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A New Version of Gamma Distribution: Properties, Estimation, and Modeling Real Datasets

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Abstract This paper introduces a novel probability distribution, known as the power-modified mixture symmetric gamma distribution. It extends the recently proposed mixture symmetric gamma distribution by modifying its support to the non-negative real line and applying a power transformation. The power-modified mixture symmetric gamma distribution includes a positive scale parameter and a positive shape parameter; the shape parameter controls skewness and tail behavior, making it a flexible model for various applications. We derive the cumulative distribution function and probability density function of the power-modified mixture symmetric gamma distribution and examine its statistical properties, including moments, shape characteristics, and hazard rate behavior. Parameter estimation employs several classical estimation methods, and assessments are made through numerical simulations. The performance of the power-modified mixture symmetric gamma distribution is evaluated using two real-world datasets. The results indicate that the power-modified mixture symmetric gamma distribution provides an excellent fit to these datasets, outperforming several competing distributions. These findings demonstrate the practical utility and robustness of the proposed distribution in modeling right-skewed positive data.

Keywords Power family, Monte Carlo simulation, mixture gamma, engineering data analysis, maximum likelihood, least-squares estimates

AMS 2010 subject classifications 11K06, 62E10

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1. Introduction

In recent years, the development of flexible probability distributions has garnered considerable attention in statistical modeling due to their ability to better capture the complexity and heterogeneity of real-world data [2, 39, 9, 34, 37]. A key strategy in this domain involves modifying existing distributions or applying transformations to derive new families with desirable properties such as skewness, kurtosis, or bounded support [33, 40, 32, 11, 28, 22].

Although the gamma distribution and its extensions have long been valued for their versatility [18, 20, 19, 46, 15], there is a demand for more adaptable variants that accommodate wider ranges of tail behaviors and asymmetry.

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Chesneau et al. [16] recently introduced the mixture symmetric gamma distribution (MSGD), a novel model characterized by a piecewise-defined cumulative distribution function (CDF) designed to model data that exhibit symmetry around zero. The CDF of the MSGD is given by:

$$G(y) = \begin{cases} \frac{1}{10}\left[4(1-\tau y)e^{\tau y} + e^{2\tau y}\right], & y < 0, \\ 1 - \frac{1}{10}\left[4(1+\tau y)e^{-\tau y} + e^{-2\tau y}\right], & y \geq 0, \end{cases} \quad \tau > 0. \tag{1}$$

Although the MSGD offers promising characteristics, many practical scenarios, such as reliability analysis, survival studies, and environmental modeling, require distributions with support restricted to the non-negative real line. Motivated by this need, we propose a modified version of the MSGD that is restricted to $y \geq 0$. The resulting model, which we refer to as the modified mixture symmetric gamma distribution (MMSGD), is defined by the following CDF:

$$W(y) = 1 - \frac{1}{5}\left[4e^{-\tau y}(\tau y + 1) + e^{-2\tau y}\right], \quad y \geq 0,\ \tau > 0. \tag{2}$$

To further enhance the modeling flexibility of the MMSGD, particularly in terms of tail behavior and skewness. We introduce a power transformation to CDF (2), it is based on the transformation $X = Y^{\frac{1}{\phi}}$, where Y is a random variable follows MMSGD. This yields a new distribution, the power modified mixture symmetric gamma distribution (PMMSGD), with a CDF defined as:

$$F(x) = W(x^{\phi}) = 1 - \frac{1}{5}e^{-2\tau x^{\phi}}\left[4e^{\tau x^{\phi}}\left(\tau x^{\phi} + 1\right) + 1\right], \quad x \geq 0,\ \tau,\ \phi > 0, \tag{3}$$

and by the derivative of Equation (3), we have the corresponding probability density function (PDF):

$$f(x) = \frac{2}{5}\tau\phi x^{\phi-1}e^{-2\tau x^{\phi}}\left(2\tau x^{\phi}e^{\tau x^{\phi}} + 1\right). \tag{4}$$

Since F depends on x only through τx^{ϕ}, we have $X \stackrel{d}{=} \tau^{-1/\phi}Z$, where Z follows the PMMSGD with $\tau = 1$. Hence τ is a scale parameter and ϕ is the only shape parameter; scale-free measures such as skewness, kurtosis and the coefficient of variation depend on ϕ alone. A graphical representation of the proposed model density is presented in Figure 1 for different parameter combinations. The plots demonstrate the model's significant flexibility, showcasing its ability to produce a variety of shapes, including right-skewed, approximately symmetric, and heavy-tailed distributions. This versatility makes it suitable for modeling diverse positive-valued data.

Recent statistical literature has witnessed significant interest in power-transformed distributions due to their enhanced flexibility in modeling real-world phenomena. Several notable contributions have emerged, spanning diverse families of distributions: the sine power-Weibull distribution [4], the new flexible power Weibull distribution [6], the McDonald generalized power Weibull distribution [38], the Nadarajah-Haghighi generalized power Weibull distribution [1], log exponential-power distribution [27], the inverse Ramos-Louzada distribution [25], the heavy-tailed power XLindley distribution [21], Poisson XLindley distribution [3], extended Gumbel Type-II distribution [45], the inverse power XLindley distribution [23], the sine power half-logistic distribution [24], extended inverse Weibull distribution [5], the gamma power half-logistic distribution [14], the Marshall-Olkin power Lomax distribution [44], new version of Lindley distribution [26], the Gull alpha power Lomax distribution [43], the exponentiated power Akash distribution [31], the novel alpha power Fréchet distribution [10].

These developments collectively demonstrate the effectiveness of transformation in enhancing distributional flexibility across multiple families. However, to our knowledge, no previous work has applied this transformation to the mixture symmetric gamma framework. Our proposed PMMSGD thus fills this gap by introducing a power-transformed variant of the MSGD, combining the benefits of mixture models with the adaptability of power transformations.

The primary goal of this paper is to investigate the statistical properties of the PMMSGD, explore its parameter estimation techniques, and demonstrate its flexibility and practical relevance through applications to real-world

datasets. This new distribution not only inherits the desirable properties of the MMSGD but also introduces an additional shape parameter,ϕ, allowing greater adaptability in modeling varying decay rates and tail heaviness.

The rest of this paper is organized as follows: Section 2 derives the fundamental properties of PMMSGD, Section 3 discusses parameter estimation, Section 4 presents simulation studies, and Section 5 demonstrates real-world applications.

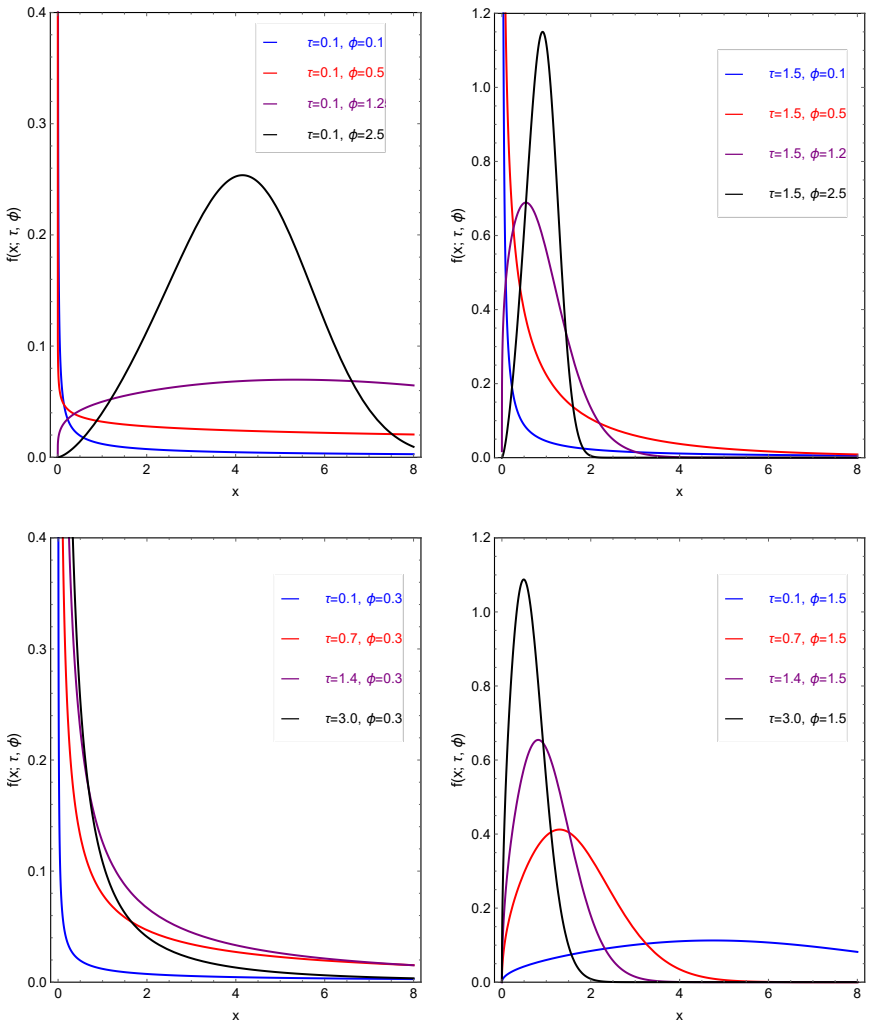


Figure 1. The PDF plots of PMMSGD.

2. Statistical properties

In this section, we will introduce statistical properties that provide important information about the distribution structure.

2.1. Survival and hazard functions

The survival function of PMMSGD can be defined in terms of the CDF, while the hazard function is given by the ratio of the PDF to the survival function, respectively, as follows:

$$S(x) = 1 - F(x) = \frac{4}{5}e^{-\tau x^{\phi}}\left(\tau x^{\phi} + 1\right) + \frac{1}{5}e^{-2\tau x^{\phi}},$$

$$h(x) = \frac{\frac{4}{5}\tau^2\phi x^{2\phi-1}e^{-\tau x^{\phi}} + \frac{2}{5}\tau\phi x^{\phi-1}e^{-2\tau x^{\phi}}}{\frac{4}{5}e^{-\tau x^{\phi}}\left(\tau x^{\phi} + 1\right) + \frac{1}{5}e^{-2\tau x^{\phi}}}. \tag{5}$$

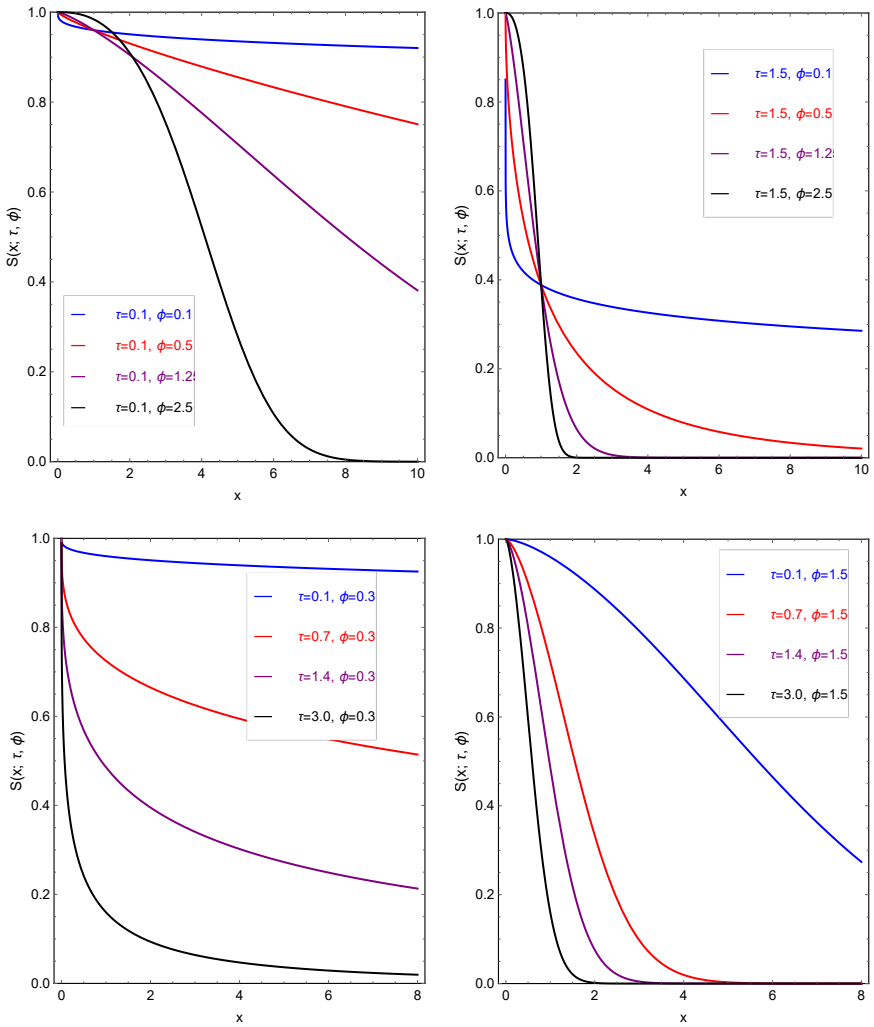


Figure 2. The survival function plots of PMMSGD.

The plots of these functions are in Figures 2 and 3 for different parameter values of PMMSGD. Figure 2 presents the reliability) function of the PMMSGD. The function plots the probability that a variable exceeds a specified value, which is crucial for applications in survival and reliability analysis. The varying rates of decay reflect the different tail behaviors governed by the parameter choices. Figure 3 explains the hazard rate function of the PMMSGD for the selected parameters. The hazard function, which represents the instantaneous failure

rate, exhibits diverse patterns including increasing, decreasing, and more complex non-monotonic behaviors. This flexibility allows the model to adapt to various real-world failure processes.

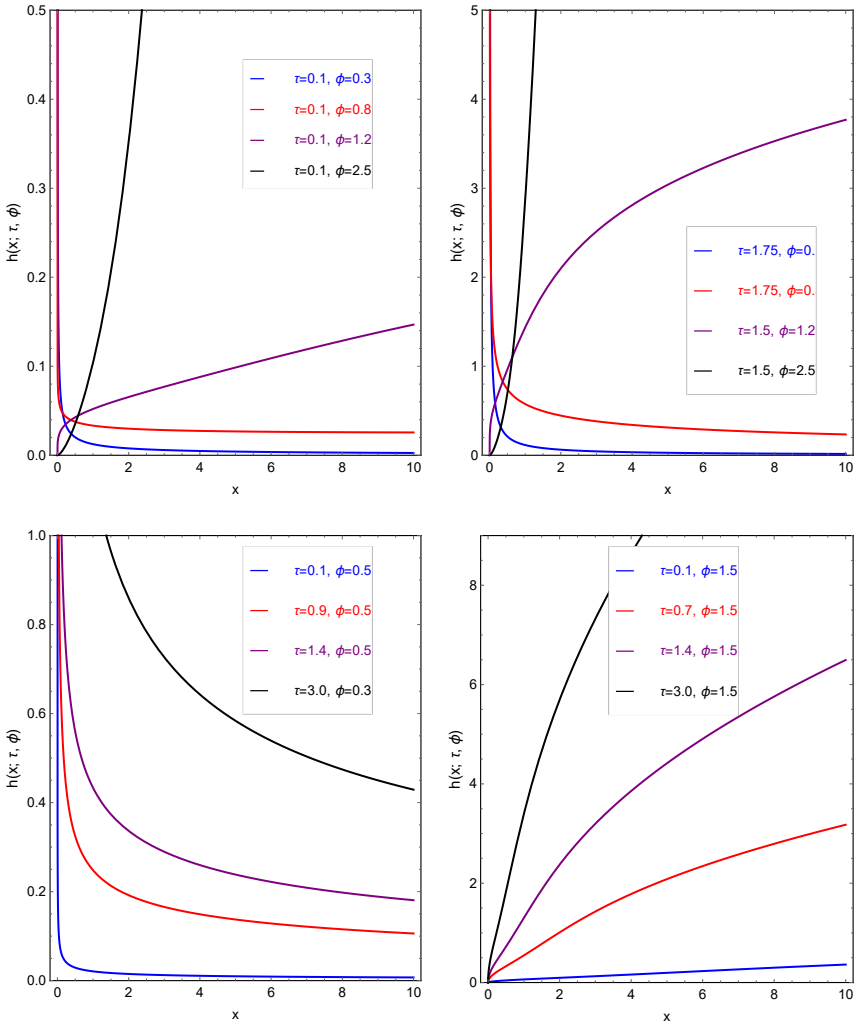


Figure 3. The hazard function plots of PMMSGD.

2.2. Moments

The moment of a distribution is utilized to compute fundamental characteristics of a distribution, including mean, variance, skewness, and kurtosis. Let X denote a random variable, and let τ and ϕ represent parameters of PMMSGD. The k-th order moment $E\left(X^k\right)$ of PMMSGD is defined as follows.

$$E\left(X^k\right) = \int_0^\infty x^k \frac{2}{5}\tau\phi x^{\phi-1} e^{-2\tau x^\phi}\left(2\tau x^\phi e^{\tau x^\phi} + 1\right)dx. \tag{6}$$

Let $u = x^\phi$ and then, we can write the rth moment of PMMSGD is given by:

$$\mu'_k = \frac{1}{5}\tau^{-\frac{k}{\phi}}\left(\frac{\phi 2^{-\frac{k}{\phi}}}{k+\phi} + 4\right)\Gamma\left(\frac{k}{\phi} + 2\right). \tag{7}$$

When we write k=1 and 2 in this non-central kth moment in Equation (6), we can write the first two moments of the PMMSGD as follows.

$$E\left(X\right)=\frac{1}{5}\tau^{-\frac{1}{\phi}}\left(\frac{\phi 2^{-\frac{1}{\phi}}}{1+\phi}+4\right)\Gamma\left(\frac{1}{\phi}+2\right),$$

and

$$E\left(X^{2}\right)=\frac{1}{5}\tau^{-\frac{2}{\phi}}\left(\frac{\phi 2^{-\frac{2}{\phi}}}{2+\phi}+4\right)\Gamma\left(\frac{2}{\phi}+2\right),$$

Since the variance is $V\left(X\right)=E\left(X^{2}\right)-\left(E\left(X\right)\right)^{2}$, we obtain the variance of PMMSGD as

$$V\left(X\right)=\frac{1}{5}\tau^{-\frac{2}{\phi}}\left(\frac{\phi 2^{-\frac{2}{\phi}}}{2+\phi}+4\right)\Gamma\left(\frac{2}{\phi}+2\right)-\left[\frac{1}{5}\tau^{-\frac{1}{\phi}}\left(\frac{\phi 2^{-\frac{1}{\phi}}}{1+\phi}+4\right)\Gamma\left(\frac{1}{\phi}+2\right)\right]^{2}. \tag{8}$$

We can obtain the coefficient of variation of PMMSGD using the non-central moment $E\left(X^{k}\right)$ as follows:

$$C.V.=\left[\frac{1}{5}\tau^{-\frac{1}{\phi}}\left(\frac{\phi 2^{-\frac{1}{\phi}}}{1+\phi}+4\right)\Gamma\left(\frac{1}{\phi}+2\right)\right]^{-1}$$

$$\times\left[\frac{1}{5}\tau^{-\frac{2}{\phi}}\left(\frac{\phi 2^{-\frac{2}{\phi}}}{2+\phi}+4\right)\Gamma\left(\frac{2}{\phi}+2\right)-\left[\frac{1}{5}\tau^{-\frac{1}{\phi}}\left(\frac{\phi 2^{-\frac{1}{\phi}}}{1+\phi}+4\right)\Gamma\left(\frac{1}{\phi}+2\right)\right]^{2}\right]^{0.5}. \tag{9}$$

2.3. Moment generating and characteristic functions

The moment generating function of PMMSGD is as follows:

$$M_{X}\left(t\right)=E\left(e^{tX}\right)=\int_{0}^{\infty}e^{tx}\frac{2}{5}\tau\phi x^{\phi-1}e^{-2\tau x^{\phi}}\left(2\tau x^{\phi}e^{\tau x^{\phi}}+1\right)dx.$$

For $\phi<1$ the tail $e^{-\tau x^{\phi}}$ is heavier than exponential, so $M_X(t)=\infty$ for every $t>0$ and the MGF does not exist. Using Taylor series and non-central moments, we have

$$M_{X}\left(t\right)=E\left(e^{tX}\right)=\sum_{k=0}^{\infty}E\left(x^{k}\right)\frac{t^{k}}{k!}$$

$$=\sum_{k=0}^{\infty}\frac{1}{5}\tau^{-\frac{k}{\phi}}\left(\frac{\phi 2^{-\frac{k}{\phi}}}{k+\phi}+4\right)\Gamma\left(\frac{k}{\phi}+2\right)\frac{t^{k}}{k!}. \tag{10}$$

Because $E(X^k)$ grows like $\Gamma(k/\phi+2)\,\tau^{-k/\phi}$, the series in Equation (10) converges for all real t when $\phi>1$, converges only for $|t|<\tau$ when $\phi=1$, and diverges for every $t\neq 0$ when $\phi<1$. The characteristic function of PMMSGD, denoted $\psi_X(t)$ to avoid confusion with the parameter ϕ, is as follows:

$$\psi_{X}\left(t\right)=E\left(e^{itX}\right)=\sum_{k=0}^{\infty}\frac{1}{5}\tau^{-\frac{k}{\phi}}\left(\frac{\phi 2^{-\frac{k}{\phi}}}{k+\phi}+4\right)\Gamma\left(\frac{k}{\phi}+2\right)\frac{\left(it\right)^{k}}{k!}. \tag{11}$$

This series has the same convergence conditions as Equation (10): it is valid for all t when $\phi>1$ and for $|t|<\tau$ when $\phi=1$. For $\phi<1$ (and for $|t|\geq\tau$ when $\phi=1$), the characteristic function, which always exists, must be computed from its integral form

$$\psi_{X}(t)=\int_{0}^{\infty}e^{itx}\frac{2}{5}\tau\phi x^{\phi-1}e^{-2\tau x^{\phi}}\left(2\tau x^{\phi}e^{\tau x^{\phi}}+1\right)dx,$$

for example by numerical integration.

3. Estimation methods

This section will present various traditional estimation methods applicable for estimating the parameters of the PMMSGD. The estimating equations derived from order statistics can be readily acquired at the minimal value for ordinary least squares estimates (OLSEs), weighted least squares estimates (WLSEs), Anderson-Darling estimates (ADEs), right-tailed Anderson-Darling estimates (RTADEs), left-tailed Anderson-Darling estimates (LTADEs), and Cramer-von Mises estimates (CVMEs). Furthermore, maximum likelihood estimates (MLEs) and maximum product of spacings estimates (MPSEs) can be derived by maximizing the objective function.

3.1. Ordinary and weighted least-squares estimates

Swain, Venkatraman, and Wilson [41] introduced the use of least squares and weighted least squares methods to estimate unknown parameters of a distribution. We can define this method for PMMSGD as follows:

$$OLSEs = \sum_{i=1}^{n}\left[F\left(x_{i:n}\right) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[\left(\frac{1}{5}e^{-2\tau x_{i:n}^{\phi}}\left[-4e^{\tau x_{i:n}^{\phi}}\left(\tau x_{i:n}^{\phi}+1\right)-1\right]+1\right) - \frac{i}{n+1}\right]^2.$$

$$WLSEs = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{i:n}\right) - \frac{i}{n+1}\right]^2$$

$$= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(\frac{1}{5}e^{-2\tau x_{i:n}^{\phi}}\left[-4e^{\tau x_{i:n}^{\phi}}\left(\tau x_{i:n}^{\phi}+1\right)-1\right]+1\right) - \frac{i}{n+1}\right]^2.$$

3.2. Maximum likelihood estimates

MLE is a method used to estimate the parameters of a statistical model. The idea is to find the parameter values that maximize the likelihood function, which measures how well the model explains the observed data. Given a random sample $X_1, X_2, \ldots, X_n$ from PMMSGD with $f\left(x;\theta\right)$, where θ is the vector of parameters to be estimated, the likelihood function $L\left(\theta\right)$ is defined as:

$$L\left(x;\tau,\phi\right) = \prod_{i=1}^{n}\frac{2}{5}\tau\phi x_i^{\phi-1}e^{-2\tau x_i^{\phi}}\left(2\tau x_i^{\phi}e^{\tau x_i^{\phi}}+1\right), \tag{12}$$

We define the log-likelihood function from Equation (12) as follows:

$$\ln\left(L\left(x;\tau,\phi\right)\right) = n\ln\left(\frac{2}{5}\right) + n\ln\left(\tau\right) + n\ln\left(\phi\right) + \left(\phi-1\right)\sum_{i=1}^{n}\ln\left(x_i\right) - 2\tau\sum_{i=1}^{n}x_i^{\phi} + \sum_{i=1}^{n}\ln\left(2\tau x_i^{\phi}e^{\tau x_i^{\phi}}+1\right).$$

When we take the derivative of the log-likelihood function with respect to the parameters, we obtain the score functions for each parameter. We can also calculate the parameter estimates by setting the score functions equal to zero. These score functions are shown as follows:

$$\frac{\partial \ln L}{\partial \tau} = \frac{n}{\tau} - 2\sum_{i=1}^{n}x_i^{\phi} + \sum_{i=1}^{n}\frac{2x_i^{\phi}e^{\tau x_i^{\phi}} + 2\tau x_i^{2\phi}e^{\tau x_i^{\phi}}}{2\tau x_i^{\phi}e^{\tau x_i^{\phi}}+1},$$

$$\frac{\partial \ln L}{\partial \phi} = \frac{n}{\phi} + \sum_{i=1}^{n}\ln x_i - 2\tau\sum_{i=1}^{n}x_i^{\phi}\ln x_i + \sum_{i=1}^{n}\frac{2\tau x_i^{\phi}\ln(x_i)\,e^{\tau x_i^{\phi}}\left(1+\tau x_i^{\phi}\right)}{2\tau x_i^{\phi}e^{\tau x_i^{\phi}}+1}.$$

3.3. Maximum product of spacing estimates

The MPSE methodology serves as a robust alternative to the MLE methodology. MPSE for PMMSGD can be defined as follows:

$$MPSEs = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F\left(x_{i:n}\right) - F\left(x_{i-1:n}\right)\right],$$

where $F(x_{i:n}) = 1 - \frac{1}{5}e^{-2\tau x_{i:n}^{\phi}}\left[4e^{\tau x_{i:n}^{\phi}}\left(\tau x_{i:n}^{\phi}+1\right)+1\right]$, $F(x_{0:n}) = 0$, and $F(x_{n+1:n}) = 1$.

3.4. Anderson-Darling estimates

The ADE method is a minimum distance estimator derived by minimizing the AD statistic. For PMMSGD, the estimates obtained using this method include ADEs, RTADEs, and LTADEs.

$$ADEs = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{i:n}\right) + \log S\left(x_{n+1-i:n}\right)\right].$$

$$RTADEs = \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_{i:n}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i:n}\right).$$

$$LTADEs = -\frac{3n}{2} + 2\sum_{i=1}^{n}F\left(x_{i:n}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F\left(x_{i:n}\right).$$

3.5. Cramer-von Mises estimates

CVMEs were first proposed by Choi and Bulgren [17]. We can obtain CVMEs for PMMSGD as follows.

$$CVMEs = \frac{1}{12n} + \sum_{i=1}^{n}\left[F\left(x_{i:n}\right) - \frac{2i-1}{2n}\right]^2.$$

4. Numerical simulation

In this section, we discuss simulation studies to examine the performance of estimation methods for PMMSGD. The objective is to assess the efficacy of different estimation techniques proposed for the PMMSGD's parameters by examining comprehensive simulation outcomes. For this purpose, we performed simulation studies in different scenarios with different sample sizes ($n = 10,\ 30,\ 50,\ 100,\ 250$) and the parameter combinations $(\tau, \phi) = (7,\ 1.5)$, $(0.7,\ 0.7)$ and $(0.5,\ 3)$. Furthermore, to produce a suitable data set for the distribution, we generated 5000 random samples by solving the PMMSGD quantile function numerically for different parameter values in the R program. In addition, we calculated the absolute bias (|Bias|), root mean square error (RMSE), and mean relative error (MRE) for all parameters using the following formulas, where $\widehat{\theta}_i$ is the estimate of θ obtained from the ith simulated sample and $N = 5000$ is the number of replications:

$$\left|\text{Bias}(\widehat{\theta})\right| = \left|\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)\right|,$$

$$\text{RMSE}(\widehat{\theta}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)^2},$$

$$\mathrm{MRE}(\widehat{\theta}) = \frac{\left|\mathrm{Bias}(\widehat{\theta})\right|}{\theta}.$$

4.1 Simulation Algorithm

1. Define initial parameter values.
2. Create a random sample of size n by solving the quantile function numerically according to different values of the distribution parameters.
3. Calculate the estimates using various estimation techniques.
4. Evaluate statistical metrics (bias, RMSE, and MRE) for each parameter using various estimation approaches.

4.2 Simulation Results

The simulation results are given in Tables 1–3, and Figure 4 displays the RMSE values. In all three scenarios, the RMSE of every method decreases as the sample size increases, with one exception: in Table 2, the RMSE of the ADE and RTADE levels off at about 0.26–0.30 from $n = 30$ onward, while their bias increases with n over the same range. Small samples are the most difficult case, particularly for τ when $\tau = 7$ (Table 1): at $n = 10$ the OLSE and CVME have RMSE values for τ above 300, and the WLSE, RTADE and LTADE above 20, whereas the MPSE (4.79) and ADE (7.47) remain much more stable. At $n = 10$, the MPSE gives the smallest RMSE for ϕ in all three scenarios. As n grows, the MLE becomes the most efficient method: at $n = 250$ it has the smallest, or very nearly the smallest, RMSE for both parameters in every scenario, with the MPSE a close second. Therefore, the MPSE is recommended for small samples, and the MLE or MPSE for moderate and large samples.

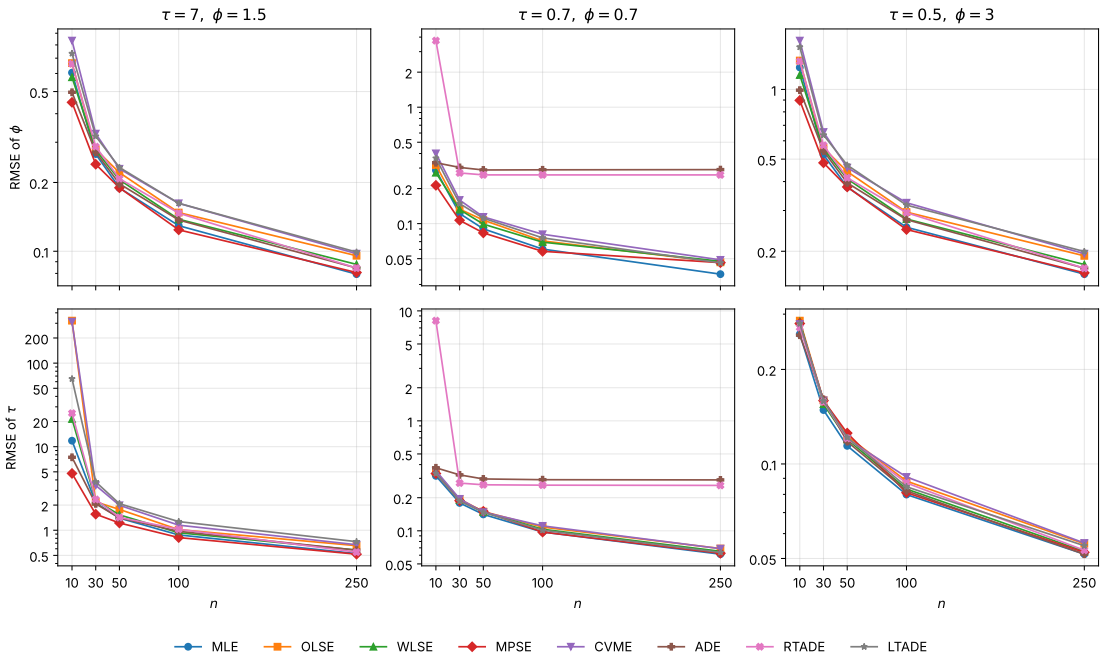


Figure 4. RMSE of the eight estimation methods versus sample size for ϕ (top) and τ (bottom) in the three simulation scenarios (logarithmic vertical axis).

Table 1. The results of simulation of $\tau = 7$ and $\phi = 1.5$

Method	n	Bias		RMSE		MRE	
		ϕ	τ	ϕ	τ	ϕ	τ
MLEs	10	0.2636	3.7292	0.6049	11.7905	0.1758	0.5327
	30	0.0740	0.7223	0.2664	2.1280	0.0494	0.1032
	50	0.0451	0.3929	0.1893	1.3964	0.0301	0.0561
	100	0.0225	0.1945	0.1294	0.8792	0.0150	0.0278
	250	0.0086	0.0669	0.0794	0.5250	0.0058	0.0096
OLSEs	10	0.0115	15.8488	0.6658	322.4380	0.0077	2.2641
	30	0.0080	0.1823	0.2792	2.1851	0.0053	0.0260
	50	0.0039	0.1256	0.2209	1.7753	0.0026	0.0179
	100	0.0030	0.0781	0.1477	1.0172	0.0020	0.0112
	250	0.0057	0.0029	0.0956	0.6535	0.0038	0.0004
WLSEs	10	0.0302	2.2266	0.5776	21.4685	0.0202	0.3181
	30	0.0080	0.3416	0.2769	2.2649	0.0054	0.0488
	50	0.0154	0.2088	0.2055	1.5285	0.0103	0.0298
	100	0.0069	0.0741	0.1381	0.9331	0.0046	0.0106
	250	0.0031	0.0364	0.0876	0.5781	0.0021	0.0052
MPSEs	10	0.1251	0.2355	0.4485	4.7861	0.0834	0.0336
	30	0.0802	0.3641	0.2403	1.5571	0.0534	0.0520
	50	0.0707	0.3326	0.1896	1.2214	0.0471	0.0475
	100	0.0371	0.2089	0.1241	0.8167	0.0248	0.0298
	250	0.0249	0.1402	0.0807	0.5169	0.0166	0.0200
CVMEs	10	0.2678	23.2335	0.8354	313.3456	0.1785	3.3191
	30	0.0857	1.0213	0.3272	3.4439	0.0571	0.1459
	50	0.0578	0.6477	0.2296	1.9888	0.0385	0.0925
	100	0.0292	0.2534	0.1621	1.1503	0.0195	0.0362
	250	0.0053	0.0530	0.0976	0.6708	0.0036	0.0076
ADEs	10	0.1021	1.7473	0.4965	7.4668	0.0681	0.2496
	30	0.0265	0.3853	0.2712	2.0695	0.0176	0.0550
	50	0.0268	0.2327	0.1974	1.4056	0.0179	0.0332
	100	0.0181	0.1587	0.1370	0.9706	0.0121	0.0227
	250	0.0012	0.0226	0.0845	0.5777	0.0008	0.0032
RTADEs	10	0.1887	3.5191	0.6617	25.1458	0.1258	0.5027
	30	0.0390	0.5628	0.2870	2.3351	0.0260	0.0804
	50	0.0273	0.2967	0.2078	1.4246	0.0182	0.0424
	100	0.0210	0.1852	0.1468	1.0266	0.0140	0.0265
	250	0.0002	0.0215	0.0844	0.5474	0.0001	0.0031
LTADEs	10	0.2023	9.9638	0.7339	64.8755	0.1348	1.4234
	30	0.0532	0.9113	0.3188	3.7904	0.0355	0.1302
	50	0.0317	0.4656	0.2327	2.0782	0.0211	0.0665
	100	0.0181	0.2407	0.1622	1.2684	0.0121	0.0344
	250	0.0053	0.0719	0.0992	0.7303	0.0035	0.0103

Table 2. The results of simulation of $\tau = 0.7$ and $\phi = 0.7$

Method	n	Bias		RMSE		MRE	
		ϕ	τ	ϕ	τ	ϕ	τ
MLEs	10	0.1269	0.0306	0.2874	0.3160	0.1813	0.0437
	30	0.0362	0.0108	0.1221	0.1793	0.0517	0.0155
	50	0.0204	0.0032	0.0902	0.1410	0.0292	0.0046
	100	0.0119	0.0049	0.0604	0.0975	0.0171	0.0069
	250	0.0044	0.0015	0.0368	0.0613	0.0063	0.0022
OLSEs	10	0.0215	0.0772	0.3184	0.3432	0.0307	0.1103
	30	0.0095	0.0342	0.1333	0.1941	0.0136	0.0489
	50	0.0152	0.0143	0.1072	0.1496	0.0218	0.0205
	100	0.0075	0.0224	0.0710	0.1081	0.0108	0.0319
	250	0.0101	0.0159	0.0465	0.0690	0.0145	0.0227
WLSEs	10	0.0277	0.0617	0.2737	0.3435	0.0396	0.0882
	30	0.0212	0.0310	0.1320	0.1903	0.0303	0.0443
	50	0.0259	0.0130	0.0990	0.1487	0.0370	0.0185
	100	0.0244	0.0079	0.0693	0.1033	0.0349	0.0113
	250	0.0246	0.0057	0.0475	0.0651	0.0351	0.0081
MPSEs	10	0.0322	0.1134	0.2132	0.3317	0.0459	0.1621
	30	0.0049	0.0520	0.1068	0.1880	0.0070	0.0743
	50	0.0049	0.0415	0.0832	0.1504	0.0070	0.0593
	100	0.0200	0.0141	0.0578	0.0974	0.0286	0.0201
	250	0.0278	0.0051	0.0462	0.0625	0.0398	0.0073
CVMEs	10	0.1442	0.0155	0.4010	0.3516	0.2061	0.0221
	30	0.0539	0.0027	0.1594	0.1952	0.0770	0.0039
	50	0.0414	0.0060	0.1142	0.1477	0.0592	0.0085
	100	0.0273	0.0034	0.0810	0.1109	0.0390	0.0049
	250	0.0154	0.0090	0.0488	0.0688	0.0221	0.0129
ADEs	10	0.0390	0.1177	0.3333	0.3741	0.0558	0.1682
	30	0.1666	0.1938	0.3038	0.3226	0.2381	0.2769
	50	0.1926	0.2023	0.2889	0.2962	0.2751	0.2890
	100	0.2383	0.2404	0.2898	0.2916	0.3404	0.3434
	250	0.2705	0.2676	0.2909	0.2903	0.3865	0.3822
RTADEs	10	0.3335	0.2986	3.7315	8.1484	0.4764	0.4266
	30	0.1218	0.1359	0.2725	0.2717	0.1740	0.1942
	50	0.1561	0.1592	0.2621	0.2621	0.2230	0.2275
	100	0.2074	0.2049	0.2620	0.2601	0.2963	0.2927
	250	0.2503	0.2461	0.2620	0.2587	0.3575	0.3515
LTADEs	10	0.0979	0.0233	0.3645	0.3326	0.1398	0.0333
	30	0.0260	0.0073	0.1486	0.1863	0.0372	0.0105
	50	0.0156	0.0067	0.1114	0.1470	0.0223	0.0095
	100	0.0087	0.0008	0.0754	0.1013	0.0124	0.0011
	250	0.0030	0.0006	0.0460	0.0636	0.0043	0.0008

Table 3. The results of simulation for $\tau = 0.5$ and $\phi = 3$

Method	n	Bias		RMSE		MRE	
		ϕ	τ	ϕ	τ	ϕ	τ
MLEs	10	0.5521	0.0293	1.2441	0.2591	0.1840	0.0585
	30	0.1462	0.0060	0.5223	0.1485	0.0487	0.0121
	50	0.0799	0.0045	0.3787	0.1142	0.0266	0.0089
	100	0.0463	0.0047	0.2540	0.0799	0.0154	0.0095
	250	0.0200	0.0014	0.1593	0.0516	0.0067	0.0028
OLSEs	10	0.0233	0.0706	1.3327	0.2860	0.0078	0.1413
	30	0.0160	0.0258	0.5585	0.1595	0.0053	0.0516
	50	0.0169	0.0191	0.4418	0.1226	0.0056	0.0382
	100	0.0243	0.0125	0.2953	0.0882	0.0081	0.0250
	250	0.0114	0.0062	0.1911	0.0557	0.0038	0.0124
WLSEs	10	0.0597	0.0577	1.1547	0.2843	0.0199	0.1153
	30	0.0336	0.0108	0.5528	0.1551	0.0112	0.0215
	50	0.0309	0.0061	0.4110	0.1198	0.0103	0.0122
	100	0.0127	0.0025	0.2757	0.0832	0.0042	0.0049
	250	0.0062	0.0010	0.1753	0.0527	0.0021	0.0021
MPSEs	10	0.2502	0.1097	0.8969	0.2801	0.0834	0.2195
	30	0.1616	0.0543	0.4820	0.1591	0.0539	0.1086
	50	0.1414	0.0447	0.3792	0.1254	0.0471	0.0895
	100	0.0743	0.0204	0.2483	0.0810	0.0248	0.0409
	250	0.0499	0.0132	0.1614	0.0523	0.0166	0.0263
CVMEs	10	0.2651	0.0059	1.6250	0.2789	0.0884	0.0118
	30	0.1713	0.0059	0.6544	0.1594	0.0571	0.0119
	50	0.1156	0.0046	0.4592	0.1201	0.0385	0.0091
	100	0.0584	0.0049	0.3241	0.0909	0.0195	0.0098
	250	0.0107	0.0001	0.1952	0.0560	0.0036	0.0002
ADEs	10	0.2043	0.0242	0.9929	0.2571	0.0681	0.0485
	30	0.0529	0.0140	0.5425	0.1605	0.0176	0.0281
	50	0.0537	0.0012	0.3948	0.1178	0.0179	0.0024
	100	0.0362	0.0020	0.2739	0.0821	0.0121	0.0041
	250	0.0024	0.0016	0.1690	0.0516	0.0008	0.0031
RTADEs	10	0.3773	0.0009	1.3230	0.2719	0.1258	0.0018
	30	0.0781	0.0127	0.5740	0.1576	0.0260	0.0254
	50	0.0547	0.0035	0.4155	0.1207	0.0182	0.0070
	100	0.0420	0.0045	0.2935	0.0870	0.0140	0.0090
	250	0.0004	0.0027	0.1688	0.0531	0.0001	0.0054
LTADEs	10	0.4539	0.0176	1.5244	0.2798	0.1513	0.0352
	30	0.0993	0.0110	0.6357	0.1587	0.0331	0.0221
	50	0.0520	0.0059	0.4712	0.1215	0.0173	0.0118
	100	0.0327	0.0001	0.3165	0.0846	0.0109	0.0002
	250	0.0121	0.0009	0.1995	0.0549	0.0040	0.0018

5. Real data analysis

This section highlights the modeling capabilities of the PMMSGD by applying it to two real-world datasets obtained from [13]. The first dataset consists of the time between failures for thirty (30) items, while the second

contains annual maximum precipitation values recorded in Karachi, Pakistan, from 1950 to 2009. Summary statistics, including minimum (min), maximum (max), mean, variance (Var), standard deviation (StDev), coefficient of variation (CoVar), coefficient of kurtosis (CoKurt), and coefficient of skewness (CoSkew), are presented to provide a descriptive overview of the datasets. Data 1 is less spread out and closer to a normal distribution. In contrast, data 2 has much larger values, greater variability, and more outliers, as shown in Table 4 and Figure 5.

Table 4. Summary and Descriptive Statistics

	N	Min	Max	Mean	Var	StDev	CoVar	CoKurt	CoSkew
Data 1	30	0.800	11.90	5.507	8.398	2.898	52.62	2.644	0.388
Data 2	59	5.600	429.3	92.7	8688.99	93.21	78.73	3.766	1.024

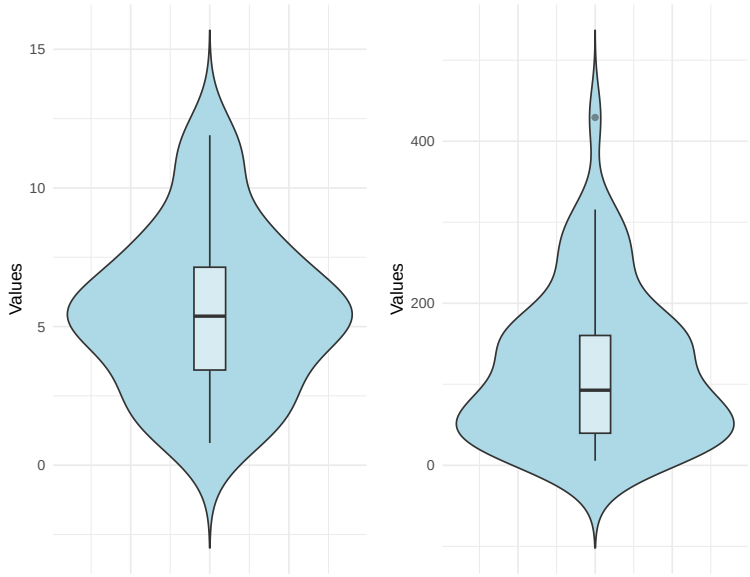


Figure 5. Violin plots of the Datasets.

To determine the modeling strengths of the PMMSGD, we compare and fit eight(8) other power distributions. They include modified alpha power transformed Burr XII distribution (MAPTBXIID) [12], power modified XLindley distribution (PMXLD) [42], generalized power Weibull distribution (GPWD) [7], generalized power Sujatha distribution (GPSD) [35], alpha power Burr XII distribution (APBXIID) [36], alpha power Weibull distribution (APWD) [30], alpha power transformed Fréchet distribution (APTFD) [29] and alpha power transformed exponential distribution (APED) [8].

We adjudge the distribution with the least value of the adequacy measures used as the best. Adequacy measures such as Akaike Information Criterion (AK1), consistent Akaike Information Criterion (CAK1), Bayesian Information Criterion (BA1), Anderson-Darling Test (ANDA1), Cramér-von Mises Test (CMT1), Kolmogorov-Smirnov Test (KST1) with their corresponding p-values in parentheses.

Our proposed PMMSGD outperforms existing well-known distributions, demonstrating the best fit for both datasets. This is confirmed by lower values of the AK1, CAK1, BA1, ANDA1, CMT1, and KST1 statistics compared to competing distributions, as shown in Tables 5 and 6.

Table 5. MLEs and Adequacy Measures (Data 1)

	Para	ℓ	AK1	CAK1	BA1	ANDA1 (p-value)	CMT1 (p-value)	KST1 (p-value)
PMMSGD	$\tau = 0.0770\ \phi = 1.7221$	-72.6510	149.3020	149.7465	152.1044	0.2375 (0.9766)	0.0385 (0.9436)	0.0905 (0.9480)
MAPTBXIID	$\alpha = 56.981\ c = 1.1948$ $k = 2.6524$	-74.8074	155.6147	156.2814	159.8183	0.5242 (0.7208)	0.0782 (0.7060)	0.1278 (0.6649)
PMXLD	$\rho = 6.2077\ \epsilon = 1.1523$	-82.1261	168.2522	168.6967	171.0546	1.8457 (0.1122)	0.3218 (0.1172)	0.2130 (0.1129)
GPWD	$\phi = 1.7661\ \epsilon = 1.8322$ $\omega = 0.0177$	-72.6055	151.2109	152.1340	155.4145	0.2434 (0.9737)	0.0408 (0.9320)	0.0931 (0.9358)
GPSD	$\alpha = 0.2045\ \tau = 1.3849$ $\omega = 0.0918$	-72.9783	151.9566	152.8797	156.1602	0.2802 (0.9520)	0.0473 (0.8958)	0.1077 (0.8409)
APBXIID	$\alpha = 276.78\ \beta = 2.1758$ $\lambda = 0.6587$	-80.1989	166.3977	167.3208	170.6013	1.6650 (0.1418)	0.3183 (0.1198)	0.2238 (0.0843)
APWD	$\alpha = 2.5719\ \lambda = 0.0475$ $\beta = 1.8087$	-72.6948	151.3896	152.3127	155.5932	0.2478 (0.9715)	0.0403 (0.9347)	0.0953 (0.9243)
APTFD	$\alpha = 31.228\ a = 1.8265$ $b = 1.6889$	-79.0937	164.1874	165.1105	168.3910	1.3121 (0.2283)	0.2149 (0.2409)	0.1885 (0.2086)
APED	$\alpha = 85.5520\ \lambda = 0.3787$	-73.6981	151.3962	151.8407	154.1986	0.3794 (0.8680)	0.0747 (0.7268)	0.1332 (0.6141)

Table 6. MLEs and Adequacy Measures (Data 2)

	Para	ℓ	AK1	CAK1	BA1	AD1 (p-value)	CMT1 (p-value)	KST1 (p-value)
PMMSGD	$\tau = 0.0098\ \phi = 1.0732$	-338.5953	681.4049	681.4385	685.3457	0.2983 (0.9392)	0.0519 (0.8667)	0.0847 (0.7912)
MAPTBXIID	$\alpha = 245.17\ c = 4.6870$ $k = 0.3464$	-342.9538	691.9077	692.5743	698.1403	0.7526 (0.5159)	0.1042 (0.5661)	0.1026 (0.5642)
PMXLD	$\rho = 33.8028\ \epsilon = 0.7915$	-352.6719	709.3439	709.5582	713.4990	2.2986 (0.0635)	0.3540 (0.0956)	0.1576 (0.1066)
GPWD	$\phi = 1.3626\ \epsilon = 0.7450$ $\omega = 0.0019$	-339.0551	684.1102	684.5465	690.3428	0.5466 (0.6991)	0.0834 (0.6740)	0.0850 (0.7869)
GPSD	$\alpha = 0.1397\ \tau = 0.6572$ $\omega = 11.012$	-339.2121	684.4243	684.8606	690.6569	0.4509 (0.7965)	0.0775 (0.7083)	0.1119 (0.4506)
APBXIID	$\alpha = 435.28\ \beta = 4.2631$ $\lambda = 0.1260$	-367.6732	741.3463	741.7827	747.5790	6.8011 (0.0004)	1.2209 (0.0007)	0.2254 (0.0050)
APWD	$\alpha = 18.212\ \lambda = 0.0030$ $\beta = 0.8574$	-339.3599	684.7199	685.1563	690.9525	0.38815 (0.8599)	0.0662 (0.7767)	0.0916 (0.7054)
APTFD	$\alpha = 34.714\ a = 19.288$ $b = 1.1408$	-347.4991	700.9982	701.4346	707.2308	1.4634 (0.1855)	0.2135 (0.2431)	0.1205 (0.3588)
APED	$\alpha = 2.2090\ \lambda = 0.0104$	-339.6966	682.6966	682.9109	686.8517	0.4723 (0.7745)	0.0759 (0.7178)	0.4043 (0.5420)

The estimated Probability-Probability and PDF plots (Figures 6 - 9) further validate the PMMSGD modeling capabilities. These visualizations clearly show that our model closely fits the empirical data better than the other distributions. This reinforces that PMMSGD is the BEST choice for analyzing these datasets.

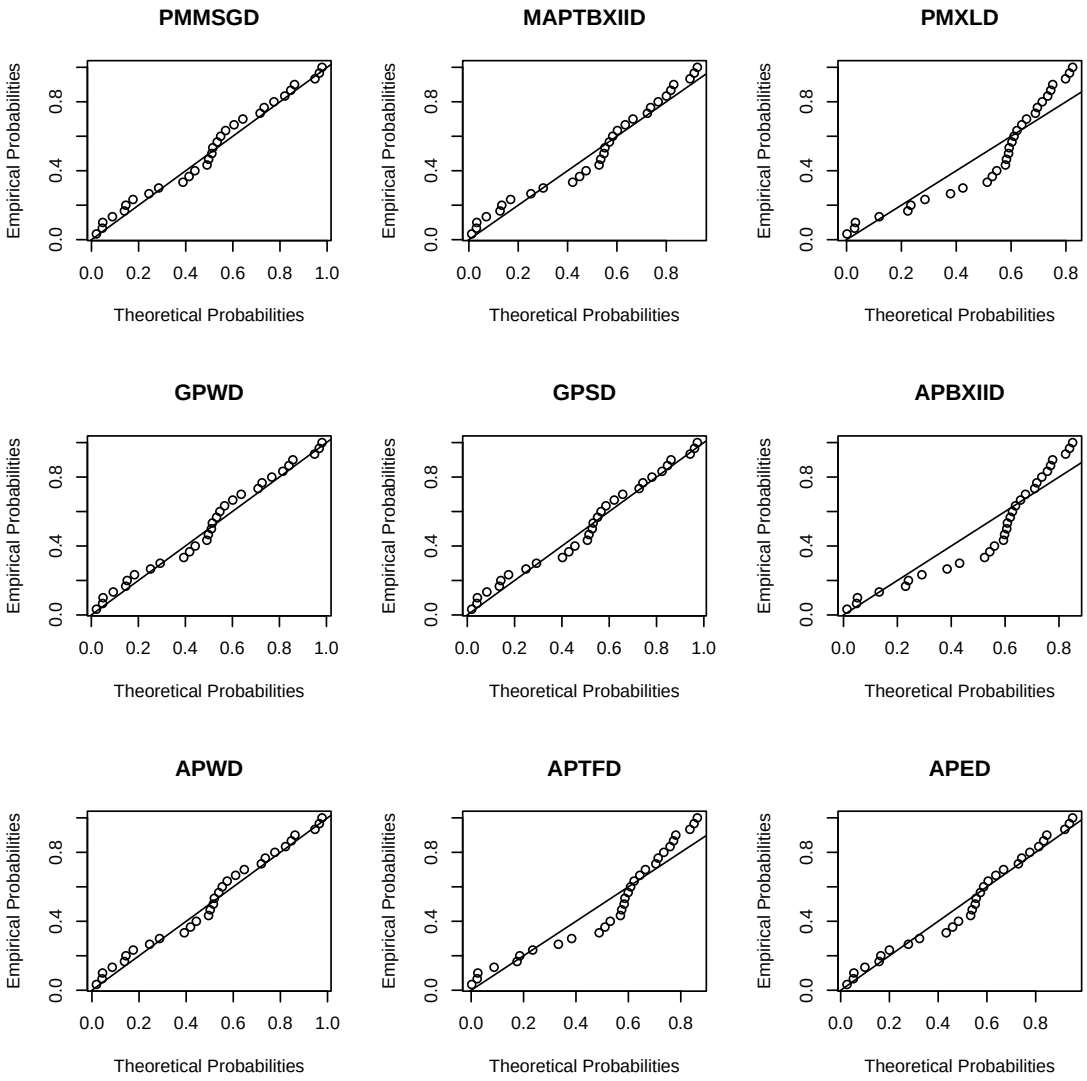


Figure 6. Estimated Probability-Probability Plots Data 1

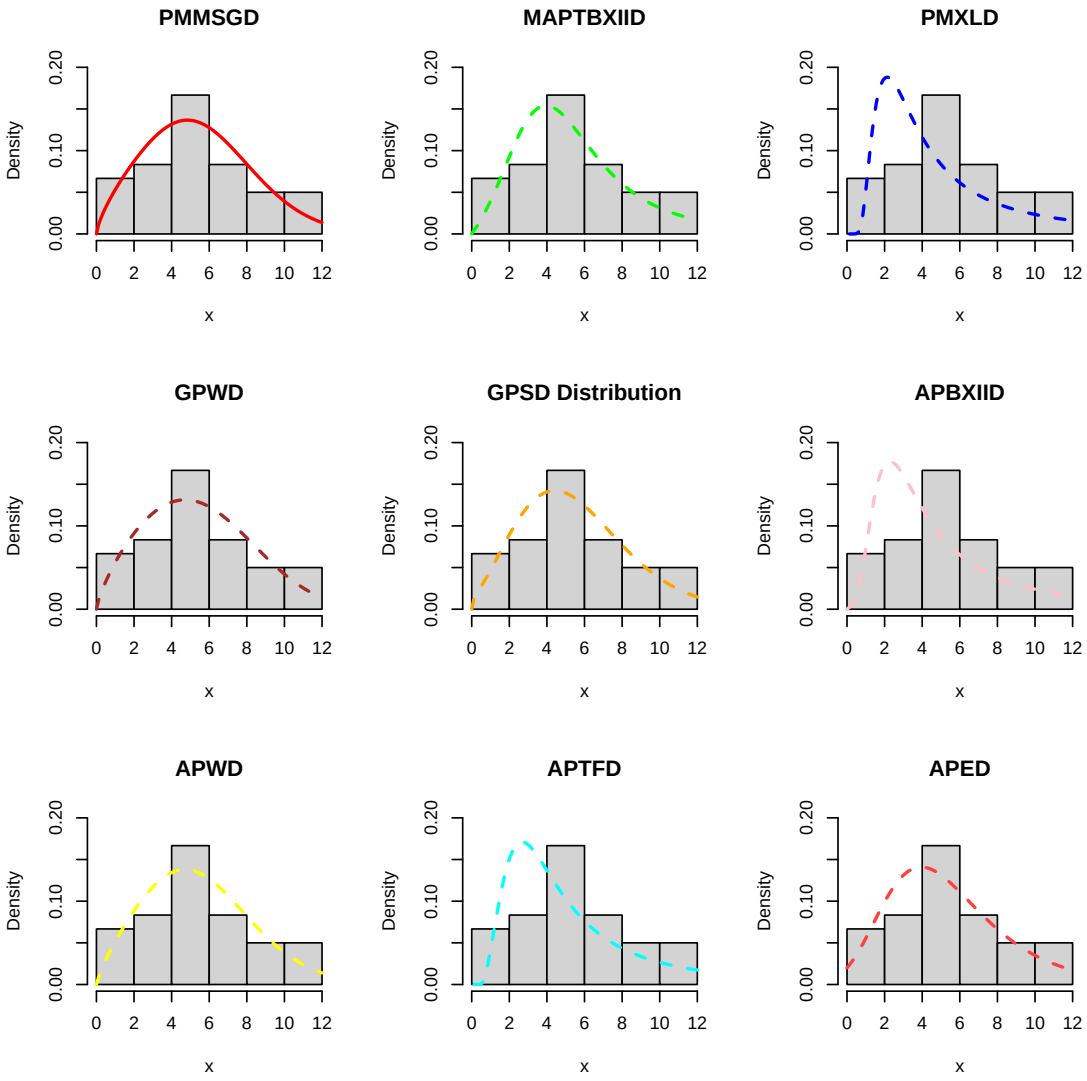


Figure 7. Estimated PDF Plots Data 1

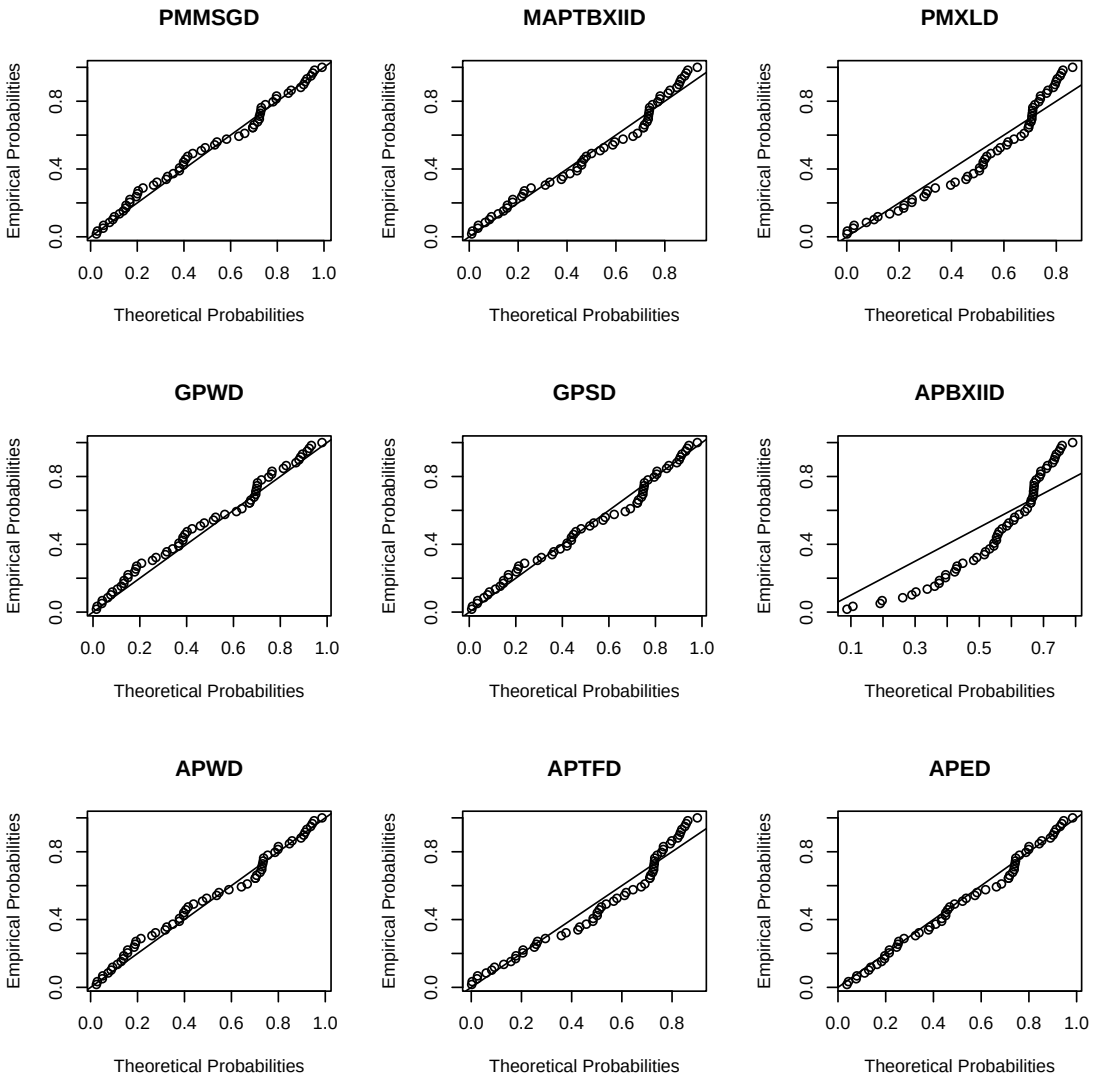


Figure 8. Estimated Probability-Probability Plots Data 2

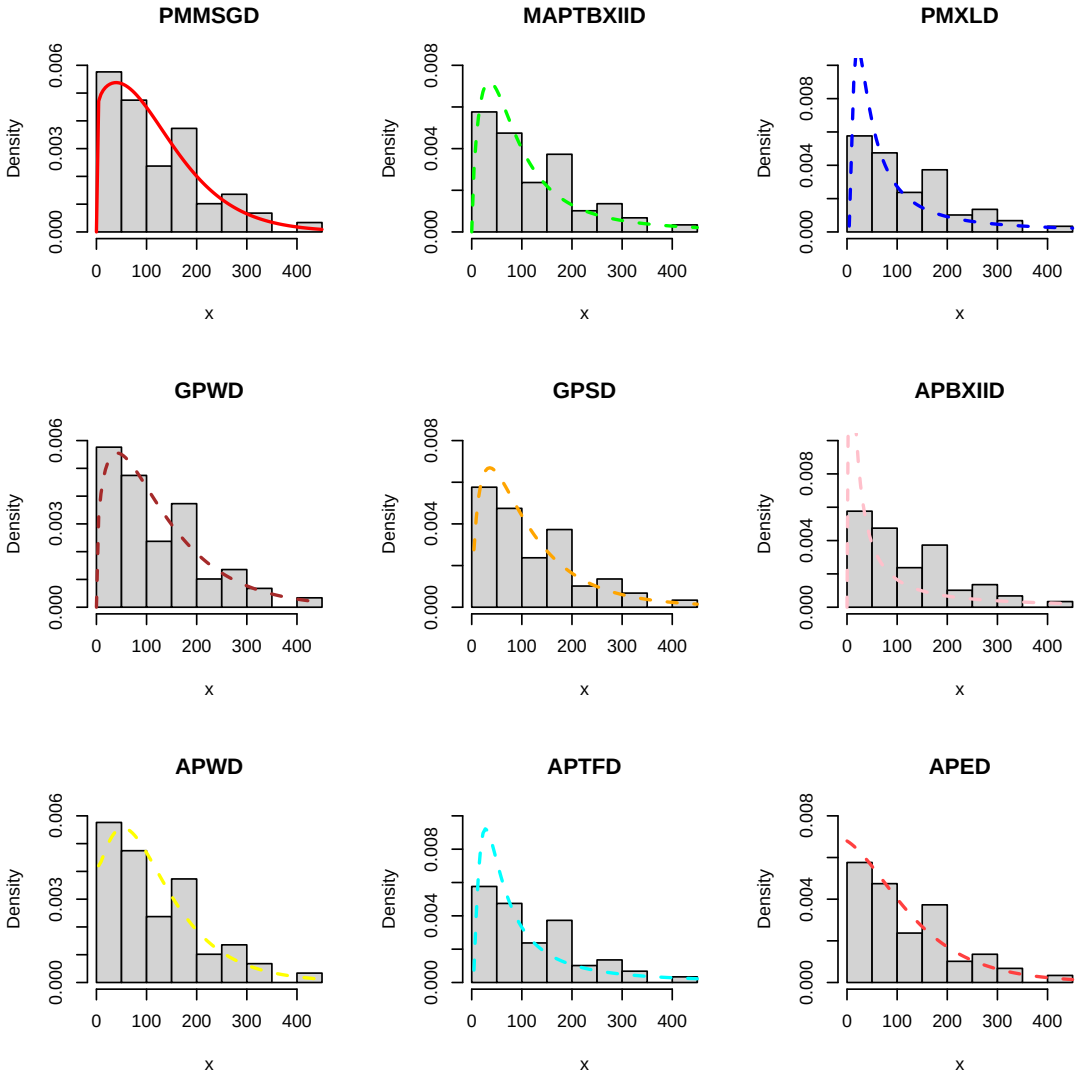


Figure 9. Estimated PDF Plots Data 2

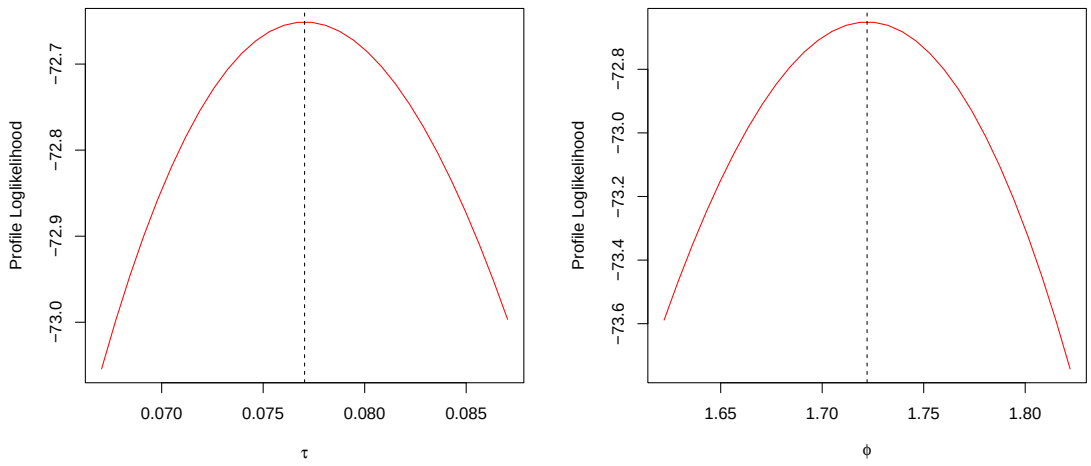


Figure 10. Log-Profile Plot Data 1

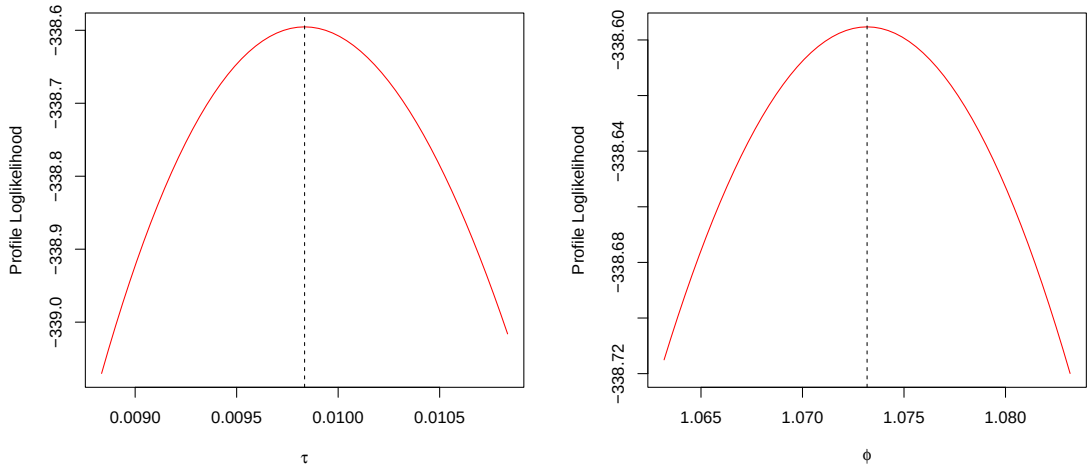


Figure 11. Log-Profile Plot Data 2

The log-likelihood profile reaches a clear peak with smooth descent, validating our parameter estimates of the PMMSGD as shown in Figures 10 and 11. No multi-modality or flat regions were detected, indicating reliable convergence.

6. Concluding remarks

In this study, we proposed a new probability distribution, the power modified mixture symmetric gamma distribution, which enhances the flexibility of the mixture symmetric gamma distribution by restricting its domain to non-negative values and incorporating a power transformation. The resulting model introduces a scale parameter

τ and a shape parameter ϕ, allowing it to capture a wide range of distributional shapes, including heavy tails (when $\phi < 1$) and skewness, making it highly adaptable to real-world data exhibiting complex behavior. In both applications, the fitted shape parameter exceeds one ($\hat{\phi} = 1.7221$ and 1.0732), so these data have lighter-than-exponential tails; assessing the model on genuinely heavy-tailed data remains a topic for future work. We derived the cumulative distribution function and the probability density function of the proposed distribution and examined key statistical properties, including its moments and hazard function. Parameter estimations were achieved using several methods, ensuring statistical efficiency in practical settings. The applicability and performance of the proposed distribution were demonstrated through two real-life datasets. Undeniably, the proposed distribution emerges as a powerful distribution for statistical modeling with promising potential in reliability engineering, environmental studies, and survival analysis. Future research could explore Bayesian inference for the proposed distribution, generalizations to multivariate forms, and applications to other domains where flexible modeling of positive continuous data is essential.

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Data Availability Statement

The data that supports the findings of this study are available within the article.

Conflicts of Interest

The authors declare no conflict of interest.

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