5388^{5}	0.5294^{3}	0.5566^{7}	0.5308^{4}	0.4972^{2}	0.6519^{8}
100	Mean	0.2647	0.2522	0.2642	0.2636	0.2663	0.2635	0.2585	0.2791
	MSE	0.0834^{3}	0.0797^{1}	0.0856^{6}	0.0845^{5}	0.0862^{7}	0.0843^{4}	0.0816^{2}	0.0974^{8}
	Bias	0.3237^{6}	0.2608^{1}	0.3210^{5}	0.3181^{4}	0.3313^{7}	0.3174^{3}	0.2923^{2}	0.3954^{8}
150	Mean	0.2452	0.2361	0.2458	0.2448	0.2474	0.2449	0.2403	0.2568
	MSE	0.0644^{3}	0.0623^{1}	0.0661^{6}	0.0652^{5}	0.0665^{7}	0.0651^{4}	0.0632^{2}	0.0746^{8}
	Bias	0.2259^{5}	0.1805^{1}	0.2288^{6}	0.2239^{3}	0.2368^{7}	0.2245^{4}	0.2017^{2}	0.2840^{8}
200	Mean	0.2355	0.2282	0.2362	0.2358	0.2374	0.2358	0.2315	0.2469
	MSE	0.0542^{3}	0.0528^{1}	0.0557^{6}	0.0549^{5}	0.0559^{7}	0.0548^{4}	0.0533^{2}	0.0621^{8}
	Bias	0.1773^{3}	0.1408^{1}	0.1809^{6}	0.1792^{5}	0.1872^{7}	0.1791^{4}	0.1577^{2}	0.2344^{8}
250	Mean	0.2246	0.2182	0.2246	0.2242	0.2256	0.2242	0.2210	0.2328
	MSE	0.0473^{3}	0.0464^{1}	0.0487^{6}	0.0480^{5}	0.0489^{7}	0.0479^{4}	0.0469^{2}	0.0539^{8}
	Bias	0.1229^{5}	0.0910^{1}	0.1230^{6}	0.1212^{4}	0.1282^{7}	0.1209^{3}	0.1050^{2}	0.1639^{8}
300	Mean	0.2192	0.2136	0.2189	0.2190	0.2198	0.2188	0.2166	0.2267
	MSE	0.0423^{3}	0.0417^{1}	0.0431^{6}	0.0428^{5}	0.0432^{7}	0.0427^{4}	0.0421^{2}	0.0473^{8}
	Bias	0.0961^{6}	0.0682^{1}	0.0945^{4}	0.0950^{5}	0.0991^{7}	0.0939^{3}	0.0832^{2}	0.1337^{8}
$\sum$Ranks		49^{4}	12^{1}	68^{6}	54^{5}	84^{7}	45^{3}	24^{2}	96^{8}

6. Real data applications

We empirically investigate the TIAL model using two separate real-world datasets. Compelling evidence of the model's usefulness can be found in data graphs showing the increased capability of the TIAL model in fitting the data against other models. The datasets used are described as follows.

Dataset I. The first dataset represents the survival durations (in months) of 121 breast cancer patients collected from a major hospital between 1929 and 1938 (see [38]). This dataset was recently analyzed by [39], and is summarized as follows: 0.3, 0.3, 4, 5, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21, 21, 21.1, 23, 23.4, 23.6, 24, 24, 27.9, 28.2, 29.1, 30, 31, 31, 32, 35, 35, 37, 37, 37, 38, 38, 38, 39, 39, 40, 40, 40, 41, 41, 41, 42, 43, 43, 43, 44, 45, 45, 46, 46, 47, 48, 49, 51, 51, 51, 52, 54, 55, 56, 57, 58, 59, 60, 60, 60, 61, 62, 65, 65, 67, 67, 68, 69, 78, 80, 83, 88, 89, 90, 93, 96, 103, 105, 109, 109, 111, 115, 117, 125, 126, 127, 129, 129, 139, 154. We standardize the data by dividing each observation by 155, ensuring compatibility

Table 5. Simulation outcomes of the TIAL model at $\alpha = 0.5$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	0.4945	0.4678	0.4894	0.4881	0.4939	0.4897	0.4805	0.5097
	MSE	0.1541^{3}	0.1512^{1}	0.1602^{6}	0.1571^{5}	0.1606^{7}	0.1561^{4}	0.1536^{2}	0.1791^{8}
	Bias	0.0110^{1}	0.0644^{8}	0.0212^{5}	0.0238^{6}	0.0122^{2}	0.0205^{4}	0.0391^{7}	0.0193^{3}
100	Mean	0.4777	0.4626	0.4752	0.4753	0.4777	0.4755	0.4695	0.4844
	MSE	0.1002^{1}	0.1005^{2}	0.1033^{7}	0.1019^{5}	0.1032^{6}	0.1014^{4}	0.1008^{3}	0.1146^{8}
	Bias	0.0447^{3}	0.0749^{8}	0.0496^{6}	0.0494^{5}	0.0446^{2}	0.0489^{4}	0.0611^{7}	0.0311^{1}
150	Mean	0.4786	0.4679	0.4761	0.4764	0.4779	0.4767	0.4726	0.4827
	MSE	0.0749^{1}	0.0756^{3}	0.0783^{7}	0.0768^{5}	0.0781^{6}	0.0765^{4}	0.0755^{2}	0.0862^{8}
	Bias	0.0429^{2}	0.0641^{8}	0.0477^{6}	0.0471^{5}	0.0441^{3}	0.0466^{4}	0.0549^{7}	0.0347^{1}
200	Mean	0.4825	0.4742	0.4810	0.4812	0.4824	0.4812	0.4777	0.4858
	MSE	0.0599^{1}	0.0605^{3}	0.0624^{7}	0.0614^{5}	0.0623^{6}	0.0613^{4}	0.0603^{2}	0.0693^{8}
	Bias	0.0351^{2}	0.0517^{8}	0.0381^{6}	0.0375^{4}	0.0353^{3}	0.0376^{5}	0.0446^{7}	0.0283^{1}
250	Mean	0.4836	0.4769	0.4814	0.4820	0.4825	0.4819	0.4796	0.4843
	MSE	0.0493^{1}	0.0498^{2}	0.0519^{7}	0.0508^{5}	0.0518^{6}	0.0507^{4}	0.0499^{3}	0.0577^{8}
	Bias	0.0328^{2}	0.0463^{8}	0.0372^{6}	0.0361^{4}	0.0349^{3}	0.0362^{5}	0.0408^{7}	0.0315^{1}
300	Mean	0.4846	0.4789	0.4829	0.4836	0.4839	0.4836	0.4812	0.4850
	MSE	0.0424^{1}	0.0430^{2}	0.0439^{7}	0.0435^{5}	0.0438^{6}	0.0431^{3}	0.0432^{4}	0.0490^{8}
	Bias	0.0308^{2}	0.0422^{8}	0.0341^{6}	0.0327^{4}	0.0322^{3}	0.0329^{5}	0.0377^{7}	0.0299^{1}
$\sum$Ranks		20^{1}	61^{7}	76^{8}	58^{5}	53^{3}	50^{2}	58^{5}	58^{5}

Table 6. Simulation outcomes of the TIAL model at $\alpha = 0.9$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	0.8814	0.8512	0.8751	0.8744	0.8801	0.8754	0.8655	0.8914
	MSE	0.1553^{1}	0.1558^{2}	0.1646^{7}	0.1602^{5}	0.1644^{6}	0.1595^{4}	0.1580^{3}	0.1873^{8}
	Bias	0.0207^{2}	0.0542^{8}	0.0277^{5}	0.0284^{6}	0.0221^{3}	0.0273^{4}	0.0383^{7}	0.0096^{1}
100	Mean	0.8902	0.8749	0.8876	0.8876	0.8902	0.8877	0.8818	0.8953
	MSE	0.0782^{1}	0.0786^{2}	0.0821^{7}	0.0805^{5}	0.0820^{6}	0.0800^{4}	0.0796^{3}	0.0946^{8}
	Bias	$0.0109^{2.5}$	0.0279^{8}	0.0137^{5}	0.0138^{6}	$0.0109^{2.5}$	0.0136^{4}	0.0202^{7}	0.0053^{1}
150	Mean	0.8950	0.8850	0.8935	0.8936	0.8952	0.8937	0.8895	0.8989
	MSE	0.0514^{1}	0.0516^{2}	0.0544^{7}	0.0529^{5}	0.0543^{6}	0.0527^{4}	0.0521^{3}	0.0626^{8}
	Bias	0.0055^{3}	0.0167^{8}	$0.0072^{5.5}$	$0.0072^{5.5}$	0.0053^{2}	0.0070^{4}	0.0116^{7}	0.0012^{1}
200	Mean	0.8995	0.8920	0.8984	0.8986	0.8996	0.8986	0.8955	0.9025
	MSE	0.0375^{1}	0.0376^{2}	0.0402^{6}	0.0391^{5}	0.0408^{7}	0.0389^{4}	0.0377^{3}	0.0470^{8}
	Bias	0.0005^{2}	0.0089^{8}	0.0018^{5}	$0.0016^{3.5}$	0.0004^{1}	$0.0016^{3.5}$	0.0050^{7}	0.0028^{6}
250	Mean	0.8997	0.8937	0.8987	0.8989	0.8997	0.8988	0.8966	0.9017
	MSE	0.0295^{2}	0.0293^{1}	0.0311^{6}	0.0303^{5}	0.0314^{7}	0.0302^{4}	0.0298^{3}	0.0362^{8}
	Bias	$0.0003^{1.5}$	0.0070^{8}	0.0014^{5}	$0.0013^{3.5}$	$0.0003^{1.5}$	$0.0013^{3.5}$	0.0038^{7}	0.0019^{6}
300	Mean	0.9004	0.8955	0.8994	0.8996	0.9002	0.8996	0.8978	0.9018
	MSE	0.0241^{2}	0.0240^{1}	$0.0252^{6.5}$	$0.0247^{4.5}$	$0.0252^{6.5}$	$0.0247^{4.5}$	0.0243^{3}	0.0293^{8}
	Bias	0.0005^{3}	0.0050^{8}	0.0007^{5}	0.0005^{3}	0.0002^{1}	0.0005^{3}	0.0025^{7}	0.0020^{6}
$\sum$Ranks		22^{1}	58^{5}	70^{8}	57^{4}	49.5^{3}	46.5^{2}	60^{6}	69^{7}

with probability models constrained to the interval $(0, 1)$, which is a common preprocessing method employed by other researchers. The modified dataset is as follows: 0.00194, 0.00194, 0.02581, 0.03226, 0.03613, 0.04, 0.04065, 0.04258, 0.04387, 0.04774, 0.04839, 0.05419, 0.05419, 0.06645, 0.07097, 0.07613, 0.07871, 0.07935, 0.0871, 0.0929, 0.0929, 0.09548, 0.1, 0.10129, 0.10452, 0.10516, 0.10645, 0.10839, 0.11097, 0.11161, 0.1129, 0.11548, 0.12774, 0.13161, 0.13484, 0.13548, 0.13548, 0.13613, 0.14839, 0.15097, 0.15226, 0.15484, 0.15484, 0.18, 0.18194, 0.18774, 0.19355, 0.2, 0.2, 0.20645, 0.22581, 0.22581, 0.23871, 0.23871, 0.23871, 0.24516, 0.24516, 0.24516, 0.25161, 0.25161, 0.25806, 0.25806, 0.25806, 0.26452, 0.26452, 0.26452, 0.27097, 0.27742,

Table 7. Simulation outcomes of the TIAL model at $\alpha = 1.4$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	1.4034	1.3682	1.3969	1.3957	1.4026	1.3968	1.3879	1.4144
	MSE	0.1282^{2}	0.1253^{1}	0.1409^{6}	0.1351^{5}	0.1413^{7}	0.1336^{4}	0.1290^{3}	0.1687^{8}
	Bias	0.0025^{4}	0.0227^{5}	0.0022^{2}	0.0030^{6}	0.0019^{1}	0.0023^{3}	0.0087^{7}	0.0103^{8}
100	Mean	1.4032	1.3853	1.4008	1.4006	1.4036	1.4006	1.3953	1.4101
	MSE	0.0623^{2}	0.0614^{1}	0.0675^{6}	0.0650^{5}	0.0676^{7}	0.0646^{4}	0.0626^{3}	0.0801^{8}
	Bias	0.0023^{4}	0.0105^{8}	0.0005^{3}	$0.0004^{1.5}$	0.0026^{5}	$0.0004^{1.5}$	0.0033^{6}	0.0072^{7}
150	Mean	1.4032	1.3912	1.4019	1.4018	1.4038	1.4018	1.3981	1.4085
	MSE	0.0416^{2}	0.0412^{1}	$0.0455^{6.5}$	0.0436^{5}	$0.0455^{6.5}$	0.0434^{4}	0.0417^{3}	0.0537^{8}
	Bias	0.0023^{5}	0.0063^{8}	$0.0014^{3.5}$	$0.0013^{1.5}$	0.0027^{6}	$0.0013^{1.5}$	$0.0014^{3.5}$	0.0061^{7}
200	Mean	1.4050	1.3960	1.4043	1.4043	1.4057	1.4042	1.4012	1.4096
	MSE	$0.0308^{2.5}$	0.0305^{1}	$0.0341^{6.5}$	0.0326^{5}	$0.0341^{6.5}$	0.0324^{4}	$0.0308^{2.5}$	0.0405^{8}
	Bias	0.0036^{6}	0.0028^{2}	$0.0031^{4.5}$	$0.0031^{4.5}$	0.0041^{7}	0.0030^{3}	0.0009^{1}	0.0068^{8}
250	Mean	1.4039	1.3967	1.4031	1.4031	1.4042	1.4030	1.4009	1.4069
	MSE	$0.0246^{2.5}$	0.0244^{1}	$0.0269^{6.5}$	0.0258^{5}	$0.0269^{6.5}$	0.0257^{4}	$0.0246^{2.5}$	0.0318^{8}
	Bias	0.0028^{6}	0.0024^{5}	$0.0022^{3.5}$	$0.0022^{3.5}$	0.0030^{7}	0.0021^{2}	0.0006^{1}	0.0049^{8}
300	Mean	1.4039	1.3979	1.4028	1.4030	1.4038	1.4029	1.4013	1.4060
	MSE	0.0203^{2}	0.0202^{1}	$0.0219^{6.5}$	0.0212^{5}	$0.0219^{6.5}$	0.0211^{4}	0.0204^{3}	0.0259^{8}
	Bias	0.0028^{7}	0.0015^{2}	0.0020^{3}	$0.0021^{4.5}$	0.0027^{6}	$0.0021^{4.5}$	0.0009^{1}	0.0043^{8}
$\sum$Ranks		45^{4}	36^{1}	47.5^{5}	50.5^{6}	72^{7}	37.5^{3}	36.5^{2}	94^{8}

Table 8. Simulation outcomes of the TIAL model at $\alpha = 1.9$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	1.9151	1.8694	1.9076	1.9061	1.9146	1.9067	1.8979	1.9287
	MSE	0.1288^{3}	0.1231^{1}	0.1472^{6}	0.1386^{5}	0.1480^{7}	0.1366^{4}	0.1287^{2}	0.1817^{8}
	Bias	0.0079^{6}	0.0161^{8}	0.0040^{4}	0.0032^{2}	0.0077^{5}	0.0035^{3}	0.0011^{1}	0.0151^{7}
100	Mean	1.9083	1.8848	1.9056	1.9053	1.9091	1.9050	1.8998	1.9167
	MSE	$0.0638^{2.5}$	0.0622^{1}	0.0718^{6}	0.0680^{5}	0.0721^{7}	0.0674^{4}	$0.0638^{2.5}$	0.0871^{8}
	Bias	0.0044^{5}	0.0080^{7}	0.0029^{4}	0.0028^{3}	0.0048^{6}	0.0026^{2}	0.0001^{1}	0.0088^{8}
150	Mean	1.9066	1.8907	1.9052	1.9049	1.9075	1.9047	1.9010	1.9130
	MSE	0.0424^{2}	0.0417^{1}	0.0483^{6}	0.0455^{5}	0.0485^{7}	0.0452^{4}	0.0426^{3}	0.0584^{8}
	Bias	0.0034^{5}	0.0049^{7}	0.0027^{4}	0.0026^{3}	0.0040^{6}	0.0025^{2}	0.0005^{1}	0.0068^{8}
200	Mean	1.9075	1.8956	1.9068	1.9067	1.9086	1.9065	1.9034	1.9131
	MSE	$0.0315^{2.5}$	0.0310^{1}	$0.0362^{6.5}$	0.0341^{5}	$0.0362^{6.5}$	0.0339^{4}	$0.0315^{2.5}$	0.0442^{8}
	Bias	0.0040^{6}	0.0023^{2}	0.0036^{5}	0.0035^{4}	0.0045^{7}	0.0034^{3}	0.0018^{1}	0.0069^{8}
250	Mean	1.9059	1.8964	1.9051	1.9050	1.9065	1.9048	1.9026	1.9096
	MSE	0.0251^{2}	0.0248^{1}	$0.0287^{6.5}$	0.0270^{5}	$0.0287^{6.5}$	0.0269^{4}	0.0252^{3}	0.0346^{8}
	Bias	0.0031^{6}	0.0019^{2}	0.0027^{5}	0.0026^{4}	0.0034^{7}	0.0025^{3}	0.0014^{1}	0.0051^{8}
300	Mean	1.9057	1.8977	1.9045	1.9046	1.9056	1.9044	1.9028	1.9082
	MSE	$0.0209^{2.5}$	0.0207^{1}	$0.0234^{6.5}$	0.0222^{5}	$0.0234^{6.5}$	0.0221^{4}	$0.0209^{2.5}$	0.0281^{8}
	Bias	$0.0030^{6.5}$	0.0012^{1}	$0.0023^{3.5}$	0.0024^{5}	$0.0030^{6.5}$	$0.0023^{3.5}$	0.0015^{2}	0.0043^{8}
$\sum$Ranks		46^{4}	33^{2}	54^{6}	51^{5}	78^{7}	33.5^{3}	25.5^{1}	95^{8}

0.27742, 0.27742, 0.28387, 0.29032, 0.29032, 0.29677, 0.29677, 0.30323, 0.30968, 0.31613, 0.32903, 0.32903, 0.32903, 0.33548, 0.34839, 0.35484, 0.36129, 0.36774, 0.37419, 0.38065, 0.3871, 0.3871, 0.3871, 0.39355, 0.4, 0.41935, 0.41935, 0.43226, 0.43226, 0.43871, 0.44516, 0.50323, 0.51613, 0.53548, 0.56774, 0.57419, 0.58065, 0.6, 0.61935, 0.66452, 0.67742, 0.70323, 0.70323, 0.71613, 0.74194, 0.75484, 0.80645, 0.8129, 0.81935, 0.83226, 0.83226, 0.89677, 0.99355.

Table 9. Simulation outcomes of the TIAL model at $\alpha = 2.5$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	2.5261	2.4625	2.5166	2.5145	2.5255	2.5145	2.5050	2.5429
	MSE	0.1504^{2}	0.1411^{1}	0.1809^{6}	0.1671^{5}	0.1822^{7}	0.1634^{4}	0.1507^{3}	0.2289^{8}
	Bias	0.0105^{6}	0.0150^{7}	0.0067^{4}	$0.0058^{2.5}$	0.0102^{5}	$0.0058^{2.5}$	0.0020^{1}	0.0172^{8}
100	Mean	2.5135	2.4802	2.5101	2.5096	2.5146	2.5089	2.5033	2.5239
	MSE	0.0744^{2}	0.0719^{1}	0.0881^{6}	0.0817^{5}	0.0885^{7}	0.0807^{4}	0.0750^{3}	0.1090^{8}
	Bias	0.0054^{5}	0.0079^{7}	0.0040^{4}	0.0039^{3}	0.0058^{6}	0.0036^{2}	0.0013^{1}	0.0096^{8}
150	Mean	2.5100	2.4875	2.5084	2.5080	2.5113	2.5075	2.5034	2.5180
	MSE	0.0492^{2}	0.0479^{1}	0.0591^{6}	0.0545^{5}	0.0593^{7}	0.0541^{4}	0.0499^{3}	0.0729^{8}
	Bias	0.0040^{5}	0.0050^{7}	0.0033^{4}	0.0032^{3}	0.0045^{6}	0.0030^{2}	0.0014^{1}	0.0072^{8}
200	Mean	2.5104	2.4933	2.5095	2.5093	2.5117	2.5088	2.5053	2.5172
	MSE	0.0365^{2}	0.0357^{1}	0.0442^{6}	0.0407^{5}	0.0443^{7}	0.0404^{4}	0.0371^{3}	0.0548^{8}
	Bias	0.0042^{6}	0.0027^{2}	0.0038^{5}	0.0037^{4}	0.0047^{7}	0.0035^{3}	0.0021^{1}	0.0069^{8}
250	Mean	2.5082	2.4945	2.5072	2.5071	2.5090	2.5066	2.5042	2.5128
	MSE	0.0292^{2}	0.0287^{1}	0.0351^{6}	0.0323^{5}	0.0352^{7}	0.0322^{4}	0.0296^{3}	0.0431^{8}
	Bias	0.0033^{6}	0.0022^{2}	0.0029^{5}	0.0028^{4}	0.0036^{7}	0.0027^{3}	0.0017^{1}	0.0051^{8}
300	Mean	2.5078	2.4963	2.5062	2.5063	2.5077	2.5060	2.5042	2.5108
	MSE	0.0243^{2}	0.0240^{1}	$0.0286^{6.5}$	0.0266^{5}	$0.0286^{6.5}$	0.0265^{4}	0.0246^{3}	0.0350^{8}
	Bias	$0.0031^{6.5}$	0.0015^{1}	$0.0025^{4.5}$	$0.0025^{4.5}$	$0.0031^{6.5}$	0.0024^{3}	0.0017^{2}	0.0043^{8}
$\sum$Ranks		46.5^{4}	32^{2}	63^{6}	51^{5}	79^{7}	39.5^{3}	25^{1}	96^{8}

Table 10. Simulation outcomes of the TIAL model at $\alpha = 3$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	3.0356	2.9532	3.0237	3.0212	3.0343	3.0205	3.0099	3.0548
	MSE	0.1810^{2}	0.1679^{1}	0.2257^{6}	0.2060^{5}	0.2276^{7}	0.2004^{4}	0.1825^{3}	0.2897^{8}
	Bias	0.0119^{6}	0.0156^{7}	0.0079^{4}	0.0071^{3}	0.0114^{5}	0.0068^{2}	0.0033^{1}	0.0183^{8}
100	Mean	3.0181	2.9742	3.0138	3.0132	3.0191	3.0120	3.0058	3.0301
	MSE	0.0893^{2}	0.0856^{1}	0.1096^{6}	0.1003^{5}	0.1101^{7}	0.0987^{4}	0.0908^{3}	0.1371^{8}
	Bias	0.0060^{5}	0.0086^{7}	0.0046^{4}	0.0044^{3}	0.0064^{6}	0.0040^{2}	0.0019^{1}	0.0100^{8}
150	Mean	3.0131	2.9831	3.0110	3.0106	3.0146	3.0097	3.0052	3.0224
	MSE	0.0587^{2}	0.0569^{1}	0.0734^{6}	0.0667^{5}	0.0736^{7}	0.0660^{4}	0.0603^{3}	0.0915^{8}
	Bias	0.0044^{5}	0.0056^{7}	0.0037^{4}	0.0035^{3}	0.0049^{6}	0.0032^{2}	0.0017^{1}	0.0075^{8}
200	Mean	3.0130	2.9902	3.0118	3.0116	3.0145	3.0108	3.0069	3.0210
	MSE	0.0437^{2}	0.0425^{1}	0.0548^{6}	0.0498^{5}	0.0550^{7}	0.0494^{4}	0.0447^{3}	0.0687^{8}
	Bias	0.0043^{6}	0.0033^{2}	$0.0039^{4.5}$	$0.0039^{4.5}$	0.0048^{7}	0.0036^{3}	0.0023^{1}	0.0070^{8}
250	Mean	3.0103	2.9919	3.0090	3.0089	3.0111	3.0082	3.0055	3.0157
	MSE	0.0349^{2}	0.0341^{1}	0.0436^{6}	0.0395^{5}	0.0437^{7}	0.0393^{4}	0.0358^{3}	0.0541^{8}
	Bias	0.0034^{6}	$0.0027^{2.5}$	$0.0030^{4.5}$	$0.0030^{4.5}$	0.0037^{7}	$0.0027^{2.5}$	0.0018^{1}	0.0052^{8}
300	Mean	3.0097	2.9943	3.0077	3.0079	3.0095	3.0074	3.0053	3.0132
	MSE	0.0291^{2}	0.0285^{1}	$0.0355^{6.5}$	0.0325^{5}	$0.0355^{6.5}$	0.0323^{4}	0.0298^{3}	0.0438^{7}
	Bias	$0.0032^{6.5}$	0.0019^{2}	$0.0026^{4.5}$	$0.0026^{4.5}$	$0.0032^{6.5}$	0.0025^{3}	0.0018^{1}	0.0044^{8}
$\sum$Ranks		46.5^{4}	33.5^{2}	62^{6}	52.5^{5}	79^{7}	38.5^{3}	24^{1}	95^{8}

Dataset II. The second dataset consists of 30 measurements, as analyzed by [40], for evaluating the tensile strength of polyester fibers. The dataset consists of the following numbers: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926.

Table 13 presents a full summary of the descriptive analysis of these datasets, and the two datasets are graphically represented in Figures 6 and 7. The first dataset shows a right-skewed distribution in both the histogram and the

Table 11. Simulation outcomes of the TIAL model at $\alpha = 3.5$.

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
50	Mean	3.5454	3.4421	3.5306	3.5278	3.5430	3.5262	3.5143	3.5669
	MSE	0.2224^{2}	0.2053^{1}	0.2851^{6}	0.2582^{5}	0.2876^{7}	0.2500^{4}	0.2259^{3}	0.3690^{8}
	Bias	0.0130^{6}	0.0166^{7}	0.0088^{4}	0.0079^{3}	0.0123^{5}	0.0075^{2}	0.0041^{1}	0.0191^{8}
100	Mean	3.5228	3.4667	3.5174	3.5169	3.5237	3.5151	3.5081	3.5364
	MSE	0.1094^{2}	0.1044^{1}	0.1382^{6}	0.1253^{5}	0.1389^{7}	0.1230^{4}	0.1123^{3}	0.1741^{8}
	Bias	0.0065^{5}	0.0095^{7}	0.0050^{4}	0.0048^{3}	0.0068^{6}	0.0043^{2}	0.0023^{1}	0.0104^{8}
150	Mean	3.5163	3.4775	3.5137	3.5133	3.5179	3.5120	3.5069	3.5270
	MSE	0.0717^{2}	0.0692^{1}	0.0924^{6}	0.0831^{5}	0.0927^{7}	0.0821^{4}	0.0745^{3}	0.1160^{8}
	Bias	0.0047^{5}	0.0064^{7}	0.0039^{4}	0.0038^{3}	0.0051^{6}	0.0034^{2}	0.0020^{1}	0.0077^{8}
200	Mean	3.5158	3.4861	3.5143	3.5141	3.5174	3.5129	3.5085	3.5249
	MSE	0.0533^{2}	0.0517^{1}	0.0690^{6}	0.0620^{5}	0.0692^{7}	0.0614^{4}	0.0552^{3}	0.0870^{8}
	Bias	0.0045^{6}	$0.0040^{3.5}$	0.0041^{5}	$0.0040^{3.5}$	0.0050^{7}	0.0037^{2}	0.0024^{1}	0.0071^{8}
250	Mean	3.5126	3.4885	3.5109	3.5108	3.5134	3.5098	3.5067	3.5187
	MSE	0.0427^{2}	0.0416^{1}	0.0548^{6}	0.0493^{5}	0.0549^{7}	0.0490^{4}	0.0442^{3}	0.0684^{8}
	Bias	0.0036^{6}	0.0033^{5}	$0.0031^{3.5}$	$0.0031^{3.5}$	0.0038^{7}	0.0028^{2}	0.0019^{1}	0.0053^{8}
300	Mean	3.5118	3.4915	3.5093	3.5096	3.5114	3.5088	3.5065	3.5157
	MSE	0.0355^{2}	0.0347^{1}	0.0446^{6}	0.0404^{5}	0.0447^{7}	0.0402^{4}	0.0368^{3}	0.0554^{8}
	Bias	0.0033^{6}	0.0029^{5}	$0.0027^{3.5}$	$0.0027^{3.5}$	0.0035^{7}	0.0022^{2}	0.0019^{1}	0.0045^{8}
$\sum$Ranks		54^{5}	40.5^{3}	60^{6}	49.5^{4}	80^{7}	36^{2}	24^{1}	96^{8}

Table 12. Partial and overall rankings of all estimation methods for the TIAL model.

Parameter	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE
$\alpha = 0.2$	4	1	6	5	7	3	2	8
$\alpha = 0.5$	1	7	8	5	3	2	5	5
$\alpha = 0.9$	1	5	8	4	3	2	6	7
$\alpha = 1.4$	4	1	5	6	7	3	2	8
$\alpha = 1.9$	4	2	6	5	7	3	1	8
$\alpha = 2.5$	4	2	6	5	7	3	1	8
$\alpha = 3$	4	2	6	5	7	3	1	8
$\alpha = 3.5$	5	3	6	4	7	2	1	8
$\sum$Ranks	27	23	51	39	48	21	19	60
Overall rank	4	3	7	5	6	2	1	8

kernel density plot, which means it is not normally distributed. The violin and box plots both reveal a concentration of the data in the lower part, with a number of outliers. The TTT and QQ plots indicate that the data are not consistent with an exponential distribution and that other models such as the Weibull or log-normal might be more appropriate. Moreover, the second dataset has a similar right-skewed distribution in both the histogram and kernel density plot, although with less variability than the first sample. The violin and box plots show a more homogeneous distribution with fewer outliers. The TTT and QQ plots show deviation from the exponential distribution, though the data appear to be closer to a normal distribution than the first dataset.

Table 13. Some descriptive statistics of the two real datasets.

Dataset	n	Mean	Median	Variance	Skewness	Kurtosis	Range	Min	Max
I	121	0.2989	0.2581	0.0518	1.0432	3.4021	0.9916	0.0019	0.9935
II	30	0.3659	0.3360	0.0721	0.5464	2.2294	0.9030	0.0230	0.9260

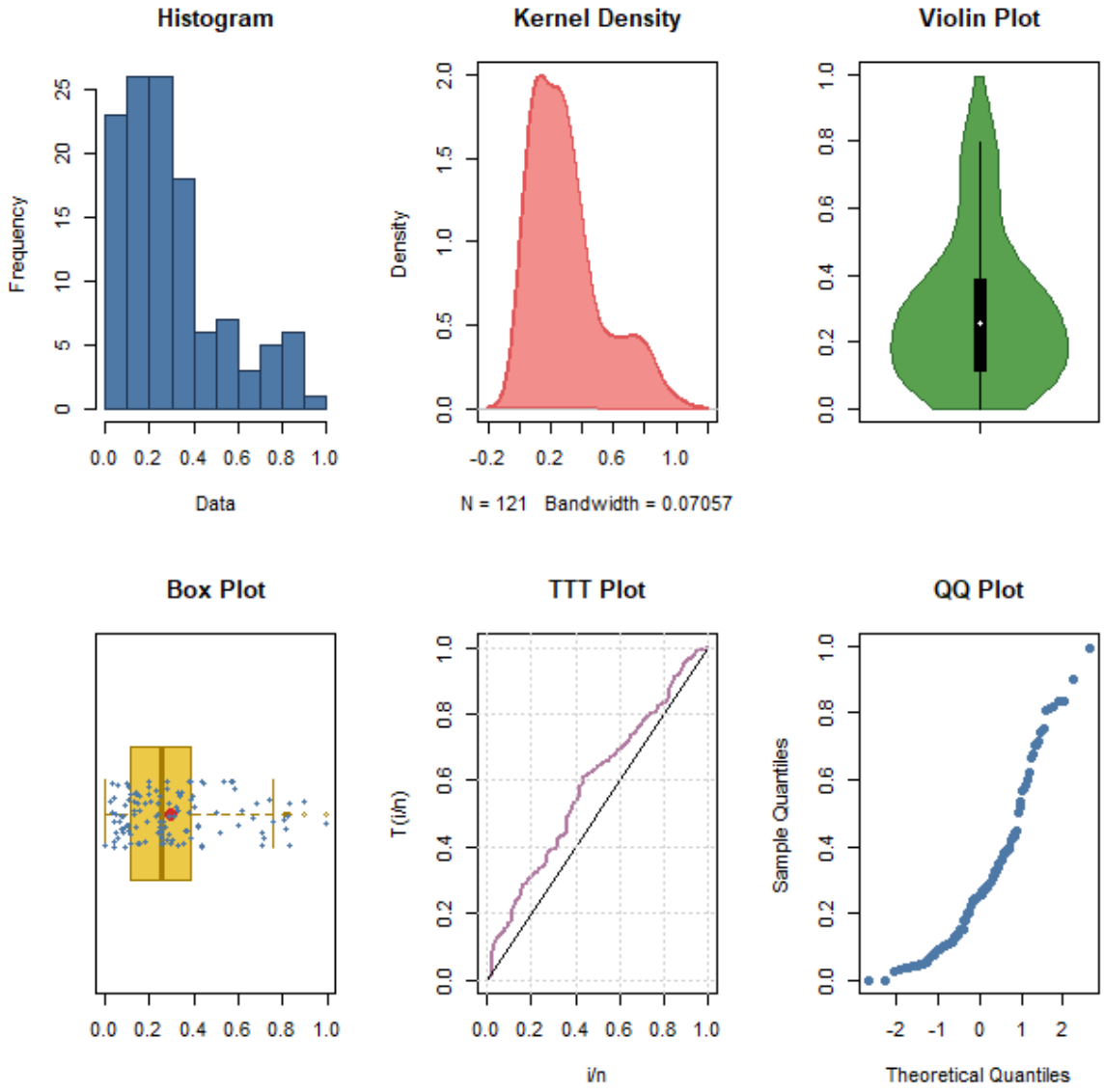


Figure 6. Visual illustration of dataset I.

A comparative analysis of the truncated induced Ailamujia (TIAL) distribution is performed with some competitive models, namely the induced Ailamujia (IAL), truncated Ailamujia (TAL), Ailamujia (AL), truncated moment exponential (TME), gamma (Ga), generalized exponential (GEx) and Burr XII (Bur XII) distributions. The maximum likelihood approach within the R programming framework was employed for parameter estimation and for assessing goodness-of-fit. Parameter estimates and their corresponding standard errors (SEs) are presented in Tables 14 and 15. Model selection was determined using the Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion (CAIC), Hannan–Quinn information criterion (HQIC), Kolmogorov–Smirnov distance (K-S), and the associated p-value (P-V). The fitted distributions were further tested by the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. From Tables 16 and 17, we can see that the TIAL model surpasses all competitors, with the minimum values of AIC, BIC, CAIC, HQIC,

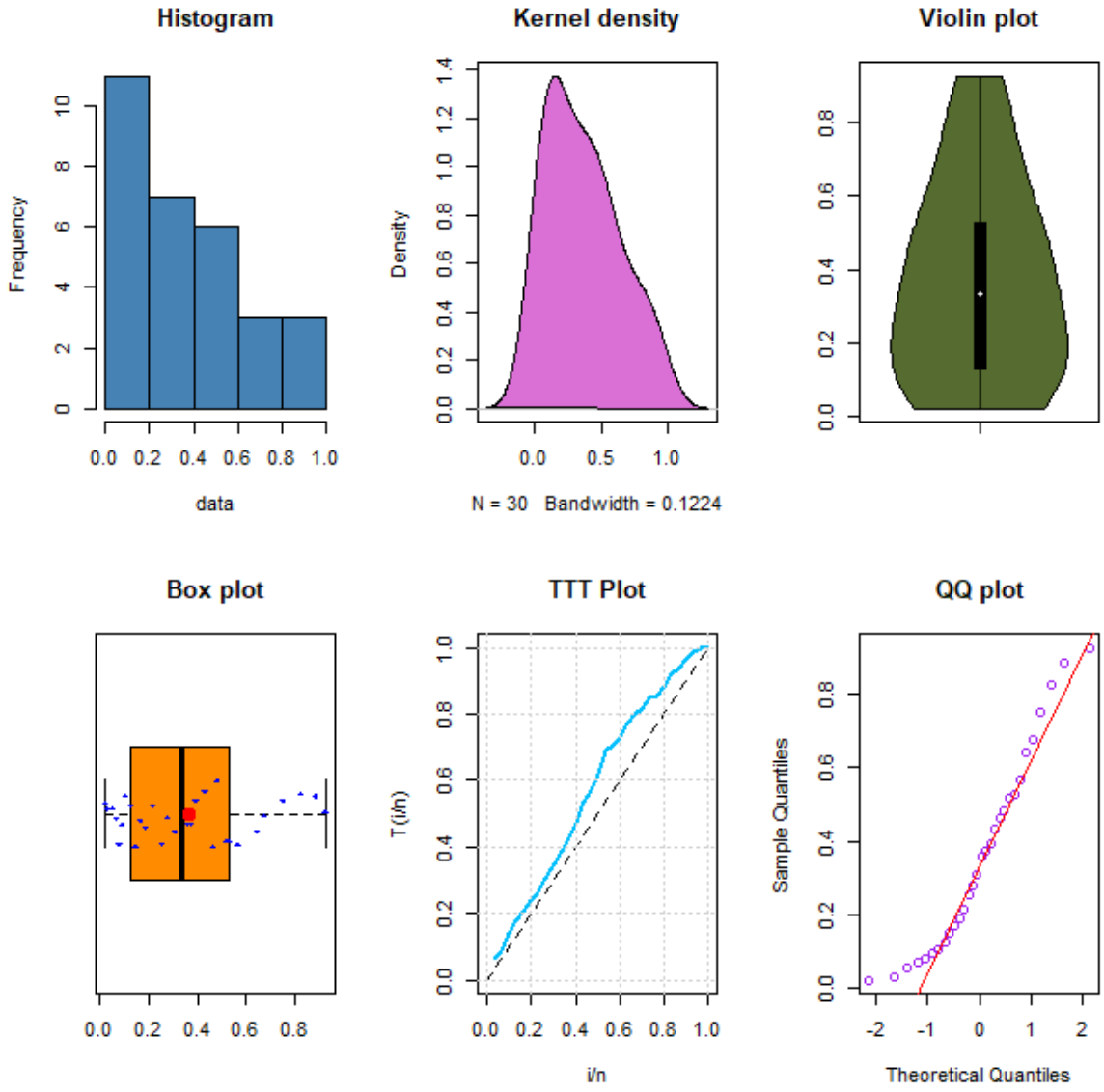


Figure 7. Visual illustration of dataset II.

K-S, W^* and A^*, and the maximum p-value. Further graphical comparisons, including fitted densities, empirical cumulative distribution functions and P–P plots, are presented in Figures 8 and 9.

Preprocessing and scaling arithmetic.

I. For dataset I (breast cancer survival times in months, range $[0.3, 154]$), observations were mapped to the unit support $(0, 1)$ via the linear scaling $x_i = y_i/c$, where $c = 155$ represents an upper bound strictly greater than the maximum observed survival duration. This linear transformation scales the physical units into proportional lifetimes while preserving the exact empirical skewness, kurtosis, and relative dispersion of the original data.

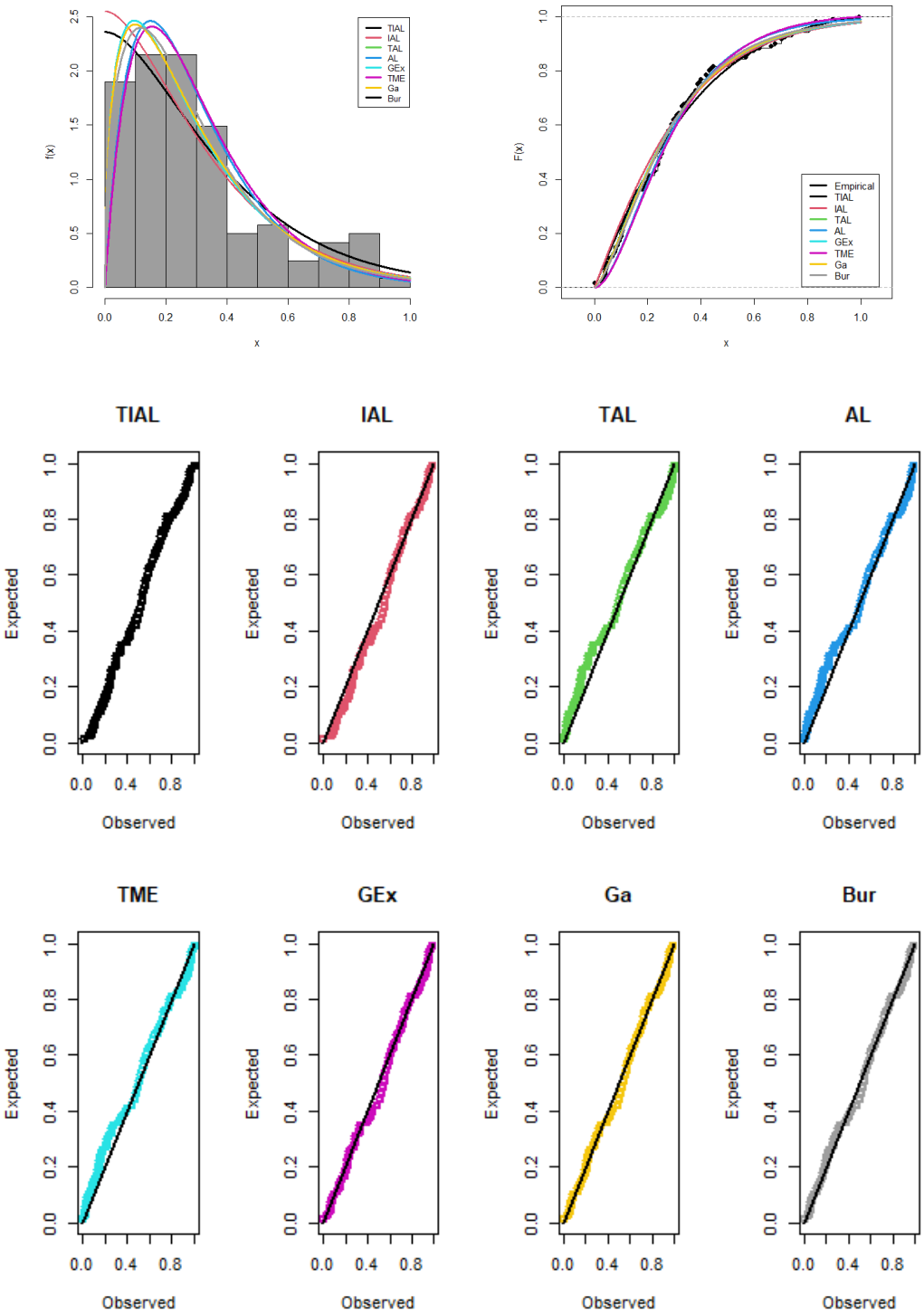


Figure 8. Fitted PDFs, CDFs and P–P plots for dataset I.

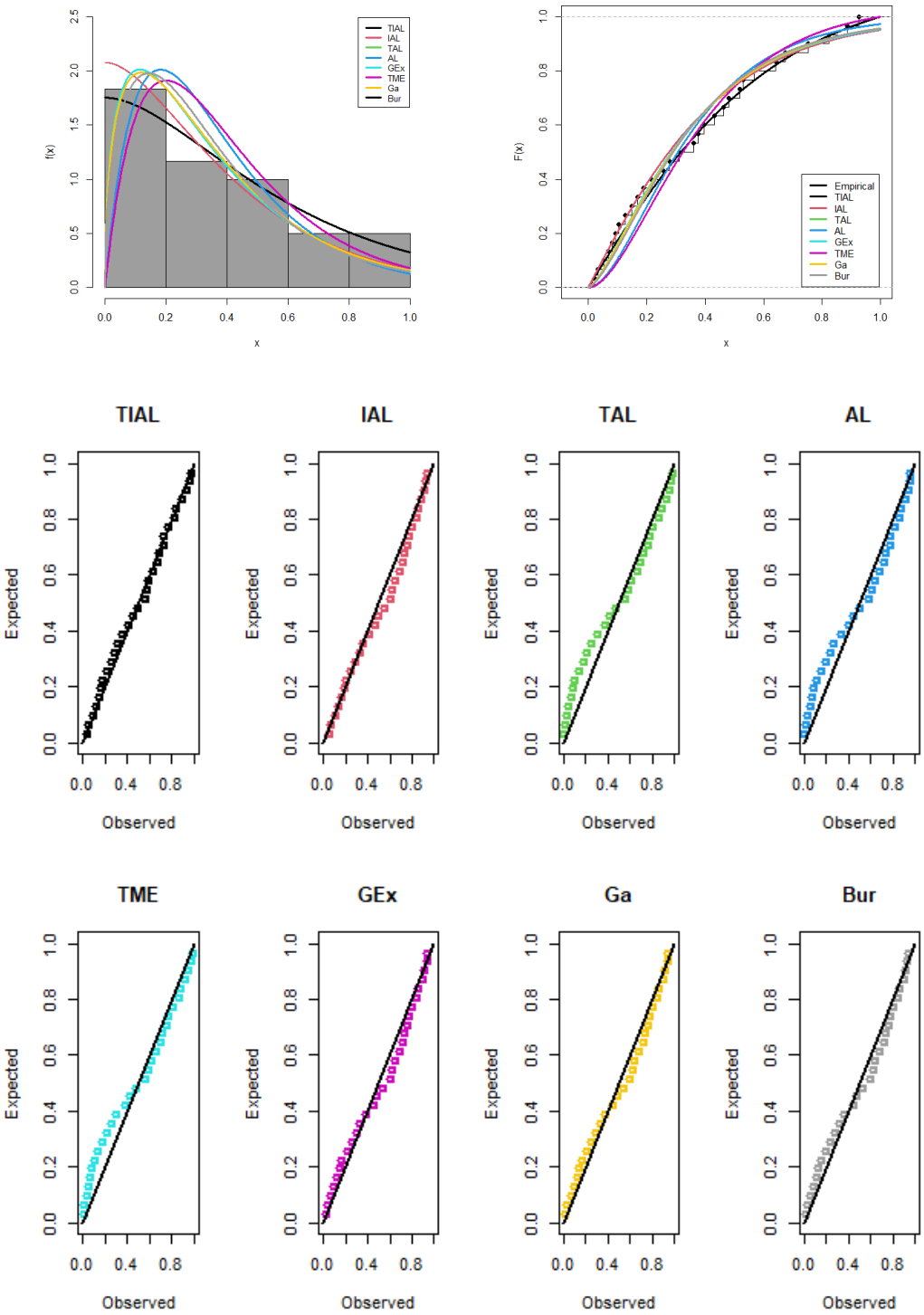


Figure 9. Fitted PDFs, CDFs and P–P plots for dataset II.

Table 14. MLEs and their SEs for the first dataset.

Model	Est. of α	SE($\hat{\alpha}$)	Est. of β	SE($\hat{\beta}$)
TIAL	2.2778	0.2385	–	–
IAL	2.5477	0.2022	–	–
TAL	3.2359	0.2299	–	–
AL	3.3453	0.2151	–	–
TME	0.1545	0.0109	–	–
Ga	1.4962	0.1751	0.1997	0.0277
GEx	1.5169	0.1928	0.2326	0.0247
Bur XII	1.4289	0.0947	5.7221	0.6591

Table 15. MLEs and their SEs for the second dataset.

Model	Est. of α	SE($\hat{\alpha}$)	Est. of β	SE($\hat{\beta}$)
TIAL	1.5516	0.4541	–	–
IAL	2.0751	0.3306	–	–
TAL	2.4902	0.4049	–	–
AL	2.7336	0.3529	–	–
TME	0.2008	0.0326	–	–
Ga	1.4853	0.3489	0.2463	0.0686
GEx	1.4907	0.3772	0.2879	0.0613
Bur XII	1.4513	0.2013	4.5068	0.9691

Table 16. Goodness-of-fit measures for the first dataset.

Model	AIC	BIC	CAIC	HQIC	K-S	P-V	W^*	A^*
TIAL	-63.089	-60.294	-63.056	-61.954	0.067	0.649	0.0571	0.4221
IAL	-56.514	-53.718	-56.481	-55.378	0.094	0.238	0.1833	1.1795
TAL	-54.748	-51.952	-54.714	-53.612	0.091	0.271	0.1832	1.6695
AL	-52.202	-49.406	-52.168	-51.066	0.083	0.374	0.1445	1.3299
TME	-54.746	-51.951	-54.713	-53.611	0.089	0.272	0.1831	1.6689
Ga	-56.899	-51.308	-56.798	-54.629	0.076	0.485	0.0611	0.4359
GEx	-56.322	-50.731	-56.219	-54.051	0.081	0.414	0.1116	0.7762
Bur XII	-56.443	-50.852	-56.342	-54.172	0.071	0.573	0.0683	0.4557

Table 17. Goodness-of-fit measures for the second dataset.

Model	AIC	BIC	CAIC	HQIC	K-S	P-V	W^*	A^*
TIAL	-4.6509	-3.2497	-4.5081	-4.2026	0.055	0.999	0.0142	0.1129
IAL	-0.3266	1.0746	-0.1838	0.1216	0.109	0.831	0.0505	0.3218
TAL	-1.1417	0.2592	-0.9974	-0.6912	0.131	0.627	0.1142	0.9993
AL	0.8714	2.2726	1.0142	1.3196	0.119	0.737	0.1018	0.8137
TME	-1.1418	0.2594	-0.9989	-0.6935	0.132	0.626	0.1139	0.9982
Ga	1.1194	3.9217	1.5638	2.0159	0.103	0.878	0.0501	0.3082
GEx	1.3378	4.1401	1.7822	2.2342	0.106	0.853	0.0512	0.3115
Bur XII	1.9544	4.7568	2.3988	2.8509	0.104	0.871	0.0612	0.3876

II. For dataset II (tensile strength measurements of polyester fibers), the raw values were already recorded as normalized ratios bounded on $(0, 1)$ in the original source literature, requiring no further preprocessing. All competing models were fitted to the exact same support for a fair goodness-of-fit evaluation.

Remark on data scope and censoring. It should be noted that both datasets evaluated in this section represent complete, uncensored observations. The proposed TIAL distribution is evaluated here as a univariate bounded probability model for standardized continuous measurements. Incorporating right-censoring schemes, time-varying risk sets, and prognostic covariate structures represents a natural extension for future research.

7. Conclusion

This paper introduced the truncated induced Ailamujia (TIAL) distribution, a continuous model designed for bounded lifetime data on $(0, 1)$. We derived key statistical properties, including moments, the quantile function, incomplete moments, order statistics, and entropy measures. The paper evaluated eight frequentist estimation techniques, including maximum likelihood, maximum product of spacings, and various minimum distance estimators (ADE, RADE, LADE, CME, LSE and WLSE). The TIAL model was verified on two genuine datasets, one related to breast cancer survival times and the other to tensile strength data. The results indicated that the proposed model performed better than a number of existing distributions in terms of several goodness-of-fit criteria, which demonstrates its flexibility and practical applicability in reliability engineering, survival analysis, and other related applied areas. Future research may extend the proposed TIAL distribution in several directions. One possible extension is the development of regression models based on the TIAL distribution for survival and reliability data with covariates. Bivariate and multivariate forms of the model may also be studied for dependent lifetime data.

Ethics approval and consent to participate

Not applicable. This study is retrospective in nature and utilizes purely anonymized secondary clinical data obtained entirely from previously published literature. Since no direct human trials or biological samples were involved, the need for formal ethics approval and informed consent was deemed unnecessary according to national regulations and institutional guidelines.

Consent for publication

Not applicable.

Compliance with guidelines and regulations

We confirm that all methods and statistical analyses performed in this study were carried out in strict accordance with the relevant guidelines and regulations.

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Data availability statement

All data generated or analysed during this study are completely included within this published article (specifically detailed in Section 6).

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