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The Weibull Power Garhy Mixture Distribution: Statistical Properties, Estimation, and Applications

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Abstract This paper investigates a constrained three-parameter Weibull Power Garhy mixture (WPGM) for positive-valued data. Under the transformation $Z = \lambda X^p$, the model is a three-component gamma mixture with shapes one, two and three and restricted weights; it contains the Weibull and the power-transformed Garhy distributions as boundary cases. We derive the density, distribution and reliability functions, ordinary and incomplete moments, quantiles, random generation, order statistics, and several inequality, information measures. Eight estimation criteria are implemented and compared in a Monte Carlo study, and explicit estimating equations are derived for all eight criteria. Two published data sets are analysed: a sample of 61 recidivism failure times (a social-science setting) and a sample of 60 electrical-appliance failure cycles (an engineering setting). For each data set, the WPGM is fitted by maximum likelihood and compared with ten competing models using information criteria, standard errors, and goodness-of-fit statistics. The WPGM attains the smallest AIC, AICc, HQIC, and ASAE, together with the smallest Kolmogorov Smirnov, Cramér von Mises, and Anderson Darling statistics, among the eleven candidate models for both data sets.

Keywords Weibull distribution; Power Garhy distribution; mixture; reliability; maximum likelihood; maximum spacing; Monte Carlo simulation.

AMS 2010 subject classifications 60E05, 62E10, 62F10, 62N05

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1. Introduction

Lifetime and material-strength analysis relies on probability models to describe survival probabilities and failure rates. The exponential, gamma, lognormal, and Weibull distributions are widely used because their properties and estimation procedures are well established. The Weibull distribution of Weibull [47] accommodates decreasing, constant, or increasing hazard rates through its shape parameter, but not a hazard rate with an interior turning point. In the rate power parameterization, its probability density function (PDF) and cumulative distribution function (CDF) are, respectively,

$$f_{\mathrm{W}}(x;\lambda,p) = p\lambda x^{p-1}e^{-\lambda x^p}, \qquad x > 0, \tag{1.1}$$

and

$$F_{\mathrm{W}}(x;\lambda,p) = 1 - e^{-\lambda x^p}, \qquad x > 0, \tag{1.2}$$

where $\lambda > 0$ and $p > 0$. The generalized gamma distribution of Stacy [42] provides another important reference model for positive data, encompassing the Weibull and gamma laws through appropriate parameter restrictions.

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More flexible models have been developed using distribution generators, weighting, and finite mixtures. Examples include the Marshall Olkin extension [29], Kumaraswamy-generated distributions [11], and the transformed transformer family [6]; see Lee et al. [28]. Weighted distribution theory was developed by Rao [37] and Patil and Rao [34], with subsequent Weibull examples including Alahmadi et al. [3] and Ramaki et al. [35]. However, additional parameters and numerous special cases do not necessarily imply a distinct family: Nadarajah [33] identified an apparently new weighted generalized gamma model as a reparameterization of the established distribution.

Finite Weibull mixtures have long been used to model heterogeneous failure times [23]. An unrestricted mixture of two Weibull distributions has five free parameters: a rate and a shape for each component, and one mixing weight. This allows the two components to have different rates and shapes. In the WPGM, the Weibull and power-transformed Garhy components share both λ and p. The resulting three-parameter model therefore attributes differences between the components to their distributional forms, while ω controls their relative contributions. General inferential issues for finite mixtures are discussed by McLachlan and Peel [30]. Exponential gamma mixtures also produce compact lifetime models, including the Lindley and generalized Lindley distributions [16, 48] and the Akash distribution [40, 41]. Gamma-mixture representations are useful because their distribution functions and moments often have tractable expressions.

Weibull gamma combinations have been studied using several distinct constructions. Mohammed et al. [32] considered an exponential gamma Weibull mixture, and Hamed [17] proposed a Weibull generalized gamma mixture. A Weibull lognormal gamma mixture was examined by Kumar and Jain [25]. In contrast to these finite mixtures, Klakattawi [24] developed a Weibull gamma distribution using a Weibull-generated transformation; a further Weibull gamma construction was investigated by Suleiman et al. [44]. Recent power-based models, including the power new XLindley distribution [15], other power-transformed lifetime families [46], and the Shanker Weibull model [31], also illustrate the usefulness of transformations and component combinations. Kalt and Abdul Sada [18] studied a three-parameter survival power Weibull model obtained by applying the survival power-G construction to a Weibull baseline. This provides a transformation-based example alongside the finite mixtures considered here.

A recent example is the one-parameter Garhy distribution proposed by Elgarhy [13]. For a nonnegative random variable Y with parameter $\lambda > 0$, the Garhy PDF and CDF are

$$g_{\mathrm{G}}(y;\lambda) = \frac{\lambda^3 y(\lambda^2+3y)}{6+\lambda^3} e^{-\lambda y}, \qquad y > 0, \tag{1.3}$$

$$G_{\mathrm{G}}(y;\lambda) = 1 - e^{-\lambda y}\left[1 + \lambda y + \frac{3\lambda^2 y^2}{6+\lambda^3}\right], \qquad y > 0, \tag{1.4}$$

The density in (1.3) is a constrained mixture of gamma distributions with shapes two and three. Its elementary CDF in (1.4) and gamma-mixture structure support tractable reliability and moment calculations. Garhy-based extensions have subsequently been investigated, including the half-logistic Garhy model [5].

The Weibull and Garhy distributions provide the components of the WPGM construction presented in Section 2; the power transformation and mixture representation are developed there.

We consider how departures from a Weibull law toward a power-transformed Garhy law can be modeled with a single mixing weight when the two components share the parameters λ and p. The mixture studied by Hamed [17] combines a Weibull component with a generalized gamma component. In the WPGM, the choice of the power-transformed Garhy component imposes a specific restriction on the mixture weights. Under $Z = \lambda X^p$, the model is a member of the family of unit-rate gamma mixtures with shapes one, two, and three (Proposition 2.1). The shape-one component has weight $1-\omega$, while the remaining weight ω is divided between the shape-two and shape-three components in the ratio $\lambda^3 : 6$. Thus, for fixed λ and p, varying ω interpolates between the Weibull law at $\omega = 0$ and the power-transformed Garhy law at $\omega = 1$. The elementary CDF, closed-form ordinary and incomplete moments, and direct sampling scheme follow from this restricted gamma-mixture representation.

The main objectives of this investigation are:

- to examine the distributional, reliability, and mathematical properties of the three-parameter WPGM model, including moments, quantiles, and order statistics;

- to investigate eight estimation criteria and assess the accuracy and numerical behavior of the resulting estimators through Monte Carlo simulation;
- to evaluate the model using two published data sets one from a social-science setting and one from an engineering setting with comparisons with established lifetime distributions based on parameter estimates, model-selection criteria, and goodness-of-fit measures.

The remainder of the article is organized as follows. 2 introduces the WPGM distribution and its reliability functions. Mathematical and statistical properties are established in 3, followed by estimation methods in 4 and the Monte Carlo study in 5. The two data applications are considered in 6, while 7 concludes the article.

2. The Weibull Power Garhy Mixture Distribution

Let Y follow the Garhy distribution with parameter $\lambda > 0$, whose PDF and CDF are given in (1.3) and (1.4), respectively. Elgarhy et al. [19] introduced the Power Garhy distribution (PGD) through the transformation $X = Y^{1/p}$, where $p > 0$. In the notation used here, its CDF and PDF are

$$\begin{aligned} F_{\mathrm{PG}}(x;\lambda,p) &= 1 - e^{-\lambda x^p}\left(1 + \lambda x^p + \frac{3\lambda^2 x^{2p}}{6+\lambda^3}\right), \\ f_{\mathrm{PG}}(x;\lambda,p) &= \frac{p\lambda^3}{6+\lambda^3}\, x^{2p-1}\left(\lambda^2 + 3x^p\right) e^{-\lambda x^p}, \end{aligned} \qquad x > 0. \tag{2.1}$$

Combining the Weibull component in (1.1) and (1.2) with the Power Garhy component in (2.1) gives the three-parameter Weibull Power Garhy mixture (WPGM), defined by

$$\begin{aligned} f_{\mathrm{WPGM}}(x;\lambda,p,\omega) &= (1-\omega) f_{\mathrm{W}}(x;\lambda,p) + \omega f_{\mathrm{PG}}(x;\lambda,p), \\ F_{\mathrm{WPGM}}(x;\lambda,p,\omega) &= (1-\omega) F_{\mathrm{W}}(x;\lambda,p) + \omega F_{\mathrm{PG}}(x;\lambda,p), \end{aligned} \qquad 0 \le \omega \le 1. \tag{2.2}$$

Here $\lambda > 0$ and $p > 0$ are the common transformed-rate and power-shape parameters, respectively, while ω is the probability assigned to the power-transformed Garhy component. Substituting the component densities into (2.2) yields the WPGM PDF

$$f(x;\lambda,p,\omega) = p\,\lambda\, x^{p-1}\, e^{-\lambda x^p}\left[(1-\omega) + \omega\,\frac{\lambda^3(\lambda x^p) + 3(\lambda x^p)^2}{6+\lambda^3}\right], \qquad x > 0. \tag{2.3}$$

The corresponding CDF is

$$F(x;\lambda,p,\omega) = 1 - e^{-\lambda x^p}\left[1 + \omega\,\lambda x^p + \frac{3\,\omega\,(\lambda x^p)^2}{6+\lambda^3}\right], \qquad x > 0. \tag{2.4}$$

Both functions are proper: the CDF in (2.4) is continuous, satisfies $F(0) = 0$ and $\lim_{x\to\infty} F(x) = 1$, and its derivative on $(0,\infty)$ is the density in (2.3). Moreover, (2.2) shows directly that the density integrates to one because it is a convex combination of two normalized component densities. Table 1 reports the distinct submodels of the WPGM and their parameter restrictions.

Proposition 2.1
If X has the WPGM density in (2.3), then the transformed variable $Z = \lambda X^p$ follows a three-component gamma mixture:

$$Z \sim \sum_{j=1}^{3} \pi_j\, \mathrm{Gamma}(j,1), \qquad \pi_1 = 1-\omega, \quad \pi_2 = \frac{\omega\lambda^3}{6+\lambda^3}, \quad \pi_3 = \frac{6\omega}{6+\lambda^3}. \tag{2.5}$$

Here $\mathrm{Gamma}(j,1)$ denotes a gamma distribution with shape j and rate one; the summation denotes a probability mixture.

Table 1. Distinct submodels of the WPGM and their parameter restrictions.

Model	Restriction	Remaining free parameters
Weibull	$\omega = 0$	λ, p
Exponential	$\omega = 0,\ p = 1$	λ
Rayleigh	$\omega = 0,\ p = 2$	λ
Garhy	$\omega = 1,\ p = 1$	λ
Power-transformed Garhy	$\omega = 1$	λ, p
Exponential Garhy mixture (EGM)	$p = 1$	λ, ω
Rayleigh Power Garhy mixture (RPGM)	$p = 2$	λ, ω

Proof
The three probabilities in (2.5) are nonnegative and satisfy

$$(1-\omega) + \frac{\omega\lambda^3}{6+\lambda^3} + \frac{6\omega}{6+\lambda^3} = 1.$$

For $t > 0$, use the inverse transformation $x = (t/\lambda)^{1/p}$ in (2.3) and the Jacobian $|dx/dt| = (p\lambda x^{p-1})^{-1}$ to obtain

$$g_Z(t) = e^{-t}\left[(1-\omega) + \frac{\omega\lambda^3}{6+\lambda^3}t + \frac{3\omega}{6+\lambda^3}t^2\right].$$

The gamma densities with unit rate and respective shapes one, two and three are e^{-t}, te^{-t}, and $t^2e^{-t}/2$. Their weighted sum using (2.5) is exactly $g_Z(t)$, which proves the representation. □

Thus, on the scale $Z = \lambda X^p$, the WPGM has a finite gamma-mixture representation with fixed shapes $1, 2, 3$ and unit rate. The restriction in the model for X is explicit in (2.5): $6\pi_2 = \lambda^3\pi_3$, where the same λ also appears in the transformation. For fixed λ, the admissible weights form the line segment joining $(1, 0, 0)$ to $(0, \lambda^3/(6+\lambda^3), 6/(6+\lambda^3))$. An unrestricted three-component mixture permits two freely varying weights at that same value of λ.

For $x > 0$, the survival function obtained from (2.4) is

$$S(x;\lambda,p,\omega) = e^{-\lambda x^p}\left[1 + \omega\lambda x^p + \frac{3\omega(\lambda x^p)^2}{6+\lambda^3}\right].$$

Hence the hazard rate function (HRF), obtained by dividing (2.3) by this survival function, is

$$h(x;\lambda,p,\omega) = \frac{p\lambda x^{p-1}\left[(1-\omega) + \omega\dfrac{\lambda^3(\lambda x^p) + 3(\lambda x^p)^2}{6+\lambda^3}\right]}{1 + \omega\lambda x^p + \dfrac{3\omega(\lambda x^p)^2}{6+\lambda^3}}, \qquad x > 0. \tag{2.6}$$

The corresponding cumulative hazard function is

$$H(x;\lambda,p,\omega) = -\log S(x;\lambda,p,\omega) = \lambda x^p - \log\left[1 + \omega\lambda x^p + \frac{3\omega(\lambda x^p)^2}{6+\lambda^3}\right], \qquad x > 0.$$

Proposition 2.2 (Behavior near zero and in the upper tail)
Let $t = \lambda x^p$ and $d = 6 + \lambda^3$. As $x \downarrow 0$,

$$F(x) \sim \begin{cases} (1-\omega)t, & 0 \le \omega < 1, \\ \dfrac{\lambda^3}{2d}\,t^2, & \omega = 1, \end{cases} \qquad h(x) \sim f(x) \sim \begin{cases} p\lambda(1-\omega)x^{p-1}, & 0 \le \omega < 1, \\ \dfrac{p\lambda^5}{d}\,x^{2p-1}, & \omega = 1. \end{cases}$$

As $x \to \infty$, the survival function satisfies

$$S(x) \sim \begin{cases} e^{-t}, & \omega = 0, \\ \dfrac{3\omega}{d}\, t^2 e^{-t}, & 0 < \omega \le 1, \end{cases} \qquad h(x) \sim p\lambda x^{p-1} \quad (0 \le \omega \le 1).$$

Proof
Near zero, the first nonzero term in the bracket in (2.3) is $1-\omega$ when $\omega < 1$ and $\lambda^3 t/d$ when $\omega = 1$. This gives the stated density equivalents. Integration gives the corresponding equivalents for $F(x)$, and $S(x) \to 1$ gives $h(x) \sim f(x)$. For $\omega > 0$, the polynomial multiplying e^{-t} in $S(x)$ has leading term $3\omega t^2/d$. The same term occurs in the density after factoring out $p\lambda x^{p-1}e^{-t}$, giving the stated tail equivalent for $h(x)$. At $\omega = 0$, the Weibull survival and hazard formulas hold exactly. □

For $\omega < 1$, the initial hazard is infinite, finite and positive, or zero according as $p < 1$, $p = 1$, or $p > 1$. At $\omega = 1$, the corresponding threshold is $p = 1/2$. In the upper tail, the hazard tends to zero for $p < 1$, to λ for $p = 1$, and to infinity for $p > 1$, for every mixing weight.

Proposition 2.3 (Identifiability)
The WPGM distribution uniquely determines the parameter vector (λ, p, ω) for $\lambda > 0$, $p > 0$, and $0 \le \omega \le 1$.

Proof
Let $S(x) = 1 - F(x)$ and $H(x) = -\log S(x)$. By (2.4), the factor multiplying $e^{-\lambda x^p}$ in $S(x)$ is a polynomial in x^p of degree at most two with constant term one and nonnegative coefficients. Its logarithm is therefore $O(\log x)$ as $x \to \infty$. Since $p > 0$, it follows that

$$H(x) = \lambda x^p + O(\log x) \sim \lambda x^p.$$

Hence the distribution determines p and then λ through

$$p = \lim_{x\to\infty} \frac{\log H(x)}{\log x}, \qquad \lambda = \lim_{x\to\infty} \frac{H(x)}{x^p}.$$

With these two parameters determined, set $x_* = \lambda^{-1/p}$, so that $\lambda x_*^p = 1$. Evaluating the survival function at x_* gives

$$\omega = \frac{e\, S(x_*) - 1}{1 + 3/(6+\lambda^3)}.$$

The denominator is strictly positive for every $\lambda > 0$, and the same formula recovers ω at both endpoints. Thus two parameter vectors giving the same distribution must agree in all three coordinates. □

At $\omega = 0$ and $\omega = 1$, the model reduces to the Weibull and power-transformed Garhy distributions, respectively. Since λ and p are shared, both parameters remain in the surviving component at either endpoint. Their identification from the tail of the survival function therefore applies throughout $0 \le \omega \le 1$. The accuracy of a finite-sample estimate is a separate issue. For example, when the generating weight was $\omega = 0.05$, the proportion of accepted fits with $\widehat{\omega} \ge 0.999$ ranged from 5.2% to 42.9% across the methods and sample sizes examined in Section 5.

Figure 1 displays the density in (2.3) and hazard function in (2.6) for four parameter settings. With $p = 0.7$ and $\omega = 0.25$, the plotted density declines from a high value near the origin and the hazard is decreasing over the displayed range. For $p = 1$, $\lambda = 1.2$, and $\omega = 0.5$, the mixing weight lies below $6/(6+\lambda^3)$, so Corollary 2.1 establishes that the hazard has a unique minimum. The curves for $p = 1.3$ and $p = 1.6$ increase over the displayed range. The densities for the latter settings are unimodal. These examples illustrate several shapes that the WPGM can represent, without implying that they exhaust the model's possible hazard patterns.

Proposition 2.4 (Monotonicity of the hazard rate)
Let $t = \lambda x^p$ and define

$$A(t) = 1 - \omega + \frac{\omega(\lambda^3 t + 3t^2)}{6+\lambda^3}, \qquad B(t) = 1 + \omega t + \frac{3\omega t^2}{6+\lambda^3}.$$

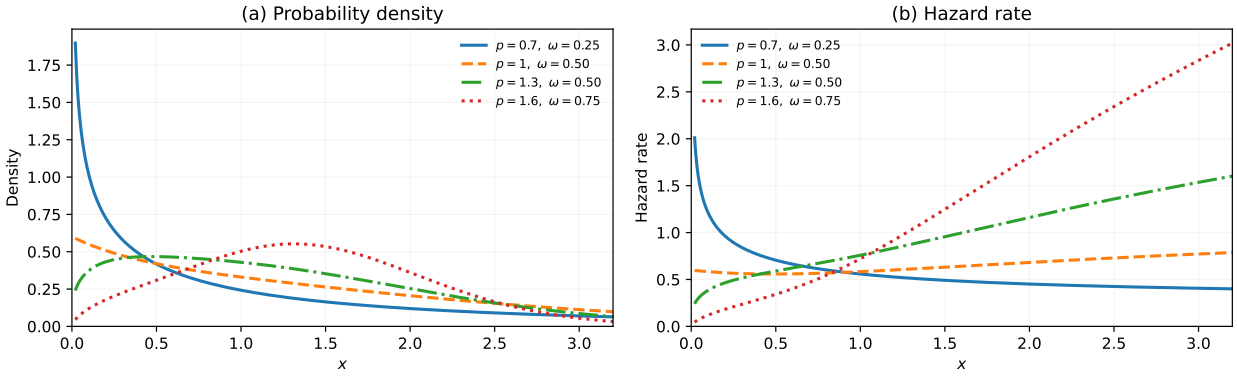

Figure 1. PDF (left) and HRF (right) of the WPGM distribution at $\lambda = 1.2$ and selected (p, ω) values.

For $x > 0$, the sign of $h'(x)$ equals the sign of

$$D(t) = (p-1)A(t)B(t) + pt\{A'(t)B(t) - A(t)B'(t)\},$$

where the primes on A and B denote differentiation with respect to t. Consequently, the hazard is nondecreasing on $(0, \infty)$ if and only if $D(t) \geq 0$ for every $t > 0$, and nonincreasing if and only if $D(t) \leq 0$ for every $t > 0$. A positive root t_0 at which D changes sign gives a turning point at $x_0 = (t_0/\lambda)^{1/p}$.

Proof
Equation (2.6) gives $h(x) = p\lambda x^{p-1}A(t)/B(t)$. Using $dt/dx = pt/x$ yields

$$h'(x) = \frac{p\lambda x^{p-2}}{B(t)^2} D(t).$$

The prefactor is positive for $x > 0$, and $t = \lambda x^p$ increases from zero to infinity. The stated conditions follow. □

Corollary 2.1 (Hazard shapes when $p = 1$)
Suppose $p = 1$ and let $\omega_c = 6/(6 + \lambda^3)$. If $\omega = 0$, the hazard is constant. If $0 < \omega < \omega_c$, the hazard decreases to a unique minimum and then increases. If $\omega_c \leq \omega \leq 1$, the hazard is strictly increasing on $(0, \infty)$.

Proof
Set $t = \lambda x$ and $b = 3\omega/(6 + \lambda^3)$. The survival function is $e^{-t}B(t)$, where $B(t) = 1 + \omega t + bt^2$. Hence

$$h(x) = \lambda\left(1 - \frac{B'(t)}{B(t)}\right), \qquad h'(x) = \frac{\lambda^2}{B(t)^2}\left[\omega^2 - 2b + 2\omega bt + 2b^2t^2\right].$$

For $\omega = 0$, $h(x) = \lambda$. For $\omega > 0$, the polynomial in brackets is strictly increasing for $t \geq 0$ and tends to infinity. Its value at zero is $\omega(\omega - \omega_c)$. It therefore has exactly one positive root if $0 < \omega < \omega_c$, and is positive for every $t > 0$ if $\omega \geq \omega_c$. □

3. Mathematical and Statistical Properties

This section examines the distributional properties of the Weibull Power Garhy mixture (WPGM). Throughout, $X \sim \text{WPGM}(\lambda, p, \omega)$, where $\lambda > 0$, $p > 0$, and $0 \leq \omega \leq 1$. The density and distribution function are given in (2.3) and (2.4). The derivations below start from these functions and evaluate the relevant integrals explicitly. We use $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt$, $\gamma(a, b) = \int_0^b t^{a-1}e^{-t}\,dt$, and $\Gamma(a, b) = \int_b^\infty t^{a-1}e^{-t}\,dt$ when the integrals are defined.

3.1. *Quantile function*

Proposition 3.1 (Existence and uniqueness of the quantile)
For every $u \in (0,1)$, there exists exactly one $q_u > 0$ such that

$$e^{-\lambda q_u^p}\left(1+\omega\lambda q_u^p+\frac{3\omega\lambda^2 q_u^{2p}}{6+\lambda^3}\right)=1-u. \tag{3.1}$$

The quantile function is continuous and strictly increasing on $(0,1)$.

Proof
For every $x>0$, both component densities in (2.2) are strictly positive. Hence their mixture is strictly positive for every $0 \le \omega \le 1$, including the endpoints, where it reduces to the density of the corresponding component. It follows that F is continuous and strictly increasing on $(0,\infty)$. Since $F(0)=0$ and $\lim_{x\to\infty} F(x)=1$, each $u\in(0,1)$ determines a unique $q_u>0$ satisfying (3.1). The inverse $u \mapsto q_u$ is therefore continuous and strictly increasing. □

Quantile-based measures. The first, second (median), and third quartiles of the WPGM distribution are $Q_1 = Q(0.25)$, $Q_2 = Q(0.50)$ and $Q_3 = Q(0.75)$. Two complementary shape coefficients are the Bowley quartile skewness and the Moors octile-based kurtosis, defined by

$$\text{Bowley} = \frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}, \qquad \text{Moors} = \frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q_3-Q_1}.$$

All these quantiles are uniquely defined by (3.1), so they can be evaluated by a bracketing root-finding algorithm without any closed-form inversion. Table 2 provides numerical values for 27 combinations of the three parameters.

3.2. *Ordinary moments and moment-based measures*

The rth raw moment of the WPGM is $\mu'_r = E(X^r)$ whenever the integral exists. Inserting the density from (2.3) directly into the expectation yields

$$\begin{aligned}
\mu'_r &= \int_0^\infty x^r f(x;\lambda,p,\omega)\,dx \\
&= p\lambda\int_0^\infty x^{r+p-1}e^{-\lambda x^p}\left[(1-\omega)+\frac{\omega\lambda^3(\lambda x^p)+3\omega(\lambda x^p)^2}{6+\lambda^3}\right]dx \\
&= p\lambda(1-\omega)\int_0^\infty x^{r+p-1}e^{-\lambda x^p}\,dx \\
&\quad+\frac{p\omega\lambda^5}{6+\lambda^3}\int_0^\infty x^{r+2p-1}e^{-\lambda x^p}\,dx+\frac{3p\omega\lambda^3}{6+\lambda^3}\int_0^\infty x^{r+3p-1}e^{-\lambda x^p}\,dx.
\end{aligned}$$

For each of the three integrals, apply the substitution $t=\lambda x^p$, so that $x=\left(\frac{t}{\lambda}\right)^{1/p}$, which gives, for $k=1,2,3$ and $r+kp>0$,

$$\begin{aligned}
\int_0^\infty x^{r+kp-1}e^{-\lambda x^p}\,dx &= \frac{\lambda^{-(r/p+k)}}{p}\int_0^\infty t^{r/p+k-1}e^{-t}\,dt \\
&= \frac{\lambda^{-(r/p+k)}}{p}\Gamma\left(k+\frac{r}{p}\right).
\end{aligned}$$

Inserting these integrations into the preceding expression and collecting the powers of λ, we obtain the rth raw moment:

$$\mu'_r=\lambda^{-r/p}\left[(1-\omega)\Gamma\left(1+\frac{r}{p}\right)+\frac{\omega\lambda^3}{6+\lambda^3}\Gamma\left(2+\frac{r}{p}\right)+\frac{3\omega}{6+\lambda^3}\Gamma\left(3+\frac{r}{p}\right)\right]. \tag{3.2}$$

Table 2. Quartiles, Bowley skewness and Moors kurtosis of the WPGM distribution.

λ	p	Q_1	Q_2	Q_3	Bowley	Moors
Mixing weight $\omega = 0.25$						
1.20	2.5	0.6417	0.9247	1.2348	0.0456	1.1848
1.60	2.5	0.5693	0.8162	1.0847	0.0421	1.1915
2.20	2.5	0.4984	0.7100	0.9376	0.0364	1.1972
1.20	3.5	0.7284	0.9456	1.1626	-0.0006	1.1927
1.60	3.5	0.6687	0.8649	1.0598	-0.0035	1.1998
2.20	3.5	0.6081	0.7830	0.9550	-0.0082	1.2063
1.20	5.0	0.8010	0.9616	1.1112	-0.0354	1.2055
1.60	5.0	0.7545	0.9034	1.0415	-0.0377	1.2127
2.20	5.0	0.7060	0.8426	0.9683	-0.0418	1.2196
Mixing weight $\omega = 0.50$						
1.20	2.5	0.7605	1.0791	1.3873	-0.0165	1.1737
1.60	2.5	0.6667	0.9388	1.2087	-0.0041	1.1832
2.20	2.5	0.5757	0.8027	1.0319	0.0049	1.1965
1.20	3.5	0.8223	1.0559	1.2635	-0.0587	1.1935
1.60	3.5	0.7486	0.9559	1.1450	-0.0461	1.2006
2.20	3.5	0.6741	0.8547	1.0227	-0.0363	1.2118
1.20	5.0	0.8720	1.0388	1.1779	-0.0905	1.2145
1.60	5.0	0.8165	0.9689	1.0994	-0.0775	1.2196
2.20	5.0	0.7588	0.8959	1.0158	-0.0672	1.2290
Mixing weight $\omega = 0.75$						
1.20	2.5	0.9309	1.2225	1.4927	-0.0380	1.2464
1.60	2.5	0.7998	1.0555	1.2987	-0.0249	1.2319
2.20	2.5	0.6752	0.8916	1.1042	-0.0090	1.2253
1.20	3.5	0.9502	1.1543	1.3313	-0.0712	1.2685
1.60	3.5	0.8525	1.0393	1.2053	-0.0592	1.2509
2.20	3.5	0.7554	0.9213	1.0733	-0.0438	1.2408
1.20	5.0	0.9649	1.1057	1.2218	-0.0962	1.2894
1.60	5.0	0.8943	1.0274	1.1396	-0.0848	1.2695
2.20	5.0	0.8217	0.9442	1.0508	-0.0699	1.2568

Proposition 3.2 (Existence of real moments)
For $r \in \mathbb{R}$, the moment $E(X^r)$ is finite if and only if

$$r > \begin{cases} -p, & 0 \le \omega < 1, \\ -2p, & \omega = 1. \end{cases}$$

In particular, every nonnegative real moment is finite.

Proof
Near zero, the density is asymptotic to a positive constant times x^{p-1} when $\omega < 1$ and x^{2p-1} when $\omega = 1$. Thus integrability of $x^r f(x)$ at zero gives the stated strict inequalities. At equality the integrand is proportional asymptotically to x^{-1}. At infinity the factor $e^{-\lambda x^p}$ ensures integrability for every real r. □

In (3.2), terms with zero mixture weight are omitted before evaluating the gamma functions.

The mean and variance follow from (3.2) by taking $r = 1, 2$:

$$\mu = \mu'_1, \qquad \sigma^2 = \mu'_2 - (\mu'_1)^2.$$

Hence the coefficient of variation, standardized skewness, and ordinary kurtosis are

$$\begin{aligned} \mathrm{CV} &= \frac{\sigma}{\mu'_1}, \\ \gamma_1 &= \frac{\mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3}{\sigma^3}, \\ \beta_2 &= \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_2 - 3(\mu'_1)^4}{\sigma^4}. \end{aligned}$$

Here β_2 denotes the ordinary, rather than excess, kurtosis. Table 3 presents these measures and the first four raw moments for the same 27 parameter configurations as Table 2. All entries are theoretical quantities computed from (3.2). For fixed λ and ω among the configurations examined, larger values of p generally reduce the coefficient of variation, while the sign and magnitude of skewness also change. These are numerical properties of the reported parameter grid, not assertions about every admissible configuration.

3.3. Incomplete moments

Incomplete moments are used to evaluate residual-life quantities, and inequality curves. For $y > 0$, the lower incomplete moment of order r is

$$\begin{aligned} \mu'_r(y) &= \int_0^y x^r f(x;\lambda,p,\omega)\,dx \\ &= p\lambda(1-\omega)\int_0^y x^{r+p-1}e^{-\lambda x^p}\,dx \\ &\quad + \frac{p\omega\lambda^5}{6+\lambda^3}\int_0^y x^{r+2p-1}e^{-\lambda x^p}\,dx + \frac{3p\omega\lambda^3}{6+\lambda^3}\int_0^y x^{r+3p-1}e^{-\lambda x^p}\,dx. \end{aligned}$$

Using $t = \lambda x^p$ now changes the upper limit to λy^p, and the three integrals become

$$\begin{aligned} \int_0^y x^{r+kp-1}e^{-\lambda x^p}\,dx &= \frac{\lambda^{-(r/p+k)}}{p}\int_0^{\lambda y^p} t^{r/p+k-1}e^{-t}\,dt \\ &= \frac{\lambda^{-(r/p+k)}}{p}\gamma\left(k+\frac{r}{p},\lambda y^p\right). \end{aligned}$$

Therefore the lower incomplete moment has the closed form

$$\begin{aligned} \mu'_r(y) = \lambda^{-r/p}\Bigg[&(1-\omega)\gamma\left(1+\frac{r}{p},\lambda y^p\right) \\ &+ \frac{\omega\lambda^3}{6+\lambda^3}\gamma\left(2+\frac{r}{p},\lambda y^p\right) + \frac{3\omega}{6+\lambda^3}\gamma\left(3+\frac{r}{p},\lambda y^p\right)\Bigg]. \end{aligned} \tag{3.3}$$

For each fixed $y > 0$, the lower incomplete moment $\mu'_r(y)$ is finite if and only if $r > -p$ when $0 \le \omega < 1$, or $r > -2p$ when $\omega = 1$. These are the conditions in Proposition 3.2, since the integral includes the origin. At $\omega = 1$, the first term in (3.3) is omitted before evaluating the incomplete gamma functions. The upper incomplete moment

Table 3. First four raw moments, variance, standard deviation and moment-based shape coefficients of the WPGM distribution.

λ	p	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	σ	γ_1	β_2	CV
Mixing weight $\omega = 0.25$										
1.20	2.5	0.9522	1.0785	1.3777	1.9262	0.1718	0.4145	0.3311	2.6982	0.4353
1.60	2.5	0.8393	0.8358	0.9378	1.1518	0.1314	0.3624	0.3327	2.7346	0.4318
2.20	2.5	0.7283	0.6270	0.6070	0.6427	0.0965	0.3106	0.3257	2.7681	0.4265
1.20	3.5	0.9455	0.9871	1.1096	1.3223	0.0930	0.3050	0.0126	2.6000	0.3226
1.60	3.5	0.8643	0.8235	0.8442	0.9170	0.0764	0.2765	0.0107	2.6315	0.3199
2.20	3.5	0.7815	0.6716	0.6201	0.6061	0.0609	0.2469	0.0005	2.6662	0.3159
1.20	5.0	0.9502	0.9522	0.9957	1.0785	0.0493	0.2221	-0.2566	2.7795	0.2338
1.60	5.0	0.8925	0.8393	0.8231	0.8358	0.0428	0.2068	-0.2610	2.8129	0.2317
2.20	5.0	0.8319	0.7283	0.6643	0.6270	0.0362	0.1904	-0.2736	2.8551	0.2289
Mixing weight $\omega = 0.50$										
1.20	2.5	1.0795	1.3520	1.8701	2.7845	0.1867	0.4320	0.0952	2.5349	0.4002
1.60	2.5	0.9434	1.0321	1.2488	1.6297	0.1421	0.3769	0.1316	2.5859	0.3995
2.20	2.5	0.8094	0.7582	0.7862	0.8806	0.1032	0.3212	0.1653	2.6563	0.3968
1.20	3.5	1.0370	1.1717	1.4086	1.7766	0.0963	0.3103	-0.2061	2.6238	0.2993
1.60	3.5	0.9420	0.9662	1.0549	1.2091	0.0788	0.2808	-0.1745	2.6519	0.2981
2.20	3.5	0.8446	0.7757	0.7580	0.7781	0.0622	0.2494	-0.1471	2.6994	0.2953
1.20	5.0	1.0151	1.0795	1.1904	1.3520	0.0491	0.2217	-0.4651	2.9455	0.2184
1.60	5.0	0.9492	0.9434	0.9723	1.0321	0.0425	0.2061	-0.4372	2.9612	0.2171
2.20	5.0	0.8796	0.8094	0.7719	0.7582	0.0357	0.1889	-0.4144	2.9992	0.2148
Mixing weight $\omega = 0.75$										
1.20	2.5	1.2068	1.6255	2.3625	3.6428	0.1691	0.4112	-0.1031	2.7619	0.3407
1.60	2.5	1.0475	1.2284	1.5598	2.1075	0.1311	0.3621	-0.0351	2.7453	0.3457
2.20	2.5	0.8904	0.8895	0.9654	1.1184	0.0967	0.3109	0.0383	2.7656	0.3492
1.20	3.5	1.1284	1.3562	1.7075	2.2309	0.0829	0.2878	-0.4144	3.0516	0.2551
1.60	3.5	1.0197	1.1089	1.2655	1.5013	0.0692	0.2630	-0.3442	2.9808	0.2580
2.20	3.5	0.9078	0.8797	0.8959	0.9500	0.0555	0.2356	-0.2710	2.9447	0.2596
1.20	5.0	1.0799	1.2068	1.3852	1.6255	0.0405	0.2013	-0.6871	3.5801	0.1864
1.60	5.0	1.0059	1.0475	1.1214	1.2284	0.0357	0.1891	-0.6139	3.4592	0.1880
2.20	5.0	0.9273	0.8904	0.8795	0.8895	0.0306	0.1748	-0.5398	3.3752	0.1885

is obtained by integrating over (y, ∞) instead:

$$\begin{aligned}\eta_r(y) &= \int_y^\infty x^r f(x;\lambda,p,\omega)\,dx \\ &= p\lambda(1-\omega)\int_y^\infty x^{r+p-1}e^{-\lambda x^p}\,dx \\ &\quad + \frac{p\omega\lambda^5}{6+\lambda^3}\int_y^\infty x^{r+2p-1}e^{-\lambda x^p}\,dx + \frac{3p\omega\lambda^3}{6+\lambda^3}\int_y^\infty x^{r+3p-1}e^{-\lambda x^p}\,dx.\end{aligned}$$

For these integrals, the transformation $t = \lambda x^p$ gives

$$\int_y^\infty x^{r+kp-1}e^{-\lambda x^p}\,dx = \frac{\lambda^{-(r/p+k)}}{p}\int_{\lambda y^p}^\infty t^{r/p+k-1}e^{-t}\,dt.$$

Consequently,

$$\eta_r(y) = \lambda^{-r/p}\Bigg[(1-\omega)\Gamma\left(1+\frac{r}{p},\lambda y^p\right) + \frac{\omega\lambda^3}{6+\lambda^3}\Gamma\left(2+\frac{r}{p},\lambda y^p\right) + \frac{3\omega}{6+\lambda^3}\Gamma\left(3+\frac{r}{p},\lambda y^p\right)\Bigg]. \tag{3.4}$$

For fixed $y > 0$, the upper incomplete moment exists for every real r, because the integration interval excludes the origin. When the ordinary moment exists, $\mu'_r(y) + \eta_r(y) = \mu'_r$ follows by splitting the expectation at y.

3.4. Lorenz, Bonferroni, Zenga and Gini measures

The Lorenz curve at population proportion $0 < u < 1$ is defined by the share of the total first moment associated with observations below q_u:

$$L(u) = \frac{1}{\mu'_1}\int_0^{q_u} x f(x)\,dx = \frac{\mu'_1(q_u)}{\mu'_1}.$$

Dividing the Lorenz ordinate by $u = F(q_u)$ gives the Bonferroni curve,

$$B(u) = \frac{\displaystyle\int_0^{q_u} x f(x)\,dx}{u\mu'_1} = \frac{\mu'_1(q_u)}{u\mu'_1}.$$

The conditional means below and above q_u are, respectively,

$$E[X \mid X \le q_u] = \frac{\mu'_1(q_u)}{u}, \qquad E[X \mid X > q_u] = \frac{\mu'_1 - \mu'_1(q_u)}{1-u}.$$

It follows from the definition of the Zenga curve that

$$Z(u) = 1 - \frac{(1-u)\mu'_1(q_u)}{u\,[\mu'_1 - \mu'_1(q_u)]}, \qquad 0 < u < 1. \tag{3.5}$$

The Zenga formula in (3.5) and the Lorenz and Bonferroni expressions all use the incomplete first moment from (3.3) with $r = 1$. For example, the share attributed to the jth population quintile is $L(j/5) - L((j-1)/5)$ for $j = 1, \ldots, 5$.

For completeness, the Gini coefficient can be calculated without numerical integration of the Lorenz curve. Let X_1 and X_2 be independent copies of X. The identity $|X_1 - X_2| = X_1 + X_2 - 2\min(X_1, X_2)$ leads to

$$\begin{aligned} G &= \frac{E|X_1 - X_2|}{2\mu'_1} = 1 - \frac{E\min(X_1, X_2)}{\mu'_1} \\ &= 1 - \frac{1}{\mu'_1}\int_0^{\infty}[1-F(x)]^2\,dx. \end{aligned}$$

Substitution of the survival function gives the integral

$$\int_0^{\infty}[1-F(x)]^2\,dx = \int_0^{\infty} e^{-2\lambda x^p}\left[1 + \omega\lambda x^p + \frac{3\omega(\lambda x^p)^2}{6+\lambda^3}\right]^2 dx.$$

Expanding the square and then substituting $t = \lambda x^p$ yields

$$\int_0^{\infty}[1-F(x)]^2\,dx = \frac{\lambda^{-1/p}}{p}\int_0^{\infty} t^{1/p-1}e^{-2t}\left[1 + 2\omega t + \left(\omega^2 + \frac{6\omega}{6+\lambda^3}\right)t^2 + \frac{6\omega^2}{6+\lambda^3}t^3 + \frac{9\omega^2}{(6+\lambda^3)^2}t^4\right]dt.$$

Applying $\int_0^\infty t^{a-1}e^{-2t}\,dt = \Gamma(a)/2^a$ term by term leads to

$$G = 1 - \frac{\lambda^{-1/p}}{p\mu_1'}\left[\frac{\Gamma(1/p)}{2^{1/p}} + \frac{2\omega\Gamma(1+1/p)}{2^{1+1/p}} + \left(\omega^2 + \frac{6\omega}{6+\lambda^3}\right)\frac{\Gamma(2+1/p)}{2^{2+1/p}} + \frac{6\omega^2}{6+\lambda^3}\frac{\Gamma(3+1/p)}{2^{3+1/p}} + \frac{9\omega^2}{(6+\lambda^3)^2}\frac{\Gamma(4+1/p)}{2^{4+1/p}}\right]. \tag{3.6}$$

The integral in (3.6) is finite for every $p > 0$.

3.5. *Extropy and weighted extropy*

For a positive continuous random variable, extropy and weighted extropy are defined by

$$J(X) = -\frac{1}{2}\int_0^\infty f(x)^2\,dx, \qquad J_w(X) = -\frac{1}{2}\int_0^\infty x f(x)^2\,dx.$$

From the WPGM density,

$$J(X) = -\frac{p^2\lambda^2}{2}\int_0^\infty x^{2p-2}e^{-2\lambda x^p}\left[(1-\omega) + \frac{\omega\lambda^3(\lambda x^p) + 3\omega(\lambda x^p)^2}{6+\lambda^3}\right]^2 dx.$$

The square in the integrand is expanded before integration. For this polynomial calculation only, let c_k denote the coefficient of t^k in

$$\left[(1-\omega) + \frac{\omega\lambda^3}{6+\lambda^3}t + \frac{3\omega}{6+\lambda^3}t^2\right]^2 = \sum_{k=0}^{4} c_k t^k,$$

where the five coefficients are explicitly

$$\begin{aligned} c_0 &= (1-\omega)^2, \\ c_1 &= \frac{2\omega(1-\omega)\lambda^3}{6+\lambda^3}, \\ c_2 &= \frac{\omega^2\lambda^6}{(6+\lambda^3)^2} + \frac{6\omega(1-\omega)}{6+\lambda^3}, \\ c_3 &= \frac{6\omega^2\lambda^3}{(6+\lambda^3)^2}, \qquad c_4 = \frac{9\omega^2}{(6+\lambda^3)^2}. \end{aligned}$$

Use $t = \lambda x^p$, $dx = \lambda^{-1/p}t^{1/p-1}dt/p$, and $x^{2p-2} = \lambda^{-2+2/p}t^{2-2/p}$. The extropy integral becomes

$$\begin{aligned} J(X) &= -\frac{p\lambda^{1/p}}{2}\int_0^\infty t^{1-1/p}e^{-2t}\sum_{k=0}^{4} c_k t^k\,dt \\ &= -\frac{p\lambda^{1/p}}{2}\sum_{k=0}^{4} c_k \int_0^\infty t^{k+1-1/p}e^{-2t}\,dt. \end{aligned}$$

Evaluating these integrals term by term gives

$$J(X) = -\frac{p\lambda^{1/p}}{2}\sum_{k=0}^{4} c_k \frac{\Gamma(k+2-1/p)}{2^{k+2-1/p}}. \tag{3.7}$$

In (3.7), the sum is taken over indices with $c_k > 0$.

For the weighted measure, the extra factor x changes the integral to

$$J_w(X) = -\frac{p^2\lambda^2}{2}\int_0^\infty x^{2p-1}e^{-2\lambda x^p}\left[(1-\omega)+\frac{\omega\lambda^3(\lambda x^p)+3\omega(\lambda x^p)^2}{6+\lambda^3}\right]^2 dx$$
$$= -\frac{p}{2}\sum_{k=0}^{4} c_k \int_0^\infty t^{k+1}e^{-2t}\,dt = -\frac{p}{2}\sum_{k=0}^{4} c_k \frac{\Gamma(k+2)}{2^{k+2}}.$$

Accordingly,

$$J_w(X) = -\frac{p}{2}\sum_{k=0}^{4} c_k \frac{\Gamma(k+2)}{2^{k+2}}. \tag{3.8}$$

Proposition 3.3 (Finiteness of extropy and weighted extropy)
For $0 \le \omega < 1$, $J(X)$ is finite if and only if $p > 1/2$. For $\omega = 1$, $J(X)$ is finite if and only if $p > 1/4$. The weighted extropy $J_w(X)$ is finite for every $p > 0$ and $0 \le \omega \le 1$.

Proof
At zero, write $f(x) \sim Cx^{a-1}$, where $C > 0$ and $a = p$ for $\omega < 1$, while $a = 2p$ for $\omega = 1$. Then $f(x)^2$ is integrable at zero exactly when $2a - 2 > -1$. For $xf(x)^2$, the condition is $2a - 1 > -1$, which holds for every $a > 0$. Both integrals converge at infinity because their integrands contain the factor $e^{-2\lambda x^p}$. □

Although λ does not occur as a separate leading factor in (3.8), it still affects the result through the coefficients c_k.

3.6. Moment-generating and characteristic functions

The moment-generating function is obtained from its defining integral rather than from a formally assumed series:

$$M_X(s) = \int_0^\infty e^{sx} f(x;\lambda,p,\omega)\,dx$$
$$= p\lambda\int_0^\infty e^{sx-\lambda x^p}x^{p-1}\left[(1-\omega)+\frac{\omega\lambda^3(\lambda x^p)+3\omega(\lambda x^p)^2}{6+\lambda^3}\right]dx.$$

With $t = \lambda x^p$, direct substitution yields the integral representation

$$M_X(s) = \int_0^\infty \exp\left[s\left(\frac{t}{\lambda}\right)^{1/p} - t\right]\left[(1-\omega)+\frac{\omega\lambda^3 t+3\omega t^2}{6+\lambda^3}\right]dt. \tag{3.9}$$

Proposition 3.4 (Domain of the exponential transform)
For real s, the expectation $M_X(s) = E(e^{sX})$ is finite on $\mathbb{R}$ if $p > 1$, on $(-\infty, \lambda)$ if $p = 1$, and on $(-\infty, 0]$ if $0 < p < 1$. The moment series $\sum_{r=0}^{\infty} E(X^r)s^r/r!$ converges absolutely for all real s if $p > 1$, for $|s| < \lambda$ if $p = 1$, and only at $s = 0$ if $0 < p < 1$.

Proof
The tail of the defining integral is determined by $\exp(sx - \lambda x^p)$ multiplied by powers of x. When $p = 1$ and $s = \lambda$, the remaining polynomial has a divergent integral. For the series, monotone convergence gives $\sum_{r\ge 0} E(X^r)|s|^r/r! = E(e^{|s|X})$, which yields the stated domains of absolute convergence. □

The characteristic function exists for all real s and follows by replacing s with $\mathrm{i}s$ in (3.9); no claim of an elementary closed form is required.

3.7. Order statistics

Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of an independent WPGM sample. The event $X_{r:n} \leq x$ occurs when at least r sample observations do not exceed x. Since their number has a binomial distribution with success probability $F(x)$,

$$F_{r:n}(x) = \sum_{j=r}^{n} \binom{n}{j} [F(x)]^j [1 - F(x)]^{n-j}.$$

Differentiating this expression, or equivalently considering exactly $r-1$ observations below x and one in $(x, x+dx)$, gives the order-statistic density

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x), \quad x > 0. \tag{3.10}$$

The functions f and F in (3.10) are the explicit WPGM functions specified in (2.3) and (2.4). The raw moment of order $q \geq 0$ may therefore be calculated from

$$E(X_{r:n}^q) = \frac{n!}{(r-1)!(n-r)!} \int_0^\infty x^q [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x)\, dx.$$

For evaluation as a finite sum, first expand $[F(x)]^{r-1} = [1 - (1 - F(x))]^{r-1}$ by the binomial theorem. After substituting $t = \lambda x^p$, the ath integral ($a = 0, \ldots, r-1$) becomes

$$\lambda^{-q/p} \int_0^\infty t^{q/p} e^{-(n-r+a+1)t} \left[(1-\omega) + \frac{\omega\lambda^3 t + 3\omega t^2}{6+\lambda^3} \right] \times \left[1 + \omega t + \frac{3\omega t^2}{6+\lambda^3} \right]^{n-r+a} dt.$$

The product of the two polynomials is finite. If $d_{a,k}$ denotes the coefficient of t^k in that product, where $m = n - r + a$ and $0 \leq k \leq 2m+2$, termwise gamma integration gives

$$\begin{aligned} E(X_{r:n}^q) = \frac{n!\lambda^{-q/p}}{(r-1)!(n-r)!} \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \\ \times \sum_{k=0}^{2(n-r+a)+2} d_{a,k} \frac{\Gamma(1+q/p+k)}{(n-r+a+1)^{1+q/p+k}}. \end{aligned} \tag{3.11}$$

The intermediate step relies on $\int_0^\infty t^{b-1} e^{-ct}\, dt = \Gamma(b)/c^b$, with $b > 0$ and $c > 0$. The finite-sum result in (3.11) holds for $q \geq 0$, so the moments of the sample minimum and maximum follow by setting $r = 1$ and $r = n$.

3.8. Laplace transform of the power-transformed variable

Proposition 3.5 (Domain of the Laplace transform of X^p)
For real s, the expectation $\mathcal{L}_{X^p}(s) = E(e^{-sX^p})$ is finite if and only if $s > -\lambda$. On this domain its value is given by the rational expression derived below.

Indeed, after the substitution $t = \lambda x^p$, the integrand is a nonzero polynomial with nonnegative coefficients multiplied by $e^{-(1+s/\lambda)t}$. Its integral is finite precisely when $1 + s/\lambda > 0$.

$$\begin{aligned} \mathcal{L}_{X^p}(s) &= \int_0^\infty e^{-sx^p} f(x;\lambda,p,\omega)\, dx \\ &= p\lambda \int_0^\infty x^{p-1} e^{-(\lambda+s)x^p} \left[(1-\omega) + \frac{\omega\lambda^4 x^p + 3\omega\lambda^2 x^{2p}}{6+\lambda^3} \right] dx. \end{aligned}$$

Putting $t = \lambda x^p$ and using the gamma integral yields

$$\begin{aligned}\mathcal{L}_{X^p}(s) &= (1-\omega)\int_0^\infty e^{-(1+s/\lambda)t}\,dt \\ &\quad + \frac{\omega\lambda^3}{6+\lambda^3}\int_0^\infty te^{-(1+s/\lambda)t}\,dt + \frac{3\omega}{6+\lambda^3}\int_0^\infty t^2e^{-(1+s/\lambda)t}\,dt \\ &= (1-\omega)\frac{\lambda}{\lambda+s} + \frac{\omega\lambda^3}{6+\lambda^3}\left(\frac{\lambda}{\lambda+s}\right)^2 + \frac{6\omega}{6+\lambda^3}\left(\frac{\lambda}{\lambda+s}\right)^3.\end{aligned}$$

Thus the Laplace transform is rational in s, in contrast with the generally non-elementary MGF in (3.9). Differentiating at zero gives

$$E(X^p) = -\mathcal{L}'_{X^p}(0) = \frac{1}{\lambda}\left[1 + \frac{\omega(\lambda^3+12)}{6+\lambda^3}\right],$$

which agrees with (3.2) when $r = p$.

4. Different Methods of Estimation

Let $X_1, \ldots, X_n$ be a random sample from the WPGM distribution with parameter vector $\boldsymbol{\theta} = (\lambda, p, \omega)^\top$ and parameter space $\Theta = (0,\infty)\times(0,\infty)\times[0,1]$. Eight methods are considered: maximum likelihood (MLE), Cramér von Mises (CME), left-tail Anderson Darling (LADE), Anderson Darling (ADE), least squares (LSE), minimum squared spacing distance (MSS), minimum spacing LINEX-type distance (MSL), and minimum Kolmogorov distance (KM). We selected these criteria to examine how parameter estimates change with the aspect of fit being emphasized. MLE provides the likelihood benchmark. CME measures the overall squared discrepancy between the empirical and fitted CDFs, while LSE matches fitted probabilities at the order statistics to their expected uniform plotting positions. ADE gives greater weight to discrepancies in both tails, and LADE focuses on the lower tail, where early failures occur. KM minimizes the largest absolute CDF discrepancy. MSS and MSL compare successive fitted probability spacings with their common reference value $1/(n+1)$. MSS uses squared deviations, whereas the asymmetric MSL loss assigns a larger penalty to an oversized spacing than to an equally undersized one. All estimates are obtained by optimizing over the same admissible parameter space. Estimating equations are derived for interior stationary points; if an estimate of ω lies at 0 or 1, the corresponding one-sided optimality condition replaces the ordinary zero-score condition.For an alpha power one-parameter Weibull model, Khalfallah [20] compared maximum likelihood, least squares, Anderson–Darling, Cramér–von Mises, and maximum product of spacings estimators. The present comparison also examines a left-tail criterion and two losses based on probability spacings.

4.1. Derivatives of the distribution function

Write $x_{1:n} \le \cdots \le x_{n:n}$ for the ordered observations. Differentiation of the WPGM distribution function is required for all criteria based on cumulative probabilities. For $\vartheta \in \{\lambda, p, \omega\}$, denote the derivative by $\mathcal{D}_\vartheta(x;\boldsymbol{\theta}) = \partial F(x;\boldsymbol{\theta})/\partial\vartheta$. The explicit expressions are

$$\begin{aligned}\mathcal{D}_\lambda(x;\boldsymbol{\theta}) &= e^{-\lambda x^p}\left\{x^p\left[1+\omega\lambda x^p + \frac{3\omega(\lambda x^p)^2}{6+\lambda^3} - \omega - \frac{6\omega\lambda x^p}{6+\lambda^3}\right] + \frac{9\omega\lambda^2(\lambda x^p)^2}{(6+\lambda^3)^2}\right\}, \\ \mathcal{D}_p(x;\boldsymbol{\theta}) &= e^{-\lambda x^p}\lambda x^p\log x\left[1+\omega\lambda x^p + \frac{3\omega(\lambda x^p)^2}{6+\lambda^3} - \omega - \frac{6\omega\lambda x^p}{6+\lambda^3}\right], \\ \mathcal{D}_\omega(x;\boldsymbol{\theta}) &= -e^{-\lambda x^p}\left[\lambda x^p + \frac{3(\lambda x^p)^2}{6+\lambda^3}\right].\end{aligned} \tag{4.1}$$

Combining these derivatives with those of $\omega\lambda x^p$ leads directly to the three expressions in (4.1). The derivative of $6+\lambda^3$ is retained throughout; neglecting it would give an incorrect estimating equation for λ. For subsequent

calculations, use

$$u_i = F(x_{i:n};\boldsymbol{\theta}) = 1 - e^{-\lambda x_{i:n}^p}\left[1 + \omega\lambda x_{i:n}^p + \frac{3\omega(\lambda x_{i:n}^p)^2}{6+\lambda^3}\right], \quad i = 1,\ldots,n,$$

and $\mathcal{D}_{\vartheta,i} = \mathcal{D}_\vartheta(x_{i:n};\boldsymbol{\theta})$. These symbols denote the fitted cumulative probabilities and their explicit derivatives; they do not replace the full PDF or CDF specifications.

4.2. Maximum likelihood estimation (MLE)

The maximum likelihood method estimates the parameter vector by maximizing the joint sampling density. From (2.3), the likelihood for positive observations is

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n}\left\{p\lambda x_i^{p-1}e^{-\lambda x_i^p}\left[(1-\omega) + \frac{\omega\left(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p}\right)}{6+\lambda^3}\right]\right\}.$$

Taking the natural logarithm gives

$$\begin{aligned}\ell(\lambda,p,\omega) = {} & n\log p + n\log\lambda + (p-1)\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i^p \\ & + \sum_{i=1}^{n}\log\left[1 - \omega + \frac{\omega(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p})}{6+\lambda^3}\right].\end{aligned} \tag{4.2}$$

For the purpose of differentiating the last term of (4.2), denote its strictly positive logarithmic argument by

$$\mathcal{B}_i = 1 - \omega + \frac{\omega(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p})}{6+\lambda^3}.$$

Its first derivatives follow from the product and quotient rules:

$$\begin{aligned}\mathcal{B}_{i,\lambda} &= \omega\left[\frac{4\lambda^3 x_i^p + 6\lambda x_i^{2p}}{6+\lambda^3} - \frac{3\lambda^2(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p})}{(6+\lambda^3)^2}\right], \\ \mathcal{B}_{i,p} &= \frac{\omega(\lambda^4 x_i^p + 6\lambda^2 x_i^{2p})\log x_i}{6+\lambda^3}, \\ \mathcal{B}_{i,\omega} &= -1 + \frac{\lambda^4 x_i^p + 3\lambda^2 x_i^{2p}}{6+\lambda^3}.\end{aligned}$$

For example, the derivative of the numerator in the mixture term with respect to λ is

$$\frac{\partial}{\partial\lambda}(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p}) = 4\lambda^3 x_i^p + 6\lambda x_i^{2p},$$

whereas its derivative with respect to p equals $(\lambda^4 x_i^p + 6\lambda^2 x_i^{2p})\log x_i$. Consequently, differentiation of (4.2) with respect to each unknown parameter leads to the score equations

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda} - \sum_{i=1}^{n}x_i^p + \sum_{i=1}^{n}\frac{\mathcal{B}_{i,\lambda}}{\mathcal{B}_i} = 0, \\ \frac{\partial\ell}{\partial p} &= \frac{n}{p} + \sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i^p\log x_i + \sum_{i=1}^{n}\frac{\mathcal{B}_{i,p}}{\mathcal{B}_i} = 0, \\ \frac{\partial\ell}{\partial\omega} &= \sum_{i=1}^{n}\frac{\mathcal{B}_{i,\omega}}{\mathcal{B}_i} = 0.\end{aligned} \tag{4.3}$$

The score equations in (4.3) have no general closed-form simultaneous solution. The MLE $\widehat{\boldsymbol{\theta}}_{\mathrm{MLE}}$ is obtained numerically by maximizing (4.2), rather than by accepting an arbitrary root of the score vector. The observed information matrix is $\mathcal{J}(\boldsymbol{\theta}) = -\nabla^2 \ell(\boldsymbol{\theta})$. Its elements can be obtained by a second differentiation, for example,

$$
\begin{aligned}
\ell_{\lambda\lambda} &= -\frac{n}{\lambda^2} + \sum_i \left\{ \frac{\mathcal{B}_{i,\lambda\lambda}}{\mathcal{B}_i} - \frac{\mathcal{B}_{i,\lambda}^2}{\mathcal{B}_i^2} \right\}, \\
\ell_{pp} &= -\frac{n}{p^2} - \lambda \sum_i x_i^p (\log x_i)^2 + \sum_i \left\{ \frac{\mathcal{B}_{i,pp}}{\mathcal{B}_i} - \frac{\mathcal{B}_{i,p}^2}{\mathcal{B}_i^2} \right\}, \\
\ell_{\lambda p} &= -\sum_i x_i^p \log x_i + \sum_i \left\{ \frac{\mathcal{B}_{i,\lambda p}}{\mathcal{B}_i} - \frac{\mathcal{B}_{i,\lambda}\mathcal{B}_{i,p}}{\mathcal{B}_i^2} \right\}, \\
\ell_{\lambda\omega} &= \sum_i \left\{ \frac{\mathcal{B}_{i,\lambda\omega}}{\mathcal{B}_i} - \frac{\mathcal{B}_{i,\lambda}\mathcal{B}_{i,\omega}}{\mathcal{B}_i^2} \right\}, \\
\ell_{p\omega} &= \sum_i \left\{ \frac{\mathcal{B}_{i,p\omega}}{\mathcal{B}_i} - \frac{\mathcal{B}_{i,p}\mathcal{B}_{i,\omega}}{\mathcal{B}_i^2} \right\}, \\
\ell_{\omega\omega} &= -\sum_i \frac{\mathcal{B}_{i,\omega}^2}{\mathcal{B}_i^2}.
\end{aligned}
$$

For full reproducibility, the six second derivatives needed in the observed-information matrix are obtained without discarding the dependence of the denominator on λ:

$$
\begin{aligned}
\mathcal{B}_{i,\lambda\lambda} &= \omega \left[\frac{12\lambda^2 x_i^p + 6x_i^{2p}}{6+\lambda^3} - \frac{6\lambda^2(4\lambda^3 x_i^p + 6\lambda x_i^{2p}) + 6\lambda(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p})}{(6+\lambda^3)^2} \right. \\
&\qquad \left. + \frac{18\lambda^4(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p})}{(6+\lambda^3)^3} \right], \\
\mathcal{B}_{i,pp} &= \frac{\omega(\lambda^4 x_i^p + 12\lambda^2 x_i^{2p})(\log x_i)^2}{6+\lambda^3}, \\
\mathcal{B}_{i,\lambda p} &= \omega \log x_i \left[\frac{4\lambda^3 x_i^p + 12\lambda x_i^{2p}}{6+\lambda^3} - \frac{3\lambda^2(\lambda^4 x_i^p + 6\lambda^2 x_i^{2p})}{(6+\lambda^3)^2} \right], \\
\mathcal{B}_{i,\lambda\omega} &= \frac{4\lambda^3 x_i^p + 6\lambda x_i^{2p}}{6+\lambda^3} - \frac{3\lambda^2(\lambda^4 x_i^p + 3\lambda^2 x_i^{2p})}{(6+\lambda^3)^2}, \\
\mathcal{B}_{i,p\omega} &= \frac{(\lambda^4 x_i^p + 6\lambda^2 x_i^{2p}) \log x_i}{6+\lambda^3}, \qquad \mathcal{B}_{i,\omega\omega} = 0.
\end{aligned}
$$

The displayed Hessian follows from differentiating the sample terms in (4.2) twice and using $\partial_{ab} \log \mathcal{B}_i = \mathcal{B}_{i,ab}/\mathcal{B}_i - \mathcal{B}_{i,a}\mathcal{B}_{i,b}/\mathcal{B}_i^2$. All these derivatives are well defined for interior parameters. At a regular interior MLE, a positive-definite and nonsingular observed information yields the covariance approximation $\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\theta}}_{\mathrm{MLE}}) = \mathcal{J}(\widehat{\boldsymbol{\theta}}_{\mathrm{MLE}})^{-1}$. The standard errors obtained from the inverse observed information rely on an interior quadratic approximation to the log-likelihood. When the fitted mixing weight is at an endpoint, the parameter constraint limits variation in ω to one direction. Near an endpoint, a symmetric Wald interval can also extend outside $[0, 1]$. For tests of $H_0 : \omega = 0$ or $H_0 : \omega = 1$, the usual χ_1^2 reference distribution requires separate justification. When $\widehat{\omega}$ is at or near an endpoint, uncertainty about the mixing weight can be assessed using the constrained profile log-likelihood

$$
\ell_{\mathrm{prof}}(w) = \sup_{\lambda>0,\, p>0} \ell(\lambda, p, w), \qquad 0 \le w \le 1.
$$

For a candidate value w_0, the likelihood-ratio statistic is

$$T(w_0) = 2\{\ell_{\mathrm{prof}}(\widehat{\omega}) - \ell_{\mathrm{prof}}(w_0)\}.$$

Its reference distribution can be approximated by a parametric bootstrap under $w = w_0$. Samples of the original size are drawn from the WPGM with w_0 fixed and (λ, p) set to their constrained maximum likelihood estimates. For each sample, the model is fitted both with $w = w_0$ and with w free on $[0, 1]$, and the likelihood-ratio statistic is recalculated. An approximate $100(1-\alpha)\%$ confidence set consists of the values w_0 for which the observed statistic does not exceed the corresponding bootstrap $(1-\alpha)$ quantile. This construction allows the confidence set to include either endpoint.

4.3. Cramér von Mises estimation (CME)

The Cramér von Mises criterion measures the quadratic distance between fitted cumulative probabilities and empirical midpoint positions. Put $b_i = (2i-1)/(2n)$ and minimize

$$\begin{aligned} C(\boldsymbol{\theta}) &= \frac{1}{12n} + \sum_{i=1}^{n}\left[u_i - \frac{2i-1}{2n}\right]^2 \\ &= \frac{1}{12n} + \sum_{i=1}^{n}\left\{1 - e^{-\lambda x_{i:n}^p}\left[1 + \omega\lambda x_{i:n}^p + \frac{3\omega(\lambda x_{i:n}^p)^2}{6+\lambda^3}\right] - \frac{2i-1}{2n}\right\}^2. \end{aligned} \tag{4.4}$$

Differentiating each squared residual in (4.4) gives

$$\frac{\partial}{\partial\vartheta}(u_i - b_i)^2 = 2(u_i - b_i)\mathcal{D}_{\vartheta,i}, \qquad \vartheta \in \{\lambda, p, \omega\}.$$

The three CME estimating equations may therefore be written as

$$\sum_{i=1}^{n}\left[u_i - \frac{2i-1}{2n}\right]\mathcal{D}_{\vartheta,i} = 0, \qquad \vartheta = \lambda, p, \omega. \tag{4.5}$$

In (4.5) the derivatives are those in (4.1). The common factor 2 has been removed because it does not affect the roots. Numerical minimization of (4.4) is used to distinguish a minimum from other stationary points.

4.4. Anderson Darling estimation (ADE)

The Anderson Darling criterion assigns relatively greater influence to both tails of the fitted CDF than an unweighted quadratic distance [7]. Its empirical objective is

$$A(\boldsymbol{\theta}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log u_i + \log\left[1 - u_{n-i+1}\right]\right\}. \tag{4.6}$$

For direct numerical evaluation of (4.6), the lower and upper terms are respectively

$$\log u_i = \log\left\{1 - e^{-\lambda x_{i:n}^p}\left[1 + \omega\lambda x_{i:n}^p + \frac{3\omega(\lambda x_{i:n}^p)^2}{6+\lambda^3}\right]\right\},$$

$$\log(1 - u_{n-i+1}) = -\lambda x_{n-i+1:n}^p + \log\left[1 + \omega\lambda x_{n-i+1:n}^p + \frac{3\omega(\lambda x_{n-i+1:n}^p)^2}{6+\lambda^3}\right].$$

The second identity is evaluated through the survival function, so that subtracting fitted cumulative probabilities close to one is unnecessary. Using the chain rule,

$$\frac{\partial}{\partial\vartheta}\log u_i = \frac{\mathcal{D}_{\vartheta,i}}{u_i}, \qquad \frac{\partial}{\partial\vartheta}\log(1 - u_{n-i+1}) = -\frac{\mathcal{D}_{\vartheta,n-i+1}}{1 - u_{n-i+1}}.$$

Setting the derivatives of (4.6) to zero gives

$$\sum_{i=1}^{n}(2i-1)\left\{\frac{\mathcal{D}_{\vartheta,i}}{u_i}-\frac{\mathcal{D}_{\vartheta,n-i+1}}{1-u_{n-i+1}}\right\}=0,\quad \vartheta=\lambda,p,\omega. \tag{4.7}$$

The negative constant factor $-1/n$ does not affect the roots. Equation (4.7) is a necessary condition only at interior differentiable minimizers of (4.6).

4.5. Left-tail Anderson Darling estimation (LADE)

The left-tail Anderson Darling approach emphasizes discrepancies near the lower endpoint. In contrast to the two-sided criterion above, its objective is

$$\begin{aligned} L(\boldsymbol{\theta}) &= -\frac{3n}{2}+2\sum_{i=1}^{n}u_i-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log u_i \\ &= -\frac{3n}{2}+2\sum_{i=1}^{n}\left\{1-e^{-\lambda x_{i:n}^{p}}\left[1+\omega\lambda x_{i:n}^{p}+\frac{3\omega(\lambda x_{i:n}^{p})^2}{6+\lambda^3}\right]\right\} \\ &\quad -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left\{1-e^{-\lambda x_{i:n}^{p}}\left[1+\omega\lambda x_{i:n}^{p}+\frac{3\omega(\lambda x_{i:n}^{p})^2}{6+\lambda^3}\right]\right\}. \end{aligned} \tag{4.8}$$

By direct differentiation of (4.8),

$$\frac{\partial L}{\partial\vartheta}=2\sum_{i=1}^{n}\mathcal{D}_{\vartheta,i}-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\mathcal{D}_{\vartheta,i}}{u_i}.$$

Therefore, the LADE estimating equations are

$$\sum_{i=1}^{n}\left[2-\frac{2i-1}{nu_i}\right]\mathcal{D}_{\vartheta,i}=0,\qquad \vartheta=\lambda,p,\omega. \tag{4.9}$$

The denominators in (4.9) are positive for positive observations and admissible parameters. In computation, $\log u_i$ and its derivative should be evaluated in a stable manner for small fitted probabilities.

4.6. Ordinary least-squares estimation (LSE)

Least-squares fitting compares fitted cumulative probabilities with the expected order-statistic positions of a standard uniform sample, $a_i=i/(n+1)$, following the approach of Swain et al. [45]. Its objective is

$$\begin{aligned} V(\boldsymbol{\theta}) &= \sum_{i=1}^{n}\left[u_i-\frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n}\left\{1-e^{-\lambda x_{i:n}^{p}}\left[1+\omega\lambda x_{i:n}^{p}+\frac{3\omega(\lambda x_{i:n}^{p})^2}{6+\lambda^3}\right]-\frac{i}{n+1}\right\}^2. \end{aligned} \tag{4.10}$$

Since $i/(n+1)$ is free of the model parameters,

$$\frac{\partial V}{\partial\vartheta}=2\sum_{i=1}^{n}\left[u_i-\frac{i}{n+1}\right]\mathcal{D}_{\vartheta,i}.$$

The LSE estimating equations are consequently

$$\sum_{i=1}^{n}\left[u_i-\frac{i}{n+1}\right]\mathcal{D}_{\vartheta,i}=0,\qquad \vartheta=\lambda,p,\omega. \tag{4.11}$$

The minimizer of (4.10) is selected among feasible candidate solutions of (4.11). The difference between (4.11) and the CME conditions (4.5) is the plotting position: $i/(n+1)$ rather than $(2i-1)/(2n)$. The two estimators therefore need not coincide.

4.7. Minimum squared spacing-distance estimation (MSS)

As in the probability-spacing approach discussed by Cheng and Amin [10], consider the $n+1$ spacings between consecutive ordered observations on the fitted probability scale:

$$\Delta_i(\boldsymbol{\theta})=F(x_{i:n};\boldsymbol{\theta})-F(x_{i-1:n};\boldsymbol{\theta}),\ \Delta_1\ =F(x_{1:n};\boldsymbol{\theta}),\ \Delta_{n+1}=1-F(x_{n:n};\boldsymbol{\theta}),\qquad i=2,\ldots,n. \tag{4.12}$$

For $2\le i\le n$, substitution of the WPGM CDF gives

$$\begin{aligned}\Delta_i=e^{-\lambda x_{i-1:n}^{p}}&\left[1+\omega\lambda x_{i-1:n}^{p}+\frac{3\omega(\lambda x_{i-1:n}^{p})^2}{6+\lambda^3}\right]\\&-e^{-\lambda x_{i:n}^{p}}\left[1+\omega\lambda x_{i:n}^{p}+\frac{3\omega(\lambda x_{i:n}^{p})^2}{6+\lambda^3}\right].\end{aligned}$$

The spacings in (4.12) satisfy $\sum_{i=1}^{n+1}\Delta_i=1$. Under a uniform distribution of fitted probabilities the reference spacing is $1/(n+1)$. The MSS criterion minimizes the squared distance from this value:

$$Q_{\mathrm{MSS}}(\boldsymbol{\theta})=\sum_{i=1}^{n+1}\left[\Delta_i(\boldsymbol{\theta})-\frac{1}{n+1}\right]^2. \tag{4.13}$$

Set $\mathcal{D}_{\vartheta,0}=\mathcal{D}_{\vartheta,n+1}=0$ and use the derivatives in (4.1). Differentiating (4.12) yields

$$\frac{\partial\Delta_i}{\partial\vartheta}=\mathcal{D}_{\vartheta,i}-\mathcal{D}_{\vartheta,i-1},\qquad i=1,\ldots,n+1.$$

Thus differentiation of (4.13) produces

$$\begin{aligned}\frac{\partial Q_{\mathrm{MSS}}}{\partial\vartheta}&=2\sum_{i=1}^{n+1}\left[\Delta_i-\frac{1}{n+1}\right](\mathcal{D}_{\vartheta,i}-\mathcal{D}_{\vartheta,i-1})\\&=2\sum_{i=1}^{n}(\Delta_i-\Delta_{i+1})\mathcal{D}_{\vartheta,i}.\end{aligned}$$

Here the second equality follows by collecting the coefficient of each $\mathcal{D}_{\vartheta,i}$; the boundary terms vanish and the constant reference spacing cancels. At an interior minimum, the MSS estimating equations reduce to

$$\sum_{i=1}^{n}(\Delta_i-\Delta_{i+1})\mathcal{D}_{\vartheta,i}=0,\qquad \vartheta=\lambda,p,\omega. \tag{4.14}$$

The conditions in (4.14) can be solved numerically, with the admissible minimizer selected by direct evaluation of the criterion in (4.13).

4.8. Minimum spacing LINEX-type distance (MSL)

A related spacing estimator applies an exponential penalty to the difference between each spacing and its uniform reference. With $r_i = \Delta_i - 1/(n+1)$, the loss is

$$Q_{\mathrm{MSL}}(\boldsymbol{\theta}) = \sum_{i=1}^{n+1} \left\{ \exp\left(\Delta_i - \frac{1}{n+1}\right) - \left(\Delta_i - \frac{1}{n+1}\right) - 1 \right\}. \tag{4.15}$$

The loss in (4.15) is nonnegative because $e^r - r - 1 \geq 0$, with equality only at $r = 0$. Its first two derivatives with respect to r are $e^r - 1$ and $e^r > 0$. By the chain rule,

$$\frac{\partial Q_{\mathrm{MSL}}}{\partial \vartheta} = \sum_{i=1}^{n+1} (e^{r_i} - 1)(\mathcal{D}_{\vartheta,i} - \mathcal{D}_{\vartheta,i-1}) = \sum_{i=1}^{n} (e^{r_i} - e^{r_{i+1}})\mathcal{D}_{\vartheta,i}.$$

Hence its three estimating equations are

$$\sum_{i=1}^{n} \left[e^{\Delta_i - 1/(n+1)} - e^{\Delta_{i+1} - 1/(n+1)} \right] \mathcal{D}_{\vartheta,i} = 0, \quad \vartheta = \lambda, p, \omega. \tag{4.16}$$

Equation (4.16) is used to verify the stationarity of an interior solution to (4.15). A Taylor expansion shows why MSS and MSL can return similar estimates:

$$e^r - r - 1 = \frac{r^2}{2} + \frac{r^3}{6} + O(r^4).$$

Thus when all spacings are close to $1/(n+1)$, the MSL objective is approximately half the MSS objective with higher-order corrections. The methods remain distinct criteria, but their numerical agreement is not evidence of independent sources of information.

4.9. Minimum Kolmogorov distance estimation (KM)

The Kolmogorov method selects the parameter vector minimizing the largest discrepancy between the fitted WPGM CDF and the empirical CDF. For continuous observations the two one-sided discrepancies at $x_{i:n}$ are $i/n - u_i$ and $u_i - (i-1)/n$. Thus the criterion is

$$K(\boldsymbol{\theta}) = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - u_i, \quad u_i - \frac{i-1}{n} \right\}. \tag{4.17}$$

Inserting the WPGM CDF into (4.17) gives

$$K(\boldsymbol{\theta}) = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - 1 + e^{-\lambda x_{i:n}^p} \left[1 + \omega \lambda x_{i:n}^p + \frac{3\omega(\lambda x_{i:n}^p)^2}{6 + \lambda^3} \right], \right.$$
$$\left. 1 - e^{-\lambda x_{i:n}^p} \left[1 + \omega \lambda x_{i:n}^p + \frac{3\omega(\lambda x_{i:n}^p)^2}{6 + \lambda^3} \right] - \frac{i-1}{n} \right\}.$$

Unlike the preceding smooth criteria, the maximum in (4.17) is generally nondifferentiable where two or more discrepancies are simultaneously active. For an interior local minimum, let $\mathcal{A}$ denote the active indices and discrepancy signs. The necessary generalized-gradient condition is

$$\mathbf{0} \in \mathrm{conv} \left\{ -\nabla_{\boldsymbol{\theta}} F(x_{i:n}; \boldsymbol{\theta}) \text{ for active } \frac{i}{n} - u_i, \quad \nabla_{\boldsymbol{\theta}} F(x_{i:n}; \boldsymbol{\theta}) \text{ for active } u_i - \frac{i-1}{n} \right\}. \tag{4.18}$$

Indeed, each discrepancy is differentiable with gradient $-\nabla F(x_{i:n}; \boldsymbol{\theta})$ or $+\nabla F(x_{i:n}; \boldsymbol{\theta})$, respectively. The generalized gradient of their finite maximum is the convex hull of the gradients of the active functions, which yields (4.18). At a boundary point, a normal-cone term for Θ must also be included. In practice, KM is computed by constrained minimax optimization of (4.17), not by pretending that a conventional smooth score equation exists.

4.10. Computational implementation and comparison

Each objective above is evaluated under the same parameter restrictions, with multi-start numerical optimization and explicit checks of the attained objective value. For λ and p, positive transformations may be used internally; ω is constrained to $[0,1]$ without allowing inadmissible values. Smooth criteria can be checked using their analytical gradients, whereas KM requires a nonsmooth or epigraph-based algorithm. Convergence flags, objective values, gradient or stationarity diagnostics, and boundary estimates should be recorded alongside parameter estimates. In each Monte Carlo replicate, all eight procedures used the same sample, numerical parameter bounds, and 12 data-dependent starting points. The solver and stationarity diagnostic were chosen to suit the criterion, including the nonsmooth KM objective. We compared the resulting parameter estimates against the same generating values using RBias and MSE. Both measures enter the aggregate ranking. RMSE is also reported and gives the same ordering as MSE within each comparison. Objective values were used only to select among solutions for the same criterion.

5. Monte Carlo Simulation Study

A Monte Carlo experiment was designed to examine the finite-sample accuracy of eight estimation procedures for the WPGM distribution. The considered estimators are MLE, ADE, CME, LADE, LSE, MSS, MSL, and KM, as defined in Section 4. The study focuses on the estimation of all three parameters, λ, p, and ω.

Six distinct parameter configurations were considered:

$$\boldsymbol{\theta}_1 = (0.50, 0.80, 0.05), \quad \boldsymbol{\theta}_2 = (0.70, 1.30, 0.55), \quad \boldsymbol{\theta}_3 = (1.00, 1.70, 0.80),$$
$$\boldsymbol{\theta}_4 = (1.20, 0.90, 0.75), \quad \boldsymbol{\theta}_5 = (1.80, 1.30, 0.35), \quad \boldsymbol{\theta}_6 = (2.00, 2.00, 0.95).$$

Here and throughout, $\boldsymbol{\theta} = (\lambda, p, \omega)^\top$. The configurations include power-shape values on both sides of one and mixing weights close to the lower and upper boundaries. At every configuration, sample sizes $n \in \{25, 50, 100, 300, 500\}$ were examined, with $B = 1000$ independent samples per design cell. The design therefore contains 30,000 samples and 240,000 estimator fits before accounting for multiple starts. The comparison assesses how accurately each procedure recovers the generating WPGM parameters, including at $n = 25$ and at the near-boundary weights $\omega = 0.05$ and $\omega = 0.95$. All generating weights lie strictly between zero and one, although fitted weights can attain either endpoint. The resulting rankings apply to point estimation under correct model specification. The design does not assess exact-boundary generation, robustness under alternative generating distributions, interval coverage, or hypothesis testing.

Random variates were generated from the transformed gamma-mixture representation in Proposition 2.1. For each simulated observation, a latent component index was first drawn with respective probabilities $1-\omega$, $\frac{\omega\lambda^3}{6+\lambda^3}$, $\frac{6\omega}{6+\lambda^3}$. Conditional on components one, two, and three, a gamma random variable with unit rate and the corresponding integer shape was generated. The WPGM observation was then obtained as $X = (Z/\lambda)^{1/p}$. This procedure generates exact WPGM random variables without requiring repeated numerical inversion of the cumulative distribution function.

All eight methods were applied to the same sample in each replicate. Samples were generated using NumPy's PCG64 generator with `SeedSequence` initialized respectively ($s = 1, \ldots, 6$ and $b = 0, \ldots, 999$). This made each sample reproducible regardless of its allocation to a worker.

The seven smooth criteria were minimized using L-BFGS-B with analytical gradients and SLSQP refinement when needed. For KM, SLSQP was applied to the epigraph formulation. The solution with the smallest objective value across starts was accepted if the optimizer reported success, the estimates and objective value were finite, neither λ nor p reached an artificial bound, and the method-specific stationarity residual was at most 10^{-5}. The 12 starts combined four multipliers of a sample-based initial value for p (0.45, 0.65, 0.85, and 1.10) with three initial values of ω (0, 0.5, and 1). The simulation used Python 3.12.14, NumPy 2.3.5, and SciPy 1.17.0, with eight worker processes.

For $\vartheta \in \{\lambda, p, \omega\}$, let $\mathcal{A}$ be the set of accepted replicates for a given method and design cell and let $B_{\mathcal{A}} = |\mathcal{A}|$. The relative absolute bias (RBias), mean squared error (MSE), root mean squared error (RMSE), and accepted

convergence rate (CR) were calculated as

$$\begin{aligned} \mathrm{RBias}(\widehat{\vartheta}) &= \frac{\left|B_{\mathcal{A}}^{-1}\sum_{b\in\mathcal{A}}(\widehat{\vartheta}^{(b)}-\vartheta_0)\right|}{|\vartheta_0|}, \\ \mathrm{MSE}(\widehat{\vartheta}) &= \frac{1}{B_{\mathcal{A}}}\sum_{b\in\mathcal{A}}(\widehat{\vartheta}^{(b)}-\vartheta_0)^2, \qquad \mathrm{RMSE}(\widehat{\vartheta}) = \sqrt{\mathrm{MSE}(\widehat{\vartheta})}, \\ \mathrm{CR} &= 100\frac{B_{\mathcal{A}}}{B}\,\%. \end{aligned} \tag{5.1}$$

All generating parameter values are positive, so RBias is defined throughout this design. Its magnitude for $\omega = 0.05$ should be interpreted in view of the small denominator. All attempted fits were recorded. Under the simulation protocol, a fit failing any acceptance check would count as a failure in the convergence summary and would be excluded from the accuracy measures for that method. No replacement sample would be generated. Additional output reports relative efficiency, defined as $\mathrm{MSE}_{\mathrm{MLE}} \,/\, \mathrm{MSE}_{\mathrm{method}}$, computed on replicates accepted by both members of each pair. Values above one favor the comparison method in terms of MSE. Rates of near-boundary estimates, $\widehat{\omega} \le 0.001$ and $\widehat{\omega} \ge 0.999$, were computed among accepted fits.

Tables 4 to 9 present the numerical results for the six configurations. Superscripts give ranks within each scenario, sample size, parameter, and criterion, with rank one assigned to the smallest unrounded value. The tables report RBias, MSE, and RMSE for each parameter. The supplementary results give the corresponding signed bias (mean estimate minus generating value) and MSE-based relative efficiency with MLE as the reference. The partial and overall sums in Table 10 include RBias and MSE only. RMSE is displayed for interpretation but is not counted a second time, because it induces the same ordering as MSE. The aggregate ranks summarize the specified grid rather than establish universal superiority.

For λ, p, and ω the mean MSE across methods and configurations decreased.

All 1,000 fits met the numerical acceptance criteria for each of the eight methods at every combination of parameter setting and sample size. Thus, the observed failure rate was 0% for every method ($\mathrm{CR} = 100\%$ across all 240,000 fits). Median elapsed time per fit ranged from 0.0078 to 0.0750 seconds across methods and design cells. To assess sensitivity to initialization, we used 20 additional starting points for each method on the first replicate of each of the 30 design cells. Across the 240 fits examined in this audit, the largest improvement in the scaled objective was 7.79×10^{-12}. In scenario 1, where the generating weight was $\omega = 0.05$, the proportion of accepted estimates with $\widehat{\omega} \ge 0.999$ ranged from 5.2% to 42.9% across methods and sample sizes. Thus, even a numerically converged fit could assign nearly all the weight to the opposite component. The simulation evaluated point estimation for interior generating weights, including 0.05 and 0.95, and did not assess confidence-set coverage at either endpoint.

According to the aggregate scores in Table 10, ADE has the smallest rank sum, followed by MSS and MLE.

To complement the tables, Figures 2 to 5 display MSE and RMSE against sample size for the second and fifth settings. Each plot distinguishes the three parameters and eight methods. All plotted values are taken from the same summaries as the tables.

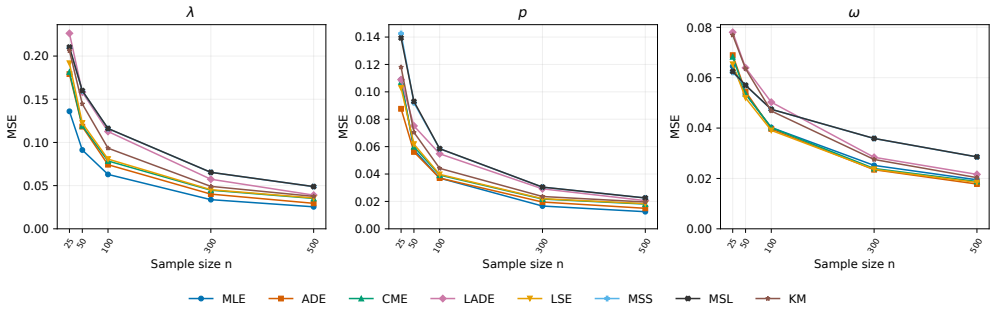

Figure 2. MSE of the eight WPGM estimators against sample size for $(\lambda, p, \omega) = (0.70, 1.30, 0.55)$.

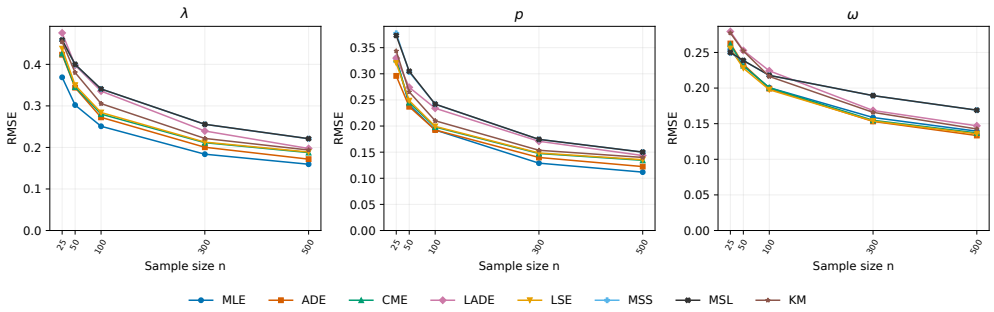

Figure 3. RMSE of the eight WPGM estimators against sample size for $(\lambda, p, \omega) = (0.70, 1.30, 0.55)$.

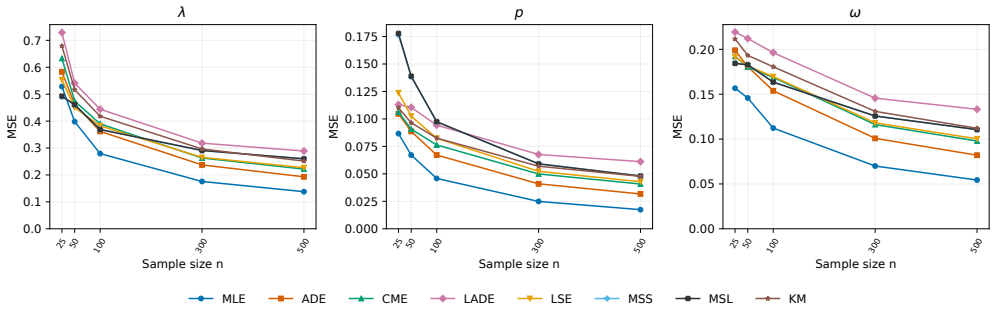

Figure 4. MSE of the eight WPGM estimators against sample size for $(\lambda, p, \omega) = (1.80, 1.30, 0.35)$.

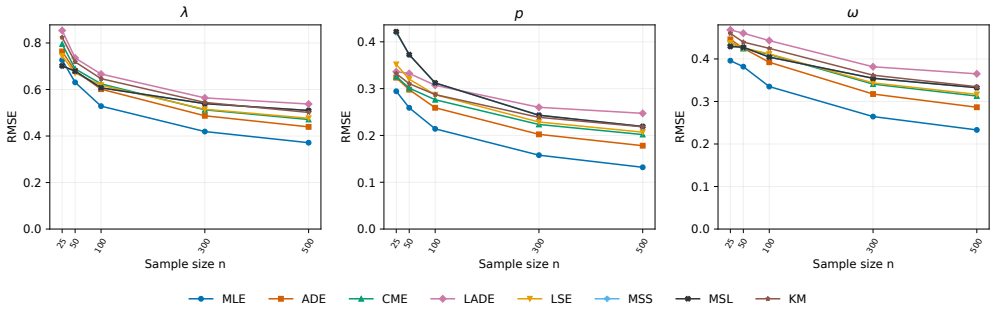

Figure 5. RMSE of the eight WPGM estimators against sample size for $(\lambda, p, \omega) = (1.80, 1.30, 0.35)$.

Table 4. The WPGM estimators for $(\lambda, p, \omega) = (0.50, 0.80, 0.05)$.

n	Method	$\widehat{\lambda}$			$\widehat{p}$			$\widehat{\omega}$			$\sum$ Ranks
		RBias	MSE	RMSE	RBias	MSE	RMSE	RBias	MSE	RMSE	
25	MLE	$1.4339^{\{1\}}$	$0.7614^{\{1\}}$	$0.8726^{\{1\}}$	$0.1371^{\{1\}}$	$0.0446^{\{1\}}$	$0.2111^{\{1\}}$	$11.8930^{\{1\}}$	$0.4497^{\{1\}}$	$0.6706^{\{1\}}$	6
	ADE	$1.7733^{\{3\}}$	$0.9974^{\{5\}}$	$0.9987^{\{5\}}$	$0.2303^{\{5\}}$	$0.0572^{\{5\}}$	$0.2391^{\{5\}}$	$13.8567^{\{5\}}$	$0.5568^{\{6\}}$	$0.7462^{\{6\}}$	29
	CME	$1.7291^{\{2\}}$	$0.9542^{\{2\}}$	$0.9768^{\{2\}}$	$0.1971^{\{2\}}$	$0.0519^{\{2\}}$	$0.2278^{\{2\}}$	$13.7324^{\{4\}}$	$0.5408^{\{4\}}$	$0.7354^{\{4\}}$	16
	LADE	$1.8718^{\{8\}}$	$1.0599^{\{8\}}$	$1.0295^{\{8\}}$	$0.2261^{\{4\}}$	$0.0551^{\{3\}}$	$0.2347^{\{3\}}$	$14.5341^{\{8\}}$	$0.5863^{\{8\}}$	$0.7657^{\{8\}}$	39
	LSE	$1.8235^{\{7\}}$	$1.0131^{\{6\}}$	$1.0065^{\{6\}}$	$0.2542^{\{6\}}$	$0.0635^{\{6\}}$	$0.2520^{\{6\}}$	$14.0935^{\{6\}}$	$0.5564^{\{5\}}$	$0.7460^{\{5\}}$	36
	MSS	$1.7855^{\{5\}}$	$0.9852^{\{3\}}$	$0.9926^{\{3\}}$	$0.2932^{\{8\}}$	$0.0859^{\{7\}}$	$0.2931^{\{7\}}$	$13.6837^{\{2\}}$	$0.5271^{\{3\}}$	$0.7260^{\{3\}}$	28
	MSL	$1.7849^{\{4\}}$	$0.9871^{\{4\}}$	$0.9935^{\{4\}}$	$0.2922^{\{7\}}$	$0.0864^{\{8\}}$	$0.2939^{\{8\}}$	$13.6853^{\{3\}}$	$0.5264^{\{2\}}$	$0.7255^{\{2\}}$	28
	KM	$1.8196^{\{6\}}$	$1.0408^{\{7\}}$	$1.0202^{\{7\}}$	$0.2071^{\{3\}}$	$0.0553^{\{4\}}$	$0.2351^{\{4\}}$	$14.4058^{\{7\}}$	$0.5838^{\{7\}}$	$0.7640^{\{7\}}$	34
50	MLE	$1.3291^{\{1\}}$	$0.6732^{\{1\}}$	$0.8205^{\{1\}}$	$0.1474^{\{1\}}$	$0.0383^{\{1\}}$	$0.1957^{\{1\}}$	$11.0561^{\{1\}}$	$0.4115^{\{1\}}$	$0.6415^{\{1\}}$	6
	ADE	$1.7029^{\{3\}}$	$0.9240^{\{3\}}$	$0.9612^{\{3\}}$	$0.2310^{\{4\}}$	$0.0520^{\{4\}}$	$0.2280^{\{4\}}$	$13.4773^{\{2\}}$	$0.5381^{\{5\}}$	$0.7336^{\{5\}}$	21
	CME	$1.6931^{\{2\}}$	$0.9078^{\{2\}}$	$0.9528^{\{2\}}$	$0.2166^{\{2\}}$	$0.0490^{\{2\}}$	$0.2214^{\{2\}}$	$13.5067^{\{3\}}$	$0.5309^{\{2\}}$	$0.7286^{\{2\}}$	13
	LADE	$1.8055^{\{8\}}$	$0.9880^{\{8\}}$	$0.9940^{\{8\}}$	$0.2377^{\{5\}}$	$0.0533^{\{5\}}$	$0.2308^{\{5\}}$	$14.1867^{\{8\}}$	$0.5712^{\{8\}}$	$0.7558^{\{8\}}$	42
	LSE	$1.7473^{\{4\}}$	$0.9382^{\{4\}}$	$0.9686^{\{4\}}$	$0.2464^{\{6\}}$	$0.0554^{\{6\}}$	$0.2354^{\{6\}}$	$13.7234^{\{6\}}$	$0.5398^{\{6\}}$	$0.7347^{\{6\}}$	32
	MSS	$1.7802^{\{7\}}$	$0.9532^{\{6\}}$	$0.9763^{\{6\}}$	$0.2857^{\{8\}}$	$0.0690^{\{7\}}$	$0.2627^{\{7\}}$	$13.7152^{\{5\}}$	$0.5317^{\{4\}}$	$0.7292^{\{4\}}$	37
	MSL	$1.7781^{\{6\}}$	$0.9520^{\{5\}}$	$0.9757^{\{5\}}$	$0.2856^{\{7\}}$	$0.0690^{\{8\}}$	$0.2627^{\{8\}}$	$13.7012^{\{4\}}$	$0.5309^{\{3\}}$	$0.7287^{\{3\}}$	33
	KM	$1.7656^{\{5\}}$	$0.9680^{\{7\}}$	$0.9839^{\{7\}}$	$0.2278^{\{3\}}$	$0.0519^{\{3\}}$	$0.2278^{\{3\}}$	$14.0313^{\{7\}}$	$0.5633^{\{7\}}$	$0.7505^{\{7\}}$	32
100	MLE	$1.3457^{\{1\}}$	$0.6993^{\{1\}}$	$0.8362^{\{1\}}$	$0.1703^{\{1\}}$	$0.0397^{\{1\}}$	$0.1992^{\{1\}}$	$10.9465^{\{1\}}$	$0.4209^{\{1\}}$	$0.6488^{\{1\}}$	6
	ADE	$1.6752^{\{3\}}$	$0.8964^{\{3\}}$	$0.9468^{\{3\}}$	$0.2327^{\{3\}}$	$0.0516^{\{3\}}$	$0.2272^{\{3\}}$	$13.2610^{\{3\}}$	$0.5317^{\{3\}}$	$0.7292^{\{3\}}$	18
	CME	$1.6619^{\{2\}}$	$0.8793^{\{2\}}$	$0.9377^{\{2\}}$	$0.2202^{\{2\}}$	$0.0494^{\{2\}}$	$0.2223^{\{2\}}$	$13.2431^{\{2\}}$	$0.5232^{\{2\}}$	$0.7233^{\{2\}}$	12
	LADE	$1.7815^{\{8\}}$	$0.9634^{\{8\}}$	$0.9816^{\{8\}}$	$0.2405^{\{6\}}$	$0.0537^{\{6\}}$	$0.2316^{\{6\}}$	$14.0392^{\{8\}}$	$0.5667^{\{8\}}$	$0.7528^{\{8\}}$	44
	LSE	$1.7097^{\{4\}}$	$0.9061^{\{4\}}$	$0.9519^{\{4\}}$	$0.2391^{\{5\}}$	$0.0532^{\{5\}}$	$0.2307^{\{5\}}$	$13.4979^{\{4\}}$	$0.5343^{\{4\}}$	$0.7310^{\{4\}}$	26
	MSS	$1.7749^{\{7\}}$	$0.9459^{\{6\}}$	$0.9726^{\{6\}}$	$0.2771^{\{8\}}$	$0.0640^{\{8\}}$	$0.2529^{\{8\}}$	$13.7579^{\{6\}}$	$0.5407^{\{6\}}$	$0.7353^{\{6\}}$	41
	MSL	$1.7725^{\{6\}}$	$0.9441^{\{5\}}$	$0.9716^{\{5\}}$	$0.2768^{\{7\}}$	$0.0639^{\{7\}}$	$0.2527^{\{7\}}$	$13.7415^{\{5\}}$	$0.5397^{\{5\}}$	$0.7346^{\{5\}}$	35
	KM	$1.7569^{\{5\}}$	$0.9518^{\{7\}}$	$0.9756^{\{7\}}$	$0.2378^{\{4\}}$	$0.0532^{\{4\}}$	$0.2306^{\{4\}}$	$13.9129^{\{7\}}$	$0.5629^{\{7\}}$	$0.7503^{\{7\}}$	34
300	MLE	$1.1273^{\{1\}}$	$0.5867^{\{1\}}$	$0.7660^{\{1\}}$	$0.1463^{\{1\}}$	$0.0336^{\{1\}}$	$0.1834^{\{1\}}$	$9.1665^{\{1\}}$	$0.3560^{\{1\}}$	$0.5966^{\{1\}}$	6
	ADE	$1.4763^{\{2\}}$	$0.7931^{\{2\}}$	$0.8906^{\{2\}}$	$0.2064^{\{2\}}$	$0.0459^{\{2\}}$	$0.2143^{\{2\}}$	$11.7093^{\{2\}}$	$0.4742^{\{2\}}$	$0.6886^{\{2\}}$	12
	CME	$1.5369^{\{3\}}$	$0.8085^{\{3\}}$	$0.8992^{\{3\}}$	$0.2087^{\{3\}}$	$0.0463^{\{3\}}$	$0.2152^{\{3\}}$	$12.2832^{\{3\}}$	$0.4874^{\{3\}}$	$0.6982^{\{3\}}$	18
	LADE	$1.6405^{\{5\}}$	$0.8857^{\{7\}}$	$0.9411^{\{7\}}$	$0.2262^{\{5\}}$	$0.0513^{\{5\}}$	$0.2266^{\{5\}}$	$12.9954^{\{7\}}$	$0.5296^{\{7\}}$	$0.7277^{\{7\}}$	36
	LSE	$1.5606^{\{4\}}$	$0.8238^{\{4\}}$	$0.9076^{\{4\}}$	$0.2169^{\{4\}}$	$0.0479^{\{4\}}$	$0.2188^{\{4\}}$	$12.4117^{\{4\}}$	$0.4940^{\{4\}}$	$0.7029^{\{4\}}$	24
	MSS	$1.6417^{\{6\}}$	$0.8756^{\{5\}}$	$0.9357^{\{5\}}$	$0.2483^{\{7\}}$	$0.0545^{\{7\}}$	$0.2335^{\{7\}}$	$12.8659^{\{5\}}$	$0.5148^{\{5\}}$	$0.7175^{\{5\}}$	35
	MSL	$1.6442^{\{7\}}$	$0.8766^{\{6\}}$	$0.9363^{\{6\}}$	$0.2486^{\{8\}}$	$0.0546^{\{8\}}$	$0.2336^{\{8\}}$	$12.8858^{\{6\}}$	$0.5155^{\{6\}}$	$0.7180^{\{6\}}$	41
	KM	$1.6835^{\{8\}}$	$0.9022^{\{8\}}$	$0.9498^{\{8\}}$	$0.2332^{\{6\}}$	$0.0516^{\{6\}}$	$0.2271^{\{6\}}$	$13.3655^{\{8\}}$	$0.5411^{\{8\}}$	$0.7356^{\{8\}}$	44
500	MLE	$1.0043^{\{1\}}$	$0.5398^{\{1\}}$	$0.7347^{\{1\}}$	$0.1372^{\{1\}}$	$0.0316^{\{1\}}$	$0.1777^{\{1\}}$	$8.0433^{\{1\}}$	$0.3226^{\{1\}}$	$0.5680^{\{1\}}$	6
	ADE	$1.3651^{\{2\}}$	$0.7418^{\{2\}}$	$0.8613^{\{2\}}$	$0.1945^{\{2\}}$	$0.0438^{\{2\}}$	$0.2093^{\{2\}}$	$10.7759^{\{2\}}$	$0.4414^{\{2\}}$	$0.6644^{\{2\}}$	12
	CME	$1.4342^{\{3\}}$	$0.7573^{\{3\}}$	$0.8703^{\{3\}}$	$0.1963^{\{3\}}$	$0.0442^{\{3\}}$	$0.2101^{\{3\}}$	$11.4419^{\{3\}}$	$0.4544^{\{3\}}$	$0.6741^{\{3\}}$	18
	LADE	$1.5041^{\{5\}}$	$0.8112^{\{5\}}$	$0.9006^{\{5\}}$	$0.2087^{\{5\}}$	$0.0479^{\{5\}}$	$0.2188^{\{5\}}$	$11.9259^{\{5\}}$	$0.4852^{\{5\}}$	$0.6966^{\{5\}}$	30
	LSE	$1.4723^{\{4\}}$	$0.7790^{\{4\}}$	$0.8826^{\{4\}}$	$0.2048^{\{4\}}$	$0.0458^{\{4\}}$	$0.2141^{\{4\}}$	$11.7072^{\{4\}}$	$0.4660^{\{4\}}$	$0.6826^{\{4\}}$	24
	MSS	$1.6151^{\{8\}}$	$0.8778^{\{8\}}$	$0.9369^{\{8\}}$	$0.2442^{\{7\}}$	$0.0540^{\{7\}}$	$0.2323^{\{7\}}$	$12.5898^{\{8\}}$	$0.5172^{\{8\}}$	$0.7191^{\{8\}}$	46
	MSL	$1.6151^{\{7\}}$	$0.8778^{\{7\}}$	$0.9369^{\{7\}}$	$0.2442^{\{8\}}$	$0.0540^{\{8\}}$	$0.2323^{\{8\}}$	$12.5896^{\{7\}}$	$0.5171^{\{7\}}$	$0.7191^{\{7\}}$	44
	KM	$1.5576^{\{6\}}$	$0.8345^{\{6\}}$	$0.9135^{\{6\}}$	$0.2161^{\{6\}}$	$0.0487^{\{6\}}$	$0.2206^{\{6\}}$	$12.3674^{\{6\}}$	$0.4987^{\{6\}}$	$0.7062^{\{6\}}$	36

Table 5. The WPGM estimators for $(\lambda, p, \omega) = (0.70, 1.30, 0.55)$.

n	Method	$\widehat{\lambda}$			$\widehat{p}$			$\widehat{\omega}$			$\sum$ Ranks
		RBias	MSE	RMSE	RBias	MSE	RMSE	RBias	MSE	RMSE	
25	MLE	$0.0599^{\{1\}}$	$0.1360^{\{1\}}$	$0.3688^{\{1\}}$	$0.0478^{\{2\}}$	$0.1089^{\{4\}}$	$0.3299^{\{4\}}$	$0.0381^{\{1\}}$	$0.0645^{\{3\}}$	$0.2540^{\{3\}}$	12
	ADE	$0.2933^{\{4\}}$	$0.1791^{\{2\}}$	$0.4232^{\{2\}}$	$0.0780^{\{4\}}$	$0.0876^{\{1\}}$	$0.2960^{\{1\}}$	$0.2269^{\{3\}}$	$0.0689^{\{6\}}$	$0.2626^{\{6\}}$	20
	CME	$0.2315^{\{2\}}$	$0.1816^{\{3\}}$	$0.4262^{\{3\}}$	$0.0372^{\{1\}}$	$0.1072^{\{3\}}$	$0.3274^{\{3\}}$	$0.1975^{\{2\}}$	$0.0683^{\{5\}}$	$0.2613^{\{5\}}$	16
	LADE	$0.4064^{\{8\}}$	$0.2263^{\{8\}}$	$0.4757^{\{8\}}$	$0.1242^{\{6\}}$	$0.1089^{\{5\}}$	$0.3300^{\{5\}}$	$0.3138^{\{8\}}$	$0.0780^{\{8\}}$	$0.2794^{\{8\}}$	43
	LSE	$0.3133^{\{5\}}$	$0.1917^{\{4\}}$	$0.4378^{\{4\}}$	$0.1062^{\{5\}}$	$0.1028^{\{2\}}$	$0.3207^{\{2\}}$	$0.2293^{\{4\}}$	$0.0653^{\{4\}}$	$0.2556^{\{4\}}$	24
	MSS	$0.3685^{\{7\}}$	$0.2096^{\{6\}}$	$0.4578^{\{6\}}$	$0.1548^{\{7\}}$	$0.1425^{\{8\}}$	$0.3776^{\{8\}}$	$0.2365^{\{6\}}$	$0.0621^{\{1\}}$	$0.2492^{\{1\}}$	35
	MSL	$0.3682^{\{6\}}$	$0.2104^{\{7\}}$	$0.4587^{\{7\}}$	$0.1561^{\{8\}}$	$0.1393^{\{7\}}$	$0.3733^{\{7\}}$	$0.2356^{\{5\}}$	$0.0625^{\{2\}}$	$0.2499^{\{2\}}$	35
	KM	$0.2800^{\{3\}}$	$0.2062^{\{5\}}$	$0.4541^{\{5\}}$	$0.0497^{\{3\}}$	$0.1180^{\{6\}}$	$0.3436^{\{6\}}$	$0.2491^{\{7\}}$	$0.0769^{\{7\}}$	$0.2773^{\{7\}}$	31
50	MLE	$0.0128^{\{1\}}$	$0.0912^{\{1\}}$	$0.3021^{\{1\}}$	$0.0250^{\{1\}}$	$0.0577^{\{2\}}$	$0.2402^{\{2\}}$	$0.0239^{\{1\}}$	$0.0535^{\{2\}}$	$0.2313^{\{2\}}$	8
	ADE	$0.1966^{\{3\}}$	$0.1183^{\{2\}}$	$0.3439^{\{2\}}$	$0.0638^{\{3\}}$	$0.0561^{\{1\}}$	$0.2369^{\{1\}}$	$0.1341^{\{2\}}$	$0.0545^{\{4\}}$	$0.2335^{\{4\}}$	15
	CME	$0.1827^{\{2\}}$	$0.1193^{\{3\}}$	$0.3454^{\{3\}}$	$0.0535^{\{2\}}$	$0.0599^{\{3\}}$	$0.2448^{\{3\}}$	$0.1388^{\{3\}}$	$0.0536^{\{3\}}$	$0.2315^{\{3\}}$	16
	LADE	$0.3084^{\{6\}}$	$0.1583^{\{6\}}$	$0.3978^{\{6\}}$	$0.1127^{\{6\}}$	$0.0751^{\{6\}}$	$0.2741^{\{6\}}$	$0.2231^{\{8\}}$	$0.0639^{\{8\}}$	$0.2528^{\{8\}}$	40
	LSE	$0.2258^{\{4\}}$	$0.1224^{\{4\}}$	$0.3499^{\{4\}}$	$0.0885^{\{5\}}$	$0.0618^{\{4\}}$	$0.2486^{\{4\}}$	$0.1558^{\{4\}}$	$0.0520^{\{1\}}$	$0.2280^{\{1\}}$	22
	MSS	$0.3360^{\{8\}}$	$0.1597^{\{7\}}$	$0.3996^{\{7\}}$	$0.1516^{\{8\}}$	$0.0920^{\{7\}}$	$0.3033^{\{7\}}$	$0.2096^{\{7\}}$	$0.0569^{\{5\}}$	$0.2385^{\{5\}}$	42
	MSL	$0.3340^{\{7\}}$	$0.1600^{\{8\}}$	$0.3999^{\{8\}}$	$0.1509^{\{7\}}$	$0.0930^{\{8\}}$	$0.3049^{\{8\}}$	$0.2073^{\{6\}}$	$0.0570^{\{6\}}$	$0.2388^{\{6\}}$	42
	KM	$0.2361^{\{5\}}$	$0.1446^{\{5\}}$	$0.3803^{\{5\}}$	$0.0699^{\{4\}}$	$0.0704^{\{5\}}$	$0.2654^{\{5\}}$	$0.1924^{\{5\}}$	$0.0636^{\{7\}}$	$0.2521^{\{7\}}$	31
100	MLE	$0.0013^{\{1\}}$	$0.0630^{\{1\}}$	$0.2510^{\{1\}}$	$0.0181^{\{1\}}$	$0.0370^{\{2\}}$	$0.1925^{\{2\}}$	$0.0333^{\{1\}}$	$0.0402^{\{4\}}$	$0.2006^{\{4\}}$	10
	ADE	$0.1101^{\{3\}}$	$0.0742^{\{2\}}$	$0.2724^{\{2\}}$	$0.0353^{\{3\}}$	$0.0370^{\{1\}}$	$0.1925^{\{1\}}$	$0.0605^{\{2\}}$	$0.0397^{\{2\}}$	$0.1992^{\{2\}}$	13
	CME	$0.1031^{\{2\}}$	$0.0785^{\{3\}}$	$0.2803^{\{3\}}$	$0.0309^{\{2\}}$	$0.0392^{\{3\}}$	$0.1980^{\{3\}}$	$0.0635^{\{3\}}$	$0.0402^{\{3\}}$	$0.2004^{\{3\}}$	16
	LADE	$0.2252^{\{6\}}$	$0.1127^{\{6\}}$	$0.3357^{\{6\}}$	$0.0869^{\{6\}}$	$0.0548^{\{6\}}$	$0.2340^{\{6\}}$	$0.1523^{\{6\}}$	$0.0503^{\{8\}}$	$0.2242^{\{8\}}$	38
	LSE	$0.1397^{\{5\}}$	$0.0807^{\{4\}}$	$0.2841^{\{4\}}$	$0.0545^{\{5\}}$	$0.0397^{\{4\}}$	$0.1993^{\{4\}}$	$0.0871^{\{4\}}$	$0.0391^{\{1\}}$	$0.1977^{\{1\}}$	23
	MSS	$0.2440^{\{7\}}$	$0.1160^{\{7\}}$	$0.3406^{\{7\}}$	$0.1055^{\{7\}}$	$0.0584^{\{7\}}$	$0.2417^{\{7\}}$	$0.1532^{\{7\}}$	$0.0474^{\{6\}}$	$0.2178^{\{6\}}$	41
	MSL	$0.2441^{\{8\}}$	$0.1162^{\{8\}}$	$0.3408^{\{8\}}$	$0.1057^{\{8\}}$	$0.0585^{\{8\}}$	$0.2419^{\{8\}}$	$0.1532^{\{8\}}$	$0.0475^{\{7\}}$	$0.2178^{\{7\}}$	47
	KM	$0.1317^{\{4\}}$	$0.0934^{\{5\}}$	$0.3055^{\{5\}}$	$0.0426^{\{4\}}$	$0.0441^{\{5\}}$	$0.2101^{\{5\}}$	$0.0908^{\{5\}}$	$0.0468^{\{5\}}$	$0.2163^{\{5\}}$	28
300	MLE	$0.0086^{\{1\}}$	$0.0338^{\{1\}}$	$0.1837^{\{1\}}$	$0.0072^{\{1\}}$	$0.0166^{\{1\}}$	$0.1289^{\{1\}}$	$0.0335^{\{4\}}$	$0.0252^{\{4\}}$	$0.1587^{\{4\}}$	12
	ADE	$0.0523^{\{2\}}$	$0.0402^{\{2\}}$	$0.2005^{\{2\}}$	$0.0192^{\{2\}}$	$0.0196^{\{2\}}$	$0.1398^{\{2\}}$	$0.0217^{\{1\}}$	$0.0235^{\{1\}}$	$0.1532^{\{1\}}$	10
	CME	$0.0578^{\{4\}}$	$0.0447^{\{3\}}$	$0.2114^{\{3\}}$	$0.0215^{\{4\}}$	$0.0217^{\{3\}}$	$0.1474^{\{3\}}$	$0.0293^{\{3\}}$	$0.0239^{\{3\}}$	$0.1547^{\{3\}}$	20
	LADE	$0.1103^{\{6\}}$	$0.0574^{\{6\}}$	$0.2396^{\{6\}}$	$0.0461^{\{6\}}$	$0.0292^{\{6\}}$	$0.1708^{\{6\}}$	$0.0668^{\{8\}}$	$0.0284^{\{6\}}$	$0.1685^{\{6\}}$	38
	LSE	$0.0716^{\{5\}}$	$0.0453^{\{4\}}$	$0.2128^{\{4\}}$	$0.0301^{\{5\}}$	$0.0220^{\{4\}}$	$0.1484^{\{4\}}$	$0.0383^{\{5\}}$	$0.0236^{\{2\}}$	$0.1537^{\{2\}}$	25
	MSS	$0.1231^{\{7\}}$	$0.0654^{\{7\}}$	$0.2557^{\{7\}}$	$0.0551^{\{7\}}$	$0.0304^{\{7\}}$	$0.1744^{\{7\}}$	$0.0642^{\{7\}}$	$0.0359^{\{7\}}$	$0.1894^{\{7\}}$	42
	MSL	$0.1231^{\{8\}}$	$0.0654^{\{8\}}$	$0.2557^{\{8\}}$	$0.0551^{\{8\}}$	$0.0304^{\{8\}}$	$0.1745^{\{8\}}$	$0.0642^{\{6\}}$	$0.0359^{\{8\}}$	$0.1894^{\{8\}}$	46
	KM	$0.0527^{\{3\}}$	$0.0491^{\{5\}}$	$0.2217^{\{5\}}$	$0.0208^{\{3\}}$	$0.0236^{\{5\}}$	$0.1536^{\{5\}}$	$0.0223^{\{2\}}$	$0.0275^{\{5\}}$	$0.1659^{\{5\}}$	23
500	MLE	$0.0031^{\{1\}}$	$0.0255^{\{1\}}$	$0.1596^{\{1\}}$	$0.0045^{\{1\}}$	$0.0125^{\{1\}}$	$0.1118^{\{1\}}$	$0.0249^{\{2\}}$	$0.0195^{\{4\}}$	$0.1395^{\{4\}}$	10
	ADE	$0.0415^{\{2\}}$	$0.0295^{\{2\}}$	$0.1718^{\{2\}}$	$0.0145^{\{2\}}$	$0.0150^{\{2\}}$	$0.1224^{\{2\}}$	$0.0164^{\{1\}}$	$0.0178^{\{1\}}$	$0.1334^{\{1\}}$	10
	CME	$0.0573^{\{4\}}$	$0.0352^{\{3\}}$	$0.1876^{\{3\}}$	$0.0206^{\{3\}}$	$0.0181^{\{3\}}$	$0.1344^{\{3\}}$	$0.0316^{\{4\}}$	$0.0188^{\{3\}}$	$0.1370^{\{3\}}$	20
	LADE	$0.0716^{\{6\}}$	$0.0390^{\{6\}}$	$0.1975^{\{6\}}$	$0.0306^{\{6\}}$	$0.0206^{\{6\}}$	$0.1435^{\{6\}}$	$0.0365^{\{5\}}$	$0.0216^{\{6\}}$	$0.1470^{\{6\}}$	35
	LSE	$0.0715^{\{5\}}$	$0.0357^{\{4\}}$	$0.1889^{\{4\}}$	$0.0280^{\{5\}}$	$0.0184^{\{4\}}$	$0.1356^{\{4\}}$	$0.0428^{\{6\}}$	$0.0184^{\{2\}}$	$0.1357^{\{2\}}$	26
	MSS	$0.1126^{\{7\}}$	$0.0489^{\{7\}}$	$0.2211^{\{7\}}$	$0.0470^{\{7\}}$	$0.0225^{\{7\}}$	$0.1501^{\{7\}}$	$0.0647^{\{8\}}$	$0.0285^{\{7\}}$	$0.1689^{\{7\}}$	43
	MSL	$0.1126^{\{8\}}$	$0.0489^{\{8\}}$	$0.2211^{\{8\}}$	$0.0470^{\{8\}}$	$0.0225^{\{8\}}$	$0.1501^{\{8\}}$	$0.0646^{\{7\}}$	$0.0286^{\{8\}}$	$0.1690^{\{8\}}$	47
	KM	$0.0568^{\{3\}}$	$0.0376^{\{5\}}$	$0.1938^{\{5\}}$	$0.0209^{\{4\}}$	$0.0196^{\{5\}}$	$0.1399^{\{5\}}$	$0.0297^{\{3\}}$	$0.0203^{\{5\}}$	$0.1426^{\{5\}}$	25

Table 6. The WPGM estimators for $(\lambda, p, \omega) = (1.00, 1.70, 0.80)$.

n	Method	$\widehat{\lambda}$			$\widehat{p}$			$\widehat{\omega}$			$\sum$ Ranks
		RBias	MSE	RMSE	RBias	MSE	RMSE	RBias	MSE	RMSE	
25	MLE	$0.2269^{\{8\}}$	$0.1893^{\{8\}}$	$0.4351^{\{8\}}$	$0.2126^{\{8\}}$	$0.4030^{\{8\}}$	$0.6348^{\{8\}}$	$0.2038^{\{8\}}$	$0.1004^{\{8\}}$	$0.3168^{\{8\}}$	48
	ADE	$0.0753^{\{5\}}$	$0.1256^{\{2\}}$	$0.3545^{\{2\}}$	$0.0796^{\{5\}}$	$0.2065^{\{2\}}$	$0.4544^{\{2\}}$	$0.0755^{\{4\}}$	$0.0566^{\{5\}}$	$0.2379^{\{5\}}$	23
	CME	$0.1272^{\{7\}}$	$0.1575^{\{7\}}$	$0.3968^{\{7\}}$	$0.1457^{\{7\}}$	$0.3156^{\{7\}}$	$0.5617^{\{7\}}$	$0.0971^{\{7\}}$	$0.0645^{\{7\}}$	$0.2540^{\{7\}}$	42
	LADE	$0.0562^{\{3\}}$	$0.1129^{\{1\}}$	$0.3359^{\{1\}}$	$0.0720^{\{4\}}$	$0.2148^{\{3\}}$	$0.4634^{\{3\}}$	$0.0569^{\{1\}}$	$0.0458^{\{1\}}$	$0.2140^{\{1\}}$	13
	LSE	$0.0672^{\{4\}}$	$0.1291^{\{4\}}$	$0.3594^{\{4\}}$	$0.0600^{\{3\}}$	$0.2025^{\{1\}}$	$0.4500^{\{1\}}$	$0.0735^{\{3\}}$	$0.0515^{\{4\}}$	$0.2270^{\{4\}}$	19
	MSS	$0.0546^{\{1\}}$	$0.1283^{\{3\}}$	$0.3583^{\{3\}}$	$0.0105^{\{1\}}$	$0.2801^{\{4\}}$	$0.5293^{\{4\}}$	$0.0937^{\{5\}}$	$0.0500^{\{3\}}$	$0.2235^{\{3\}}$	17
	MSL	$0.0558^{\{2\}}$	$0.1295^{\{5\}}$	$0.3599^{\{5\}}$	$0.0128^{\{2\}}$	$0.2941^{\{5\}}$	$0.5423^{\{5\}}$	$0.0940^{\{6\}}$	$0.0496^{\{2\}}$	$0.2228^{\{2\}}$	22
	KM	$0.0943^{\{6\}}$	$0.1546^{\{6\}}$	$0.3931^{\{6\}}$	$0.1282^{\{6\}}$	$0.2972^{\{6\}}$	$0.5452^{\{6\}}$	$0.0608^{\{2\}}$	$0.0573^{\{6\}}$	$0.2394^{\{6\}}$	32
50	MLE	$0.1926^{\{8\}}$	$0.1584^{\{8\}}$	$0.3979^{\{8\}}$	$0.1597^{\{8\}}$	$0.2726^{\{8\}}$	$0.5221^{\{8\}}$	$0.1888^{\{8\}}$	$0.0957^{\{8\}}$	$0.3094^{\{8\}}$	48
	ADE	$0.0828^{\{4\}}$	$0.1056^{\{4\}}$	$0.3249^{\{4\}}$	$0.0748^{\{4\}}$	$0.1556^{\{2\}}$	$0.3944^{\{2\}}$	$0.0829^{\{4\}}$	$0.0549^{\{4\}}$	$0.2343^{\{4\}}$	22
	CME	$0.1496^{\{7\}}$	$0.1374^{\{7\}}$	$0.3707^{\{7\}}$	$0.1397^{\{7\}}$	$0.2348^{\{7\}}$	$0.4845^{\{7\}}$	$0.1238^{\{7\}}$	$0.0664^{\{7\}}$	$0.2578^{\{7\}}$	42
	LADE	$0.0779^{\{3\}}$	$0.0928^{\{1\}}$	$0.3046^{\{1\}}$	$0.0720^{\{3\}}$	$0.1502^{\{1\}}$	$0.3876^{\{1\}}$	$0.0734^{\{1\}}$	$0.0431^{\{1\}}$	$0.2076^{\{1\}}$	10
	LSE	$0.1113^{\{5\}}$	$0.1169^{\{5\}}$	$0.3419^{\{5\}}$	$0.0903^{\{5\}}$	$0.1730^{\{5\}}$	$0.4159^{\{5\}}$	$0.1049^{\{6\}}$	$0.0576^{\{5\}}$	$0.2400^{\{5\}}$	31
	MSS	$0.0449^{\{1\}}$	$0.1034^{\{2\}}$	$0.3215^{\{2\}}$	$0.0120^{\{1\}}$	$0.1654^{\{3\}}$	$0.4067^{\{3\}}$	$0.0758^{\{2\}}$	$0.0443^{\{2\}}$	$0.2104^{\{2\}}$	11
	MSL	$0.0453^{\{2\}}$	$0.1037^{\{3\}}$	$0.3220^{\{3\}}$	$0.0122^{\{2\}}$	$0.1662^{\{4\}}$	$0.4076^{\{4\}}$	$0.0762^{\{3\}}$	$0.0444^{\{3\}}$	$0.2106^{\{3\}}$	17
	KM	$0.1172^{\{6\}}$	$0.1309^{\{6\}}$	$0.3618^{\{6\}}$	$0.1214^{\{6\}}$	$0.2159^{\{6\}}$	$0.4647^{\{6\}}$	$0.0868^{\{5\}}$	$0.0591^{\{6\}}$	$0.2430^{\{6\}}$	35
100	MLE	$0.1847^{\{8\}}$	$0.1331^{\{7\}}$	$0.3648^{\{7\}}$	$0.1346^{\{7\}}$	$0.1863^{\{6\}}$	$0.4316^{\{6\}}$	$0.1851^{\{8\}}$	$0.0890^{\{8\}}$	$0.2984^{\{8\}}$	44
	ADE	$0.1252^{\{4\}}$	$0.1031^{\{4\}}$	$0.3211^{\{4\}}$	$0.0935^{\{4\}}$	$0.1372^{\{4\}}$	$0.3703^{\{4\}}$	$0.1220^{\{4\}}$	$0.0611^{\{4\}}$	$0.2472^{\{4\}}$	24
	CME	$0.1724^{\{7\}}$	$0.1336^{\{8\}}$	$0.3655^{\{8\}}$	$0.1415^{\{8\}}$	$0.2045^{\{8\}}$	$0.4523^{\{8\}}$	$0.1526^{\{7\}}$	$0.0717^{\{7\}}$	$0.2677^{\{7\}}$	45
	LADE	$0.0890^{\{3\}}$	$0.0813^{\{1\}}$	$0.2851^{\{1\}}$	$0.0701^{\{3\}}$	$0.1206^{\{3\}}$	$0.3473^{\{3\}}$	$0.0838^{\{1\}}$	$0.0422^{\{1\}}$	$0.2055^{\{1\}}$	12
	LSE	$0.1441^{\{5\}}$	$0.1158^{\{5\}}$	$0.3404^{\{5\}}$	$0.1094^{\{5\}}$	$0.1605^{\{5\}}$	$0.4006^{\{5\}}$	$0.1325^{\{6\}}$	$0.0620^{\{5\}}$	$0.2490^{\{5\}}$	31
	MSS	$0.0629^{\{2\}}$	$0.0889^{\{3\}}$	$0.2982^{\{3\}}$	$0.0249^{\{2\}}$	$0.1124^{\{2\}}$	$0.3353^{\{2\}}$	$0.0900^{\{3\}}$	$0.0470^{\{3\}}$	$0.2169^{\{3\}}$	15
	MSL	$0.0623^{\{1\}}$	$0.0884^{\{2\}}$	$0.2973^{\{2\}}$	$0.0243^{\{1\}}$	$0.1115^{\{1\}}$	$0.3340^{\{1\}}$	$0.0890^{\{2\}}$	$0.0464^{\{2\}}$	$0.2155^{\{2\}}$	9
	KM	$0.1479^{\{6\}}$	$0.1310^{\{6\}}$	$0.3619^{\{6\}}$	$0.1276^{\{6\}}$	$0.1962^{\{7\}}$	$0.4430^{\{7\}}$	$0.1267^{\{5\}}$	$0.0678^{\{6\}}$	$0.2604^{\{6\}}$	36
300	MLE	$0.1177^{\{5\}}$	$0.0802^{\{5\}}$	$0.2832^{\{5\}}$	$0.0787^{\{5\}}$	$0.0955^{\{5\}}$	$0.3091^{\{5\}}$	$0.1241^{\{6\}}$	$0.0590^{\{7\}}$	$0.2429^{\{7\}}$	33
	ADE	$0.0901^{\{4\}}$	$0.0648^{\{4\}}$	$0.2545^{\{4\}}$	$0.0642^{\{4\}}$	$0.0818^{\{4\}}$	$0.2860^{\{4\}}$	$0.0892^{\{4\}}$	$0.0407^{\{4\}}$	$0.2019^{\{4\}}$	24
	CME	$0.1423^{\{7\}}$	$0.0946^{\{7\}}$	$0.3076^{\{7\}}$	$0.1087^{\{8\}}$	$0.1358^{\{7\}}$	$0.3685^{\{7\}}$	$0.1307^{\{7\}}$	$0.0549^{\{6\}}$	$0.2342^{\{6\}}$	42
	LADE	$0.0820^{\{3\}}$	$0.0582^{\{1\}}$	$0.2413^{\{1\}}$	$0.0598^{\{3\}}$	$0.0783^{\{3\}}$	$0.2798^{\{3\}}$	$0.0779^{\{3\}}$	$0.0330^{\{1\}}$	$0.1818^{\{1\}}$	14
	LSE	$0.1282^{\{6\}}$	$0.0867^{\{6\}}$	$0.2944^{\{6\}}$	$0.0952^{\{6\}}$	$0.1199^{\{6\}}$	$0.3463^{\{6\}}$	$0.1197^{\{5\}}$	$0.0499^{\{5\}}$	$0.2235^{\{5\}}$	34
	MSS	$0.0502^{\{1\}}$	$0.0604^{\{2\}}$	$0.2457^{\{2\}}$	$0.0281^{\{1\}}$	$0.0709^{\{1\}}$	$0.2664^{\{1\}}$	$0.0639^{\{1\}}$	$0.0342^{\{2\}}$	$0.1850^{\{2\}}$	8
	MSL	$0.0503^{\{2\}}$	$0.0604^{\{3\}}$	$0.2458^{\{3\}}$	$0.0281^{\{2\}}$	$0.0710^{\{2\}}$	$0.2664^{\{2\}}$	$0.0640^{\{2\}}$	$0.0342^{\{3\}}$	$0.1850^{\{3\}}$	14
	KM	$0.1423^{\{8\}}$	$0.1004^{\{8\}}$	$0.3168^{\{8\}}$	$0.1082^{\{7\}}$	$0.1358^{\{8\}}$	$0.3685^{\{8\}}$	$0.1327^{\{8\}}$	$0.0624^{\{8\}}$	$0.2498^{\{8\}}$	47
500	MLE	$0.0762^{\{5\}}$	$0.0541^{\{5\}}$	$0.2327^{\{5\}}$	$0.0497^{\{5\}}$	$0.0584^{\{3\}}$	$0.2416^{\{3\}}$	$0.0834^{\{5\}}$	$0.0417^{\{5\}}$	$0.2041^{\{5\}}$	28
	ADE	$0.0672^{\{4\}}$	$0.0486^{\{2\}}$	$0.2204^{\{2\}}$	$0.0483^{\{4\}}$	$0.0594^{\{5\}}$	$0.2438^{\{5\}}$	$0.0676^{\{4\}}$	$0.0304^{\{2\}}$	$0.1743^{\{2\}}$	21
	CME	$0.1217^{\{8\}}$	$0.0792^{\{7\}}$	$0.2813^{\{7\}}$	$0.0913^{\{8\}}$	$0.1104^{\{8\}}$	$0.3323^{\{8\}}$	$0.1146^{\{8\}}$	$0.0480^{\{7\}}$	$0.2190^{\{7\}}$	46
	LADE	$0.0628^{\{3\}}$	$0.0441^{\{1\}}$	$0.2099^{\{1\}}$	$0.0468^{\{3\}}$	$0.0587^{\{4\}}$	$0.2422^{\{4\}}$	$0.0592^{\{3\}}$	$0.0239^{\{1\}}$	$0.1547^{\{1\}}$	15
	LSE	$0.1089^{\{6\}}$	$0.0716^{\{6\}}$	$0.2677^{\{6\}}$	$0.0799^{\{6\}}$	$0.0969^{\{6\}}$	$0.3114^{\{6\}}$	$0.1041^{\{6\}}$	$0.0435^{\{6\}}$	$0.2087^{\{6\}}$	36
	MSS	$0.0431^{\{1\}}$	$0.0496^{\{3\}}$	$0.2227^{\{3\}}$	$0.0244^{\{1\}}$	$0.0569^{\{1\}}$	$0.2385^{\{1\}}$	$0.0575^{\{1\}}$	$0.0311^{\{3\}}$	$0.1763^{\{3\}}$	10
	MSL	$0.0439^{\{2\}}$	$0.0500^{\{4\}}$	$0.2237^{\{4\}}$	$0.0249^{\{2\}}$	$0.0576^{\{2\}}$	$0.2399^{\{2\}}$	$0.0582^{\{2\}}$	$0.0313^{\{4\}}$	$0.1769^{\{4\}}$	16
	KM	$0.1163^{\{7\}}$	$0.0814^{\{8\}}$	$0.2853^{\{8\}}$	$0.0885^{\{7\}}$	$0.1100^{\{7\}}$	$0.3316^{\{7\}}$	$0.1098^{\{7\}}$	$0.0504^{\{8\}}$	$0.2244^{\{8\}}$	44

Table 7. The WPGM estimators for $(\lambda, p, \omega) = (1.20, 0.90, 0.75)$.

n	Method	$\widehat{\lambda}$			$\widehat{p}$			$\widehat{\omega}$			$\sum$ Ranks
		RBias	MSE	RMSE	RBias	MSE	RMSE	RBias	MSE	RMSE	
25	MLE	$0.1737^{\{8\}}$	$0.2105^{\{8\}}$	$0.4588^{\{8\}}$	$0.1761^{\{8\}}$	$0.0979^{\{8\}}$	$0.3129^{\{8\}}$	$0.1780^{\{8\}}$	$0.0868^{\{8\}}$	$0.2947^{\{8\}}$	48
	ADE	$0.0435^{\{5\}}$	$0.1408^{\{2\}}$	$0.3752^{\{2\}}$	$0.0485^{\{5\}}$	$0.0492^{\{1\}}$	$0.2218^{\{1\}}$	$0.0444^{\{5\}}$	$0.0520^{\{5\}}$	$0.2280^{\{5\}}$	23
	CME	$0.0891^{\{7\}}$	$0.1746^{\{6\}}$	$0.4179^{\{6\}}$	$0.1125^{\{7\}}$	$0.0763^{\{5\}}$	$0.2763^{\{5\}}$	$0.0707^{\{7\}}$	$0.0605^{\{7\}}$	$0.2460^{\{7\}}$	39
	LADE	$0.0106^{\{1\}}$	$0.1345^{\{1\}}$	$0.3667^{\{1\}}$	$0.0275^{\{2\}}$	$0.0520^{\{3\}}$	$0.2280^{\{3\}}$	$0.0109^{\{1\}}$	$0.0412^{\{1\}}$	$0.2031^{\{1\}}$	9
	LSE	$0.0388^{\{4\}}$	$0.1444^{\{3\}}$	$0.3800^{\{3\}}$	$0.0272^{\{1\}}$	$0.0497^{\{2\}}$	$0.2228^{\{2\}}$	$0.0428^{\{3\}}$	$0.0494^{\{4\}}$	$0.2223^{\{4\}}$	17
	MSS	$0.0162^{\{2\}}$	$0.1500^{\{4\}}$	$0.3874^{\{4\}}$	$0.0337^{\{4\}}$	$0.0783^{\{6\}}$	$0.2798^{\{6\}}$	$0.0460^{\{6\}}$	$0.0442^{\{3\}}$	$0.2102^{\{3\}}$	25
	MSL	$0.0168^{\{3\}}$	$0.1530^{\{5\}}$	$0.3912^{\{5\}}$	$0.0303^{\{3\}}$	$0.0844^{\{7\}}$	$0.2905^{\{7\}}$	$0.0436^{\{4\}}$	$0.0437^{\{2\}}$	$0.2091^{\{2\}}$	24
	KM	$0.0543^{\{6\}}$	$0.1769^{\{7\}}$	$0.4206^{\{7\}}$	$0.0913^{\{6\}}$	$0.0720^{\{4\}}$	$0.2683^{\{4\}}$	$0.0267^{\{2\}}$	$0.0567^{\{6\}}$	$0.2382^{\{6\}}$	31
50	MLE	$0.1601^{\{8\}}$	$0.1766^{\{8\}}$	$0.4202^{\{8\}}$	$0.1353^{\{8\}}$	$0.0579^{\{8\}}$	$0.2406^{\{8\}}$	$0.1823^{\{8\}}$	$0.0882^{\{8\}}$	$0.2970^{\{8\}}$	48
	ADE	$0.0532^{\{4\}}$	$0.1151^{\{4\}}$	$0.3393^{\{4\}}$	$0.0502^{\{5\}}$	$0.0335^{\{2\}}$	$0.1831^{\{2\}}$	$0.0634^{\{4\}}$	$0.0524^{\{5\}}$	$0.2289^{\{5\}}$	24
	CME	$0.1014^{\{7\}}$	$0.1435^{\{6\}}$	$0.3788^{\{6\}}$	$0.0994^{\{7\}}$	$0.0476^{\{6\}}$	$0.2183^{\{6\}}$	$0.1013^{\{7\}}$	$0.0628^{\{7\}}$	$0.2506^{\{7\}}$	40
	LADE	$0.0312^{\{3\}}$	$0.1036^{\{1\}}$	$0.3219^{\{1\}}$	$0.0316^{\{3\}}$	$0.0326^{\{1\}}$	$0.1806^{\{1\}}$	$0.0409^{\{2\}}$	$0.0434^{\{3\}}$	$0.2083^{\{3\}}$	13
	LSE	$0.0613^{\{5\}}$	$0.1183^{\{5\}}$	$0.3439^{\{5\}}$	$0.0483^{\{4\}}$	$0.0347^{\{3\}}$	$0.1862^{\{3\}}$	$0.0697^{\{6\}}$	$0.0504^{\{4\}}$	$0.2245^{\{4\}}$	27
	MSS	$0.0145^{\{1\}}$	$0.1132^{\{2\}}$	$0.3364^{\{2\}}$	$0.0208^{\{2\}}$	$0.0393^{\{4\}}$	$0.1983^{\{4\}}$	$0.0406^{\{1\}}$	$0.0428^{\{1\}}$	$0.2070^{\{1\}}$	11
	MSL	$0.0155^{\{2\}}$	$0.1142^{\{3\}}$	$0.3380^{\{3\}}$	$0.0205^{\{1\}}$	$0.0396^{\{5\}}$	$0.1991^{\{5\}}$	$0.0418^{\{3\}}$	$0.0433^{\{2\}}$	$0.2081^{\{2\}}$	16
	KM	$0.0755^{\{6\}}$	$0.1451^{\{7\}}$	$0.3809^{\{7\}}$	$0.0847^{\{6\}}$	$0.0485^{\{7\}}$	$0.2202^{\{7\}}$	$0.0693^{\{5\}}$	$0.0609^{\{6\}}$	$0.2468^{\{6\}}$	37
100	MLE	$0.1518^{\{8\}}$	$0.1495^{\{8\}}$	$0.3866^{\{8\}}$	$0.1060^{\{8\}}$	$0.0382^{\{8\}}$	$0.1954^{\{8\}}$	$0.1824^{\{8\}}$	$0.0833^{\{8\}}$	$0.2885^{\{8\}}$	48
	ADE	$0.0790^{\{4\}}$	$0.1041^{\{4\}}$	$0.3226^{\{4\}}$	$0.0559^{\{4\}}$	$0.0268^{\{4\}}$	$0.1637^{\{4\}}$	$0.0943^{\{4\}}$	$0.0520^{\{4\}}$	$0.2281^{\{4\}}$	24
	CME	$0.1115^{\{7\}}$	$0.1264^{\{6\}}$	$0.3555^{\{6\}}$	$0.0871^{\{7\}}$	$0.0351^{\{6\}}$	$0.1873^{\{6\}}$	$0.1200^{\{7\}}$	$0.0619^{\{7\}}$	$0.2487^{\{7\}}$	40
	LADE	$0.0412^{\{3\}}$	$0.0872^{\{1\}}$	$0.2953^{\{1\}}$	$0.0268^{\{3\}}$	$0.0243^{\{1\}}$	$0.1558^{\{1\}}$	$0.0528^{\{1\}}$	$0.0394^{\{1\}}$	$0.1985^{\{1\}}$	10
	LSE	$0.0837^{\{5\}}$	$0.1077^{\{5\}}$	$0.3281^{\{5\}}$	$0.0564^{\{5\}}$	$0.0279^{\{5\}}$	$0.1669^{\{5\}}$	$0.0949^{\{5\}}$	$0.0526^{\{5\}}$	$0.2292^{\{5\}}$	30
	MSS	$0.0333^{\{1\}}$	$0.0912^{\{2\}}$	$0.3020^{\{2\}}$	$0.0046^{\{2\}}$	$0.0245^{\{2\}}$	$0.1565^{\{2\}}$	$0.0558^{\{2\}}$	$0.0396^{\{2\}}$	$0.1989^{\{2\}}$	11
	MSL	$0.0342^{\{2\}}$	$0.0916^{\{3\}}$	$0.3026^{\{3\}}$	$0.0042^{\{1\}}$	$0.0246^{\{3\}}$	$0.1568^{\{3\}}$	$0.0567^{\{3\}}$	$0.0398^{\{3\}}$	$0.1994^{\{3\}}$	15
	KM	$0.0959^{\{6\}}$	$0.1314^{\{7\}}$	$0.3624^{\{7\}}$	$0.0795^{\{6\}}$	$0.0378^{\{7\}}$	$0.1944^{\{7\}}$	$0.0993^{\{6\}}$	$0.0606^{\{6\}}$	$0.2462^{\{6\}}$	38
300	MLE	$0.1060^{\{5\}}$	$0.0917^{\{6\}}$	$0.3028^{\{6\}}$	$0.0704^{\{5\}}$	$0.0203^{\{5\}}$	$0.1425^{\{5\}}$	$0.1244^{\{6\}}$	$0.0534^{\{8\}}$	$0.2312^{\{8\}}$	35
	ADE	$0.0854^{\{4\}}$	$0.0747^{\{4\}}$	$0.2734^{\{4\}}$	$0.0594^{\{4\}}$	$0.0178^{\{4\}}$	$0.1335^{\{4\}}$	$0.0955^{\{4\}}$	$0.0403^{\{4\}}$	$0.2008^{\{4\}}$	24
	CME	$0.1235^{\{8\}}$	$0.0973^{\{7\}}$	$0.3119^{\{7\}}$	$0.0922^{\{8\}}$	$0.0260^{\{7\}}$	$0.1612^{\{7\}}$	$0.1303^{\{8\}}$	$0.0494^{\{6\}}$	$0.2222^{\{6\}}$	44
	LADE	$0.0695^{\{3\}}$	$0.0674^{\{3\}}$	$0.2595^{\{3\}}$	$0.0463^{\{3\}}$	$0.0171^{\{3\}}$	$0.1307^{\{3\}}$	$0.0773^{\{3\}}$	$0.0343^{\{3\}}$	$0.1851^{\{3\}}$	18
	LSE	$0.1113^{\{6\}}$	$0.0900^{\{5\}}$	$0.3000^{\{5\}}$	$0.0795^{\{6\}}$	$0.0230^{\{6\}}$	$0.1515^{\{6\}}$	$0.1196^{\{5\}}$	$0.0463^{\{5\}}$	$0.2152^{\{5\}}$	33
	MSS	$0.0377^{\{1\}}$	$0.0671^{\{1\}}$	$0.2590^{\{1\}}$	$0.0134^{\{1\}}$	$0.0157^{\{1\}}$	$0.1252^{\{1\}}$	$0.0516^{\{1\}}$	$0.0341^{\{1\}}$	$0.1846^{\{1\}}$	6
	MSL	$0.0377^{\{2\}}$	$0.0671^{\{2\}}$	$0.2591^{\{2\}}$	$0.0134^{\{2\}}$	$0.0157^{\{2\}}$	$0.1253^{\{2\}}$	$0.0516^{\{2\}}$	$0.0341^{\{2\}}$	$0.1847^{\{2\}}$	12
	KM	$0.1216^{\{7\}}$	$0.1022^{\{8\}}$	$0.3196^{\{8\}}$	$0.0920^{\{7\}}$	$0.0274^{\{8\}}$	$0.1656^{\{8\}}$	$0.1265^{\{7\}}$	$0.0503^{\{7\}}$	$0.2243^{\{7\}}$	44
500	MLE	$0.0699^{\{5\}}$	$0.0602^{\{5\}}$	$0.2453^{\{5\}}$	$0.0416^{\{5\}}$	$0.0117^{\{5\}}$	$0.1080^{\{5\}}$	$0.0874^{\{5\}}$	$0.0375^{\{7\}}$	$0.1937^{\{7\}}$	32
	ADE	$0.0491^{\{4\}}$	$0.0460^{\{2\}}$	$0.2145^{\{2\}}$	$0.0327^{\{4\}}$	$0.0108^{\{1\}}$	$0.1040^{\{1\}}$	$0.0573^{\{4\}}$	$0.0236^{\{2\}}$	$0.1537^{\{2\}}$	17
	CME	$0.0894^{\{7\}}$	$0.0717^{\{7\}}$	$0.2677^{\{7\}}$	$0.0649^{\{7\}}$	$0.0186^{\{7\}}$	$0.1363^{\{7\}}$	$0.0976^{\{7\}}$	$0.0357^{\{6\}}$	$0.1889^{\{6\}}$	41
	LADE	$0.0426^{\{3\}}$	$0.0420^{\{1\}}$	$0.2049^{\{1\}}$	$0.0271^{\{3\}}$	$0.0111^{\{2\}}$	$0.1052^{\{2\}}$	$0.0494^{\{2\}}$	$0.0198^{\{1\}}$	$0.1406^{\{1\}}$	12
	LSE	$0.0820^{\{6\}}$	$0.0671^{\{6\}}$	$0.2590^{\{6\}}$	$0.0576^{\{6\}}$	$0.0171^{\{6\}}$	$0.1308^{\{6\}}$	$0.0908^{\{6\}}$	$0.0334^{\{5\}}$	$0.1827^{\{5\}}$	35
	MSS	$0.0328^{\{1\}}$	$0.0524^{\{3\}}$	$0.2290^{\{3\}}$	$0.0113^{\{1\}}$	$0.0113^{\{3\}}$	$0.1062^{\{3\}}$	$0.0488^{\{1\}}$	$0.0277^{\{3\}}$	$0.1663^{\{3\}}$	12
	MSL	$0.0336^{\{2\}}$	$0.0528^{\{4\}}$	$0.2298^{\{4\}}$	$0.0118^{\{2\}}$	$0.0114^{\{4\}}$	$0.1067^{\{4\}}$	$0.0497^{\{3\}}$	$0.0279^{\{4\}}$	$0.1670^{\{4\}}$	19
	KM	$0.0953^{\{8\}}$	$0.0777^{\{8\}}$	$0.2788^{\{8\}}$	$0.0695^{\{8\}}$	$0.0199^{\{8\}}$	$0.1412^{\{8\}}$	$0.1029^{\{8\}}$	$0.0390^{\{8\}}$	$0.1975^{\{8\}}$	48

Table 8. The WPGM estimators for $(\lambda, p, \omega) = (1.80, 1.30, 0.35)$.

n	Method	$\widehat{\lambda}$			$\widehat{p}$			$\widehat{\omega}$			$\sum$ Ranks
		RBias	MSE	RMSE	RBias	MSE	RMSE	RBias	MSE	RMSE	
25	MLE	$0.2406^{\{1\}}$	$0.5283^{\{3\}}$	$0.7269^{\{3\}}$	$0.0707^{\{1\}}$	$0.0866^{\{1\}}$	$0.2944^{\{1\}}$	$0.7670^{\{1\}}$	$0.1567^{\{1\}}$	$0.3959^{\{1\}}$	8
	ADE	$0.3157^{\{5\}}$	$0.5832^{\{5\}}$	$0.7637^{\{5\}}$	$0.1677^{\{4\}}$	$0.1046^{\{2\}}$	$0.3234^{\{2\}}$	$1.0691^{\{6\}}$	$0.1990^{\{6\}}$	$0.4460^{\{6\}}$	28
	CME	$0.3165^{\{6\}}$	$0.6329^{\{6\}}$	$0.7956^{\{6\}}$	$0.1340^{\{2\}}$	$0.1063^{\{3\}}$	$0.3260^{\{3\}}$	$1.0337^{\{2\}}$	$0.1923^{\{5\}}$	$0.4385^{\{5\}}$	24
	LADE	$0.3620^{\{8\}}$	$0.7289^{\{8\}}$	$0.8537^{\{8\}}$	$0.1872^{\{5\}}$	$0.1130^{\{5\}}$	$0.3361^{\{5\}}$	$1.1818^{\{8\}}$	$0.2193^{\{8\}}$	$0.4683^{\{8\}}$	42
	LSE	$0.3012^{\{4\}}$	$0.5537^{\{4\}}$	$0.7441^{\{4\}}$	$0.1904^{\{6\}}$	$0.1237^{\{6\}}$	$0.3518^{\{6\}}$	$1.0630^{\{5\}}$	$0.1917^{\{4\}}$	$0.4379^{\{4\}}$	29
	MSS	$0.2631^{\{3\}}$	$0.4905^{\{1\}}$	$0.7003^{\{1\}}$	$0.2318^{\{8\}}$	$0.1764^{\{7\}}$	$0.4200^{\{7\}}$	$1.0433^{\{4\}}$	$0.1847^{\{3\}}$	$0.4298^{\{3\}}$	26
	MSL	$0.2608^{\{2\}}$	$0.4926^{\{2\}}$	$0.7019^{\{2\}}$	$0.2305^{\{7\}}$	$0.1778^{\{8\}}$	$0.4217^{\{8\}}$	$1.0386^{\{3\}}$	$0.1843^{\{2\}}$	$0.4293^{\{2\}}$	24
	KM	$0.3375^{\{7\}}$	$0.6791^{\{7\}}$	$0.8241^{\{7\}}$	$0.1460^{\{3\}}$	$0.1102^{\{4\}}$	$0.3320^{\{4\}}$	$1.1075^{\{7\}}$	$0.2116^{\{7\}}$	$0.4599^{\{7\}}$	35
50	MLE	$0.2037^{\{1\}}$	$0.3979^{\{1\}}$	$0.6308^{\{1\}}$	$0.0969^{\{1\}}$	$0.0671^{\{1\}}$	$0.2590^{\{1\}}$	$0.6891^{\{1\}}$	$0.1458^{\{1\}}$	$0.3818^{\{1\}}$	6
	ADE	$0.2775^{\{5\}}$	$0.4623^{\{5\}}$	$0.6800^{\{5\}}$	$0.1690^{\{4\}}$	$0.0887^{\{2\}}$	$0.2978^{\{2\}}$	$0.9662^{\{3\}}$	$0.1807^{\{3\}}$	$0.4251^{\{3\}}$	22
	CME	$0.2783^{\{6\}}$	$0.4743^{\{6\}}$	$0.6887^{\{6\}}$	$0.1581^{\{2\}}$	$0.0907^{\{3\}}$	$0.3011^{\{3\}}$	$0.9636^{\{2\}}$	$0.1802^{\{2\}}$	$0.4245^{\{2\}}$	21
	LADE	$0.3314^{\{8\}}$	$0.5410^{\{8\}}$	$0.7355^{\{8\}}$	$0.2042^{\{6\}}$	$0.1105^{\{6\}}$	$0.3325^{\{6\}}$	$1.1506^{\{8\}}$	$0.2120^{\{8\}}$	$0.4605^{\{8\}}$	44
	LSE	$0.2763^{\{2\}}$	$0.4502^{\{2\}}$	$0.6710^{\{2\}}$	$0.1880^{\{5\}}$	$0.1025^{\{5\}}$	$0.3202^{\{5\}}$	$0.9932^{\{4\}}$	$0.1815^{\{4\}}$	$0.4260^{\{4\}}$	22
	MSS	$0.2764^{\{4\}}$	$0.4602^{\{3\}}$	$0.6784^{\{3\}}$	$0.2333^{\{7\}}$	$0.1385^{\{7\}}$	$0.3721^{\{7\}}$	$1.0365^{\{7\}}$	$0.1829^{\{6\}}$	$0.4277^{\{6\}}$	34
	MSL	$0.2763^{\{3\}}$	$0.4610^{\{4\}}$	$0.6790^{\{4\}}$	$0.2333^{\{8\}}$	$0.1389^{\{8\}}$	$0.3726^{\{8\}}$	$1.0362^{\{6\}}$	$0.1828^{\{5\}}$	$0.4275^{\{5\}}$	34
	KM	$0.2967^{\{7\}}$	$0.5166^{\{7\}}$	$0.7187^{\{7\}}$	$0.1673^{\{3\}}$	$0.0965^{\{4\}}$	$0.3106^{\{4\}}$	$1.0256^{\{5\}}$	$0.1933^{\{7\}}$	$0.4396^{\{7\}}$	33
100	MLE	$0.1477^{\{1\}}$	$0.2792^{\{1\}}$	$0.5284^{\{1\}}$	$0.0777^{\{1\}}$	$0.0458^{\{1\}}$	$0.2141^{\{1\}}$	$0.5142^{\{1\}}$	$0.1122^{\{1\}}$	$0.3350^{\{1\}}$	6
	ADE	$0.2309^{\{2\}}$	$0.3618^{\{2\}}$	$0.6015^{\{2\}}$	$0.1449^{\{2\}}$	$0.0671^{\{2\}}$	$0.2591^{\{2\}}$	$0.8155^{\{2\}}$	$0.1538^{\{2\}}$	$0.3922^{\{2\}}$	12
	CME	$0.2509^{\{3\}}$	$0.3898^{\{6\}}$	$0.6244^{\{6\}}$	$0.1548^{\{3\}}$	$0.0763^{\{3\}}$	$0.2762^{\{3\}}$	$0.8934^{\{3\}}$	$0.1687^{\{5\}}$	$0.4108^{\{5\}}$	23
	LADE	$0.2935^{\{8\}}$	$0.4441^{\{8\}}$	$0.6664^{\{8\}}$	$0.1903^{\{6\}}$	$0.0943^{\{6\}}$	$0.3071^{\{6\}}$	$1.0487^{\{8\}}$	$0.1963^{\{8\}}$	$0.4431^{\{8\}}$	44
	LSE	$0.2531^{\{4\}}$	$0.3825^{\{5\}}$	$0.6184^{\{5\}}$	$0.1713^{\{5\}}$	$0.0824^{\{4\}}$	$0.2870^{\{4\}}$	$0.9172^{\{4\}}$	$0.1696^{\{6\}}$	$0.4118^{\{6\}}$	28
	MSS	$0.2567^{\{6\}}$	$0.3686^{\{3\}}$	$0.6071^{\{3\}}$	$0.2007^{\{7\}}$	$0.0973^{\{7\}}$	$0.3120^{\{7\}}$	$0.9473^{\{6\}}$	$0.1635^{\{4\}}$	$0.4043^{\{4\}}$	33
	MSL	$0.2566^{\{5\}}$	$0.3687^{\{4\}}$	$0.6072^{\{4\}}$	$0.2009^{\{8\}}$	$0.0975^{\{8\}}$	$0.3122^{\{8\}}$	$0.9472^{\{5\}}$	$0.1634^{\{3\}}$	$0.4043^{\{3\}}$	33
	KM	$0.2698^{\{7\}}$	$0.4180^{\{7\}}$	$0.6466^{\{7\}}$	$0.1662^{\{4\}}$	$0.0825^{\{5\}}$	$0.2873^{\{5\}}$	$0.9634^{\{7\}}$	$0.1805^{\{7\}}$	$0.4249^{\{7\}}$	37
300	MLE	$0.0737^{\{1\}}$	$0.1757^{\{1\}}$	$0.4192^{\{1\}}$	$0.0506^{\{1\}}$	$0.0249^{\{1\}}$	$0.1578^{\{1\}}$	$0.2635^{\{1\}}$	$0.0700^{\{1\}}$	$0.2645^{\{1\}}$	6
	ADE	$0.1491^{\{2\}}$	$0.2372^{\{2\}}$	$0.4870^{\{2\}}$	$0.1004^{\{2\}}$	$0.0409^{\{2\}}$	$0.2023^{\{2\}}$	$0.5241^{\{2\}}$	$0.1008^{\{2\}}$	$0.3175^{\{2\}}$	12
	CME	$0.1824^{\{3\}}$	$0.2628^{\{3\}}$	$0.5126^{\{3\}}$	$0.1204^{\{3\}}$	$0.0499^{\{3\}}$	$0.2234^{\{3\}}$	$0.6452^{\{3\}}$	$0.1161^{\{3\}}$	$0.3408^{\{3\}}$	18
	LADE	$0.2112^{\{8\}}$	$0.3181^{\{8\}}$	$0.5640^{\{8\}}$	$0.1469^{\{8\}}$	$0.0676^{\{8\}}$	$0.2601^{\{8\}}$	$0.7613^{\{8\}}$	$0.1456^{\{8\}}$	$0.3816^{\{8\}}$	48
	LSE	$0.1860^{\{4\}}$	$0.2652^{\{4\}}$	$0.5150^{\{4\}}$	$0.1280^{\{4\}}$	$0.0523^{\{4\}}$	$0.2287^{\{4\}}$	$0.6630^{\{4\}}$	$0.1181^{\{4\}}$	$0.3436^{\{4\}}$	24
	MSS	$0.1944^{\{6\}}$	$0.2910^{\{6\}}$	$0.5394^{\{6\}}$	$0.1441^{\{7\}}$	$0.0592^{\{7\}}$	$0.2434^{\{7\}}$	$0.7026^{\{6\}}$	$0.1259^{\{6\}}$	$0.3548^{\{6\}}$	38
	MSL	$0.1935^{\{5\}}$	$0.2908^{\{5\}}$	$0.5393^{\{5\}}$	$0.1439^{\{6\}}$	$0.0591^{\{6\}}$	$0.2431^{\{6\}}$	$0.7001^{\{5\}}$	$0.1256^{\{5\}}$	$0.3545^{\{5\}}$	32
	KM	$0.2009^{\{7\}}$	$0.2965^{\{7\}}$	$0.5445^{\{7\}}$	$0.1335^{\{5\}}$	$0.0569^{\{5\}}$	$0.2386^{\{5\}}$	$0.7156^{\{7\}}$	$0.1309^{\{7\}}$	$0.3618^{\{7\}}$	38
500	MLE	$0.0374^{\{1\}}$	$0.1379^{\{1\}}$	$0.3713^{\{1\}}$	$0.0355^{\{1\}}$	$0.0174^{\{1\}}$	$0.1319^{\{1\}}$	$0.1426^{\{1\}}$	$0.0543^{\{1\}}$	$0.2331^{\{1\}}$	6
	ADE	$0.1103^{\{2\}}$	$0.1931^{\{2\}}$	$0.4394^{\{2\}}$	$0.0808^{\{2\}}$	$0.0317^{\{2\}}$	$0.1781^{\{2\}}$	$0.3917^{\{2\}}$	$0.0820^{\{2\}}$	$0.2864^{\{2\}}$	12
	CME	$0.1433^{\{3\}}$	$0.2218^{\{3\}}$	$0.4710^{\{3\}}$	$0.1016^{\{3\}}$	$0.0407^{\{3\}}$	$0.2019^{\{3\}}$	$0.5114^{\{3\}}$	$0.0978^{\{3\}}$	$0.3127^{\{3\}}$	18
	LADE	$0.1833^{\{8\}}$	$0.2889^{\{8\}}$	$0.5375^{\{8\}}$	$0.1328^{\{8\}}$	$0.0611^{\{8\}}$	$0.2472^{\{8\}}$	$0.6659^{\{8\}}$	$0.1332^{\{8\}}$	$0.3650^{\{8\}}$	48
	LSE	$0.1494^{\{4\}}$	$0.2270^{\{4\}}$	$0.4764^{\{4\}}$	$0.1081^{\{4\}}$	$0.0428^{\{4\}}$	$0.2070^{\{4\}}$	$0.5349^{\{4\}}$	$0.1002^{\{4\}}$	$0.3166^{\{4\}}$	24
	MSS	$0.1620^{\{6\}}$	$0.2599^{\{6\}}$	$0.5098^{\{6\}}$	$0.1214^{\{6\}}$	$0.0480^{\{6\}}$	$0.2191^{\{6\}}$	$0.5785^{\{5\}}$	$0.1106^{\{5\}}$	$0.3325^{\{5\}}$	34
	MSL	$0.1620^{\{5\}}$	$0.2599^{\{7\}}$	$0.5098^{\{7\}}$	$0.1214^{\{7\}}$	$0.0480^{\{7\}}$	$0.2191^{\{7\}}$	$0.5785^{\{6\}}$	$0.1106^{\{6\}}$	$0.3325^{\{6\}}$	38
	KM	$0.1689^{\{7\}}$	$0.2514^{\{5\}}$	$0.5014^{\{5\}}$	$0.1182^{\{5\}}$	$0.0477^{\{5\}}$	$0.2184^{\{5\}}$	$0.6031^{\{7\}}$	$0.1120^{\{7\}}$	$0.3347^{\{7\}}$	36

Table 9. The WPGM estimators for $(\lambda, p, \omega) = (2.00, 2.00, 0.95)$.

n	Method	$\widehat{\lambda}$			$\widehat{p}$			$\widehat{\omega}$			$\sum$ Ranks
		RBias	MSE	RMSE	RBias	MSE	RMSE	RBias	MSE	RMSE	
25	MLE	$0.2054^{\{8\}}$	$0.4422^{\{8\}}$	$0.6650^{\{8\}}$	$0.3100^{\{8\}}$	$0.8722^{\{8\}}$	$0.9339^{\{8\}}$	$0.2927^{\{8\}}$	$0.1753^{\{8\}}$	$0.4187^{\{8\}}$	48
	ADE	$0.1198^{\{4\}}$	$0.2524^{\{3\}}$	$0.5024^{\{3\}}$	$0.1769^{\{4\}}$	$0.4434^{\{1\}}$	$0.6659^{\{1\}}$	$0.1820^{\{4\}}$	$0.0966^{\{6\}}$	$0.3108^{\{6\}}$	22
	CME	$0.1284^{\{5\}}$	$0.2814^{\{7\}}$	$0.5305^{\{7\}}$	$0.2467^{\{7\}}$	$0.6855^{\{7\}}$	$0.8279^{\{7\}}$	$0.1961^{\{5\}}$	$0.1007^{\{7\}}$	$0.3173^{\{7\}}$	38
	LADE	$0.0973^{\{1\}}$	$0.2352^{\{1\}}$	$0.4850^{\{1\}}$	$0.2116^{\{5\}}$	$0.5453^{\{5\}}$	$0.7385^{\{5\}}$	$0.1632^{\{2\}}$	$0.0755^{\{1\}}$	$0.2747^{\{1\}}$	15
	LSE	$0.1160^{\{3\}}$	$0.2362^{\{2\}}$	$0.4860^{\{2\}}$	$0.1594^{\{3\}}$	$0.4613^{\{2\}}$	$0.6792^{\{2\}}$	$0.1813^{\{3\}}$	$0.0859^{\{3\}}$	$0.2931^{\{3\}}$	16
	MSS	$0.1414^{\{6\}}$	$0.2690^{\{5\}}$	$0.5186^{\{5\}}$	$0.1028^{\{2\}}$	$0.5151^{\{3\}}$	$0.7177^{\{3\}}$	$0.2084^{\{7\}}$	$0.0925^{\{5\}}$	$0.3041^{\{5\}}$	28
	MSL	$0.1416^{\{7\}}$	$0.2695^{\{6\}}$	$0.5191^{\{6\}}$	$0.1022^{\{1\}}$	$0.5153^{\{4\}}$	$0.7178^{\{4\}}$	$0.2083^{\{6\}}$	$0.0923^{\{4\}}$	$0.3039^{\{4\}}$	28
	KM	$0.1040^{\{2\}}$	$0.2525^{\{4\}}$	$0.5025^{\{4\}}$	$0.2282^{\{6\}}$	$0.6087^{\{6\}}$	$0.7802^{\{6\}}$	$0.1616^{\{1\}}$	$0.0815^{\{2\}}$	$0.2856^{\{2\}}$	21
50	MLE	$0.1881^{\{8\}}$	$0.3942^{\{8\}}$	$0.6279^{\{8\}}$	$0.2349^{\{8\}}$	$0.5543^{\{8\}}$	$0.7445^{\{8\}}$	$0.2664^{\{8\}}$	$0.1689^{\{8\}}$	$0.4110^{\{8\}}$	48
	ADE	$0.1177^{\{2\}}$	$0.2370^{\{5\}}$	$0.4868^{\{5\}}$	$0.1504^{\{3\}}$	$0.3247^{\{4\}}$	$0.5698^{\{4\}}$	$0.1721^{\{2\}}$	$0.0985^{\{5\}}$	$0.3138^{\{5\}}$	21
	CME	$0.1325^{\{7\}}$	$0.2652^{\{7\}}$	$0.5150^{\{7\}}$	$0.1952^{\{7\}}$	$0.4173^{\{6\}}$	$0.6460^{\{6\}}$	$0.1959^{\{7\}}$	$0.1097^{\{7\}}$	$0.3313^{\{7\}}$	41
	LADE	$0.1060^{\{1\}}$	$0.2029^{\{1\}}$	$0.4505^{\{1\}}$	$0.1738^{\{5\}}$	$0.3601^{\{5\}}$	$0.6001^{\{5\}}$	$0.1624^{\{1\}}$	$0.0825^{\{1\}}$	$0.2873^{\{1\}}$	14
	LSE	$0.1223^{\{4\}}$	$0.2338^{\{4\}}$	$0.4835^{\{4\}}$	$0.1516^{\{4\}}$	$0.3239^{\{3\}}$	$0.5691^{\{3\}}$	$0.1833^{\{4\}}$	$0.0960^{\{4\}}$	$0.3098^{\{4\}}$	23
	MSS	$0.1260^{\{5\}}$	$0.2312^{\{2\}}$	$0.4809^{\{2\}}$	$0.0854^{\{2\}}$	$0.2966^{\{2\}}$	$0.5446^{\{2\}}$	$0.1844^{\{5\}}$	$0.0875^{\{3\}}$	$0.2958^{\{3\}}$	19
	MSL	$0.1261^{\{6\}}$	$0.2314^{\{3\}}$	$0.4810^{\{3\}}$	$0.0850^{\{1\}}$	$0.2963^{\{1\}}$	$0.5444^{\{1\}}$	$0.1844^{\{6\}}$	$0.0875^{\{2\}}$	$0.2957^{\{2\}}$	19
	KM	$0.1222^{\{3\}}$	$0.2536^{\{6\}}$	$0.5036^{\{6\}}$	$0.1950^{\{6\}}$	$0.4281^{\{7\}}$	$0.6543^{\{7\}}$	$0.1813^{\{3\}}$	$0.1008^{\{6\}}$	$0.3175^{\{6\}}$	31
100	MLE	$0.1782^{\{8\}}$	$0.3777^{\{8\}}$	$0.6146^{\{8\}}$	$0.1995^{\{8\}}$	$0.4466^{\{8\}}$	$0.6683^{\{8\}}$	$0.2562^{\{8\}}$	$0.1746^{\{8\}}$	$0.4179^{\{8\}}$	48
	ADE	$0.1078^{\{3\}}$	$0.2251^{\{5\}}$	$0.4745^{\{5\}}$	$0.1296^{\{3\}}$	$0.2700^{\{4\}}$	$0.5196^{\{4\}}$	$0.1589^{\{3\}}$	$0.0993^{\{5\}}$	$0.3151^{\{5\}}$	23
	CME	$0.1225^{\{7\}}$	$0.2467^{\{7\}}$	$0.4966^{\{7\}}$	$0.1647^{\{7\}}$	$0.3372^{\{7\}}$	$0.5807^{\{7\}}$	$0.1812^{\{7\}}$	$0.1100^{\{7\}}$	$0.3317^{\{7\}}$	42
	LADE	$0.0869^{\{1\}}$	$0.1673^{\{1\}}$	$0.4090^{\{1\}}$	$0.1348^{\{4\}}$	$0.2485^{\{3\}}$	$0.4985^{\{3\}}$	$0.1338^{\{1\}}$	$0.0723^{\{1\}}$	$0.2689^{\{1\}}$	11
	LSE	$0.1167^{\{6\}}$	$0.2303^{\{6\}}$	$0.4799^{\{6\}}$	$0.1411^{\{5\}}$	$0.2892^{\{5\}}$	$0.5378^{\{5\}}$	$0.1741^{\{6\}}$	$0.1036^{\{6\}}$	$0.3219^{\{6\}}$	34
	MSS	$0.1084^{\{4\}}$	$0.1974^{\{2\}}$	$0.4443^{\{2\}}$	$0.0847^{\{1\}}$	$0.2292^{\{1\}}$	$0.4787^{\{1\}}$	$0.1618^{\{4\}}$	$0.0819^{\{2\}}$	$0.2862^{\{2\}}$	14
	MSL	$0.1089^{\{5\}}$	$0.1980^{\{3\}}$	$0.4449^{\{3\}}$	$0.0851^{\{2\}}$	$0.2294^{\{2\}}$	$0.4789^{\{2\}}$	$0.1626^{\{5\}}$	$0.0823^{\{3\}}$	$0.2869^{\{3\}}$	20
	KM	$0.1066^{\{2\}}$	$0.2165^{\{4\}}$	$0.4653^{\{4\}}$	$0.1506^{\{6\}}$	$0.2974^{\{6\}}$	$0.5454^{\{6\}}$	$0.1587^{\{2\}}$	$0.0950^{\{4\}}$	$0.3083^{\{4\}}$	24
300	MLE	$0.1600^{\{8\}}$	$0.3717^{\{8\}}$	$0.6097^{\{8\}}$	$0.1518^{\{7\}}$	$0.3260^{\{8\}}$	$0.5710^{\{8\}}$	$0.2324^{\{8\}}$	$0.1778^{\{8\}}$	$0.4216^{\{8\}}$	47
	ADE	$0.1209^{\{5\}}$	$0.2686^{\{5\}}$	$0.5183^{\{5\}}$	$0.1225^{\{3\}}$	$0.2515^{\{3\}}$	$0.5015^{\{3\}}$	$0.1755^{\{5\}}$	$0.1270^{\{5\}}$	$0.3563^{\{5\}}$	26
	CME	$0.1359^{\{7\}}$	$0.2846^{\{7\}}$	$0.5335^{\{7\}}$	$0.1518^{\{8\}}$	$0.3098^{\{7\}}$	$0.5566^{\{7\}}$	$0.1975^{\{7\}}$	$0.1338^{\{7\}}$	$0.3658^{\{7\}}$	43
	LADE	$0.1123^{\{3\}}$	$0.2298^{\{3\}}$	$0.4794^{\{3\}}$	$0.1323^{\{4\}}$	$0.2617^{\{4\}}$	$0.5115^{\{4\}}$	$0.1648^{\{3\}}$	$0.1093^{\{3\}}$	$0.3306^{\{3\}}$	20
	LSE	$0.1335^{\{6\}}$	$0.2768^{\{6\}}$	$0.5261^{\{6\}}$	$0.1436^{\{6\}}$	$0.2912^{\{6\}}$	$0.5396^{\{6\}}$	$0.1944^{\{6\}}$	$0.1305^{\{6\}}$	$0.3613^{\{6\}}$	36
	MSS	$0.0991^{\{1\}}$	$0.1909^{\{1\}}$	$0.4369^{\{1\}}$	$0.0831^{\{1\}}$	$0.1726^{\{2\}}$	$0.4155^{\{2\}}$	$0.1462^{\{1\}}$	$0.0867^{\{1\}}$	$0.2945^{\{1\}}$	7
	MSL	$0.0991^{\{2\}}$	$0.1909^{\{2\}}$	$0.4369^{\{2\}}$	$0.0831^{\{2\}}$	$0.1726^{\{1\}}$	$0.4154^{\{1\}}$	$0.1462^{\{2\}}$	$0.0867^{\{2\}}$	$0.2945^{\{2\}}$	11
	KM	$0.1188^{\{4\}}$	$0.2489^{\{4\}}$	$0.4989^{\{4\}}$	$0.1370^{\{5\}}$	$0.2794^{\{5\}}$	$0.5286^{\{5\}}$	$0.1728^{\{4\}}$	$0.1162^{\{4\}}$	$0.3409^{\{4\}}$	26
500	MLE	$0.1523^{\{8\}}$	$0.3718^{\{8\}}$	$0.6098^{\{8\}}$	$0.1379^{\{7\}}$	$0.2961^{\{8\}}$	$0.5441^{\{8\}}$	$0.2250^{\{8\}}$	$0.1808^{\{8\}}$	$0.4252^{\{8\}}$	47
	ADE	$0.1253^{\{6\}}$	$0.3015^{\{7\}}$	$0.5491^{\{7\}}$	$0.1182^{\{3\}}$	$0.2496^{\{3\}}$	$0.4996^{\{3\}}$	$0.1856^{\{6\}}$	$0.1463^{\{7\}}$	$0.3824^{\{7\}}$	32
	CME	$0.1326^{\{7\}}$	$0.3013^{\{6\}}$	$0.5489^{\{6\}}$	$0.1398^{\{8\}}$	$0.2946^{\{7\}}$	$0.5428^{\{7\}}$	$0.1957^{\{7\}}$	$0.1433^{\{6\}}$	$0.3785^{\{6\}}$	41
	LADE	$0.1207^{\{4\}}$	$0.2647^{\{4\}}$	$0.5145^{\{4\}}$	$0.1329^{\{6\}}$	$0.2708^{\{6\}}$	$0.5203^{\{6\}}$	$0.1794^{\{4\}}$	$0.1277^{\{4\}}$	$0.3573^{\{4\}}$	28
	LSE	$0.1217^{\{5\}}$	$0.2712^{\{5\}}$	$0.5207^{\{5\}}$	$0.1267^{\{5\}}$	$0.2650^{\{5\}}$	$0.5148^{\{5\}}$	$0.1800^{\{5\}}$	$0.1289^{\{5\}}$	$0.3591^{\{5\}}$	30
	MSS	$0.0963^{\{1\}}$	$0.2041^{\{1\}}$	$0.4518^{\{1\}}$	$0.0819^{\{1\}}$	$0.1673^{\{2\}}$	$0.4090^{\{2\}}$	$0.1453^{\{1\}}$	$0.0975^{\{1\}}$	$0.3123^{\{1\}}$	7
	MSL	$0.0963^{\{2\}}$	$0.2041^{\{2\}}$	$0.4518^{\{2\}}$	$0.0819^{\{2\}}$	$0.1673^{\{1\}}$	$0.4090^{\{1\}}$	$0.1453^{\{2\}}$	$0.0975^{\{2\}}$	$0.3123^{\{2\}}$	11
	KM	$0.1139^{\{3\}}$	$0.2507^{\{3\}}$	$0.5007^{\{3\}}$	$0.1264^{\{4\}}$	$0.2595^{\{4\}}$	$0.5094^{\{4\}}$	$0.1685^{\{3\}}$	$0.1185^{\{3\}}$	$0.3442^{\{3\}}$	20

Table 10. Partial and overall ranks for the eight WPGM estimation methods.

Parameter (λ, p, ω)	n	MLE	ADE	CME	LADE	LSE	MSS	MSL	KM
$(0.50, 0.80, 0.05)$	25	6	29	16	39	36	28	28	34
	50	6	21	13	42	32	37	33	32
	100	6	18	12	44	26	41	35	34
	300	6	12	18	36	24	35	41	44
	500	6	12	18	30	24	46	44	36
$(0.70, 1.30, 0.55)$	25	12	20	16	43	24	35	35	31
	50	8	15	16	40	22	42	42	31
	100	10	13	16	38	23	41	47	28
	300	12	10	20	38	25	42	46	23
	500	10	10	20	35	26	43	47	25
$(1.00, 1.70, 0.80)$	25	48	23	42	13	19	17	22	32
	50	48	22	42	10	31	11	17	35
	100	44	24	45	12	31	15	9	36
	300	33	24	42	14	34	8	14	47
	500	28	21	46	15	36	10	16	44
$(1.20, 0.90, 0.75)$	25	48	23	39	9	17	25	24	31
	50	48	24	40	13	27	11	16	37
	100	48	24	40	10	30	11	15	38
	300	35	24	44	18	33	6	12	44
	500	32	17	41	12	35	12	19	48
$(1.80, 1.30, 0.35)$	25	8	28	24	42	29	26	24	35
	50	6	22	21	44	22	34	34	33
	100	6	12	23	44	28	33	33	37
	300	6	12	18	48	24	38	32	38
	500	6	12	18	48	24	34	38	36
$(2.00, 2.00, 0.95)$	25	48	22	38	15	16	28	28	21
	50	48	21	41	14	23	19	19	31
	100	48	23	42	11	34	14	20	24
	300	47	26	43	20	36	7	11	26
	500	47	32	41	28	30	7	11	20
$\sum$ Ranks		764	596	895	825	821	756	812	1011
Overall Rank		3	1	7	6	5	2	4	8

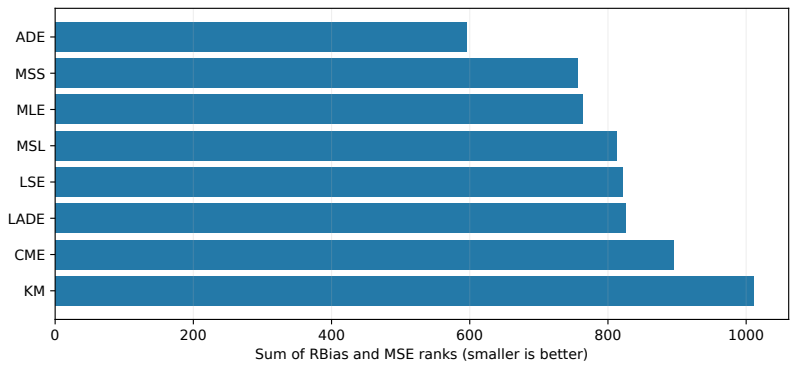

Figure 6. Overall ranks based on RBias and MSE across the specified simulation grid.

The graphical summary in Figure 6 agrees with the ordering in Table 10.

6. Real Data Analysis

We illustrate the WPGM model using two published samples: recidivism failure times and electrical-appliance failure-cycle measurements. Each sample is analyzed in its original units. We fit the WPGM and ten established distributions to the same observations and compare their in-sample fit. The comparisons are specific to these two samples and the candidate models considered. Predictive performance on independent observations was not assessed. Table 11 gives the densities, parameterizations, and sources of the ten competing models.

Table 11. WPGM and the ten competing models.

Code	Distribution and parameters	Density $f(x)$	Source
WPGM	Weibull–Power Garhy mixture (λ, p, ω)	$p\lambda x^{p-1}\, e^{-z}\left[(1-\omega)+\omega\,\dfrac{\lambda^3 z+3z^2}{6+\lambda^3}\right]$	Present work
WD	Weibull (λ, p)	$p\lambda x^{p-1}\, e^{-\lambda x^p}$	[47]
GaD	Gamma (α, β)	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\beta x}$	[27]
LND	Lognormal (μ, σ)	$\dfrac{1}{x\sigma\sqrt{2\pi}}\, \exp\left[-\dfrac{(\log x-\mu)^2}{2\sigma^2}\right]$	[27]
LLD	Log-logistic (α, s)	$\dfrac{\alpha}{s}\, \dfrac{(x/s)^{\alpha-1}}{[\,1+(x/s)^\alpha\,]^2}$	[14]
NWGD	New Weibull–Gamma (β, α, θ)	$\dfrac{\theta}{\theta+\Gamma(\alpha)}\left[\beta\theta(\theta x)^{\beta-1} e^{-(\theta x)^\beta}+(\theta x)^{\alpha-1} e^{-\theta x}\right]$	[44]
PGD	Power Garhy (λ, p)	$\dfrac{p\lambda x^{p-1}\, e^{-z}}{6+\lambda^3}\,(\lambda^3 z+3z^2)$	[19]
GRD	Garhy (λ)	$\dfrac{\lambda^3 x(\lambda^2+3x)}{6+\lambda^3}\, e^{-\lambda x}$	[13]
LD	Lindley (θ)	$\dfrac{\theta^2}{1+\theta}\,(1+x)\, e^{-\theta x}$	[16]
AkD	Akash (θ)	$\dfrac{\theta^3}{\theta^2+2}\,(1+x^2)\, e^{-\theta x}$	[40]
XgD	xgamma (θ)	$\dfrac{\theta^2}{1+\theta}\left(1+\dfrac{\theta x^2}{2}\right) e^{-\theta x}$	[39]

Several candidates in Table 11 provide direct comparisons with the WPGM construction. The WD and PGD models are its two-parameter boundary cases at $\omega = 0$ and $\omega = 1$, respectively. The GRD further fixes $p = 1$ in PGD. The NWGD provides a comparison with a different three-parameter mixture. In the recidivism data, WPGM had a lower AIC than WD, PGD, and NWGD by 2.89, 11.95, and 7.74, respectively. The corresponding differences for the electrical-appliance data were 2.15, 8.65, and 3.94 (Tables 13 and 15). Across both samples, the improvement in AIC was smallest relative to the Weibull boundary model. Unrestricted finite mixtures were discussed in the Introduction and were not fitted in these applications.

For each distribution, the unknown parameters are estimated by MLE under the corresponding parameter constraints, and the associated standard errors (SEs) are based on the observed information at the reported estimates.For the WPGM parameters, we report approximate 95% Wald confidence intervals calculated on the log scale for λ and p and on the logit scale for ω. We obtained the transformed standard errors by the delta method and back-transformed the interval endpoints to the original parameter scales. We assess the fitted models by their maximized log-likelihood, Akaike information criterion (AIC) [2], small-sample corrected AIC (AICc), Hannan Quinn information criterion (HQIC), Bayesian Information Criterion (BIC) to assess the effect of a stronger penalty for model complexity, and average scaled absolute error (ASAE). For a model with k free parameters, the information criteria are calculated as:

$$\mathrm{AIC} = -2\ell(\widehat{\boldsymbol{\vartheta}}) + 2k, \quad \mathrm{AICc} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1}, \quad \mathrm{HQIC} = -2\ell(\widehat{\boldsymbol{\vartheta}}) + 2k\log\log n, \quad \mathrm{BIC} = -2\ell(\widehat{\boldsymbol{\vartheta}}) + k\log n.$$

Following the quantile-based criterion of Castillo and Hadi [9], the ASAE is defined by

$$\mathrm{ASAE} = \frac{1}{n\big(x_{(n)} - x_{(1)}\big)} \sum_{i=1}^{n} \left| x_{(i)} - Q\left(\frac{i}{n+1}; \widehat{\boldsymbol{\vartheta}}\right)\right|,$$

where $x_{(i)}$ denotes the ith order statistic and $Q(\cdot; \widehat{\boldsymbol{\vartheta}})$ is the estimated quantile function, computed numerically when necessary. Lower values of AIC, AICc, HQIC, and ASAE indicate a preferable fit under the respective criteria.

The Kolmogorov Smirnov (D_n), Cramér von Mises (W^*), and Anderson Darling (A^*) statistics are also calculated from the fitted distribution. With $u_i = F(x_{(i)}; \widehat{\boldsymbol{\vartheta}})$, their definitions are

$$\begin{aligned} D_n &= \max_{1\le i\le n} \left\{ \frac{i}{n} - u_i,\ u_i - \frac{i-1}{n} \right\}, \\ W^* &= \frac{1}{12n} + \sum_{i=1}^{n} \left(u_i - \frac{2i-1}{2n} \right)^2, \\ A^* &= -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log u_i + \log\big(1 - u_{n+1-i}\big) \right\}. \end{aligned}$$

The p-values supplied with these statistics.

6.1. Recidivism failure-time data

The first application concerns 61 recidivism failure times, measured in days, from the data associated with the correctional-program analysis of Stollmack and Harris [43]. The observations range from 1 to 556 days and are recorded {1, 6, 9, 29, 30, 34, 39, 41, 44, 45, 49, 56, 84, 89, 91, 100, 103, 104, 115, 119, 124, 138, 141, 146, 156, 162, 168, 183, 185, 198, 209, 217, 217, 228, 233, 238, 241, 252, 258, 271, 275, 276, 279, 282, 305, 313, 329, 331, 334, 336, 336, 362, 384, 404, 408, 422, 438, 441, 465, 486, 556}. This sample represents time-to-event behavior in a social-science setting and therefore provides a different form of heterogeneity from the engineering lifetime data considered in the second application.

The MLE and their SEs for all eleven models are given in Table 12. The WPGM estimates are $\widehat{\lambda} = 0.0033$, $\widehat{p} = 1.2051$, and $\widehat{\omega} = 0.6130$. The approximate 95% confidence intervals were $(0.00033,\ 0.033)$ for λ, $(0.885,\ 1.641)$ for p, and $(0.307,\ 0.850)$ for ω.

The goodness-of-fit results are reported in Table 13. The WPGM attains the smallest AIC (768.6099), AICc (769.0310), HQIC (771.0917), and ASAE (0.0211) among the eleven distributions. The WPGM also has the smallest Kolmogorov Smirnov (0.0610), Cramér von Mises (0.0418), and Anderson Darling (0.3211) statistics. Its nominal p-values are 0.9769, 0.9251, and 0.9212, respectively. WPGM also had the lowest BIC (774.94), although the two-parameter Weibull model was close (775.72). The difference was larger within the three-parameter comparison, where NWGD had a BIC of 782.69.

Figure 7 summarizes the sample distribution and compares the observed order statistics with the WPGM quantiles. The eleven estimated densities are displayed against the data histogram in Figure 8, while Figure 9 contrasts the fitted distribution functions with the empirical CDF. The probability probability (PP) comparisons in Figure 10 provide a complementary assessment across the probability scale. Together, the figures show that WPGM follows the central mass and overall empirical distribution closely, whereas some of the simpler competitors display more noticeable departures.

6.2. Electrical-appliance failure-cycle data

The second sample contains 60 complete electrical-appliance failure-cycle measurements from the lifetime-data example of Lawless [26, 27]. The observations range from 14 to 9701 cycles as {14, 34, 59, 61, 69, 80, 123, 142, 165, 210, 381, 464, 479, 556, 574, 839, 917, 969, 991, 1064, 1088, 1091, 1174, 1270, 1275, 1355, 1397, 1477, 1578, 1649, 1702, 1893, 1932, 2001, 2161, 2292, 2326, 2337, 2628, 2785, 2811, 2886, 2993, 3122, 3248, 3715, 3790,

Table 12. MLEs and SEs of the distributions for the recidivism failure-time data.

Model	Parameter	MLE	SE
WPGM	λ	0.0033	0.0039
	p	1.2051	0.1899
	ω	0.6130	0.1541
WD	λ	0.0006	0.0005
	p	1.3785	0.1477
GaD	α	1.4340	0.2356
	β	0.0068	0.0013
LND	μ	4.9677	0.1469
	σ	1.1477	0.1039
LLD	α	1.7433	0.1896
	s	170.2390	21.4130
NWGD	β	0.3629	0.3746
	α	1.5921	0.3508
	θ	0.0075	0.0018
PGD	λ	0.0909	0.0343
	p	0.6641	0.0676
GRD	λ	0.0142	0.0010
LD	θ	0.0094	0.0009
AkD	θ	0.0142	0.0010
XgD	θ	0.0138	0.0010

Table 13. Log-likelihood, information criteria, ASAE and goodness-of-fit measures for the recidivism data.

Model	$\ell(\hat{\boldsymbol{\theta}})$	AIC	AICc	HQIC	BIC	ASAE	D_n	$p_D^\dagger$	W^*	$p_W^\dagger$	A^*	$p_A^\dagger$
WPGM	**-381.3050**	**768.6099**	**769.0310**	**771.0917**	**774.94**	**0.0211**	**0.0610**	**0.9769**	**0.0418**	**0.9251**	**0.3211**	**0.9212**
WD	-383.7480	771.4959	771.7028	773.1505	775.72	0.0352	0.0960	0.6278	0.1116	0.5323	0.7425	0.5237
GaD	-385.4857	774.9714	775.1783	776.6259	779.19	0.0510	0.1137	0.4093	0.1805	0.3092	1.0460	0.3337
LD	-386.8943	775.7886	775.8564	776.6158	777.90	0.0402	0.0991	0.5871	0.1466	0.4009	1.2848	0.2373
NWGD	-385.1769	776.3538	776.7749	778.8356	782.69	0.0460	0.1069	0.4880	0.1478	0.3971	0.9180	0.4029
PGD	-388.2817	780.5634	780.7703	782.2180	784.79	0.0604	0.1200	0.3429	0.2096	0.2498	1.3270	0.2237
XgD	-390.3877	782.7754	782.8432	783.6027	784.89	0.0484	0.1461	0.1480	0.2542	0.1831	3.0188	0.0269
LLD	-392.8825	789.7650	789.9719	791.4196	793.99	0.1373	0.1220	0.3241	0.1936	0.2807	1.6434	0.1458
AkD	-397.4525	796.9050	796.9728	797.7323	799.02	0.0525	0.1502	0.1276	0.2909	0.1434	3.8548	0.0103
GRD	-398.1869	798.3737	798.4415	799.2010	800.48	0.0525	0.1502	0.1274	0.2910	0.1432	3.8886	0.0100
LND	-397.9891	799.9783	800.1851	801.6328	804.20	0.1581	0.1417	0.1727	0.4468	0.0542	2.6560	0.0413

3857, 3912, 4100, 4106, 4116, 4315, 4510, 4584, 5267, 5299, 5583, 6065, 9701}, exhibiting substantially greater spread and a longer upper tail than the recidivism measurements.

Table 14 lists the estimated parameters and SEs. The WPGM estimates are $\widehat{\lambda} = 0.0033$, $\widehat{p} = 0.8542$, and $\widehat{\omega} = 0.6305$, reflecting a different power and tail pattern from the first sample. The approximate 95% confidence intervals were $(0.00039,\ 0.028)$ for λ, $(0.642,\ 1.137)$ for p, and $(0.334,\ 0.853)$ for ω.

As shown in Table 15, WPGM achieves the smallest AIC (1045.0179), AICc (1045.4465), HQIC (1047.4756), and ASAE (0.0174). WPGM also provides the lowest values of $D_n = 0.0638$, $W^* = 0.0405$, and $A^* = 0.3586$ among the eleven candidates. WPGM had the lowest BIC (1051.30), although the two-parameter Weibull (1051.35) and gamma (1051.55) models gave similar values. Among the two three-parameter candidates, the difference was larger, with BIC values of 1051.30 for WPGM and 1055.24 for NWGD. The corresponding nominal p-values are 0.9673, 0.9320, and 0.8882, respectively.

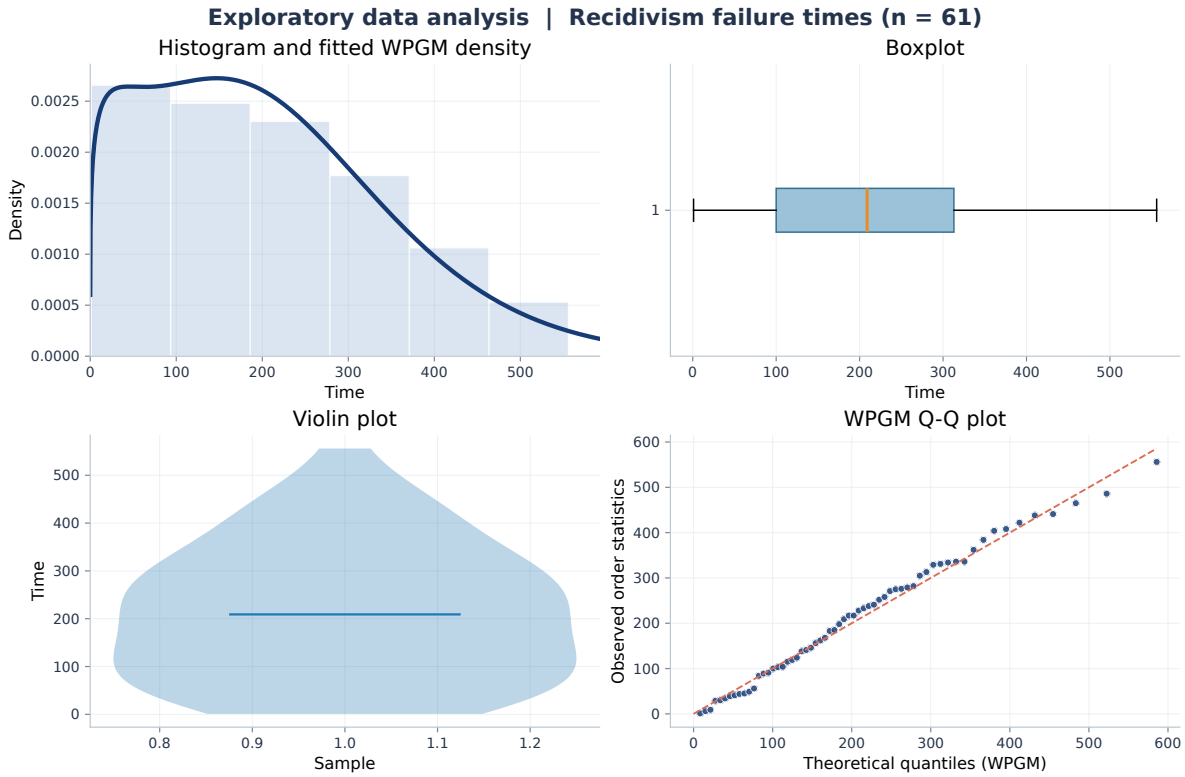

Figure 7. Exploratory plots for the recidivism failure-time data.

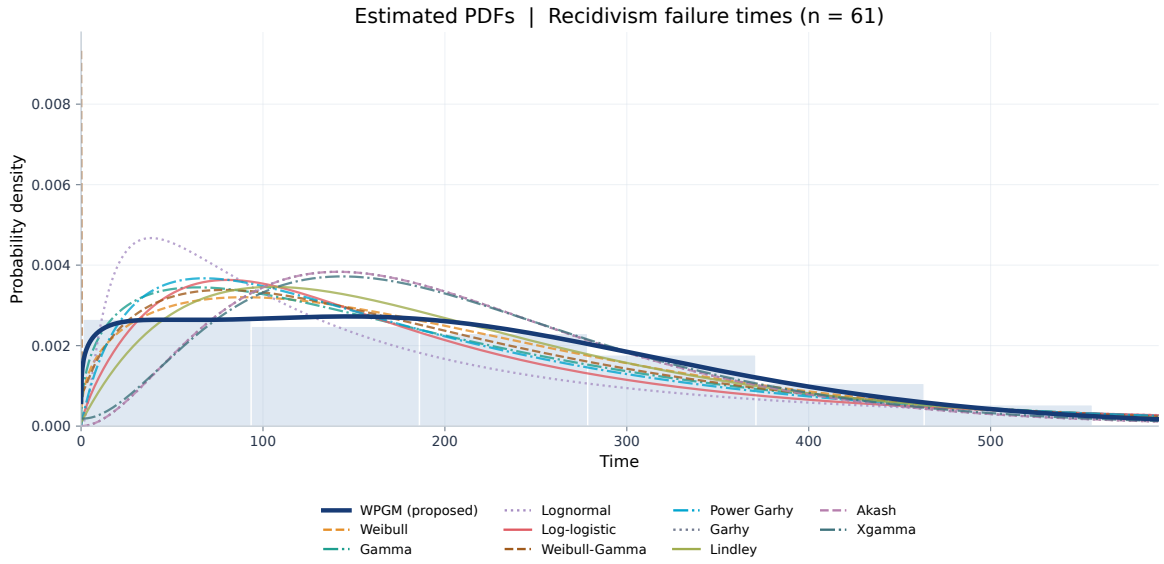

Figure 8. Estimated PDF for WPGM and the ten competing models, superimposed on the recidivism data histogram.

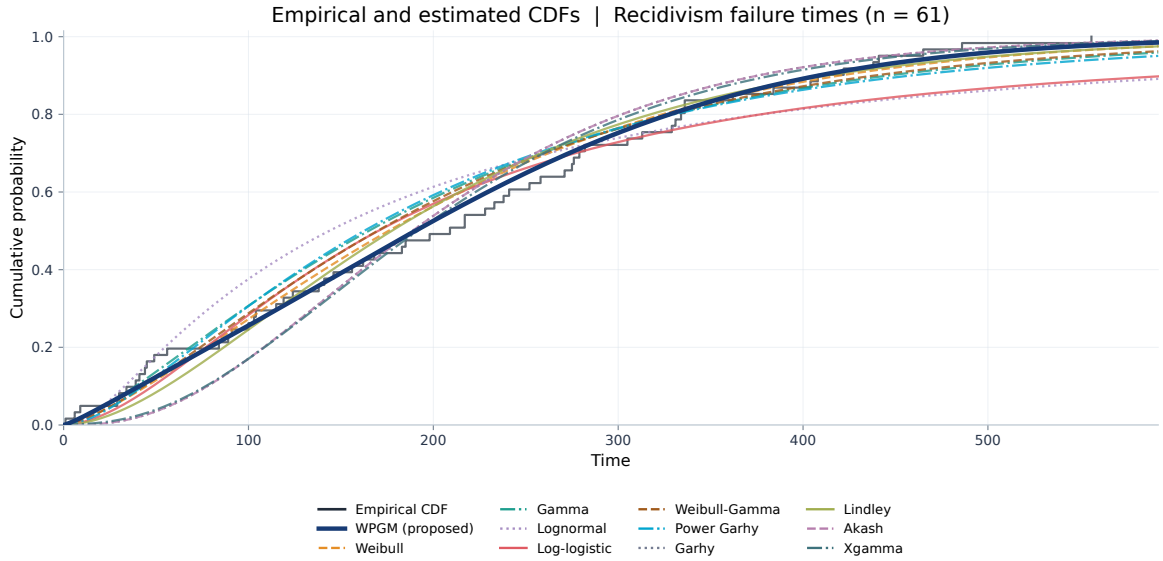

Figure 9. Empirical CDF and the eleven fitted CDFs for the recidivism data.

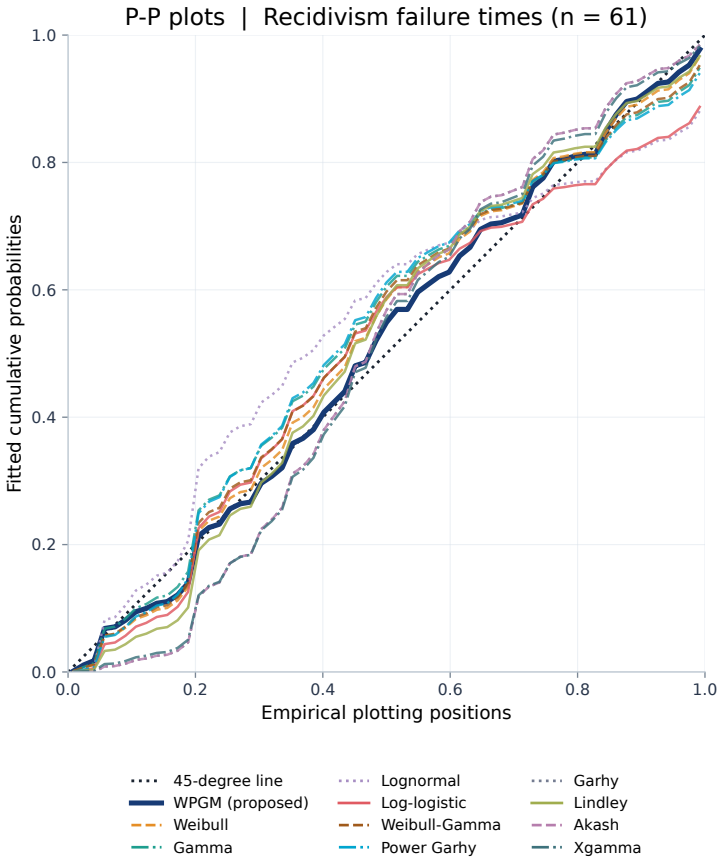

Figure 10. P P plots of the eleven fitted distributions for the recidivism data.

Table 14. MLEs and SEs of the eleven fitted distributions for the electrical-appliance failure-cycle data.

Model	Parameter	MLE	SE
WPGM	λ	0.0033	0.0036
	p	0.8542	0.1245
	ω	0.6305	0.1453
WD	λ	0.0005	0.0004
	p	1.0009	0.1066
GaD	α	0.9307	0.1487
	β	0.0004	0.0001
LND	μ	7.0675	0.1858
	σ	1.4392	0.1314
LLD	α	1.2920	0.1427
	s	1445.2660	248.7836
NWGD	β	1.9737	21.2596
	α	0.9302	0.1486
	θ	0.0004	0.0001
PGD	λ	0.0768	0.0301
	p	0.4915	0.0496
GRD	λ	0.0014	0.0001
LD	θ	0.0009	0.0001
AkD	θ	0.0014	0.0001
XgD	θ	0.0013	0.0001

Table 15. Log-likelihood, information criteria, ASAE and goodness-of-fit measures for the electrical-appliance data.

Model	$\ell(\hat{\boldsymbol{\theta}})$	AIC	AICc	HQIC	BIC	ASAE	D_n	$p_D^\dagger$	W^*	$p_W^\dagger$	A^*	$p_A^\dagger$
WPGM	**-519.5090**	**1045.0179**	**1045.4465**	**1047.4756**	**1051.30**	**0.0174**	**0.0638**	**0.9673**	**0.0405**	**0.9320**	**0.3586**	**0.8882**
WD	-521.5824	1047.1648	1047.3754	1048.8033	1051.35	0.0268	0.0777	0.8618	0.1014	0.5796	0.7922	0.4861
GaD	-521.6793	1047.3586	1047.5691	1048.9970	1051.55	0.0290	0.0897	0.7198	0.1198	0.4975	0.7761	0.4980
NWGD	-521.4778	1048.9557	1049.3843	1051.4133	1055.24	0.0290	0.0897	0.7201	0.1197	0.4980	0.7753	0.4986
PGD	-524.8328	1053.6656	1053.8761	1055.3040	1057.85	0.0461	0.1096	0.4664	0.2081	0.2527	1.4186	0.1972
LLD	-529.6660	1063.3320	1063.5425	1064.9704	1067.52	0.1339	0.1189	0.3649	0.1942	0.2793	1.8102	0.1173
LND	-531.0312	1066.0624	1066.2730	1067.7009	1070.25	0.1167	0.1654	0.0751	0.4253	0.0617	2.5009	0.0497
LD	-535.7766	1073.5533	1073.6222	1074.3725	1075.65	0.0398	0.1524	0.1232	0.3114	0.1254	5.2094	0.0023
XgD	-556.1387	1114.2774	1114.3463	1115.0966	1116.37	0.0588	0.2057	0.0124	0.6957	0.0128	11.7385	0.0000
AkD	-560.4816	1122.9632	1123.0321	1123.7824	1125.06	0.0597	0.2047	0.0131	0.6819	0.0138	12.3096	0.0000
GRD	-560.4886	1122.9772	1123.0462	1123.7965	1125.07	0.0597	0.2047	0.0131	0.6819	0.0138	12.3105	0.0000

The exploratory diagnostics in Figure 11 highlight the substantial dispersion and positive skewness of the observed failure cycles. The estimated PDFs in Figure 12 and the empirical-versus-fitted CDFs in Figure 13 show the ability of WPGM to accommodate the main features of the sample. Figure 14 further illustrates the agreement of the fitted probabilities with their empirical plotting positions.

Across the two applications, WPGM has the lowest AIC, AICc, HQIC, and ASAE and the smallest three empirical-distribution-function statistics among the eleven selected models. This pattern is consistent with the benefit of combining a Weibull component with a power-transformed Garhy component.

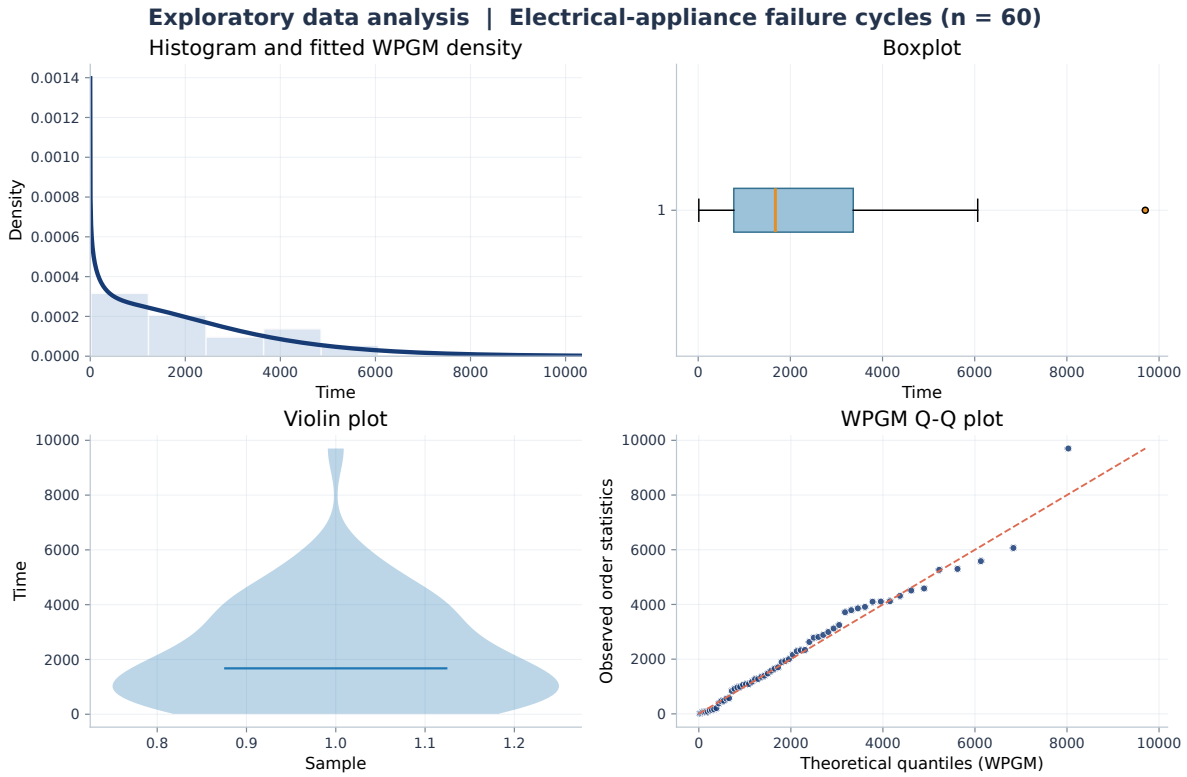

Figure 11. Exploratory plots for the electrical-appliance failure-cycle data.

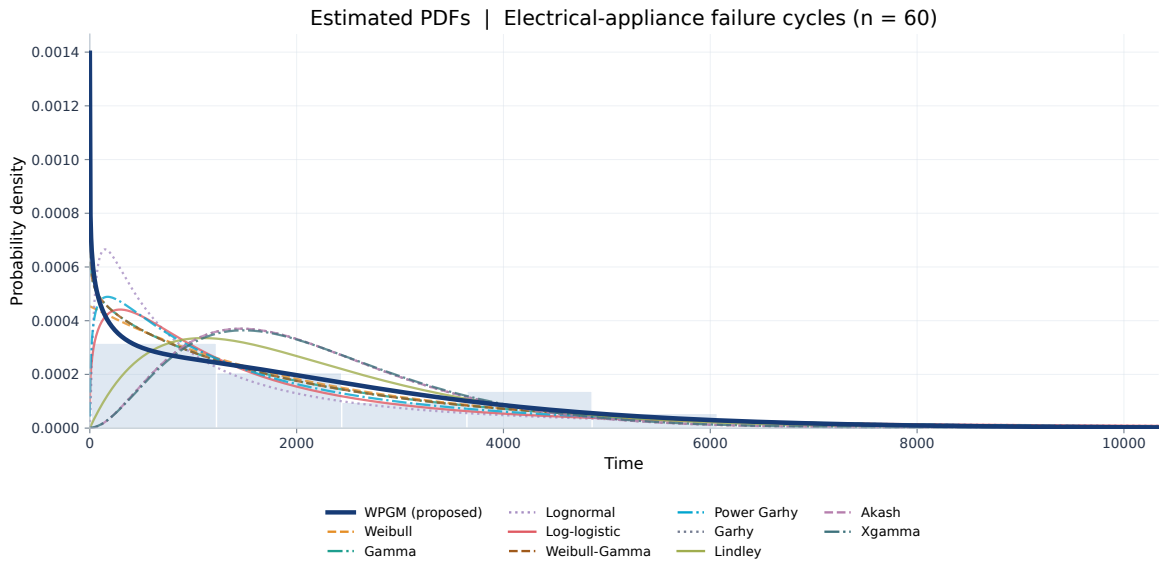

Figure 12. Estimated PDF of WPGM and the ten competing models for the electrical-appliance failure cycles.

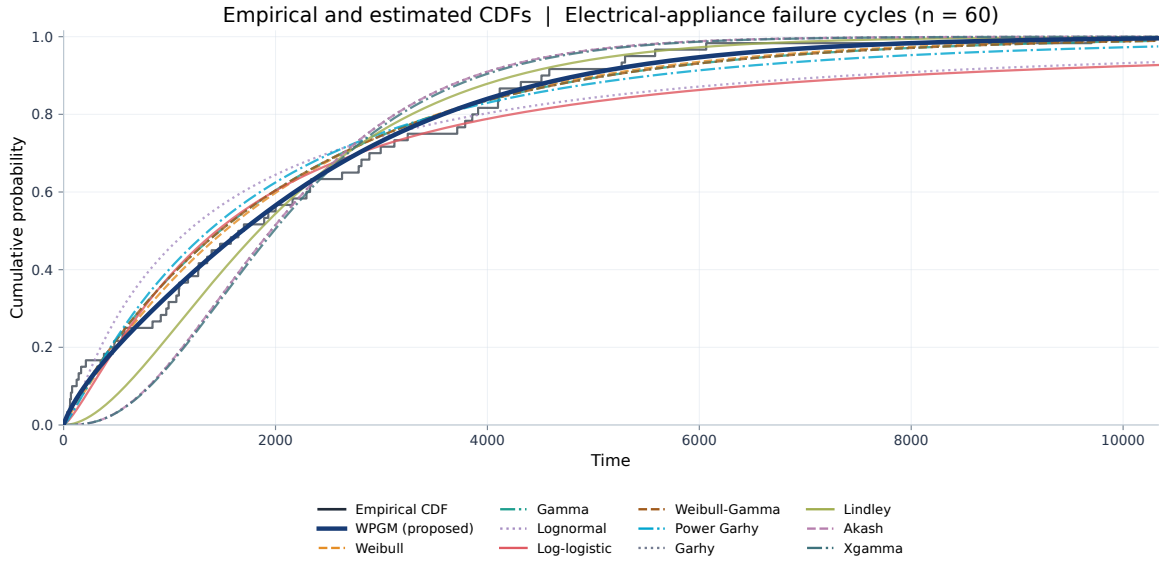

Figure 13. Empirical and fitted CDF of the eleven models for the electrical-appliance failure-cycle data.

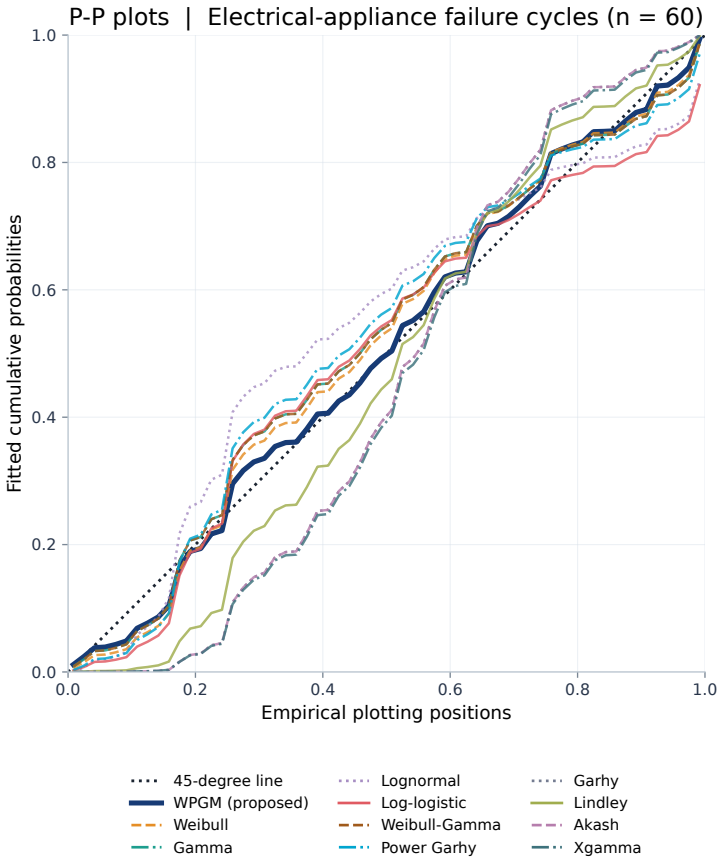

Figure 14. P P plots of the eleven fitted models for the electrical-appliance failure-cycle data.

7. Concluding Remarks

The three-parameter WPGM interpolates between the Weibull and power-transformed Garhy laws through the mixing weight ω. On the transformed scale, ω determines the combined weight of the shape-two and shape-three gamma components, and λ determines how that weight is divided. This restriction gives a direct sampling scheme and yields an elementary CDF and closed-form ordinary and incomplete moments through the gamma-mixture representation. In addition, we derived the quantile function, moment- and quantile-based shape measures, inequality and Bonferroni Lorenz curves, a Gini-type measure, extropy and weighted extropy, order statistics, and the Laplace transform of X^p. These derivations rely only on the corresponding density and distribution functions, so the resulting formulas do not require additional numerical integration beyond their stated domains. Eight estimation criteria (MLE, CME, LADE, ADE, LSE, MSS, MSL, and KM) were formally developed. Estimating equations were derived for all eight methods at interior stationary points, together with explicit treatment of the constrained parameter space $\lambda > 0$, $p > 0$, $0 \leq \omega \leq 1$, and of the boundary behavior of $\widehat{\omega}$. Observed-information expressions were provided for the MLE case, and the standard-error approximation was restricted to interior solutions.
A Monte Carlo experiment with six parameter configurations, five sample sizes $n \in \{25, 50, 100, 300, 500\}$, and $B = 1000$ replications per cell was conducted. The resulting design comprises eight procedures. On average, estimation accuracy improved with sample size across the reported design. In the aggregate ranking, ADE performed best, followed by MSS and MLE. Two published data sets were analysed: recidivism failure times ($n = 61$, social-science setting) and electrical-appliance failure cycles ($n = 60$, engineering setting). The WPGM attained the smallest AIC, AICc, HQIC, and ASAE, together with the smallest Kolmogorov Smirnov, Cramér von Mises, and Anderson Darling statistics, among the eleven candidate models for both data sets. The empirical comparisons describe fit to the two observed samples. The gains may be sample-specific, and predictive performance on independent observations remains untested. Neither application establishes that the fitted WPGM captures a multimodal density or a bathtub-shaped hazard. The eight estimation procedures also optimize different criteria. ADE had the lowest aggregate rank under the RBias and MSE measures used here, but this ranking applies to point estimation over the six parameter configurations and five sample sizes examined.

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