STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–16.
Published online in International Academic Press (www.IAPress.org)

Wavelets, unitary representations and Schauder bases in the numerical approximation of partial differential equations

Andrés Felipe Camelo Muñoz*, Carlos Alberto Ramírez Vanegas, Guillermo Villa Martínez

Department of Mathematics, Universidad Tecnológica de Pereira, Colombia

Abstract This paper develops a functional and harmonic framework for the numerical approximation of partial differential equations by wavelet methods and makes explicit its connection with a Haar-wavelet collocation discretization. The starting point is the approximation of functions through Schauder bases and finite-dimensional projections in Banach and Hilbert spaces. This structure is then linked to the continuous wavelet transform through unitary representations of locally compact groups, with particular emphasis on the real affine group and its dyadic discretization. The resulting construction provides a direct transition from the orbit of an affine representation to a finite-dimensional Haar approximation space used in the numerical scheme. As a model problem, we consider the one-dimensional viscous Burgers equation. A fully discrete Haar-wavelet collocation formulation is derived, including the treatment of the boundary conditions, spatial collocation, time discretization, and the nonlinear algebraic problem arising at each time level. The Cole–Hopf solution is used as an independent reference for numerical validation. The revised formulation is designed to permit systematic spatial and temporal refinement, error measurement in L^2 and L^∞, and comparison with standard numerical discretizations. In this way, the paper establishes an explicit bridge between the representation-theoretic origin of wavelet systems and their use as finite-dimensional approximation spaces for partial differential equations.

Keywords Affine group; Burgers equation; Collocation methods; Continuous wavelet transform; Haar wavelets; Harmonic analysis; Partial differential equations; Schauder bases; Unitary representations; Wavelet methods

AMS 2010 subject classifications 42C40, 43A65, 46B15, 65M70, 35Q35.

DOI: 10.19139/soic-2310-5070-5040

1. Introduction

Wavelet-based numerical methods play an important role in the approximation of solutions to partial differential equations because they combine spatial localization, multiscale structure, and efficient representations of functions with nonhomogeneous behavior. Since the foundational work of Meyer and Daubechies, wavelet theory has proved particularly effective in signal analysis, numerical approximation, and data processing, largely because it captures both global features and local phenomena within a unified framework [1, 2, 3, 4]. In computational settings, these properties lead to approximation schemes with good spatial resolution, structured matrices, and efficient Galerkin, collocation, and operational-matrix formulations [5, 6, 7, 8].

The role of wavelets in partial differential equations is not merely algorithmic. From the functional-analytic point of view, approximating a solution by finite linear combinations of basis functions may be viewed as a projection procedure onto finite-dimensional subspaces. In this context, Schauder bases provide an appropriate language for reconstruction in Banach spaces and for the abstract formulation of convergent approximation methods [9, 10, 11, 12]. In particular, Palomares, Pasadas, Ramírez, and Ruiz Galán showed that this viewpoint

*Correspondence to: Andrés Felipe Camelo (Email: afkamelo@utp.edu.co). Department of Mathematics, Universidad Tecnológica de Pereira. Barrio Álamos, Pereira, Risaralda, Colombia.

ISSN 2310-5070 (online) ISSN 2311-004X (print)
Copyright © 202x International Academic Press

can be used to justify numerical schemes for differential equations through inverse operators and basis-induced projections [13].

Continuous wavelet theory also admits a deeper interpretation in terms of harmonic analysis and unitary representations of locally compact groups. In this setting, the wavelet transform appears as a family of matrix coefficients associated with a unitary representation. Folland developed the general framework of analysis on groups and their L^p spaces, while Deitmar and Echterhoff presented the wavelet transform as a natural construction arising from matrix coefficients of unitary representations [14, 15]. Fuhr further showed that continuous wavelet transforms can be treated within abstract harmonic analysis as square-integrable representations equipped with reproducing formulas [16]. Thus, Haar measure, cyclic functions, and admissibility are not auxiliary notions, but part of the mathematical structure underlying continuous reconstruction.

The basic example is the real affine group $ax + b$, whose unitary representation on $L^2(\mathbb{R})$ yields the classical continuous wavelet transform. On this basis, Daubechies developed a robust theory of orthonormal wavelets, while Mallat introduced multiresolution analysis as the bridge between the continuous theory and the discrete systems used in applications [2, 3]. Christensen, in turn, showed that these discrete systems can be studied within the theory of frames and Riesz bases, which is especially useful for questions of stability and reconstruction in numerical analysis [17]. The passage from the continuous to the discrete setting should therefore be viewed not as a purely computational step, but as a structural transition supported by harmonic analysis and basis theory.

The numerical use of wavelets for differential equations has developed along several complementary directions. Cohen established a systematic wavelet framework for adaptive and nonlinear numerical approximation [5], while Urban developed wavelet constructions for the numerical treatment of elliptic partial differential equations [6]. More recently, Zhou presented a broad computational framework for nonlinear problems based on wavelet discretization [7], and Qasim and Al-Rawi considered operational and collocation-based wavelet techniques for partial differential equations [8]. These contributions show that wavelet methods are particularly effective when the multiscale structure of the basis is combined with a finite-dimensional discretization adapted to the differential operator.

Two recent contributions published in *Statistics, Optimization & Information Computing* are especially relevant to the present work. Camelo, Ramírez, and González studied the representation of functions in Hilbert spaces using Haar wavelet series and showed, through numerical examples, that increasing the resolution level improves the approximation of functions represented in the Haar basis [18]. Their formulation provides the computational starting point for interpreting Haar expansions as finite-dimensional approximation spaces. More recently, Camelo, Ramírez, and Villa developed a representation-theoretic framework for constructing Schauder bases in Hilbert spaces from unitary representations of locally compact groups [19]. In particular, they emphasized the affine group, the associated continuous wavelet transform, admissibility, dyadic discretization, and the construction of orthonormal—hence Schauder—systems through multiresolution analysis. These results provide the functional and harmonic ingredients required to relate abstract group-generated systems to concrete wavelet bases.

The present paper builds on these two complementary viewpoints but addresses a different question. The first of the above works focuses on the approximation of functions by Haar wavelet series, whereas the second develops the abstract connection between unitary group actions and Schauder bases. Neither work develops the resulting structure into a complete numerical discretization of a partial differential equation. The gap addressed here is therefore the transition from a representation-theoretic wavelet system to a finite-dimensional differential scheme. Our aim is to make this passage explicit and to show how the abstract orbit generated by the affine representation leads, after dyadic discretization and finite truncation, to the approximation space used in a Haar-wavelet collocation method.

The motivation of the present work is consequently not to introduce a new wavelet family, but to establish a mathematically transparent connection between the functional-analytic construction of the basis and its use in a numerical PDE scheme. The procedure may be summarized as follows: a unitary affine representation generates a continuous family of analyzing functions; restriction to a dyadic parameter set produces a discrete wavelet system; truncation of this system defines a finite-dimensional approximation space; and collocation on that space transforms the differential problem into a finite-dimensional algebraic system. In this sense, the numerical scheme is derived

as the final stage of the same approximation structure introduced at the functional and harmonic levels, rather than as an unrelated computational construction.

Accordingly, the main contributions of this paper are threefold. First, we make explicit the transition

$$\begin{aligned} \text{affine unitary representation} &\longrightarrow \text{dyadic wavelet system}, \\ \text{dyadic wavelet system} &\longrightarrow \text{finite-dimensional Haar space}, \\ \text{finite-dimensional Haar space} &\longrightarrow \text{discrete differential problem}. \end{aligned}$$

thereby clarifying the mathematical link between the representation-theoretic construction and the numerical basis. Second, we derive a fully specified Haar-wavelet collocation formulation for the one-dimensional viscous Burgers equation, including the reconstruction of lower-order derivatives from the Haar expansion of the highest-order spatial derivative, the incorporation of homogeneous boundary conditions, the temporal discretization, and the nonlinear algebraic system arising at each time level. Third, we use the Cole–Hopf solution as an independent benchmark and formulate a numerical validation strategy based on spatial and temporal refinement, L^2- and L^∞-errors, observed convergence rates, and comparison with a standard numerical discretization.

The one-dimensional viscous Burgers equation is particularly suitable for this purpose. It combines nonlinear convection with viscous diffusion and is therefore a standard test problem for nonlinear numerical schemes [20, 21]. At the same time, the Cole–Hopf transformation provides an analytical representation against which the wavelet approximation can be evaluated independently. This separation between the numerical approximation and the reference solution makes it possible to identify the effects of wavelet resolution, temporal discretization, nonlinear iteration, and boundary treatment on the overall error.

The paper is organized as follows. Section 2 introduces the functional and harmonic preliminaries required for the approximation framework, including Schauder bases, locally compact groups, Haar measure, and unitary representations. Section 3 develops the continuous wavelet transform and specializes the construction to the affine group. Section 4 establishes the passage from the affine representation to the discrete Haar approximation space and derives the corresponding collocation formulation for the Burgers equation. Section 5 presents the fully discrete algorithm and its numerical implementation. Section 6 reports the numerical experiments, including comparison with the Cole–Hopf reference solution, spatial and temporal refinement studies, error measurements, observed convergence rates, and comparison with a standard numerical method. The final section summarizes the main conclusions and limitations of the approach.

2. Mathematical preliminaries

This section collects the functional and harmonic background needed for the wavelet formulation used later in the numerical approximation of partial differential equations. From the functional point of view, approximating a solution by finite linear combinations of basis functions naturally leads to the theory of expansions in Banach spaces. The classical works of Schauder introduced the notion of a basis in normed spaces, while Banach established the general framework of complete linear spaces in which this theory is developed [10, 9]. In a more modern setting, texts by Rudin, Conway, and Brezis show how bases, projections, and the structure of Hilbert spaces provide the basic tools for the abstract study of approximation problems and linear operators [11, 22, 12].

Definition 2.1. Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A sequence $\{x_n\}_{n\geq 1} \subset X$ is called a *Schauder basis* if for every $x \in X$ there exists a unique sequence of scalars $\{a_n\}_{n\geq 1} \subset \mathbb{K}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n,$$

with convergence in the norm of X.

A Schauder basis allows one to reconstruct every element of the space from its coefficients with respect to an ordered system. If $\{x_n\}_{n\geq 1}$ is a Schauder basis of X, the uniqueness of the expansion defines, for each $n \in \mathbb{N}$, the

coordinate functional

$$x_n^* : X \to \mathbb{K}, \qquad x_n^*(x) = a_n,$$

where $\{a_n\}$ are the coefficients of x with respect to the basis. The associated partial sum operators are then given by

$$P_N x = \sum_{n=1}^{N} x_n^*(x)\, x_n, \qquad N \in \mathbb{N}.$$

Proposition 2.2. Let $\{x_n\}_{n\geq 1}$ be a Schauder basis of a Banach space X, and let

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n$$

be the corresponding expansion. Then:

1. each coordinate functional $x_n^* : X \to \mathbb{K}$ is linear and bounded;
2. each partial sum operator
$$P_N x = \sum_{n=1}^{N} x_n^*(x) x_n$$
is linear and bounded;
3. for every $x \in X$,
$$P_N x \longrightarrow x \qquad \text{in } X.$$
Moreover,
$$\sup_{N\geq 1} \|P_N\| < \infty.$$

These properties are standard consequences of the theory of Schauder bases; see, for instance, [11, 22]. In particular, the boundedness of the coordinate functionals and partial sum operators is a structural property of a Schauder basis and will be used below in the approximation argument.

For the purposes of this paper, the main point is that a Schauder basis allows one to replace an infinite-dimensional problem by a sequence of finite-dimensional truncated problems. This idea appears explicitly in the work of Palomares, Pasadas, Ramírez, and Ruiz Galán, who showed that projections associated with a basis can be used to justify approximation schemes for differential equations through inverse operators [13].

Proposition 2.3. Let X and Y be Banach spaces, and let

$$D : X \to Y$$

be a bounded linear operator that is injective and has closed range $D(X)$. Assume that $D(X)$, endowed with the norm inherited from Y, admits a Schauder basis $\{y_n\}_{n\geq 1}$, with associated partial sum operators P_N.

If $x_0 \in X$ satisfies

$$Dx_0 = y_0, \qquad y_0 \in D(X),$$

then

$$x_0 = \lim_{N\to\infty} D^{-1}(P_N y_0),$$

and

$$\|x_0 - D^{-1}(P_N y_0)\|_X \leq \|D^{-1}\|\, \|y_0 - P_N y_0\|_Y.$$

Proof
Since $D(X)$ is closed in Y, it is a Banach space. The operator

$$D : X \to D(X)$$

is therefore a bounded linear bijection. By the bounded inverse theorem,

$$D^{-1} : D(X) \to X$$

is bounded.

Because $\{y_n\}_{n\geq 1}$ is a Schauder basis of $D(X)$,

$$P_N y_0 \in D(X)$$

for every N, and

$$P_N y_0 \longrightarrow y_0 \qquad \text{in } Y.$$

Hence,

$$D^{-1}(P_N y_0) \longrightarrow D^{-1}(y_0) = x_0.$$

Moreover,

$$\begin{aligned} \|x_0 - D^{-1}(P_N y_0)\|_X &= \|D^{-1}(y_0 - P_N y_0)\|_X \\ &\leq \|D^{-1}\| \, \|y_0 - P_N y_0\|_Y, \end{aligned}$$

which proves the result. □

The proposition provides an abstract justification for projection-based approximations of differential problems whenever the differential operator has a bounded inverse on its range and that range admits a Schauder basis. In this setting, the approximation error of the solution is controlled by the truncation error of the corresponding basis expansion. This principle motivates the use of finite wavelet spaces in the numerical formulation developed later.

Theorem 2.4. Every separable Hilbert space has a countable orthonormal basis. In particular, every orthonormal basis of a separable Hilbert space is a Schauder basis.

Proof
The existence of a countable orthonormal basis in a separable Hilbert space is classical. If $\{e_n\}_{n\geq 1}$ is such a basis and $x \in H$, then

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

with convergence in norm by Parseval's identity. Uniqueness follows from orthonormality. □

The Hilbert-space setting will be the natural framework for the wavelet theory developed later, since discrete wavelet systems are usually constructed as orthonormal systems or, more generally, as Riesz bases or stable frames [2, 17].

2.1. Locally compact groups and Haar measure

In its abstract formulation, continuous wavelet theory arises from the study of matrix coefficients of unitary representations of locally compact groups. This connection was systematically developed in harmonic analysis by authors such as Hewitt and Ross, Reiter and Stegeman, Folland, and, more recently, Deitmar and Echterhoff [23, 24, 14, 15]. We now recall the basic framework of locally compact groups and Haar measure.

Definition 2.5. A *locally compact group* is a Hausdorff topological group G such that every point of G has a compact neighborhood.

The importance of this class of groups lies in the existence of an intrinsic integration theory. The fundamental result is Haar's theorem, presented in detail in the classical texts of Hewitt–Ross, Reiter–Stegeman, and Folland [23, 24, 14].

Theorem 2.6 (Haar measure)**.** Every locally compact group G admits a nontrivial regular Borel measure μ that is left-invariant, in the sense that

$$\mu(gE) = \mu(E) \qquad \text{for every } g \in G$$

and every Borel set $E \subset G$. Moreover, this measure is unique up to a positive multiplicative constant.

Proof
The full proof belongs to the general theory of locally compact groups and relies on compactness, regularity, and invariant functional arguments. We use this theorem here as a basic tool. □

Definition 2.7. A measure μ as in the previous theorem is called a *left Haar measure* on G.

The existence of Haar measure makes it possible to define the spaces $L^p(G)$, and in particular the Hilbert space $L^2(G)$, which is the natural codomain for many transforms associated with unitary representations. In both the classical treatment of Folland and the more abstract perspective of Deitmar and Echterhoff, this step is essential in order to interpret such transforms as operators between Hilbert spaces [14, 15].

Definition 2.8. Let G be a locally compact group endowed with a left Haar measure μ. The *left regular representation* of G on $L^2(G)$ is defined by

$$(\lambda(g)f)(x) = f(g^{-1}x), \qquad f \in L^2(G),\ x \in G.$$

Proposition 2.9. The map

$$\lambda : G \to \mathcal{U}(L^2(G))$$

is a strongly continuous unitary representation.

Proof
Left invariance of Haar measure gives

$$\|\lambda(g)f\|^2_{L^2(G)} = \int_G |f(g^{-1}x)|^2 \, d\mu(x) = \int_G |f(x)|^2 \, d\mu(x) = \|f\|^2_{L^2(G)}.$$

Hence $\lambda(g)$ is unitary for every $g \in G$. The representation property

$$\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$$

is immediate. Strong continuity follows from the continuity of translations in $L^2(G)$. □

2.2. Unitary representations

The abstract construction of wavelet systems relies on unitary representations more general than the regular representation. This point of view appears clearly in the works of Folland, Fuhr, and Deitmar–Echterhoff, where the continuous wavelet transform is interpreted as a family of matrix coefficients associated with a unitary representation [14, 16, 15].

Definition 2.10. Let G be a locally compact group and let $\mathcal{H}$ be a Hilbert space. A map

$$\pi : G \to \mathcal{U}(\mathcal{H})$$

is called a *strongly continuous unitary representation* if

1. $\pi(e) = I_{\mathcal{H}}$;

2. $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for every $g_1, g_2 \in G$;
3. for every $v \in \mathcal{H}$, the map $g \mapsto \pi(g)v$ is continuous in the topology of $\mathcal{H}$.

Definition 2.11. Let $(\pi, \mathcal{H})$ be a unitary representation of G. Given $v \in \mathcal{H}$, the family

$$\mathcal{O}_\pi(v) = \{\pi(g)v : \ g \in G\}$$

is called the *orbit* of v under π. The function v is called *cyclic* if

$$\overline{\text{span } \mathcal{O}_\pi(v)} = \mathcal{H}.$$

The notion of cyclic function is central in representation theory, since it expresses that the action of the group on a single element generates the whole space densely. In the wavelet setting, this is particularly relevant because it suggests that the orbit of a single generating function may yield a system rich enough for reconstruction and approximation. This idea appears both in Fuhr's abstract harmonic analysis and in the formulation of the continuous wavelet transform given by Deitmar and Echterhoff [16, 15]. From the numerical point of view, it means that the later discretization of a suitably chosen orbit may produce expansion systems capable of approximating solutions to partial differential equations.

3. Continuous wavelet transform

Wavelet theory admits formulations at different levels of abstraction. In this section we adopt a point of view rooted in harmonic analysis and representation theory, in which the continuous wavelet transform is viewed as a family of matrix coefficients associated with a unitary representation. From this perspective, generalized multiresolution structures can be described in terms of representations, invariant subspaces, and dilation systems. This approach has been developed in several recent works, including those of Baggett, Furst, Merrill, and Packer, and of Bownik and Jahan, where wavelet constructions are studied in settings more general than the classical Euclidean one [25, 26].

Definition 3.1. Let (X, μ) and (T, ν) be measure spaces. Assume that $\phi \in L^2(X, \mu)$ and that for each $t \in T$ there is an associated element $\pi_t \phi \in L^2(X, \mu)$. The transformation

$$W_\phi : L^2(X, \mu) \to \mathbb{C}^T, \qquad W_\phi(\psi)(t) = \langle \psi, \pi_t \phi \rangle,$$

is called the *wavelet transform associated with* ϕ.

The main question is to determine under which assumptions this transform is injective and when it admits a reconstruction formula. Recent work shows that this question is not confined to the classical setting $L^2(\mathbb{R})$. For instance, Pandey, Dixit, Ado, and Singh extended the continuous wavelet transform to Sobolev spaces over locally compact abelian groups and proved corresponding approximation results in that framework [27]. This supports the view that the continuous wavelet transform should be understood as a construction naturally attached to harmonic analysis on groups.

Definition 3.2. Let G be a locally compact group and let $(\pi, \mathcal{H})$ be a unitary representation of G on a Hilbert space $\mathcal{H}$. Given $\phi \in \mathcal{H}$, the *generalized wavelet transform* associated with (π, ϕ) is defined by

$$W_\phi : \mathcal{H} \to \mathbb{C}^G, \qquad W_\phi(\psi)(g) = \langle \psi, \pi(g)\phi \rangle,$$

for every $\psi \in \mathcal{H}$ and $g \in G$.

Thus, the continuous wavelet transform is a matrix coefficient of the representation π. Its properties therefore depend on the structure of the representation and on the choice of the admissible function ϕ. The same point of view appears in recent work on systems more general than classical wavelets. For example, Bartolucci, De Mari, De Vito, and Odone showed that the affine Radon transform intertwines wavelet and shearlet representations unitarily, thereby illustrating the flexibility of the representation-theoretic framework [28].

Proposition 3.3. The transformation W_ϕ is injective if and only if ϕ is cyclic for the representation π.

Proof
Assume first that W_ϕ is injective. If the linear span of $\{\pi(g)\phi : \ g \in G\}$ were not dense in $\mathcal{H}$, then there would exist a nonzero function $\psi \in \mathcal{H}$ orthogonal to that subspace. Hence

$$\langle \psi, \pi(g)\phi \rangle = 0 \qquad \text{for every } g \in G,$$

so $W_\phi(\psi) = 0$. Since W_ϕ is injective, this implies $\psi = 0$, a contradiction. Therefore, ϕ is cyclic.

Conversely, assume that ϕ is cyclic and that $W_\phi(\psi) = 0$. Then

$$\langle \psi, \pi(g)\phi \rangle = 0 \qquad \text{for every } g \in G.$$

Thus ψ is orthogonal to the linear span of the orbit of ϕ, whose closure is $\mathcal{H}$. Hence $\psi = 0$, and W_ϕ is injective. □

Injectivity alone, however, is not enough for reconstruction or for a Hilbert-space formulation of the transform. For that, one needs the matrix coefficients associated with W_ϕ to belong to $L^2(G)$. In more general settings, this issue has been studied from several directions. Hemmat, Thirulogasanthar, and Krzyżak showed, for example, that discretizations of continuous wavelet transforms in nonclassical settings may still lead to stable discrete frames and bases in Hilbert spaces [29]. This is the role of admissibility.

Definition 3.4. Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G. A vector $\phi \in \mathcal{H}$ is called *admissible* if, for every $\psi \in \mathcal{H}$, the coefficient function

$$g \longmapsto \langle \psi, \pi(g)\phi \rangle$$

belongs to $L^2(G)$. Equivalently, the associated wavelet transform

$$W_\phi(\psi)(g) = \langle \psi, \pi(g)\phi \rangle$$

defines a bounded operator

$$W_\phi : \mathcal{H} \to L^2(G).$$

Admissibility ensures that the matrix coefficients generated by ϕ are square-integrable and that the wavelet transform can be treated as an operator between Hilbert spaces. In the case of square-integrable representations, this property leads to norm identities and reconstruction formulas under the corresponding representation-theoretic assumptions [16, 15]. This distinction between admissibility and reconstruction is important in the present context because the latter requires additional structural properties of the representation.

Theorem 3.5. Let $(\pi, \mathcal{H})$ be a square-integrable unitary representation of a locally compact group G, and let $\phi \in \mathcal{H}$ be an admissible vector. Assume that there exists a constant $C_\phi > 0$ such that

$$\langle W_\phi(\psi), W_\phi(\eta) \rangle_{L^2(G)} = C_\phi \, \langle \psi, \eta \rangle_{\mathcal{H}}$$

for every $\psi, \eta \in \mathcal{H}$. Then:

1. $W_\phi : \mathcal{H} \to L^2(G)$ is continuous and injective;
2. $$W_\phi^* W_\phi = C_\phi I_{\mathcal{H}};$$
3. for every $\psi \in \mathcal{H}$,
$$\psi = \frac{1}{C_\phi} W_\phi^* W_\phi(\psi).$$

Proof
Taking $\eta = \psi$ gives

$$\|W_\phi(\psi)\|^2_{L^2(G)} = C_\phi \|\psi\|^2_{\mathcal{H}},$$

and therefore W_ϕ is continuous and injective. Moreover, by the definition of the adjoint,

$$\langle W_\phi^* W_\phi(\psi), \eta\rangle_{\mathcal{H}} = \langle W_\phi(\psi), W_\phi(\eta)\rangle_{L^2(G)}.$$

Hence,

$$\langle W_\phi^* W_\phi(\psi), \eta\rangle_{\mathcal{H}} = C_\phi \langle \psi, \eta\rangle_{\mathcal{H}},$$

for every $\eta \in \mathcal{H}$. Therefore,

$$W_\phi^* W_\phi = C_\phi I_{\mathcal{H}},$$

and the reconstruction identity follows immediately. □

The basic example is the affine group on $\mathbb{R}^n$. Before introducing its algebraic structure, we recall the elementary operators involved in the formulation of the continuous wavelet transform. These operators also arise in more general settings. Hemmat, Thirulogasanthar, and Krzyżak showed that affine-type actions preserve frame and reconstruction properties even in quaternionic contexts [29], while Bartolucci, De Mari, De Vito, and Odone related the same structure of translations and dilations to shearlet systems through intertwining operators [28].

Definition 3.6. Let $\phi \in L^2(\mathbb{R}^n)$ and let $b \in \mathbb{R}^n$. The *translation operator* T_b is defined by

$$(T_b\phi)(x) = \phi(x - b), \qquad x \in \mathbb{R}^n.$$

Definition 3.7. Let $\phi \in L^2(\mathbb{R}^n)$ and let $a \in \mathbb{R}_+$. The *dilation operator* D_a is defined by

$$(D_a\phi)(x) = a^{-n/2}\phi\left(\frac{x}{a}\right), \qquad x \in \mathbb{R}^n.$$

Definition 3.8. Let $\phi \in L^2(\mathbb{R}^n)$ and let $c \in \mathbb{R}^n$. The *modulation operator* E_c is defined by

$$(E_c\phi)(x) = e^{2\pi i\, x\cdot c}\,\phi(x), \qquad x \in \mathbb{R}^n.$$

These operators are unitary on $L^2(\mathbb{R}^n)$ and form the basic building blocks of many continuous transforms in harmonic analysis. In the wavelet setting, the transform is generated by translations and dilations, while modulation becomes relevant in more general anisotropic frameworks [28].

The admissibility condition in this setting requires that $\phi \in L^2(\mathbb{R}^n)$ produce a square-integrable wavelet transform over the affine group, namely,

$$\int_G \left|\langle \phi, \pi(a,b)\phi\rangle\right|^2 d\mu(a,b) < \infty,$$

where the left Haar measure is given by

$$d\mu(a,b) = \frac{db\,da}{a^{n+1}}, \qquad a > 0,\ b \in \mathbb{R}^n.$$

Under this hypothesis, the associated continuous wavelet transform

$$W_\phi(\psi)(a,b) = \langle \psi, \pi(a,b)\phi\rangle, \qquad \psi \in L^2(\mathbb{R}^n),$$

is well defined as an operator on $L^2(G)$ and admits a reconstruction formula.

Although this formulation can be developed in arbitrary dimension, in this paper we restrict attention to the case $n = 1$, that is, to the positive real affine group

$$G = \mathbb{R}_+ \rtimes \mathbb{R},$$

also called the $ax + b$ group, with $a > 0$.

In this setting, the passage from the continuous to the discrete theory may be interpreted as a concrete form of multiresolution analysis, in the spirit of the generalized MRA framework studied by Baggett, Furst, Merrill, and Packer [25]. Related results of Bownik and Jahan show that this point of view extends beyond the classical Euclidean setting when the scale and translation structure is suitably adapted [26].

For the numerical formulation developed in the next section, the relevant step is the restriction of the continuous affine action to dyadic scales and translations. In one dimension, this leads to wavelet functions of the form

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \qquad j, k \in \mathbb{Z},$$

which are generated by discrete dilations and translations. When ψ is chosen as the Haar mother wavelet, the resulting family provides the piecewise-constant multiresolution structure used in the collocation scheme. Finite truncation at a fixed resolution level J then produces the finite-dimensional approximation space employed in the numerical discretization of the Burgers equation.

4. Formulation of the solution using wavelets

Consider the one-dimensional Burgers model problem

$$u_t + u\,u_x - \nu u_{xx} = 0, \qquad 0 < x < 1, \quad t > 0, \tag{1}$$

subject to the boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \qquad t > 0,$$

$$u(x,0) = \sin(\pi x), \qquad 0 < x < 1.$$

The goal of this section is to describe how a wavelet discretization of the spatial operator leads to an algebraic system at each time step. The central idea of the method is to expand the highest-order spatial derivative in a wavelet basis and to reconstruct the lower-order derivatives and the function itself by integration. This strategy makes it possible to translate the partial differential equation into a nonlinear algebraic system for the coefficients of the expansion.

4.1. Expansion of the highest-order derivative

Let $\{h_i\}_{i=1}^N$ be a Haar wavelet family defined on $[0,1]$. We propose the approximation

$$u_{xx}(x,t) \approx \sum_{i=1}^{N} a_i(t)\, h_i(x),$$

where the functions $a_i(t)$ are time-dependent coefficients to be determined.

Integrating once with respect to x, we obtain the approximation for the first derivative

$$u_x(x,t) \approx \sum_{i=1}^{N} a_i(t)\, p_{i,1}(x) + c_1(t),$$

where

$$p_{i,1}(x) = \int_0^x h_i(s)\, ds.$$

Integrating once more, we obtain an approximation for the solution

$$u(x,t) \approx \sum_{i=1}^{N} a_i(t)\, p_{i,2}(x) + x\, c_1(t) + c_2(t),$$

with

$$p_{i,2}(x) = \int_0^x p_{i,1}(s)\, ds.$$

The boundary conditions allow us to determine $c_1(t)$ and $c_2(t)$. Indeed, from $u(0,t) = 0$ it follows that

$$c_2(t) = 0.$$

Moreover, imposing $u(1,t) = 0$, we obtain

$$\sum_{i=1}^{N} a_i(t)\, p_{i,2}(1) + c_1(t) = 0,$$

and therefore

$$c_1(t) = -\sum_{i=1}^{N} a_i(t)\, p_{i,2}(1).$$

Substituting this expression into the previous formula yields

$$u(x,t) \approx \sum_{i=1}^{N} a_i(t) \Big[p_{i,2}(x) - x\, p_{i,2}(1) \Big].$$

Similarly,

$$u_x(x,t) \approx \sum_{i=1}^{N} a_i(t) \Big[p_{i,1}(x) - p_{i,2}(1) \Big].$$

These expressions show that once the coefficients $\{a_i(t)\}_{i=1}^{N}$ are determined, the solution u, its derivative u_x, and the second derivative u_{xx} are simultaneously reconstructed.

At a prescribed resolution level J, we take

$$N = 2^J,$$

so that the number of Haar basis functions coincides with the number of spatial collocation points introduced below. This choice produces a square algebraic system for the coefficients $\{a_i(t)\}_{i=1}^{N}$ at each time level.

4.2. Time discretization and spatial collocation

Let

$$0 = t_0 < t_1 < \cdots < t_M = T$$

be a uniform partition of the time interval, with

$$\Delta t = t_n - t_{n-1}.$$

At each time level t_n, the coefficients $a_i(t_n)$ are determined by combining the Haar spatial approximation with a first-order implicit discretization in time.

The time derivative at t_n is approximated by the backward difference

$$u_t(x,t_n) \approx \frac{u(x,t_n) - u(x,t_{n-1})}{\Delta t}.$$

The use of the backward difference is convenient because the nonlinear convective and diffusive terms are evaluated at the new time level t_n.

The spatial collocation points are chosen as

$$x_\ell = \left(\ell - \frac{1}{2}\right)\Delta x, \qquad \Delta x = 2^{-J}, \qquad \ell = 1, \ldots, N,$$

with

$$N = 2^J.$$

Substituting the approximations of u, u_x, and u_{xx} into (1), and evaluating the resulting equation at each collocation point x_ℓ, yields

$$\frac{u(x_\ell, t_n) - u(x_\ell, t_{n-1})}{\Delta t} + u(x_\ell, t_n)\, u_x(x_\ell, t_n) - \nu \sum_{i=1}^{N} a_i(t_n) h_i(x_\ell) = 0,$$

for

$$\ell = 1, \ldots, N.$$

Using the reconstruction formulas obtained in the previous subsection, this system can be written entirely in terms of the unknown coefficient vector

$$\boldsymbol{a}^{(n)} = \begin{bmatrix} a_1(t_n) & \cdots & a_N(t_n) \end{bmatrix}^T.$$

Thus, at each time step, the spatially discretized Burgers equation is reduced to a nonlinear algebraic system of N equations for the N unknown Haar coefficients.

4.3. Nonlinear iteration

The nonlinear term $u\, u_x$ is treated by Picard iteration. Let $u^{(m)}(x, t_n)$ denote the approximation at iteration m. The nonlinear product is linearized as

$$u(x, t_n) u_x(x, t_n) \approx u^{(m)}(x, t_n)\, u_x^{(m+1)}(x, t_n).$$

Consequently, given the coefficient vector $\boldsymbol{a}^{(n,m)}$, the next iterate $\boldsymbol{a}^{(n,m+1)}$ is obtained by solving the resulting linear algebraic system.

The iteration is initialized with the solution from the previous time level,

$$\boldsymbol{a}^{(n,0)} = \boldsymbol{a}^{(n-1)},$$

and is terminated when

$$\frac{\|\boldsymbol{a}^{(n,m+1)} - \boldsymbol{a}^{(n,m)}\|_2}{1 + \|\boldsymbol{a}^{(n,m+1)}\|_2} < \varepsilon,$$

where ε is a prescribed tolerance. In the numerical experiments reported below, the tolerance and maximum number of iterations are stated explicitly together with the remaining computational parameters.

4.4. Initial approximation

At $t_0 = 0$, the initial condition

$$u(x, 0) = \sin(\pi x)$$

is imposed at the collocation points. The initial Haar coefficient vector is obtained by requiring the reconstructed approximation

$$u_h(x, 0) = \sum_{i=1}^{N} a_i(0)\, [p_{i,2}(x) - x\, p_{i,2}(1)]$$

to satisfy

$$u_h(x_\ell, 0) = \sin(\pi x_\ell), \qquad \ell = 1, \ldots, N.$$

This provides the initial coefficient vector required to start the time-stepping procedure.

4.5. Functional and harmonic interpretation of the scheme

The preceding discretization provides the concrete link with the functional and harmonic framework introduced in Sections 2 and 3. From the functional point of view, truncation at level J replaces the infinite-dimensional problem by an approximation in the finite-dimensional space generated by the first $N = 2^J$ Haar functions. The corresponding coefficients determine the projection of the highest-order spatial derivative onto this space.

From the harmonic point of view, the Haar family is obtained from dyadic translations and dilations of a generating function, which may be interpreted as a discrete subset of the orbit associated with the affine representation. Thus, the passage from the continuous affine action to the finite Haar family provides the structural connection between the representation-theoretic construction and the numerical collocation scheme.

The numerical method therefore uses three consecutive levels of approximation: the affine representation generates the wavelet system, dyadic discretization produces the Haar family, and finite truncation combined with collocation converts the Burgers equation into a finite nonlinear algebraic problem.

5. Numerical results

This section presents numerical reference results for problem (1) with viscosity

$$\nu = 0.01$$

and initial condition

$$u(x, 0) = \sin(\pi x).$$

The solution used as a reference was obtained from the exact representation induced by the Cole–Hopf transformation. In this context, this solution plays the role of a *benchmark*: it does not replace the wavelet implementation, but rather provides a quantitative reference against which later approximations may be compared.

The Cole–Hopf transformation leads to a solution of the form

$$u(x,t) = 2\pi\nu \, \frac{\sum_{n\geq 1} n A_n e^{-n^2\pi^2\nu t} \sin(n\pi x)}{A_0/2 + \sum_{n\geq 1} A_n e^{-n^2\pi^2\nu t} \cos(n\pi x)},$$

where the coefficients $\{A_n\}$ come from the cosine expansion of the transformed data. In practice, this series was truncated to a finite number of terms and evaluated on a uniform mesh in space and time.

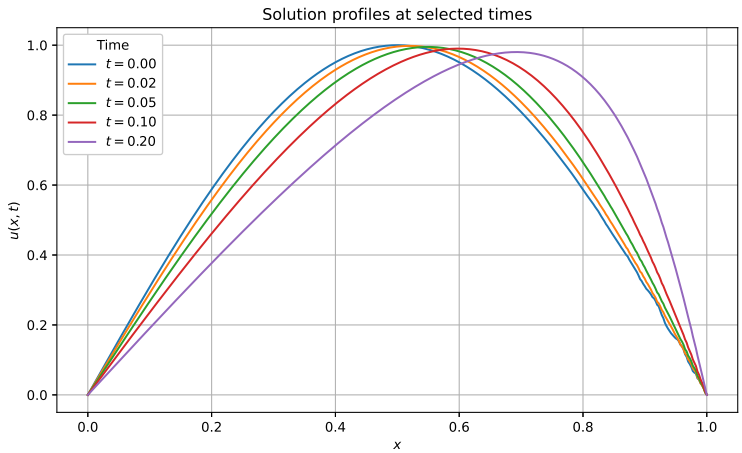


Figure 1. Profiles of the Burgers solution for different times.

5.1. Temporal profiles

Figure 1 shows several spatial profiles of the solution for increasing times. One observes the regular evolution of the profile and the dissipative effect induced by the viscous term.

5.2. Heat map

Figure 2 presents a heat map of $u(x,t)$ over the space–time domain. This representation makes it possible to visualize simultaneously the global evolution of the profile and the progressive smoothing of the solution.

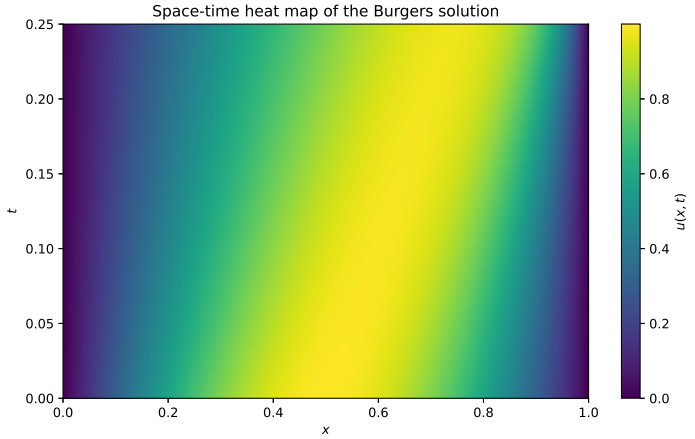


Figure 2. Heat map of the solution $u(x,t)$.

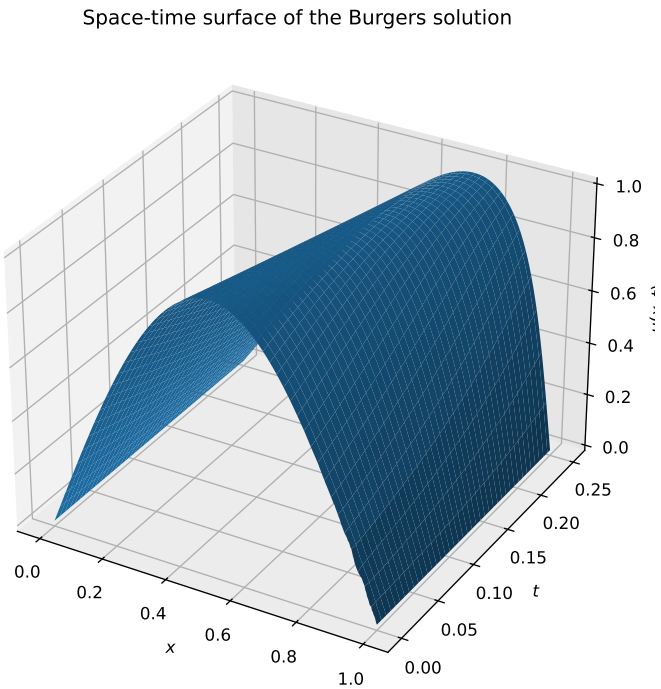


Figure 3. Surface of the Burgers solution.

5.3. *Surface of the solution*

Figure 3 shows the surface associated with $u(x,t)$, which makes it possible to appreciate jointly the spatial and temporal dependence of the solution.

5.4. *Reference profile at a fixed time*

Figure 4 presents the spatial profile of the solution at time

$$t = 0.02.$$

This profile is especially useful as a comparison curve for future collocation implementations using Haar wavelets, CAS, or Haar$*$CAS convolutions.

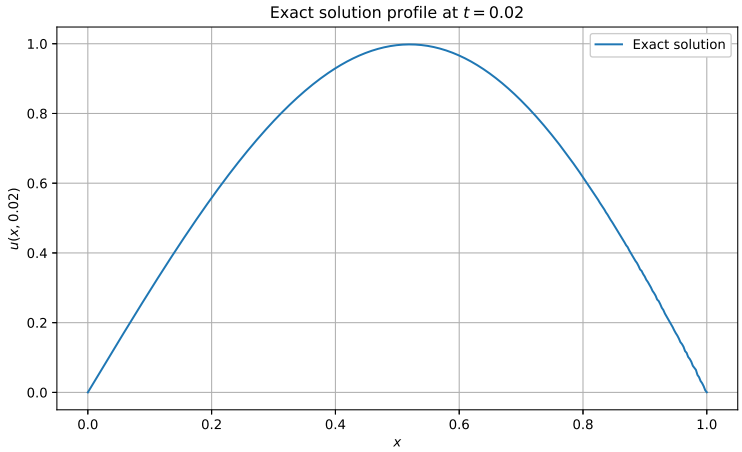


Figure 4. Reference spatial profile of the solution at $t = 0.02$.

The preceding results thus constitute a validation basis for the wavelet schemes developed in this work. Their main purpose is to provide a reference solution with which to measure pointwise and norm errors, as well as to study convergence behavior when the discretization is refined.

5.5. *Error table*

To quantify the difference between the exact solution and the solution approximated by Haar wavelets, the discrete errors in the L^∞ and L^2 norms are used, defined by

$$\|e\|_{L^\infty} \approx \max_{1\le i\le M} |u(x_i,t) - u_h(x_i,t)|,$$

and

$$\|e\|_{L^2} \approx \left(\int_0^1 |u(x,t) - u_h(x,t)|^2 \, dx \right)^{1/2}.$$

In the numerical experiments the parameters

$$J = 4, \qquad N_t = 100, \qquad \Delta t = \frac{T}{N_t} = 0.002,$$

with

$$T = 0.2, \qquad \nu = 0.01,$$

were used. Here J denotes the wavelet level, so that the number of basis functions employed in the spatial discretization is

$$2^J = 16.$$

Table 1 reports the errors obtained at several representative times.

Table 1. Errors between the exact solution and the solution approximated by Haar wavelets.

t	L^∞ error	L^2 error
0.02	1.0387972226e-01	1.4412752244e-02
0.05	1.1511680475e-01	1.4866607239e-02
0.10	1.3954572059e-01	1.6469653699e-02
0.20	2.2847966967e-01	2.4551544342e-02

5.6. Convergence comment

From a qualitative point of view, the approximation by Haar wavelets reproduces satisfactorily the global shape of the exact solution and captures the dissipative effect characteristic of the viscous Burgers equation. Moreover, as the wavelet level J increases, the number of basis functions used in the spatial discretization grows, thereby increasing the local resolution of the scheme.

From a quantitative point of view, one expects that, upon simultaneously refining the spatial and temporal discretizations, that is, increasing J and decreasing Δt, the errors in the L^∞ and L^2 norms decrease. In practice, this behavior may be studied by comparing the errors for different wavelet levels while keeping N_t fixed, or by refining both the spatial and temporal discretizations simultaneously.

It should be noted, however, that in the case of Haar wavelets the convergence is usually moderate, due to the lack of smoothness of the basis. Even so, the method retains several relevant advantages: structural simplicity, ease of implementation, spatial localization, and low computational cost. These characteristics justify its use as a starting point in the formulation of wavelet schemes for partial differential equations.

Acknowledgement

The authors are grateful for the support and academic environment of the Universidad Tecnológica de Pereira, whose infrastructure and guidance made this work possible. We also thank colleagues and students for valuable feedback during internal seminars.

REFERENCES

1. Y. Meyer, *Wavelets and Operators*. Cambridge: Cambridge University Press, 1992.
2. I. Daubechies, *Ten Lectures on Wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Philadelphia: SIAM, 1992.
3. S. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*. Amsterdam: Academic Press, 3 ed., 2009.
4. P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*. Cambridge: Cambridge University Press, 1997.
5. A. Cohen, *Numerical Analysis of Wavelet Methods*, vol. 32 of *Studies in Mathematics and its Applications*. Amsterdam: Elsevier, 2003.
6. K. Urban, *Wavelet Methods for Elliptic Partial Differential Equations*. Oxford: Oxford University Press, 2009.
7. Y.-H. Zhou, *Wavelet Numerical Method and Its Applications in Nonlinear Problems*, vol. 6 of *Engineering Applications of Computational Methods*. Singapore: Springer, 2021.
8. A. M. Qasim and E. S. Al-Rawi, "New wavelet method for solving partial differential equations," *Turkish Journal of Computer and Mathematics Education*, vol. 12, no. 13, pp. 3951–3959, 2021.
9. S. Banach, *Théorie des opérations linéaires*. Warsaw: Monografje Matematyczne, 1932.
10. J. Schauder, "Studien über lineare Funktionalgleichungen," *Mathematische Zeitschrift*, vol. 26, pp. 47–65, 1927.
11. W. Rudin, *Functional Analysis*. New York: McGraw-Hill, 2 ed., 1991.
12. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York: Springer, 2010.

13. A. Palomares, M. Pasadas, V. Ramírez, and M. R. Galán, "Schauder bases in banach spaces: Application to numerical solutions of differential equations," *Computers & Mathematics with Applications*, vol. 44, no. 5–6, pp. 619–622, 2002.
14. G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*. New York: Wiley, 2 ed., 1999.
15. A. Deitmar and S. Echterhoff, *Principles of Harmonic Analysis*. Universitext, Cham: Springer, 2 ed., 2014.
16. H. Führ, *Abstract Harmonic Analysis of Continuous Wavelet Transforms*, vol. 1863 of *Lecture Notes in Mathematics*. Berlin: Springer, 2005.
17. O. Christensen, *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis, Cham: Birkhäuser, 2 ed., 2016.
18. A. F. Camelo, C. A. Ramirez, and J. R. Gonzalez, "Function representation in hilbert spaces using haar wavelet series," *Statistics, Optimization & Information Computing*, vol. 13, no. 6, pp. 2477–2486, 2025.
19. A. F. Camelo, C. A. Ramirez, and G. V. Martinez, "On the construction of schauder bases in hilbert spaces via unitary representations," *Statistics, Optimization & Information Computing*, vol. 15, no. 4, pp. 2872–2889, 2026.
20. G. B. Whitham, *Linear and Nonlinear Waves*. New York: Wiley, 1974.
21. L. C. Evans, *Partial Differential Equations*, vol. 19 of *Graduate Studies in Mathematics*. Providence: American Mathematical Society, 2 ed., 2010.
22. J. B. Conway, *A Course in Functional Analysis*. New York: Springer, 2 ed., 1990.
23. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Volume I: Structure of Topological Groups, Integration Theory, Group Representations*, vol. 115 of *Grundlehren der mathematischen Wissenschaften*. Berlin: Springer, 2 ed., 1979.
24. H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups*, vol. 22 of *London Mathematical Society Monographs. New Series*. Oxford: Oxford University Press, 2 ed., 2000.
25. L. W. Baggett, V. Furst, K. D. Merrill, and J. A. Packer, "Classification of generalized multiresolution analyses," *Journal of Functional Analysis*, vol. 258, no. 12, pp. 4210–4228, 2010.
26. M. Bownik and Q. Jahan, "Wavelets on compact abelian groups," *Applied and Computational Harmonic Analysis*, vol. 49, no. 2, pp. 471–494, 2020.
27. C. P. Pandey, M. M. Dixit, M. Ado, and S. K. Singh, "The continuous wavelet transform in sobolev spaces over locally compact abelian group and its approximation properties," *International Journal of Analysis and Applications*, vol. 21, p. 139, 2023.
28. F. Bartolucci, F. D. Mari, E. D. Vito, and F. Odone, "The radon transform intertwines wavelets and shearlets," *Applied and Computational Harmonic Analysis*, vol. 47, no. 3, pp. 822–847, 2019.
29. A. A. Hemmat, K. Thirulogasanthar, and A. Krzyżak, "Discretization of quaternionic continuous wavelet transforms," *Journal of Geometry and Physics*, vol. 117, pp. 36–49, 2017.