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Inverse Akash Distribution: Properties and Estimation with Application

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Abstract The statistical modeling of heavy tailed positive skewed life data is of great importance in reliability and survival analysis. Although the Akash distribution offers a simple one parameter structure, its limited flexibility may restrict its ability to capture highly positive skewed, heavy-tailed data and complex non-monotone hazard rate patterns. These limitations motivate the developments and investigation of its inverse counterpart as a more flexible alternative for modeling such data. The research aims to study the inverse Akash (*Inv. Akash*) distribution, which is an alternative framework for modeling this type of data. The statistical properties of model are examined, and estimation of its parameter is conducted using maximum likelihood (*ML*) method and Bayesian technique based on importance sampling under a weakly informative *Gamma* prior based on a square error loss function. A simulation study is performed for different parameter values and sample sizes to assess the finite-sample performance of the competing estimators using bias, mean squared error (*MSE*). The efficiency and suitability of the importance sampling procedure are additionally evaluated using the effective sample size (*ESS*) and related diagnostic measures. The results demonstrated that the performance of the estimators depends on both the sample and the underlying parameter value, with the Bayesian estimators showing competitive performance under appropriate prior specifications, improved estimation accuracy in several simulation settings. The (*ESS*) number of samples indicates that the importance sampling procedure provides a sufficiently effective sample for reliable Bayesian inference when the proposed distribution is appropriately selected distribution. Furthermore, applications to real datasets demonstrate the practical an applicability and flexibility of distribution as a useful and computationally tractable alternative for modeling positively skewed data and provide a comprehensive comparison between classical and Bayesian estimators.

Keywords Inverse Akash distribution, weakly informative prior, Importance sampling algorithm, effective sample size.

AMS 2010 subject classifications 62F15, 62E15, 65C05

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1. Introduction

Inverse distributions are important statistical distributions that have received increasing attention in recent studies due to their high flexibility in representing data with positive skewness, long tails and outliers. Their importance lies in their ability to model time dependent variables, such as speed or failure rate (which is the inverse of age) as well as their effectiveness in describing non-traditional risk functions in reliability and survival analysis. They also show better agreement with many realistic datasets compared to some traditional distributions, making them commonly used in finance, insurance, engineering and medical science. Furthermore, Some inverse distributions are particularly important in Bayesian statistics, especially in constructing conjugate prior distributions and developing modern Bayesian models, which has contributed to their wider application in contemporary statistical research.

The need for flexible statistical distributions has encouraged researchers to developed extensions and transformations of existing distributions. These distributions aim to represent diverse hazard function shapes,

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particularly those based on the Lindley distribution introduced by [1], which doesn't have enough flexibility in lifetime data analysis because it has only single scale parameter. Several studies have proposed alternatives to Lindley distribution that have increased its flexibility. These studies based on constructing many probability distributions inspired by the Lindley distribution as [2] proposed a three-parameter Lindley distribution which it has a decreasing exponential density curve. [3] developed an acceptance sampling plan under truncated lifetime test, based on different levels of consumer confidence and calculated the minimum appropriate sample size that determines the average age. This calculation based on three parameter-Lindley distribution where the acceptable sample sizes are smaller than the sample sizes of some other distributions. [4] introduced another three-parameter generalized Lindley distribution which is a mixture of Weibull and generalized Gamma distribution. This distribution has high flexibility, as the probability density function curve takes more than one form, including monotonic decreasing and unimodal. It has the ability to represent different types of medical, engineering, risk analysis and life testing. [5] presented a new two-parameter Lindley distribution in which the hazard rate function increases and then stabilizes at a near - constant level. This distribution is suitable for modeling data where the failure rate increases over time and then stabilize. It is also useful in survival analysis, reliability engineering, risk analysis, insurance science and quality assurance studies.

Some probability distributions with only one scale parameter, inspired by the Lindley distribution have been presented; [6] introduced a one parameter Akash distribution which is more flexible than Lindley distribution, showing a faster increase in hazard rate and a faster decrease in average remaining life with changing parameters. It is more suitable for modeling reliability data with increasing failures. [7] proposed a one parameter Akash distributions, She discussed the conditions for excessive, equal and insufficient dispersions of the distribution. This distribution is more flexible than Lindley distribution in modeling lifetime data due to accommodate a wider range of density and hazard rate shapes. [8] introduced Aradhana distribution which exhibits who has a unimodal probability density function (pdf) and an increasing hazard rate (hr) function, making it suitable for modeling lifetime data with an increasing risk of failures. Rama distribution proposed by [9]. [10] proposed a two parameter Rama distribution which possesses a flexible (pdf) and an increasing (hr) function controlled by both the shape and scale parameters, making it more flexible than one parameter Rama distribution.

Due to mathematical simplicity and flexibility in modeling lifetime data, the Akash distribution was selected as a baseline distribution for applying the inverse transformation method to derive a new distribution with the enhanced statistical and reliability properties and to evaluate, its performance in fitting lifetime data compared with the existing competing distributions. [11] found the density function of Inverse Akash distribution using transformation method. He discussed some properties represented by survival and hazard functions, order statistics, entropy measure, stress strength reliability and the maximum likelihood estimator[12]. [13] introduced an extension of this distribution.

The aim of this paper is to consider the inverse Akash distribution in order to extend Akash distribution by producing another model for positive-valued data with different skewness, tail and hazard rate behavior, and 50 presenting greater flexibility in statistical modeling and inference.

The reminder of this paper is organized as follows: section1 consists of the derivation of the pdf of $Inv.Akash$ distribution with the behavior of the pdf and hazard rate functions. Section3 deals with its statistical properties. Section4 included the parameter estimation using maximum likelihood and Bayesian methods. Section5 deals with a simulation study and real data sets to illustrate the application based on two estimation methods. The important conclusions illustrated in Section5.

This paper organized as follows: Section2 consist of the derivation of inverse Akash (*Inv. Akash*) distribution with its statistical properties. Section3 deals with parameter estimation by using maximum likelihood and Bayesian methods of estimation. Section4 contains a simulation study and real data sets to illustrate the application based on two estimation methods. The important conclusions illustrated in Section5..

2. Inverse Akash distribution

. First, consider a random variable x follow Akash (θ) with a (pdf) and cumulative distribution function (cdf) defined by [6]:

$$f_X(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x} \quad ;x,\theta > 0 \tag{1}$$

$$F_X(x;\theta) = 1 - \left(1 + \frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x} \quad ;x,\theta > 0 \tag{2}$$

and the hazard rate function (hr) is:

$$h_X(x;\theta) = \frac{\theta^3(1+x^2)}{\theta^2x^2+2\theta x+\theta^2+2} \quad ;x,\theta > 0 \tag{3}$$

The (hr) defined in (3) has a bathtub shaped only, so that, this distribution can represent a bathtub hazard but in its original form it cannot represent, for example; an increasing hazard function only, only decreasing hazard, inverted bathtub shape, or a hazard function with more than one inflection point. For this reason we studied the inverted distribution to provide greater flexibility in (pdf) and (hr) functions. We define a one to one transformation $(Y = 1/X)$. The random variable (Y) defined on the support $(0,\infty)$. The (cdf) of (Y) is:

$$F_Y(y) = p(Y < y) = p\left(\frac{1}{X} < y\right) = p\left(X > \frac{1}{y}\right) = 1 - F_X\left(\frac{1}{y}\right)$$

From (2), by putting $\left(X = \frac{1}{y}\right)$ and making some mathematical simplifications, the (cdf) of (Y) defined by:

$$F_Y(y;\theta) = \left(1 + \frac{\theta}{y}\frac{\left(\frac{\theta}{y}+2\right)}{\theta^2+2}\right)e^{-\frac{\theta}{y}} \quad ;y,\theta > 0 \tag{4}$$

Taking the derivative to both sides of (4) with respect to (y), the (pdf) of (Y) defined by:

$$f_Y(y;\theta) = \frac{\theta^3(y^2+1)}{y^4(\theta^2+2)}e^{-\frac{\theta}{y}} \;;,\; y,\theta > 0 \tag{5}$$

The random variable (Y) whose (cdf) and (pdf) defined in (4) and (5), The distribution named as inverse Akash distribution and denoted by $(Y \sim Inv.Akash(\theta))$, where (θ) is a scale parameter. The reliability and hazard rate functions of (Y) defined by:

$$R_Y(y;\theta) = 1 - \left(1 + \frac{\frac{\theta}{y}\left(\frac{\theta}{y}+2\right)}{\theta^2+2}\right)e^{-\frac{\theta}{y}} \tag{6}$$

$$h_Y(y;\theta) = \frac{\frac{\theta^3(y^2+1)}{y^4(\theta^2+2)}e^{-\frac{\theta}{y}}}{1 - \left(1 + \frac{\frac{\theta}{y}\left(\frac{\theta}{y}+2\right)}{\theta^2+2}\right)e^{-\frac{\theta}{y}}} \tag{7}$$

The $(pdf), (cdf)$ reliability and hazard functions of $Inv.Akash$ distribution at different values of the scale parameter are plotted in Figures (1, 2, 3).

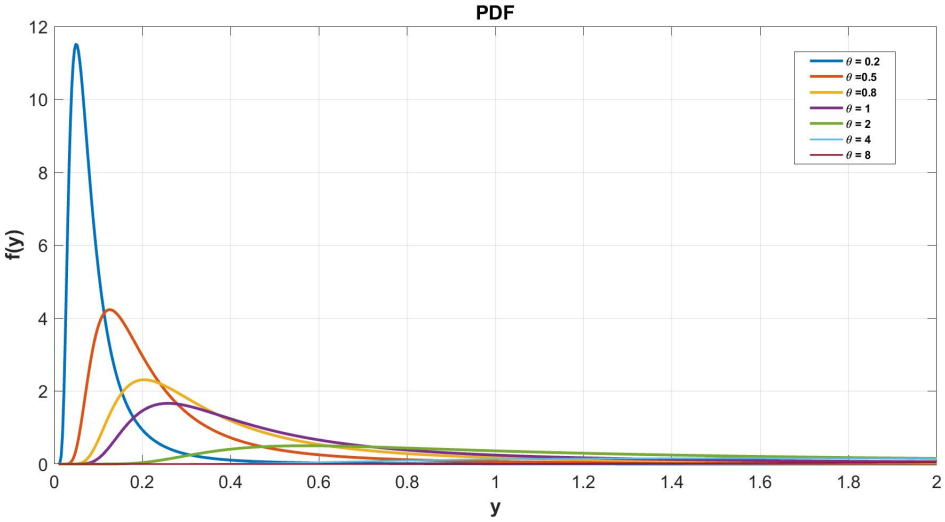

Figure 1. The (pdf) plot of Inv.Akash distribution at different values of scale parameter θ

Figure 1 shows that the pdf has a positive right skew for all considered values of θ with most of the density concentrated near small values of y and the right tail extends toward larger values of y. The pdf is a unimodal where the location and height of the mode vary considerably with θ; the density has sharper and higher peak for small values of θ, while the peak becomes lower and more dispersion as θ increases. This indicates that θ plays an important role in controlling the degree of skewness, the pdf location, and the length of the right tail, thereby increasing the flexibility of the distribution in modeling positive skewed data.

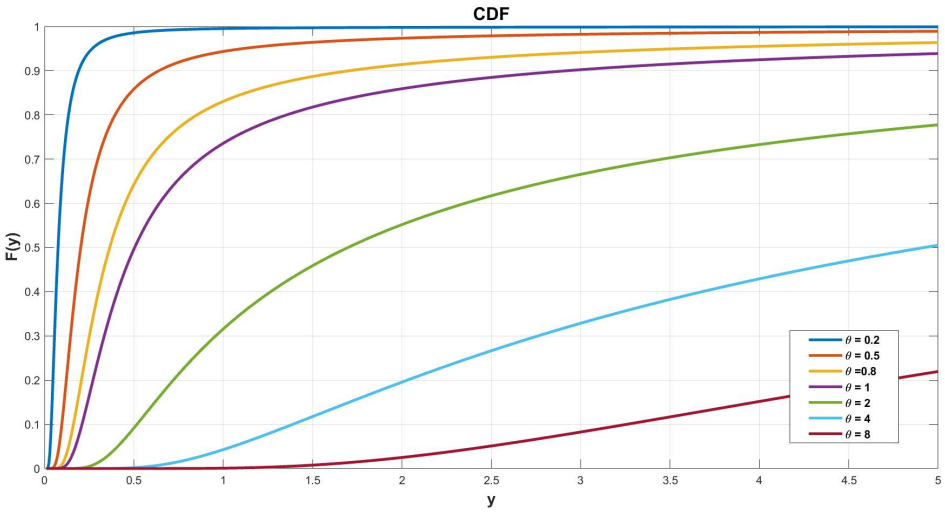

Figure 2. The (cdf) and reliability functions plots at different values of θ

It is seen from Figure 2 the illustration the complementary behavior of cdf and reliability functions of $Inv.Akash$ distribution. The cdf increases monotonically from zero to one whereas the reliability function decreases monotonically from one to zero. It has become that the increasing θ leads to a. slower accumulation and a greater extension of reliability function at large values of θ. This indicates a wider dispersion and a longer right tail, while smaller values of θ are associated with a faster probability accumulation and shorter right tail. This behavior is consistent with positive skewness exhibited by the pdf, as θ provides considerable flexibility in controlling the properties of inverse Akash distribution, particularly the probability density spread and tail behavior.

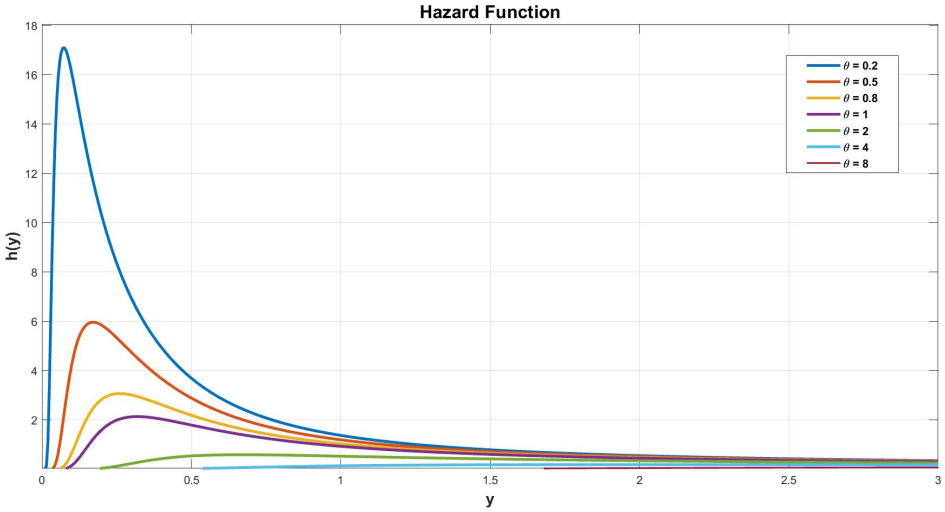

Figure 3. The hazard function plot of Inv.Akash distribution at different values of θ

Figure 3 shows that the hazard function curve is a unimodal non-monotonic type pattern which is an upside-down U shape pattern (inverted bathtub) characterized by an initially increasing hazard followed by a decreasing hazard rate. The height and position of the peak of hazard rate change with changes in the values of θ, demonstrating it's significant influence on the failure rate behavior. This flexible structure of risk function allows for the modeling lifetimes characterized by an initial high failure rate followed by a decreasing failure rate. The $Inv.Akash$ distribution is flexible in accommodating data with positively skewed and positive-valued data, together with its flexible tail behavior and non-monotonic hazard rate, make it a favorable candidate for modeling economic data such as income, expenditure, investment amounts, and financial losses. It is also can be considered a suitable model for waiting-time data, owing positive values, right skewed nature, flexible tail behavior. Also the inverse Akash may also be provide a useful probabilistic model for environmental pollution data, particularly positive pollutant concentrations exhibiting pronounced right skewness. The following is a mathematical discussion of the behavior of both the (pdf) and (hr) functions defined in (4) and (7).

The following is a mathematical discussion of the behavior of both the (pdf) and (hr) functions defined in (4) and (7). Taking the first derivative of (4) with respect to (y) and solving $\left(\frac{df_Y(y)}{dy} = 0\right)$ with respect to y where the critical region is the solution of the following equation with respect to y.

$$2y^3 - \theta y^2 + 4y - \theta = 0 \tag{8}$$

Consider y_0 is a numerical solution of (8), y_0 is the mode if $\left(\frac{d^2 f_Y(y)}{dy^2}\Big|_{y=y_0} < 0\right)$.

It is seen from table (1) that the mode increases as θ increases, indicating that θ has a significant influence on the location of the peak of the pdf curve of $Inv.Akash$ distribution. The critical values of $h_Y(y)$ is a numerical solution of the following equation with respect to y.

$$\frac{N'(y)D(y) - N(y)D'(y)}{(D(y))^2} = 0$$

Where:

$$N(y) = -y\theta(y^2+1)^3 e^{-\frac{\theta}{y}} \quad ; \quad N'(y) = -\theta e^{-\frac{\theta}{y}}\left(6y^2(y^2+1)^2 + \frac{\theta}{y}(y^2+1)^3 + (y^2+1)^3\right)$$

$$D(y) = (\theta^2+2)^3\left(y^2(\theta^2+2)\left(e^{-\frac{\theta}{y}} - 1\right) + e^{-\frac{\theta}{y}}(\theta^2 + 2y\theta)\right) = (\theta^2+2)^3 G(y)$$

Table 1. Increasing, decreasing regions and mode at different values of θ.

	regions		
$\boldsymbol{\theta}$	$\boldsymbol{y < y_0}$	$\boldsymbol{y > y_0}$	***mode***
0.2	*increasing*	*decreasing*	0.05006
0.5	*increasing*	*decreasing*	0.12598
0.8	*increasing*	*decreasing*	0.20408
1	*increasing*	*decreasing*	0.25984
2	*increasing*	*decreasing*	0.56984
4	*increasing*	*decreasing*	1.54369
8	*increasing*	*decreasing*	3.7511

$$G(y) = \left(y^2(\theta^2+2)\left(e^{-\frac{\theta}{y}}-1\right) + e^{-\frac{\theta}{y}}(\theta^2+2y\theta)\right)$$

$$D'(y) = (\theta^2+2)^3 G'(y)$$

$$G'(y) = \left(\frac{d}{dy}\left(y^2(\theta^2+2)\left(e^{-\frac{\theta}{y}}-1\right)\right) + \frac{d}{dy}\left(e^{-\frac{\theta}{y}}(\theta^2+2y\theta)\right)\right)$$

$$G'(y) = \left(\left((\theta^2+2)\left(2y\left(e^{-\frac{\theta}{y}}-1\right)+\theta e^{-\frac{\theta}{y}}\right)\right) + \theta e^{-\frac{\theta}{y}}\left(\frac{\theta}{y^2}(\theta+2y)+2\right)\right)$$

$$h'_Y(y) = \frac{\theta e^{-\frac{\theta}{y}}}{(\theta^2+2)^3 G(y)^2}\left(G(y)A(y) - y(y^2+1)^3 B(y)\right) = 0 \tag{9}$$

where:

$$A(y) = 6y^2(y^2+1)^2 + (y^2+1)^3\left(1+\frac{\theta}{y}\right) \quad \text{and} \quad B(y) = G'(y)$$

$$B(y) = (\theta^2+2)\left(2y\left(e^{-\frac{\theta}{y}}-1\right)+\theta e^{-\frac{\theta}{y}}\right) + \theta e^{-\frac{\theta}{y}}\left(\frac{\theta(\theta+2y)}{y^2}+2\right)$$

Consider y_0^* be the critical value which satisfy's (9). By using y^*, the increasing and decreasing regions have been identified at different values of θ. The result were placed in table (2) .

Table 2. Critical values, increasing and decreasing regions at different values of scale parameter.

	regions		
$\boldsymbol{\theta}$	$\boldsymbol{y < y_0^*}$	$\boldsymbol{y > y_0^*}$	***Critical value*** $\boldsymbol{y_0^*}$
0.2	++	–	0.07294
0.5	++	–	0.17101
0.8	++	–	0.26012
1	++	–	0.31905
2	++	–	0.66946
4	++	–	1.94892
8	++	–	0.01

From table(1) above it is seen that as θ increases, the critical value y_0^* at which the hr function attains its maximum shift toward larger values of y. The hr function increasing before y_0^* and decreasing after y_0^* so that the hr is a unimodal pattern. Thus increasing θ delays the occurrence of the maximum hazard level and shifts the peak of the hazard function to the right. This shows the important behavior demonstrates the important role of θ in controlling the position and pattern the time hazard rate, providing additional flexibility for modeling lifetime data with a clear peak in hazard function. Also it is observed an interesting behavior when $\theta = 8$, the value of y_0^* decreases compared with the preceding values, so that suggesting a non-monotonic effect of θ on the location of maximum hazard rate and the hazard function remains a unimodal which demonstrates the flexibility of $Inv.Akash$ in modeling variations in location and magnitude of hazard peak.

3. statistical properties.

Some statistical properties has been presented as:

3.1. Moments.

The r-th moment around zero for $Inv.Akash$ distribution does not exist. Therefore, we need alternative measures to the mean and variance of the distribution. These measures are the quantiles.

3.2. Quartiles.

The quantile function of $InvAkash$ is the value of Y_q satisfying the following equation.

$$\left(1 + \frac{\theta\left(\frac{\theta}{y_q} + 2\right)}{y_q(\theta^2 + 2)}\right) e^{-\frac{\theta}{y_q}} = q \tag{10}$$

The numerical solution of (10) with respect to Y_q can easily found the first quantile $Y_{0.25}$, second quantile $Y_{0.5}$ and third quantile $Y_{0.75}$ by replacing q in (10) by $(0.25, 0.5, 0.75)$ respectively. These three quartiles were calculated at different values of θ. Since quartiles are important in describing the distribution location by the second quartile which represents the median, while its spread is evaluated through the interquartile range IQR defined by:

$$IQR = Y_{0.75} - Y_{0.25}$$

Table 3. $(Y_{0.25}, Y_{0.5}, Y_{0.75})$, IQR of $Inv.Akash$ distribution at different values of θ.

$\boldsymbol{\theta}$	$\boldsymbol{y_{0.25}}$	$\boldsymbol{y_{0.5}}$	$\boldsymbol{y_{0.75}}$	$\boldsymbol{IQR}$
0.2	0.0514	0.0757	0.11857	0.06735
0.5	0.1336	0.20255	0.33903	0.20544
0.8	0.22838	0.36461	0.69281	0.46438
1	0.30183	0.50421	1.06165	0.75982
2	0.83245	1.69777	4.3405	3.50806
4	2.37814	4.92186	12.14213	9.764
8	5.47927	11.07241	26.84522	21.36596

We see from table(3) the median increases as θ increases. This indicates that the $Inv.Akash$ distribution shifts toward larger values and that θ has a positive effect on the median. Also we can see that $Y_{0.75} - Y_{0.5} > Y_{0.5} - Y_{0.25}$ at all values of θ, this indicates that the $Inv.Akash$ distribution is a positive skew with a relatively long tail. It is also observed from column IQR that θ affects the degree of dispersion, as dispersion becomes clearly evident when $\theta \geq 2$. However, when $\theta \leq 1$, the distribution becomes less dispersed.

3.3. Characteristic function.

The Characteristic function (cf) of Y defined by:

$$\phi_Y^{(t)} = E\left(e^{itY}\right) = \int_0^\infty \frac{\theta^3(y^2+1)}{y^4(\theta^2+2)} e^{-\left(\frac{\theta}{y} - ity\right)} dy \tag{11}$$

Define a one to one transformation $(Z = \theta/y)$ so that:

$$\phi_Y^{(t)} = \frac{\theta^3}{(\theta^2+2)} \left(\int_0^\infty \frac{1}{\theta} e^{-\left(Z + \frac{it\theta}{Z}\right)} dZ + \int_0^\infty \frac{Z^2}{\theta^3} e^{-\left(Z + \frac{it\theta}{Z}\right)} dZ \right)$$

$$\phi_Y^{(t)} = \frac{\theta^2}{(\theta^2+2)} \int_0^\infty e^{-\left(Z + \frac{it\theta}{Z}\right)} dZ + \frac{1}{(\theta^2+2)} \int_0^\infty Z^2 e^{-\left(Z + \frac{it\theta}{Z}\right)} dZ \tag{12}$$

The two integrals in (12) are the integral representation of modified Bessel function of second kind [14] , so that:

$$\phi_Y^{(t)} = \frac{2\theta^2}{(\theta^2+2)} (-it\theta)^{\frac{1}{2}} K_1(2\sqrt{-it\theta}) + \frac{2}{(\theta^2+2)} (-it\theta)^{\frac{3}{2}} K_3(2\sqrt{-it\theta}) \tag{13}$$

With condition $(-it\theta > 0)$, where $K_1(.)$ and $K_3(.)$ are the modified Bessel function of the second kind of orders one and three respectively. $\sqrt{-1}$ in Bessel function in (13) has a principal square root equal to $\left(\frac{1-i}{\sqrt{2}}\right)$ [15]. The representation of (cf), defined in (13) provides the analytical device for characterizing inverse Akash distribution and searching its probabilistic properties, including the moments around zero also it can be used to investigate the distributions of linear and non-linear functions of $Inv.Akash$ variable.

4. Parameters estimation.

The parameter of $Inv.Akash$ distribution was estimated by maximum likelihood (ML) method and Bayesian (B) approach under square error loss function with an informative prior distribution for θ.

4.1. Maximum likelihood method.

Consider a complete random sample of size (n) drawn from $Inv.Akash$ θ. The likelihood function is given by:

$$L(\theta) = \left(\frac{\theta^3}{(\theta^2+2)}\right)^n \prod_{i=1}^{n} \left(\frac{y_i^2+1}{y_i^4}\right) e^{-\theta \sum_{i=1}^{n} \frac{1}{y_i}} \tag{14}$$

By using factorization method to (14), $L(\theta)$ is given by:

$$L(\theta) = h(y)m(T(y);\theta)$$

where $h(y) = \prod_{i=1}^{n} \left(\frac{y_i^2+1}{y_i^4}\right)$, $m(T(y);\theta) = \left(\frac{\theta^3}{(\theta^2+2)}\right)^n e^{-\theta T(y)}$, and $T(y) = \sum_{i=1}^{n} \frac{1}{y_i}$, $h(y)$ does not depend on θ. By using the *Neyman-Fisher factorization theorem* [16] confirms that $T(y)$ is sufficient statistic forθ. The natural logarithm (14) is given by:

$$ln(L(\theta)) = n\left(ln(\theta^3) - ln(\theta^2+2)\right) + \sum_{i=1}^{n} ln\left(\frac{y_i^2+1}{y_i^4}\right) - \theta \sum_{i=1}^{n} \frac{1}{y_i} \tag{15}$$

Taking the first derivative to both sides of (15) with respect to θ and equating it to zero.

$$T(y)\theta^3 - n\theta^2 + 2T(y)\theta - 6n = 0 \tag{16}$$

The (ML) estimator for θ is the positive root of (16). There are various numerical methods for solving (16) such as the interior point algorithm [17] or the numerical $Newton - Raphson$ method, or the R package (root Solve) [18] and this the method that will be employed in this research:

$$\frac{d^2 ln(L(\theta))}{d\theta^2} = -n\frac{\theta^4 + 16\theta^2 + 12}{\theta^2(\theta^2+2)^2} < 0 \tag{17}$$

for all $\theta > 0$, so that the root satisfies (17) is a maximum likelihood estimator $(\hat{\theta}_{ML})$. $\frac{d^2 ln(L(\theta))}{d\theta^2}$ is strictly negative for every $\theta > 0$,so $ln(L(\theta))$ is strictly concave up, therefore $(\hat{\theta}_{ML})$ is a unique global maximizer of $L(\theta)$. Under the standard regularity conditions for (ML) estimators including identifiability, differentiability and integrability conditions, also a unique maximizer of so $ln(L(\theta))$ [19], $\hat{\theta}_{ML}$ is consistent for the true parameter value θ_0 in probability. Equivalently, for every $\epsilon > 0$.

$$\lim_{n\to\infty} P_{\theta_0}\left(\left|\hat{\theta}_{ML} - \theta_0\right| > \epsilon\right) = 0$$

So, the probability that $\hat{\theta}_{ML}$ differs from θ_0 by more than any positive constant amount tends to zero as the sample size is large.

For sufficiently large n, $\hat{\theta}_{ML} \xrightarrow{D} N\left(\theta_0, (I_n(\theta_0))^{-1}\right)$, where $I_n(\theta_0) = \frac{\partial^2 ln(L(\theta))}{\partial\theta^2}\Big|_{\theta=\theta_0} = \frac{n(\theta_0^4+16\theta_0^2+12)}{\theta_0^2(\theta_0^2+2)^2}$, So the asymptotic variance of $\hat{\theta}_{ML}$ equal to $(I_n(\theta_0))^{-1}$. Therefore, $\hat{\theta}_{ML}$ is asymptotically efficient in usual regular parametric situation, because it's asymptotic variance achieves Fisher information bound.

4.2. Bayesian estimation.

In this part, a development of Bayesian estimators of the pdf and cdf of $Inv.Akash$ distribution have been done by using the importance sampling technique. It is well known that Bayesian estimator for θ is an estimator which minimizes the posterior risk [20, 21]. It is well-known that Bayesian estimation method may produce an appropriate approach if there is a strong prior information about parameter θ. Let the prior of θ is $Gamma(a,b)$ where:

$$p(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta} \quad ; \quad a,b,\theta > 0 \tag{18}$$

Now, Combining the sample information defined in (14) with prior information in (18) using Bayes theorem, the posterior distribution of $\theta|\underline{Y}$ is given by:

$$p(\theta|\underline{Y}) \propto \frac{1}{(\theta^2+2)^n}\theta^{3n+a-1}e^{-(b+T(y))\theta} \tag{19}$$

where $T(y)$ is a sufficient statistic for θ and$T(y) = \sum_{i=1}^{n} \frac{1}{y_i}$ also $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)'$ is a vector of sample observations. The resulting posterior distribution defined in (19) doesn't have a closed-form representation, the posterior expectations are needed for the evaluation of Bayesian estimation. These expectations are evaluated numerically by using the importance sampling technique. Under quadratic loss function, Bayesian estimator of θ is given by:

$$\hat{\theta}_B = E_\theta\left(\theta|\underline{y}\right) \tag{20}$$

Bayesian estimator for $f(y|\theta)$ is :

$$\hat{f}_B(y|\theta) = E_\theta\left(f(y|\theta)|\underline{y}\right) \tag{21}$$

and Bayesian estimator for $F(y|\theta)$ is given by:

$$\hat{F}_B(y|\theta) = E_\theta\left(F(y|\theta)|\underline{y}\right) \tag{22}$$

However, all the posterior expectations defined in (20), (21), (22) are difficult and do not have a closed form which is why we use importance sampling [22] to approximate them numerically. Posterior distribution of $\theta|\underline{Y}$ defined in (19) can be re-written as:

$$p(\theta|\underline{Y}) \propto Gamma\Big(3n+a, \big(b+T(y)\big)\Big) \times K(\theta) \tag{23}$$

Where:

$$K(\theta) = \frac{1}{(\theta^2+2)^n} \tag{24}$$

The steps of an acceptance sampling algorithm are applied to find Bayesian estimators of θ, pdf, cdf of $Inv.Akash$ distribution defined in (20), (21), (22) respectively according to the following steps:
Step (1): The proposal distribution $V(\theta)$ is chosen so that it close to the Kernel of posterior distribution defined in (23), it was chosen to be similar to the Kernel of Gamma distribution. It is given by:

$$V(\theta) \propto \theta^{3n+a-1}e^{-(b+T(y))\theta} \tag{25}$$

This choice is very practical because it makes calculating importance sampling weights very simple.
Step (2): Generate $(\theta_1, \theta_2, \theta_3, \ldots, \theta_M)$ from $V(\theta)$ where M is the number of generated values of θ.
Step (3): Derivation of importance weights.
Define the importance weight of a sample j for $j = 1, 2, 3, \ldots, M$ as:

$$u_j \propto \frac{p(\theta|Y)}{V(\theta)} \tag{26}$$

Put (23) and (25) onto (26), so that:

$$u_j = \frac{1}{\left(\theta_j^2+2\right)^n} \tag{27}$$

Making normalization to u_j, the normalized importance weight is given by:

$$u_j^* = \frac{u_j}{\sum_{j=1}^{M} u_j} \quad ; j = 1, 2, 3, \cdots, M \tag{28}$$

Step (4): under square error loss function, the approximate Bayesian estimators for θ, $f_Y(y;\theta)$ and $F_Y(y;\theta)$ given by:

$$\hat{\theta}_B = \frac{\sum_{j=1}^{M} \theta_j \left(\theta_j^2+2\right)^{-n}}{\sum_{j=1}^{M} \left(\theta_j^2+2\right)^{-n}} \tag{29}$$

$$\hat{f}_{YB}(y) = \sum_{j=1}^{M} \left(\theta_j^2+2\right)^{-n} \left(\frac{\theta_j^3(y^2+2)}{y^4(\theta_j^2+2)}\right) e^{-\frac{\theta_j}{y}} \left(\sum_{j=1}^{M} \left(\theta_j^2+2\right)^{-n}\right)^{-1} \tag{30}$$

and

$$\hat{F}_{YB}(y) = \sum_{j=1}^{M} \left(\theta_j^2+2\right)^{-n} \left(1-\left(1+\frac{\theta_j\left(\frac{\theta_j}{y}+2\right)}{y\left(\theta_j^2+2\right)}\right)\right) e^{-\frac{\theta_j}{y}} \left(\sum_{j=1}^{M} \left(\theta_j^2+2\right)^{-n}\right)^{-1} \tag{31}$$

Step (5): Examining the quality of importance sampling: This is a very important step because proving that importance sampling works efficiently. It is roughly assessed for importance sampling approximation by calculating the effective sample size (ESS) based on normalized importance weights (u_j^*) associated with $j-th$ generated sample θ_j. The ESS [23] is calculated according to the following formula:

$$ESS = \frac{1}{\sum_{j=1}^{M} \left(u_j^*\right)^2} \tag{32}$$

When standard weights u_j^* for $j = 1, 2, \ldots, M$ are closely to discrete uniform distribution, the ESS is high and close to M. When u_j^* is concentrated in few samples, the ESS decreases. When most of u_j^* concentrated in a single weight, the ESS becomes very small and approaches to 1. A low ESS indicates weight degeneracy which is described as the condition in which few samples have significant weights while the weights of most samples are close to zero, which decreases the efficiency of the method [24].

5. Application

. In this section, a simulation experiment was conducted to evaluate the performance of ML method and Bayesian methods of estimation. Also, the analysis of real data set for illustrative purposes. The R package is used to perform all numerical calculations.

5.1. Simulation experiment

. This subsection presents a simulation experiment. The inverse function was used on (10) to generate random samples of sizes (n = 25, 50, 100) of $Inv.Akash$ distribution. For the purpose of studying the performance of estimation methods and verifying their efficiency various conditions. Different values of (θ = 0.2, 0.5, 0.8, 1, 2, 4, 8) were selected. The experiment was repeated (N = 1000) times. The parameter of $Inv.Akash$, it's pdf and cdf were estimated using both ML and Bayesian estimation methods. The informative prior distribution was chosen to determine its effect on estimation accuracy, compare it with maximum likelihood estimator, evaluation the performance of Bayesian estimator in the presence of informative prior and studying the sensitivity of the estimator with different values of hyperparameters was assessed we used four types of weakly informative $Gamma$ priors ($Gamma$ (2,1), $Gamma$ (0.5,0.5), $Gamma$ (0.25,0.5) and $Gamma$ (2,2)). This choice helps to provide a flexible and appropriate prior distribution while allowing the monitored data to control subsequent inference. In the acceptance sampling, M = 1000 samples were generated from $V(\theta)$ defined in (25). The parameter θ, pdf and cdf were estimated based on (20), (21), and (22) respectively. The ML and Bayesian estimators were evaluated at different values of θ and sample size using bias and mean square error (MSE) criterion. The performance and reliability of the Bayesian estimators were also assessed under different values of θ and sample size, taking into account the four assessed under different values of θ and sample size, taking into account the four types of

weakly informative Gamma priors to study the sensitivity and robustness of the estimators for selecting a weakly informative prior distribution. The results of parameter estimators are shown in tables (4), (5), (6) and (7) and used bridge sampling package in R language.

Table 4. ML and Bayesian estimators of $Inv.Akash$ parameter with a weakly informative $Gamma(2,1)$ at different values of θ and sample sizes.

n	θ	$\hat{\theta}_{ML}$	$\hat{\theta}_{B}$	$bias_{ML}$	$bias_{B}$	MSE_{ML}	MSE_{B}
	0.2	0.2069	0.2028	0.0069	0.0028	0.0006	0.0002
	0.5	0.4884	0.4954	-0.0160	-0.0046	0.0023	0.0007
	0.8	0.8032	0.8023	0.0032	0.0013	0.0068	0.002
25	1	1.0218	1.0087	0.218	0.0087	0.0134	0.004
	2	2.0451	0.018	0.0451	0.018	0.0805	0.0241
	4	4.0803	4.0321	0.0803	0.0321	0.6074	0.1822
	8	7.634	7.3536	-0.366	-0.1464	1.4284	0.4235
	0.2	0.2011	0.2004	0.0011	0.0004	0.0002	0.0001
	0.5	0.4989	0.4996	-0.0011	-0.0004	0.0013	0.0004
	0.8	0.8156	0.8063	0.0156	0.0063	0.0064	0.0019
50	1	1.057	1.0228	0.057	0.0228	0.0105	0.0032
	2	2.029	2.0077	0.0192	0.0077	0.0333	0.0100
	4	4.0627	4.0251	0.0627	0.0251	0.2493	0.0748
	8	8.5416	8.2166	0.5416	0.2166	1.6372	0.4912
	0.2	0.1999	0.1999	-0.0001	-0.0001	0.0001	0.0000
	0.5	0.5037	0.5015	0.0037	0.0015	0.001	0.0003
	0.8	0.8097	0.8039	0.0097	0.0039	0.0024	0.0007
100	1	1.0025	1.001	0.0025	0.001	0.004	0.0012
	2	1.9808	1.9923	-0.0192	-0.0077	0.0123	0.0037
	4	3.981	3.9924	-0.019	-0.0076	0.0913	0.0274
	8	7.9651	7.986	-0.0349	-0.014	0.4892	0.1468

Table 5. ML and Bayesian estimators of $Inv.Akash$ parameter with a weakly informative $Gamma(0.5,0.5)$ at different values of θ and sample sizes.

n	θ	$\hat{\theta}_{ML}$	$\hat{\theta}_{B}$	$bias_{ML}$	$bias_{B}$	MSE_{ML}	MSE_{B}
	0.2	0.2065	0.2026	0.0065	0.0026	0.0007	0.0002
	0.5	0.4888	0.4955	-0.0012	-0.0045	0.0016	0.0005
	0.8	0.7921	0.7969	-0.0079	-0.0031	0.0039	0.0012
25	1	0.9994	0.9998	-0.0006	-0.0002	0.0113	0.0034
	2	2.0657	2.0263	0.0657	0.0263	0.0598	0.0179
	4	4.2020	4.0808	0.2020	0.0808	0.3835	0.1151
	8	7.7781	7.9112	-0.2219	-0.0888	1.8233	0.5470
	0.2	0.1993	0.1997	-0.0007	-0.0003	0.0003	0.0001
	0.5	0.5003	0.5001	0.0003	0.0001	0.0015	0.0004
	0.8	0.8098	0.8039	0.0098	0.0039	0.0047	0.0014
50	1	1.0137	1.0055	0.0137	0.0055	0.0071	0.0021
	2	2.0352	2.0141	0.0352	0.0141	0.0525	0.0158
	4	4.0945	4.0378	0.0945	0.0378	0.1850	0.0555
	8	8.3537	8.1413	0.3537	0.1415	1.1815	0.3544
	0.2	0.2024	0.201	0.0024	0.001	0.0001	0.0000
	0.5	0.4962	0.4985	-0.0038	-0.0015	0.0009	0.0003
	0.8	0.7888	0.7955	-0.0112	-0.0045	0.0022	0.0007
100	1	0.9996	0.9998	-0.0004	-0.0002	0.0034	0.0010
	2	1.9939	1.9975	-0.0061	-0.0025	0.0189	0.0057
	4	3.9156	3.9662	-0.0344	-0.0338	0.1115	0.0335
	8	7.8828	7.9531	-0.1172	-0.0469	0.4581	0.1374

Table 6. ML and Bayesian estimators of $Inv.Akash$ parameter with a weakly informative $Gamma(0.25, 0.5)$ at different values of θ and sample sizes.

n	θ	$\hat{\theta}_{ML}$	$\hat{\theta}_B$	$bias_{ML}$	$bias_B$	MSE_{ML}	MSE_B
	0.2	0.2032	0.2023	0.0032	0.0013	0.0006	0.0002
	0.5	0.5024	0.5009	0.0024	0.009	0.003	0.0009
	0.8	0.7958	0.7983	-0.0042	-0.0017	0.0086	0.0026
25	1	1.0263	1.0105	0.0263	0.0105	0.0164	0.0049
	2	2.0276	2.0121	0.0276	0.0111	0.0771	0.0231
	4	4.0269	4.0107	0.0269	0.0107	0.3142	0.0942
	8	7.8222	7.9289	-0.1778	-0.0711	1.5256	0.4577
	0.2	0.203	0.2012	0.003	0.0012	0.0002	0.0001
	0.5	0.5075	0.503	0.0075	0.003	0.0025	0.0008
	0.8	0.7932	0.7973	-0.0068	-0.0027	0.0041	0.0012
50	1	0.9915	0.9966	-0.0085	-0.0034	0.0055	0.0017
	2	2.0141	2.0056	0.0141	0.0056	0.0321	0.0096
	4	3.9485	3.9792	-0.0515	-0.0206	0.141	0.0423
	8	8.336	8.1344	0.336	0.1344	0.8284	0.2485
	0.2	0.2019	0.2008	0.0019	0.0008	0.0002	0.0001
	0.5	0.4968	0.4987	-0.0032	-0.0013	0.0007	0.0002
	0.8	0.8011	0.8004	0.0011	0.0004	0.0024	0.0007
100	1	1.0021	1.0008	0.0021	0.0008	0.0049	0.0015
	2	2.0109	2.0043	0.0109	0.0043	0.023	0.0069
	4	4.045	4.018	0.045	0.018	0.0675	0.0202
	8	8.2267	8.0907	0.2267	0.0907	0.6382	0.1915

Table 7. ML and Bayesian estimators of $Inv.Akash$ parameter with a weakly informative $Gamma(2, 2)$ at different values of θ and sample sizes.

n	θ	$\hat{\theta}_{ML}$	$\hat{\theta}_B$	$bias_{ML}$	$bias_B$	MSE_{ML}	MSE_B
	0.2	0.2034	0.2013	0.0034	0.0013	0.0007	0.0002
	0.5	0.5122	0.5049	0.0122	0.0049	0.0033	0.001
	0.8	0.7937	0.7975	-0.0063	-0.0025	0.0082	0.0025
25	1	0.9939	0.9976	-0.0061	-0.0024	0.0093	0.0028
	2	2.0784	2.0313	0.0784	0.0313	0.06	0.013
	4	4.0904	4.0362	0.0904	0.0362	0.3446	0.1034
	8	8.4435	8.1774	0.4435	0.1774	3.6378	1.0913
	0.2	0.2011	0.2005	0.0011	0.0005	0.0002	0.0001
	0.5	0.5039	0.5016	0.0039	0.0016	0.0015	0.0005
	0.8	0.8037	0.8015	0.0037	0.0015	0.0044	0.0013
50	1	1.0026	1.0011	0.0026	0.0011	0.0072	0.0022
	2	2.0249	2.0059	0.0149	0.0059	0.0239	0.0072
	4	4.0098	4.0039	0.0098	0.0039	0.2379	0.0714
	8	8.0679	8.0271	0.0679	0.0271	0.9877	0.2953
	0.2	0.2037	0.2015	0.0037	0.0015	0.0001	0.0000
	0.5	0.5030	0.5012	0.0030	0.0012	0.0006	0.0002
	0.8	0.8084	0.8034	0.0084	0.0034	0.0013	0.0004
100	1	0.9996	0.9998	-0.0004	-0.0002	0.0034	0.001
	2	1.9911	1.9964	-0.0089	-0.0036	0.0207	0.0062
	4	3.9679	3.9871	-0.0321	-0.0129	0.0628	0.0188
	8	8.0232	8.0093	0.0232	0.3989	0.0093	0.1197

The results in tables (4), (5), (6) and (7) showed that the ML estimator and Bayesian estimator calculated using the importance sampling method are getting closer to the true values of parameter as the sample size increases. There are clear decreases in both $bias$ and MSE was observed when moving from n=25 to 50 and then 100.

However Bayesian estimator generally performed better than ML estimator, achieving smaller MSE values in all of the cases studied. This superiority was more pronounced in small sample sizes and large values of θ. For example, when n=25 , θ=8, the MSE decreased from approximately 1.8233 for the ML estimator to 0.547 for Bayesian estimator using a weakly informative $Gamma$ (2, 2) prior in table (7). A similar was observed when using a weakly informative prior $Gamma$ (0.5, 0.5) in table (5) with the MSE decreasing from approximately 1.5256 to 0.4577, also the same pattern has emerged when using $Gamma$ (0.25, 0.5) and $Gamma$ (2, 1) cases shown in tables (6) and (4) respectively. These results indicate that the incorporating the prior information into Bayesian technique leads to a reduction in the variance of an estimation also an improvement in their accuracy, this is especially true in small samples and with high parameter values. As the sample size increases, the estimators converge due to the growing influence of sample information, compared to the prior information, explaining the advantage of the Bayesian estimator having because it has the smallest MSE in the majority of the cases. Accordingly, the Bayesian importance sampling estimator provides more efficient and stable estimators than ML. estimators within the studied $Inv.Akash$ distribution.

The efficiency and stability of the importance sampling algorithm were evaluated using the standard effective sample size ratio $D = ESS/M$. The sensitivity of Bayesian estimators to θ was assessed using a relative coefficient of variation (RCV) defined as:

$$RCV(\hat{\theta}_B) = \frac{max(\hat{\theta}_B) - min(\hat{\theta}_B)}{\bar{\hat{\theta}}_B} \times 100 \tag{33}$$

where $\bar{\hat{\theta}}_B = \sum_{i=1}^{4} \hat{\theta}_{iB}/4$. The RCV evaluated based on the estimators obtained by using four weakly informative $Gamma$ priors.

Table 8. The effective sample size ratio D and Relative coefficient of variation of Bayesian estimators of θ

n	θ	D_1	D_2	D_3	D_4	RCV
25	0.2	0.68	0.785	0.853	0.723	0.25%
	0.5	0.753	0.855	0.887	0.856	0.9%
	0.8	0.818	0.852	0.907	0.900	0.7%
	1	0.826	0.889	0.881	0.854	1.3%
	2	0.71	0.728	0.694	0.709	0.9%
	4	0.73	0.305	0.378	0.131	1.7%
	8	0.851	0.152	0.013	0.2236	3.4%
50	0.2	0.811	0.829	0.85	0.821	0.3%
	0.5	0.851	0.884	0.866	0.87	0.7%
	0.8	0.893	0.904	0.911	0.867	1.2%
	1	0.857	0.902	0.89	0.904	2.6%
	2	0.713	0.683	0.65	0.715	0.1%
	4	0.374	0.38	0.373	0.195	0.8%
	8	0.948	0.148	0.161	0.326	0.9%
100	0.2	0.791	0.795	0.862	0.846	0.8%
	0.5	0.873	0.862	0.892	0.87	0.5%
	0.8	0.877	0.867	0.89	0.898	1.3%
	1	0.88	0.889	0.899	0.885	0.2%
	2	0.732	0.69	0.734	0.709	0.6%
	4	0.323	0.329	0.346	0.254	1.3%
	8	0.117	0.188	0.174	0.533	0.2%

The results shown in table (8) indicate that the algorithm generally exhibits good efficiency, with many D values exceeding 0.80, corresponding to ESS exceeding 800 out of 1000 samples generated from the proposal distribution. However some cases exhibited low D values, suggesting reduced computational efficiency for certain

parameter values and weakly informative prior $Gamma$ distributions. Conversely, the RCV values were generally low, ranging approximately between 0.1% and 3.4%, indicating low sensitivity of the Bayesian estimators to variations in the studied sample sizes and parameter of the $Inv.Akash$ distribution. The RCV values for n = 100 case showed relatively higher stability, with an RCV of approximately 0.007, which is lower than the RCV values for n = 25, 50 cases. Overall, the results indicate that the importance sampling algorithm has a good degree of efficiency and stability, with some variation in efficiency depending on the value of θ, the sample sizes and weakly informative Gamma priors. When the goal is to select the best prior Gamma distribution, it is not permissible to rely on a single measure, because (MSE, D, RCV) measure different aspects.

The first measures of the accuracy of θ_B, the second measures the efficiency of the importance sampling and the third measures the stability of the estimator and its sensitivity to the prior distribution. The best prior distribution is chosen, which gives the lowest MSE, highest D, and lowest RCV together. Also since most values of $RCV < 5\%$ so that Bayesian estimators are insensitive to changes in weakly informative prior distributions.

Note that D_1, D_2, D_3, D_4 were calculated based on weakly informative $Gamma(2,1)$, $Gamma(0.5, 0.5)$, $Gamma(0.25, 0.5)$ and $Gamma(2,2)$ respectively.

The ML estimator of the pdf and cdf of Y have been estimated and also estimated by Bayesian importance sampling based on (30) and (31) respectively. The curves of real and estimated pdf and cdf were plotted using two estimation methods in a single plane, as shown in figure(4).

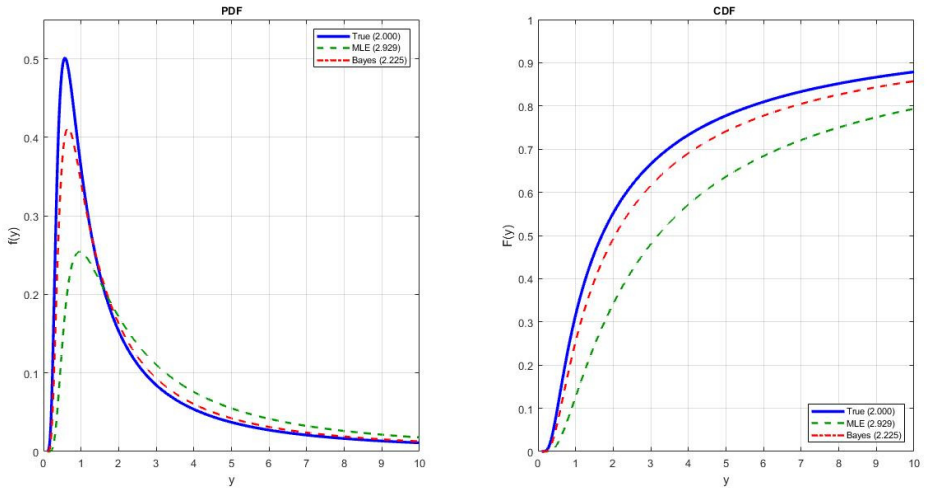

Figure 4. The ML and Bayesian estimators and real curves of pdf, cdf of $Inv.Akash$ distribution

5.2. Real data application

. This section uses a real data set to illustrate the importance of $Inv.Akash$ distribution in modeling reliability and engineering maintenance. This data represents repair times (in hours) for airborne communication transceivers. The sample consists of (40) values of observations displayed in table (9) [25].

Table 9. Active repair times in (hours)

0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8
1	1	1	1	1.1	1.3	1.5	1.5
1.5	1.5	2	2	2.2	2.5	2.7	3
3	3.3	4	4.5	5.4	5.4	7	7.5
8.8	9	10.2	22	24.5	0.7	4	4.7

Some descriptive statistics for above data were displayed in table (10).

Table 10. Descriptive statistics for active repair times data

$Mean$	$Median$	Q_1	Q_3	$Sample\ Variane$	$skewness$	$Kurtosis$
4.0125	2.1	1	4.85	26.6801	2.7171	7.5433

Table (10) shows that a high skewness values indicates positive skewness and a pronounced right tail in the data, while excessive kurtosis suggests tail that is relatively heavier than a normal distribution. To understand the nature and behavior of the data, it was graphically represented using a TTT plot, box plot and a histogram, as shown in Figure (5).

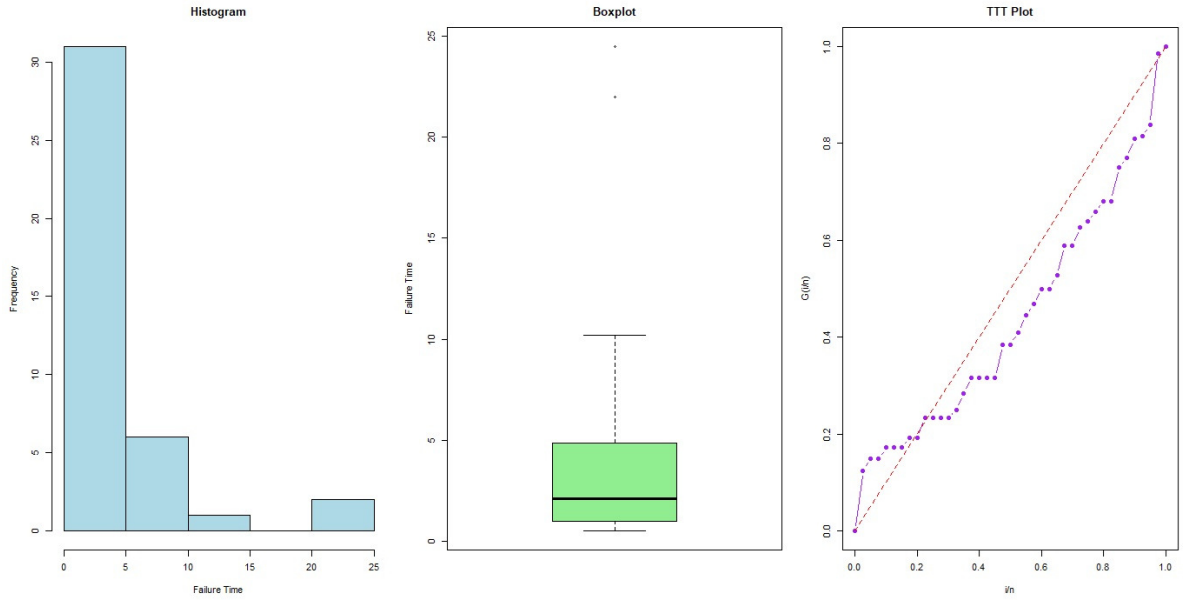

Figure 5. TTT plot, Box plot and histogram of dataset.

The histogram revealed an asymmetric distribution with a right skew which is consistent to the positive value of skewness coefficient. The box plot showed a spread data with values at the right end, reflecting the right long tail. The TTT curve in TTT plot exhibits a transition from a predominantly concave pattern in the early portion of the sample to a more convex pattern in the later portion. This concave to convex behavior indicates that the hazard rate may be non-monotonic and may be an upside-down bathtub shape. This provides an initial indication of the suitability of inverse distributions for the data.

The data were fitted to the $InvAkash$, $Inv.Rayleigh$ [26], One parameter $Inv.Weibull$ [27], and one parameter $Inv.Lomax$ [28] distributions where the pdf and cdf of last three distributions defined below.

1-Inv. Rayleigh distribution

$$f_Y(y) = \left(\frac{2\theta}{y^3}\right) e^{-\frac{\theta}{y^2}} \quad ;\theta, y > 0 \tag{34}$$

$$F_Y(y) = e^{-\frac{\theta}{y^2}} \tag{35}$$

2-One parameter Inv. Weibull distribution

$$f_Y(y) = \theta y^{-(\theta+1)} e^{-\left(\frac{1}{y}\right)^{\theta}} \quad ;\theta, y > 0 \tag{36}$$

$$F_Y(y) = e^{-\left(\frac{1}{y}\right)^{\theta}} \tag{37}$$

3-One parameter Inv. Lomax distribution

$$f_Y(y) = \left(\frac{\theta}{y^2}\right)\left(1+\frac{1}{y}\right)^{-(\theta+1)} \quad ;\theta, y > 0 \tag{38}$$

$$F_Y(y) = \left(1+\frac{1}{y}\right)^{-\theta} \tag{39}$$

The parameters of these four distributions were fitted by ML method and the fit quality was assessed in table (11) using Kolmogorov-Smirnov (KS) test, $-2logL$,AIC and BIC criteria. The results showed that the Inv.Akash distribution performed the best according to KS test, regarding the lowest value (KS=0.1934) and the highest $p-value$ (p=0.5922), indicating that the hypothesis of fitting distribution was not rejected at significant level of 0.05.

Table 11. Goodness of fit statistics for active repair times data.

Model	***Estimatoin Method***	$\hat{\boldsymbol{\theta}}$	$\boldsymbol{-2logL}$	$\boldsymbol{KS}$	$\boldsymbol{p-value}$	$\boldsymbol{AIC}$	$\boldsymbol{BIC}$
Inv. Akash	*ML*	3.1634	-96.48	0.1934	0.5922	191.08	192.77
	Bayes	2.3730	—	—	—	182.60	183.96
Inv. Rayleigh	*ML*	0.4291	-119.05	0.4713	0.0000	240.11	241.8
	Bayes	1.7046	—	—	—	201.84	203.19
One parameter Inv. Weibull	*ML*	2.1996	-106.18	0.3544	0.0001	214.35	216.04
	Bayes	1.2259	—	—	—	187.45	188.89
One parameter Inv. Lomax	*ML*	3.2141	-97.38	0.2564	0.0104	196.76	198.45
	Bayes	0.2567	—	—	—	190.07	191.42

Also $Inv.Akash$ has smallest AIC and BIC with largest $log\ L$. The parameter of $Inv.Akash$ was estimated by Bayesian method using an acceptance sampling algorithm under four weakly informative $Gamma$ priors. The Bayesian estimators for functions of θ and a mean square error of that estimator did not change because of its insensitivity to changes in the prior distribution of θ or changes in hyperparameter.

values. The Bayesian estimator of the parameter of $Inv.Akash$ distribution under using importance sampling algorithm is: θ_B = 2.3730, and the value of $D = ESS/1000$ = 0.947. This means that the effective sample size was 947 out of 1000 draws, corresponding to D ratio of 0.947. This indicates high sampling efficiency and suggests that the importance standard weights are well behaved with limited weight variability and no substantial weight degeneracy. Also the RCV=0.1% which is less than 5%, this means the relative dispersion of the parameter estimator or the estimator of function of parameter have high degree of stability and accuracy. Bayesian estimators of the pdf and cdf of $InvAkash$ at all values of sample observations and the plotted with the two real pdf and cdf curves on the same plane.

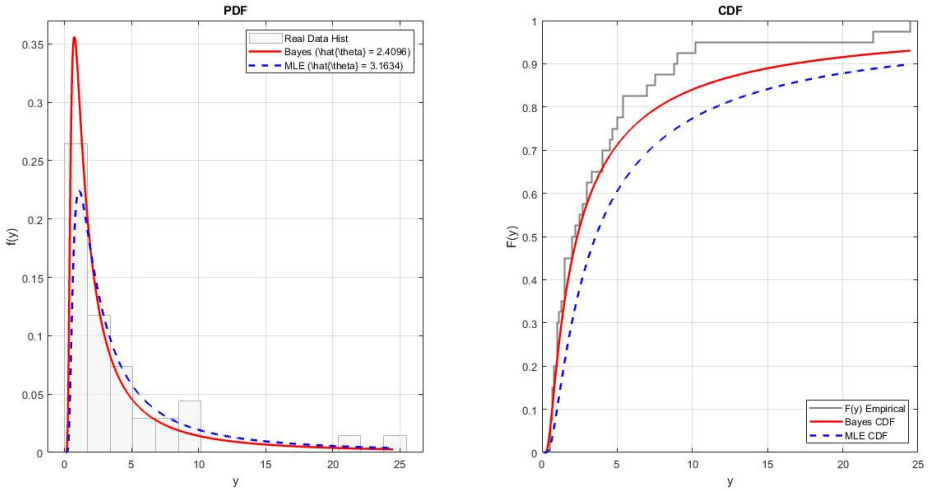

Figure 6. Real, ML and Bayesian estimator curves of *pdf* and *cdf* of $Inv.Akash$ distribution.

6. Contributions.

This article investigated some statistical properties and parameter estimation of $inv.Akash$ distribution using ML and Bayesian methods under squared error loss function. The pdf is unimodal while the hr function exhibits a non-monotonic upside-down bathtub shape. Other statistical properties including quartiles, mode and characteristic function were examined. A simulation study was conducted for different parameter values and sample sizes using four $Gamma$ prior distributions. The performance of the estimators was evaluated using bias, mean squared error (MSE) and effective sample size diagnostics. The results indicated that Bayesian estimation generally outperformed to ML estimation in terms of bias and MSE under the considered simulation settings. The effective sample size and relative computational variability diagnostics were also used to assess the reliability of the Bayesian estimates. Finally, a real dataset was fitted using $inv.Akash$, $inv.Rayleigh$, $inv.Weibull$ and $inv.Lomax$ distributions. Based on the reported Kolmogorov-Smirnov test, p-value,AIC and BIC results, the $inv.Akash$ distribution provided the best fit among the models considered. These findings suggest that the $inv.Akash$ distribution may be a useful alternative for modeling positive lifetime data with a non-monotonic hazard rate.

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