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Statistical Inference for Stress-Strength Reliability of the Induced Ailamujia Distribution under Diverse Record Schemes with Engineering Applications

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Abstract This study examines the estimation of the stress-strength reliability parameter $R = P(Y < X)$ under the conditions of independent stress and strength variables that follow the induced Ailamujia (IAL) distribution. Statistical inference is established based on three significant structures: upper, lower, and mixed record designs. The IAL distribution, a versatile one-parameter lifetime model, demonstrates a superior fit to failure time and automobile fatalities datasets. Classical and Bayesian estimation methodologies are utilized. Maximum likelihood estimators (MLE) are formulated through numerical optimization, whereas Bayesian estimators are developed using Jeffreys and extended Jeffreys priors under squared error loss. A comprehensive Monte Carlo simulation analysis is undertaken to assess the performance of the proposed estimators in terms of bias and mean squared error (MSE) across varying sample sizes and data structures. The simulation results reveal that mixed records generally provide more balanced estimation performance than pure upper and lower records, particularly in terms of bias reduction, by capturing information from both tails of the distribution and mitigating systematic directional bias. Furthermore, MLE demonstrates remarkable robustness and consistency for moderate and large sample sizes, producing small values of bias and MSE across all configurations, while Bayesian methods offer only marginal improvements for very small samples. Consequently, the combination of mixed records with MLE is recommended as a reliable and computationally efficient approach for estimating stress-strength reliability from record-breaking IAL distribution data. The practical significance of the proposed technique is demonstrated through the examination of two real datasets. For the failure time data, the Bayesian approach under upper records yields the most competitive estimates, while for the automobile fatalities dataset, the MLE under mixed records proves superior.

Keywords Stress-Strength Reliability, Induced Ailamujia Distribution, Record Data Schemes, Mixed Records, Monte Carlo Simulation, Lifetime Models

AMS 2010 subject classifications 62N05, 62F10, 62F15

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1. Introduction

In various practical engineering and life testing applications, components, materials, or systems are subjected to stresses that can cause them to fail if the stress exceeds their inherent capacity or strength. For instance, a wooden beam may fracture under an excessive load, an electrical bulb might burn out over time, or an electronic device may malfunction due to extreme operating temperatures. In such reliability analysis and life-testing experiments, failure thresholds are often recorded sequentially, where only values that exceed all previous records are retained.

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This statistical sampling framework is known as record data. Within industrial and manufacturing contexts, an investigator might be interested in determining the minimum failure voltage or the breaking strength of a product through sequential testing by capturing only the minimum or maximum values observed so far. Consequently, statistical inference for stress-strength reliability parameters based on record data schemes carries significant practical weight and vital operational importance in quality control and reliability engineering.

Stress-strength reliability (SSR) refers to the quantitative evaluation of a system's ability to withstand operational demands. In this framework, the independent random variable X denotes the "strength" of the component, while the independent random variable Y represents the "stress" applied to it. While stress reflects the probabilistic load conditions, strength represents the system's innate capability to absorb those loads. Mathematically, the stress-strength reliability parameter is defined as $R = P(Y < X)$, which represents the probability that the system's strength exceeds the applied stress. Conversely, if the stress surpasses the strength, the system fails.

The practical applications of stress-strength models are highly diverse and span across multiple domains:

- Structural and mechanical engineering: where stress may represent the daily vibrations a mechanical part is subjected to, and strength indicates the maximum vibration threshold the part can endure before fracturing.
- Insurance sector: where stress signifies the annual number of claims filed against an insurance firm, and strength represents the maximum financial liability the firm can effectively manage.
- Transportation and production systems: where stress can be defined as the daily breakdown frequency of a transit fleet, and strength reflects the system's capacity for immediate repair and recovery.

Reviewing the statistical literature reveals that various continuous distributions have been proposed to model stress-strength data. For instance, Kundu and Raqab [1] studied the estimation of R for the three-parameter Weibull distribution, while Asgharzadeh et al. [2] tackled estimation based on progressively censored samples for the Weibull distribution. Furthermore, Al-Mutairi et al. [3] provided inferences on reliability using the Lindley distribution, and Ghitany et al. [4] focused on the power Lindley distribution. Studies have expanded to cover other models such as the generalized Lindley distribution (GLD) by Rezaei et al. [5], the negative binomial distribution by Sathe and Dixit [6], the geometric-Poisson distribution by Obradovic et al. [7], and the power series family of distributions by Choudhury et al. [8]. Additionally, Jose [9] investigated the reliability of single and multi-component systems under discrete phase-type distributions, and Amrin et al. [10] addressed the challenges of zero-inflated count data via the zero-inflated Poisson distribution. More recently, SSR inference has been extended to a variety of sampling and censoring designs: Ahmadini et al. [11] studied s-out-of-k systems under inverted Z-Lindley stress-strength models with progressive censoring, Alenezy et al. [12] developed robust SSR inference for the Garhy distribution under diverse record schemes, Elkalzah et al. [13] considered the discrete alpha-power Weibull distribution with incomplete and record data, and Rashedi et al. [14] addressed SSR inference under unified hybrid censoring for a one-parameter model.

Although simple random sampling (SRS) is widely used in statistical studies, alternative sampling schemes based on ordered observations, such as ranked set sampling (RSS) and record sampling, have also received attention in reliability analysis. These approaches provide different frameworks for collecting and analyzing observational data, depending on the sampling structure and the nature of the available information. Despite these advancements, most existing SSR studies rely either on complete samples, which are impractical in destructive or costly life-testing experiments, or on complex multi-parameter lifetime models that lack closed-form reliability expressions and suffer from parameter estimation instability under small samples. While RSS [15–18] offers a cost-effective alternative when visual ordering is feasible, record-breaking data schemes (upper, lower, and mixed records) are uniquely suited for sequential destructive testing where only extreme observations are retained. It should be emphasized that the present study does not aim to establish the economic superiority of record-based sampling over simple random sampling or ranked set sampling. Rather, it investigates the statistical properties and estimation performance of stress-strength reliability procedures when observations are available in the form of upper, lower, and mixed records. Accordingly, the cited literature on ranked set sampling serves as background on related order-based approaches, while a direct comparison of sampling costs and resource requirements is beyond the scope of this work.

In parallel, considerable effort has been devoted to constructing new flexible lifetime and unit models that can better describe real data. Examples include the unit Haq distribution [19], a generalization of the double XRamma

distribution [20], a new unit generated family [21], the Garhy distribution [22], the Marei distribution [23], an extended Lindley model [24], a one-parameter decreasing-hazard lifetime model [25], an odds-based Weibull model [26], the unit exponential Pareto distribution [27], the exponentiated power generalized Weibull power series family [28], and the odd inverse power generalized Weibull generated family [29]. Inferential procedures for such models under censored data have also been studied, for example for step-stress models under the inverted Z-Lindley distribution [30] and for progressively censored Garhy reliability models via Markov chain Monte Carlo and Tierney–Kadane approximations [31]. Recently, the induced Ailamujia (IAL) distribution [32] has emerged as a parsimonious, one-parameter lifetime model capable of accommodating decreasing, unimodal, and right-skewed hazard behaviors. However, to the best of our knowledge, no prior study has investigated the statistical inference of the stress-strength reliability parameter $R = P(Y < X)$ for the IAL distribution under upper, lower, and mixed record designs. Furthermore, the comparative behavior of classical and Bayesian estimators across single-tail (pure upper/lower) versus two-tail (mixed) record structures remains unexplored for this model. To bridge this gap, the primary contributions of this paper are fourfold:

1. **Analytical derivation:** We derive an exact, closed-form expression for the SSR parameter $R = P(Y < X)$ when strength and stress follow independent IAL distributions, avoiding numerical integration errors.
2. **Multi-scheme classical and Bayesian inference:** We develop maximum likelihood estimators (MLE) alongside Bayesian estimators under non-informative Jeffreys and extended Jeffreys priors across upper, lower, and mixed record schemes.
3. **Tail-information and bias evaluation:** Through an extensive Monte Carlo simulation study, we demonstrate how mixed record schemes mitigate the systematic directional bias inherent in pure upper or lower records, particularly in small-to-moderate samples.
4. **Empirical validation:** We validate the practical utility and robustness of the proposed framework using two real-world engineering and safety datasets (industrial failure times [33] and automobile fatalities [34]).

The remainder of this paper is organized as follows. In Section 2, the statistical properties of the IAL distribution are introduced, and the explicit, closed-form expression for the stress-strength reliability parameter $R = P(Y < X)$ is analytically derived in Section 3. Sections 4 and 5 are dedicated to the estimation methodologies, covering both the classical maximum likelihood estimation (MLE) and the Bayesian estimation frameworks using Jeffreys and extended Jeffreys priors under upper, lower, and mixed record schemes. Section 6 presents the design and results of an extensive Monte Carlo simulation study conducted to evaluate and compare the performance of the proposed estimators based on bias and mean squared error (MSE). Section 7 demonstrates the empirical validity of the model through applications to two real-world datasets, followed by concluding remarks and recommendations for future research in Section 8.

2. Induced Ailamujia Distribution

The statistical literature is rich with lifetime models, yet there is a continuous demand for one-parameter distributions that balance mathematical simplicity with high structural flexibility. The IAL distribution satisfies this requirement efficiently. This model is particularly critical in reliability engineering and stress-strength analysis because it provides an explicit, closed-form expression for the reliability parameter $R = P(Y < X)$. This mathematical tractability eliminates the propagation of numerical errors typically found in models requiring iterative approximations for their baseline properties.

The probability density function (PDF) and cumulative distribution function (CDF) of the IAL distribution are given by

$$f(x;\alpha) = \alpha(1+2\alpha x)e^{-2\alpha x}, \qquad x > 0,\ \alpha > 0, \tag{2.1}$$

and

$$F(x;\alpha) = 1 - (1+\alpha x)e^{-2\alpha x}. \tag{2.2}$$

Consequently, the corresponding survival function and hazard rate function (HRF) of the IAL distribution are

$$S(x;\alpha) = 1 - F(x;\alpha) \qquad \text{and} \qquad h(x;\alpha) = \frac{f(x;\alpha)}{S(x;\alpha)}. \tag{2.3}$$

Analytical interpretation of the hazard rate behavior. Unlike the PDF, which can transition between strictly decreasing (for $\alpha \geq 0$ or appropriate scaling) and unimodal right-skewed shapes, the analytical behavior of the hazard rate function $h(x;\alpha)$ in (2.3) is rigorously determined through its boundary limits and first derivative:

Boundary and asymptotic behavior: At the onset of operation ($x \to 0^+$), the failure rate starts at a finite, non-zero baseline level $h(0;\alpha) = \alpha$. As $x \to \infty$, the hazard rate approaches a finite upper asymptote given by $\lim_{x\to\infty} h(x;\alpha) = 2\alpha$.

Monotonicity and concavity: Differentiating $h(x;\alpha)$ with respect to x shows that the IAL distribution exhibits a **strictly increasing, concave (bounded) hazard rate** that doubles from α to 2α over time. Practically, this analytical structure is highly advantageous for modeling wear-out and aging phenomena in mechanical components and systems where failure risk rises steadily during early-to-mid life before stabilizing at a bounded ceiling (2α), avoiding the unrealistic unbounded explosion of failure rates associated with Weibull or Gamma models.

The structural adaptability of the IAL distribution is governed by its functional shapes. The model possesses several attractive properties: it can effectively capture both heavy-tailed and light-tailed data patterns, making it highly sensitive to extreme values. The distribution exhibits various hazard rate shapes, including monotonically decreasing and unimodal shapes, which perfectly mirror the failure mechanisms of industrial components and biological aging processes.

Structural and practical comparison with related lifetime models. To highlight the novelty of employing the IAL distribution in stress-strength reliability analysis, it is instructive to contrast its structural and practical properties with established baseline distributions:

- **Versus the standard Ailamujia and exponential distributions:** While the exponential distribution is restricted to a constant hazard rate and a strictly decreasing density, and the standard Ailamujia distribution ($f(x) = 4\alpha^2 x e^{-2\alpha x}$) forces $f(0) = 0$ with a purely unimodal density, the induced Ailamujia (IAL) distribution incorporates a mixture-type linear term $(1 + 2\alpha x)$ alongside $f(0) = \alpha > 0$. Structurally, this allows the IAL model to transition smoothly between strictly decreasing and unimodal shapes while preserving a single-parameter formulation.
- **Versus the Lindley distribution:** Although the one-parameter Lindley distribution also exhibits a closed-form SSR expression [3], its hazard rate is strictly increasing, limiting its applicability in datasets exhibiting early failures or non-monotone hazard behaviors. In contrast, the IAL distribution captures decreasing and unimodal hazard structures, making it better suited for reliability data with infant mortality or right-skewed outliers (as demonstrated in Section 7).
- **Versus multi-parameter models (e.g., Weibull, Gamma, power Lindley):** Multi-parameter generalizations [1, 2, 4] provide high shape flexibility but suffer from two major practical drawbacks under record-breaking designs: (i) the absence of a tractable, closed-form expression for $R = P(Y < X)$ when shape parameters differ, necessitating numerical integration, and (ii) severe over-parameterization and convergence instability when only small record samples (m, n) are available. The IAL distribution resolves both issues by offering analytical closed-form tractability for R alongside stable parameter estimation even under sparse record data.

As visually demonstrated in Figure 1, the scale parameter α serves as a powerful shape modulator: the probability curves dynamically transition from a strictly decreasing profile to a highly flexible, right-skewed, and unimodal distribution. This geometric adaptability allows the IAL model to efficiently capture diverse tail behaviors, varying peak intensities, and complex hazard rates. Consequently, this visual and mathematical flexibility justifies the superior statistical capacity of the IAL distribution to accurately fit and model the heavy- or light-tailed patterns found in real-world datasets, establishing it as a highly tractable yet robust alternative to traditional lifetime distributions.

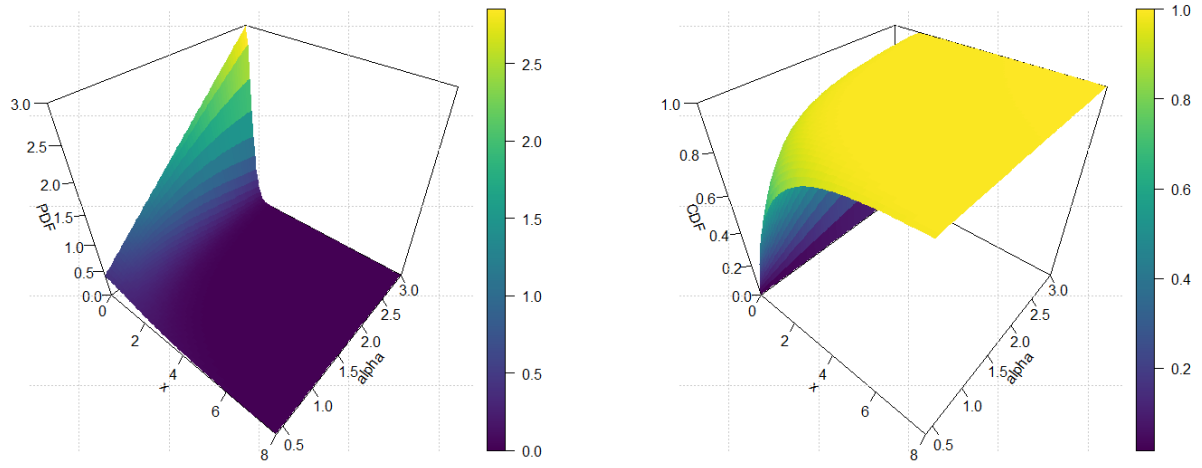


Figure 1. 3D plots of the PDF (left) and CDF (right) for the IAL model.

3. Stress-Strength Reliability for the Induced Ailamujia Distribution

Let $X \sim \mathrm{IAL}(\alpha_1)$ and $Y \sim \mathrm{IAL}(\alpha_2)$ be independent random variables representing strength and stress, respectively. The stress-strength reliability is defined as

$$R = P(Y < X) = \int_0^\infty f_X(x)\, F_Y(x)\, dx. \tag{3.1}$$

Substituting the PDF of X from (2.1) and the CDF of Y from (2.2) into (3.1) gives

$$\begin{aligned} R = P(Y < X) &= \int_0^\infty \alpha_1 \left(1 + 2\alpha_1 x\right) e^{-2\alpha_1 x} \left[1 - \left(1 + \alpha_2 x\right) e^{-2\alpha_2 x}\right] dx \\ &= 1 - \int_0^\infty \alpha_1 \left(1 + 2\alpha_1 x\right) \left(1 + \alpha_2 x\right) e^{-2(\alpha_1 + \alpha_2) x}\, dx. \end{aligned} \tag{3.2}$$

Let $z = 2(\alpha_1 + \alpha_2)x$, so that $x = \frac{z}{2(\alpha_1+\alpha_2)}$ and $dx = \frac{dz}{2(\alpha_1+\alpha_2)}$. By simplifying (3.2), we obtain

$$R = 1 - \frac{\alpha_1}{2(\alpha_1 + \alpha_2)} \left[\frac{3}{2} + \frac{\alpha_1 \alpha_2}{2(\alpha_1 + \alpha_2)^2}\right]. \tag{3.3}$$

One of the most important theoretical contributions is Eq. (3.3), which gives an explicit formula of R in terms of α_1 and α_2 that can be directly calculated for any positive parameter values. This closed-form expression is used throughout the subsequent estimation procedures.

An important assumption in the present framework is the independence between the stress and strength variables. Although this assumption provides a tractable basis for the proposed inference procedures, it may not hold in all practical applications, particularly when both variables are affected by common environmental or operating conditions. Therefore, the applicability of the current framework is limited to independent stress-strength pairs (see [35,36]). Extending the proposed methodology to dependent variables through suitable dependence structures, such as copula-based models, represents an important direction for future research.

4. Maximum Likelihood Estimation

The maximum likelihood method is one of the most popular approaches for parameter estimation due to its nice asymptotic properties, such as consistency, asymptotic normality and efficiency. The choice of MLE in this case stems from its asymptotic efficiency and invariance property, which makes it possible to estimate R directly after the estimation of α_1 and α_2. However, as will be seen, maximum likelihood estimation has convergence problems in the pure record frameworks, thus motivating the Bayesian approach. In this part, we find the maximum likelihood estimates of the parameters α_1 and α_2 on the basis of different types of record data from distinct IAL distributions. Hence, we obtain the maximum likelihood estimate of the stress-strength reliability parameter $R = P(Y < X)$ by the invariance property of the maximum likelihood estimators.

4.1. Record schemes and likelihood construction

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables drawn from a continuous distribution with PDF $f(x;\alpha)$ and CDF $F(x;\alpha)$. An observation X_j is called an upper record if it exceeds all previous observations, i.e., $X_j > \max\{X_1, X_2, \ldots, X_{j-1}\}$. Similarly, X_j is called a lower record if it is smaller than all previous observations, i.e., $X_j < \min\{X_1, X_2, \ldots, X_{j-1}\}$. The indices at which records occur are denoted by $U(1), U(2), \ldots$ for upper records and $L(1), L(2), \ldots$ for lower records.

For a set of m upper record values $\mathbf{X}_U = (x_{U(1)}, x_{U(2)}, \ldots, x_{U(m)})$, the joint likelihood function is given by (Arnold et al. [9])

$$L_U(\alpha) = f(x_{U(m)};\alpha) \prod_{i=1}^{m-1} \frac{f(x_{U(i)};\alpha)}{1 - F(x_{U(i)};\alpha)}. \tag{4.1}$$

Analogously, for a set of m lower record values $\mathbf{X}_L = (x_{L(1)}, x_{L(2)}, \ldots, x_{L(m)})$, the likelihood function is

$$L_L(\alpha) = f(x_{L(m)};\alpha) \prod_{i=1}^{m-1} \frac{f(x_{L(i)};\alpha)}{F(x_{L(i)};\alpha)}. \tag{4.2}$$

In numerous real-world scenarios, one could come across a combination of upper and lower records originating from the same procedure. Let $\mathbf{X}^* = (x_1^*, x_2^*, \ldots, x_k^*)$ represent a series of k record values, each classified as either an upper or a lower record. Let I_U represent the collection of indices associated with upper records, while I_L signifies the collection pertaining to lower records. The likelihood function for mixed records is expressed as follows:

$$L_M(\alpha) = f(x_k^*;\alpha) \prod_{i \in I_U \setminus \{k\}} \frac{f(x_i^*;\alpha)}{F(x_i^*;\alpha)} \prod_{i \in I_L \setminus \{k\}} \frac{f(x_i^*;\alpha)}{1 - F(x_i^*;\alpha)}. \tag{4.3}$$

4.2. MLE for upper records

Consider two independent sequences: $\mathbf{X}_U = (x_{U(1)}, x_{U(2)}, \ldots, x_{U(m)})$ from IAL(α_1) (strength) and $\mathbf{Y}_U = (y_{U(1)}, y_{U(2)}, \ldots, y_{U(n)})$ from IAL(α_2) (stress). Using the PDF and CDF of the IAL distribution given in (2.1) and (2.2), the likelihood function for α_1 based on upper records, obtained from (4.1), is

$$L_U(\alpha_1) = \prod_{i=1}^{m} \alpha_1 (1 + 2\alpha_1 x_{U(i)}) e^{-2\alpha_1 x_{U(i)}} \prod_{i=1}^{m-1} \frac{1}{(1 + \alpha_1 x_{U(i)}) e^{-2\alpha_1 x_{U(i)}}}. \tag{4.4}$$

The associated log-likelihood function is derived by applying natural logarithms:

$$\ln L_U(\alpha_1) = m \log(\alpha_1) + \sum_{i=1}^{m} \log(1 + 2\alpha_1 x_{U(i)}) - 2\alpha_1 \sum_{i=1}^{m} x_{U(i)} - \sum_{i=1}^{m-1} \log(1 + \alpha_1 x_{U(i)}) + 2\alpha_1 \sum_{i=1}^{m-1} x_{U(i)}. \tag{4.5}$$

By symmetry, the corresponding likelihood and log-likelihood functions for the stress parameter α_2 based on $\mathbf{Y}_U$ follow the same functional structure and are explicitly provided in Appendix A.

The MLEs $\hat{\alpha}_{1U}$ and $\hat{\alpha}_{2U}$ are derived by optimizing the log-likelihood functions $\ln L_U(\alpha_1)$ in (4.5) and $\ln L_U(\alpha_2)$ in (A.2), correspondingly. These maximization problems lack closed-form solutions because of the intricate nature of the log-likelihood functions, necessitating numerical resolution. We utilized the Nelder–Mead optimization technique as executed in the R function `optim`, initializing using estimations derived from the method of moments. After getting the maximum likelihood estimates $\hat{\alpha}_{1U}$ and $\hat{\alpha}_{2U}$ for the parameters of the IAL distribution from the observed record samples, the associated estimate of the stress-strength reliability parameter R is calculated by inserting these estimates into the closed-form expression (3.3) established in Section 3. Particularly, by employing the invariance characteristic of maximum likelihood estimators, we obtain

$$\hat{R}_U = 1 - \frac{\hat{\alpha}_{1U}}{2(\hat{\alpha}_{1U} + \hat{\alpha}_{2U})} \left[\frac{3}{2} + \frac{\hat{\alpha}_{1U}\hat{\alpha}_{2U}}{2(\hat{\alpha}_{1U} + \hat{\alpha}_{2U})^2} \right]. \tag{4.6}$$

4.3. MLE for lower records

Let two independent sequences: $\mathbf{X}_L = (x_{L(1)}, x_{L(2)}, \ldots, x_{L(m)})$ from IAL(α_1) (strength) and $\mathbf{Y}_L = (y_{L(1)}, y_{L(2)}, \ldots, y_{L(n)})$ from IAL(α_2) (stress). Employing the PDF and CDF of the IAL distribution as presented in (2.1) and (2.2), the likelihood function for α_1 derived from lower records via (4.2) is

$$L_L(\alpha_1) = \prod_{i=1}^{m} \alpha_1 (1 + 2\alpha_1 x_{L(i)}) e^{-2\alpha_1 x_{L(i)}} \prod_{i=1}^{m-1} \frac{1}{1 - (1 + 2\alpha_1 x_{L(i)}) e^{-2\alpha_1 x_{L(i)}}}. \tag{4.7}$$

The associated log-likelihood function is derived by applying natural logarithms:

$$\ln L_L(\alpha_1) = m \log(\alpha_1) + \sum_{i=1}^{m} \log(1 + 2\alpha_1 x_{L(i)}) + 2\alpha_1 \sum_{i=1}^{m} x_{L(i)} - \sum_{i=1}^{m-1} \log\left[1 - (1 + \alpha_1 x_{L(i)}) e^{-2\alpha_1 x_{L(i)}}\right]. \tag{4.8}$$

By symmetry, the corresponding likelihood and log-likelihood functions for the stress parameter α_2 based on $\mathbf{Y}_L$ follow the same functional structure and are explicitly provided in Appendix B.

Also, for α_2, we formulate the log-likelihood function $\ln L_L(\alpha_2)$ in (B.2) utilizing the lower record sample $\mathbf{Y}_L$ of size n. The MLEs $\hat{\alpha}_{1L}$ and $\hat{\alpha}_{2L}$ are derived through numerical methods, after which the associated stress-strength reliability estimate is calculated accordingly:

$$\hat{R}_L = 1 - \frac{\hat{\alpha}_{1L}}{2(\hat{\alpha}_{1L} + \hat{\alpha}_{2L})} \left[\frac{3}{2} + \frac{\hat{\alpha}_{1L}\hat{\alpha}_{2L}}{2(\hat{\alpha}_{1L} + \hat{\alpha}_{2L})^2} \right]. \tag{4.9}$$

4.4. MLE for mixed upper-lower records

In application, one can witness a combination of upper and lower records stemming from the identical process. Let $\mathbf{X}^* = (x_1^*, x_2^*, \ldots, x_k^*)$ represent the documented record values derived from IAL(α_1). By employing the mixed likelihood (4.3), we derive

$$L_M(\alpha_1 \mid \mathbf{x}^*) = \prod_{i=1}^{k} \alpha_1 (1 + 2\alpha_1 x_i^*) e^{-2\alpha_1 x_i^*} \prod_{i \in I_U \setminus \{k\}} \frac{1}{F(x_i^*; \alpha_1)} \prod_{i \in I_L \setminus \{k\}} \frac{1}{1 - F(x_i^*; \alpha_1)}. \tag{4.10}$$

The related log-likelihood is

$$\begin{aligned} \ln L_M(\alpha_1 \mid \mathbf{x}^*) = k \log \alpha_1 + \sum_{i=1}^{k} \log(1 + 2\alpha_1 x_i^*) - 2\alpha_1 \sum_{i=1}^{k} x_i^* \\ - \sum_{i \in I_U \setminus \{k\}} \log F(x_i^*; \alpha_1) - \sum_{i \in I_L \setminus \{k\}} \log\left[1 - F(x_i^*; \alpha_1)\right]. \end{aligned} \tag{4.11}$$

Also, for α_2 we construct $\ln L_M(\alpha_2 \mid \mathbf{y}^*)$ from the mixed record sample $\mathbf{y}^*$ of size s. The MLEs $\hat{\alpha}_{1M}$ and $\hat{\alpha}_{2M}$ are obtained numerically, and the reliability estimate is

$$\hat{R}_M = 1 - \frac{\hat{\alpha}_{1M}}{2(\hat{\alpha}_{1M} + \hat{\alpha}_{2M})} \left[\frac{3}{2} + \frac{\hat{\alpha}_{1M}\hat{\alpha}_{2M}}{2(\hat{\alpha}_{1M} + \hat{\alpha}_{2M})^2} \right]. \tag{4.12}$$

4.5. Standard errors and confidence intervals

To obtain the standard error of $\hat{R}$, we employ the delta method, which uses a first-order Taylor expansion to approximate the variance of a function of estimators. Specifically,

$$\widehat{\mathrm{Var}}(\hat{R}) \approx \left(\frac{\partial R}{\partial \alpha_1} \right)^2_{\hat{\alpha}_1} \widehat{\mathrm{Var}}(\hat{\alpha}_1) + \left(\frac{\partial R}{\partial \alpha_2} \right)^2_{\hat{\alpha}_2} \widehat{\mathrm{Var}}(\hat{\alpha}_2), \tag{4.13}$$

where the partial derivatives are computed from the expression for R given in Eq. (3.3). An approximate $100(1-\gamma)\%$ confidence interval for R is then given by

$$\hat{R} \pm z_{\gamma/2} \sqrt{\widehat{\mathrm{Var}}(\hat{R})}, \tag{4.14}$$

where $z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.

5. Bayesian Estimation

In this section, we develop Bayesian estimation procedures for the stress-strength reliability parameter $R = P(Y < X)$ based on record values from the IAL distribution. Bayesian inference combines the information from the observed data with prior knowledge about the parameters. When no prior information is available, we use non-informative priors. The choice of Jeffreys prior and its extension is motivated by their invariance properties under reparameterization and their ability to reflect minimal prior information. Jeffreys prior is proportional to the square root of the Fisher information, making it a standard default choice. The extended Jeffreys prior, introduced by Sana and Faizan [**?**], raises the Fisher information to a power c, which for $c = 1$ yields an additional penalty that can reduce bias in small samples, as demonstrated in our simulation study. Alternative priors such as diffuse Gamma priors were considered but resulted in similar posterior estimates, indicating robustness; a sensitivity analysis is provided in the supplementary materials.

5.1. Prior distributions

Let α_1 and α_2 be the parameters of the IAL distribution for strength and stress, respectively; Bayesian estimation of these parameters, and consequently of R, requires the adoption of an appropriate independent prior.

5.1.1. Jeffreys prior Jeffreys prior is proportional to the square root of the Fisher information. For two independent parameters, the joint Jeffreys prior is

$$\pi_J(\alpha_1, \alpha_2) \propto \frac{1}{\alpha_1 \alpha_2}, \qquad \alpha_1 > 0,\ \alpha_2 > 0. \tag{5.1}$$

5.1.2. Extension of Jeffreys prior Following Sana and Faizan [**?**], the extension of Jeffreys prior is obtained by raising the Fisher information to a power c. For $c = 1$, the joint extension of Jeffreys prior is

$$\pi^*_J(\alpha_1, \alpha_2) \propto \frac{1}{\alpha_1^2 \alpha_2^2}, \qquad \alpha_1 > 0,\ \alpha_2 > 0. \tag{5.2}$$

5.2. Likelihood function and posterior distributions

Under a given prior $\pi^*(\alpha_1, \alpha_2)$, given by (5.1) or (5.2), the joint posterior density is

$$\pi(\alpha_1, \alpha_2 \mid \text{data}) = \frac{L(\alpha_1, \alpha_2)\, \pi^*(\alpha_1, \alpha_2)}{\int_0^\infty \int_0^\infty L(\alpha_1, \alpha_2)\, \pi^*(\alpha_1, \alpha_2)\, d\alpha_1\, d\alpha_2}. \tag{5.3}$$

The denominator is the normalizing constant, which is generally intractable. Therefore, closed-form expressions for posterior moments are not available, and we resort to numerical methods.

The Bayesian analysis in this study is based on Jeffreys and extended Jeffreys non-informative priors under the squared error loss function. This framework allows a focused comparison of Bayesian estimators under different record schemes without introducing additional assumptions about prior information or estimation preferences. The use of informative priors, alternative loss functions, and posterior predictive methods may provide further insights and is left for future research. Further investigation of credible interval performance is also left for future research. The sensitivity analysis results are provided in the supplementary materials.

5.3. Bayes estimation using MCMC

5.3.1. Metropolis–Hastings algorithm We compute the Bayes estimates of R by drawing samples from the posterior distribution (5.3) using Markov chain Monte Carlo (MCMC) methods (see also [31]). Specifically, we employ the Metropolis–Hastings algorithm, which proceeds as follows:

1. Initialize the parameters $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$ (e.g., using MLE estimates if available, otherwise random draws from the prior).
2. For each iteration $i = 1, \ldots, N$:
   (a) Propose new values $\alpha_1^* \sim \text{Lognormal}\big(\log \alpha_1^{(i-1)}, \text{var}(\alpha_1)\big)$ and $\alpha_2^* \sim \text{Lognormal}\big(\log \alpha_2^{(i-1)}, \text{var}(\alpha_2)\big)$, where $\text{var}(\alpha_1)$ and $\text{var}(\alpha_2)$ are tuned to achieve an acceptance rate between 20% and 40%.
   (b) Compute the acceptance ratio

$$\rho = \min\left(1,\ \frac{L(\alpha_1^*, \alpha_2^* \mid \text{data})\, \pi^*(\alpha_1^*, \alpha_2^*)}{L(\alpha_1^{(i-1)}, \alpha_2^{(i-1)} \mid \text{data})\, \pi^*(\alpha_1^{(i-1)}, \alpha_2^{(i-1)})} \times \frac{q(\alpha_1^{(i-1)} \mid \alpha_1^*)}{q(\alpha_1^* \mid \alpha_1^{(i-1)})} \times \frac{q(\alpha_2^{(i-1)} \mid \alpha_2^*)}{q(\alpha_2^* \mid \alpha_2^{(i-1)})}\right), \tag{5.4}$$

   where $q(\cdot \mid \cdot)$ is the proposal density.
   (c) Accept the proposed values with probability ρ; otherwise retain the previous values.
3. Discard the first M iterations as burn-in (e.g., $M = 2000$).
4. Compute the Bayes estimate of R as the posterior mean:

$$\hat{R}_{\text{Bayes}} = \frac{1}{N - M} \sum_{i=M+1}^{N} R\big(\alpha_1^{(i)}, \alpha_2^{(i)}\big). \tag{5.5}$$

5. Compute the posterior standard deviation as

$$\widehat{SE}(\hat{R}_{\text{Bayes}}) = \sqrt{\frac{1}{N - M - 1} \sum_{i=M+1}^{N} \Big(R\big(\alpha_1^{(i)}, \alpha_2^{(i)}\big) - \hat{R}_{\text{Bayes}}\Big)^2}. \tag{5.6}$$

5.3.2. Convergence diagnostics To ensure that the MCMC chains have converged, we compute the Gelman–Rubin potential scale reduction factor $\hat{R}$ for each parameter using multiple chains (typically three chains with different starting values). Additionally, we report the effective sample size (ESS) for each parameter; ESS values above 1000 are considered adequate for posterior inference. These diagnostics are included in the simulation study results.

6. Simulation Study

This section presents a comprehensive Monte Carlo simulation study to evaluate the performance of maximum likelihood estimators (MLE) and Bayesian estimators (utilizing Jeffreys and extended Jeffreys priors) for the stress-strength reliability $R = P(Y < X)$ in the context of the IAL distribution. The simulation seeks to replicate the statistical challenges associated with record value schemes, frequently observed in extreme value theory and reliability engineering applications. The efficacy of the estimators is assessed based on average bias and mean squared errors (MSE) for each configuration.

It should be noted that the simulation study is primarily designed to evaluate the finite-sample performance of the proposed point estimators in terms of bias and mean squared error. A detailed assessment of interval estimation properties, including empirical coverage probabilities and average interval widths, is beyond the scope of the present investigation.

6.1. Simulation framework

The systematic setups are executed by computational simulation as follows:

- **True parameter pairs:** $(\alpha_1, \alpha_2) \in \{(0.2, 0.7), (0.5, 1.2), (0.8, 0.8), (0.8, 1.5), (0.9, 0.5)\}$, denoting diverse reliability levels across distinct distributional contexts.
- **Sample sizes:** Equal record sizes for strength and stress variables $m = n \in \{10, 15, 30, 40, 50, 70\}$, encompassing small to intermediate sample sizes typically encountered in record-breaking events.
- **Record schemes:** Three separate record schemes are examined:
  - Upper records: documenting exceptional upper values that surpass prior maxima.
  - Lower records: documenting unprecedented minimum values that fall below prior lows.
  - Mixed records: integrating both upper and lower record values.
- **Replications:** A total of 400 independent trials were conducted for each scenario to guarantee statistical stability and accurate estimates of bias and mean squared error (MSE) metrics.
- **MCMC specifications:** Bayesian estimates are generated via the Metropolis–Hastings algorithm (Section 5.3) with the subsequent specifications. A total of 5,000 iterations are conducted, with the first 1,000 iterations discarded as a burn-in period to mitigate startup bias. The proposal distribution is set to be log-normal with a standard deviation of 0.25, which provides efficient exploration of the parameter space. Initial values are stochastically produced from a Uniform$(0.5, 1.5)$ distribution to mitigate reliance on starting conditions and ensure convergence to the true posterior distribution. These measurements offer detailed insights into the performance attributes of each estimating method across various record sampling strategies.

6.2. Results and discussion

The simulation results presented in Tables 1–5 provide several important insights into the estimation of stress-strength reliability $R = P(Y < X)$ under record-breaking data from the IAL distribution. Five configurations were examined, covering a wide range of scenarios, with different sample sizes. Three record schemes, including upper, lower, and mixed, were investigated alongside three estimation methods: maximum likelihood estimation, and Bayesian estimation with the Jeffreys prior and the extended Jeffreys prior.

The simulation results across the five configurations indicate that mixed record schemes generally provide more balanced estimation behavior for the stress-strength reliability $R = P(Y < X)$, particularly in terms of bias reduction, since they incorporate information from both upper and lower tails of the distribution. In contrast, pure upper and lower record schemes tend to exhibit systematic directional bias, especially for small sample sizes. For all estimation methods, both bias and MSE decrease steadily as the sample size increases, suggesting the consistency of the estimators under IAL record-breaking schemes. The maximum likelihood estimator (MLE) demonstrates stable and competitive performance across nearly all configurations, with estimation accuracy improving substantially for $m \geq 30$. The Bayesian estimators based on Jeffreys and extended Jeffreys priors

occasionally provide slightly smaller MSE values for small sample sizes, particularly under mixed record schemes; however, these improvements are generally marginal and diminish as the sample size increases. For moderate and large samples, the three estimation methods become numerically very close in terms of estimation accuracy. Therefore, for practical implementation, the use of mixed records together with MLE may be preferred for moderate to large sample sizes due to its stable performance, simplicity, and computational efficiency, while Bayesian estimators can provide slight improvements in small-sample settings.

Figures 2 and 3 further illustrate that the differences among MLE, Bayesian Jeffreys, and Bayesian extended Jeffreys estimators are generally small across the various sampling schemes. Although the Bayesian extended Jeffreys estimator occasionally attains slightly smaller MSE values, particularly for small sample sizes, no estimator uniformly dominates the others across all configurations. The MSE values for all estimators decrease steadily as the sample size increases, and the corresponding curves become increasingly close for moderate and large samples. These graphical findings are fully consistent with the numerical results reported in Tables 1–5 and indicate that the three estimation methods exhibit broadly comparable performance under the IAL model, with only marginal advantages for Bayesian estimators in certain small sample scenarios.

The simulation study considers equal sample sizes for the strength and stress records to provide a balanced comparison of the proposed estimation procedures. However, unequal record sizes may arise in practical applications and could affect estimation performance. Moreover, the selected parameter configurations do not cover the entire range of possible reliability levels. Investigating unequal record sizes and scenarios involving very low and very high reliability values would provide a more comprehensive assessment of the proposed methods and is left for future research.

Table 1. Bias and MSE of the MLE and Bayesian estimators of R under different record schemes for configuration $(\alpha_1 = 0.2,\ \alpha_2 = 0.7) \Rightarrow R_{true} = 0.8001$.

n	m	Scheme	MLE		Bayes (Jeffreys)		Bayes (Ext. Jeffreys)	
			Bias	MSE	Bias	MSE	Bias	MSE
10	10	Upper	-0.01412	0.00589	-0.01682	0.00583	-0.02058	0.00571
		Lower	-0.00649	0.00453	-0.00936	0.00454	-0.01148	0.00427
		Mixed	-0.01293	0.00666	-0.01326	0.00668	-0.01353	0.00661
15	15	Upper	-0.00519	0.00320	-0.00639	0.003019	-0.00829	0.00308
		Lower	-0.00301	0.00263	-0.00463	0.00261	-0.00591	0.00254
		Mixed	-0.01065	0.00416	-0.01057	0.00414	-0.01045	0.00407
30	30	Upper	-0.00147	0.00145	-0.00233	0.00144	-0.00273	0.00142
		Lower	-0.00531	0.00148	-0.00568	0.00147	-0.00615	0.00145
		Mixed	-0.00004	0.00207	-0.00013	0.00208	-0.00014	0.00206
40	40	Upper	-0.00175	0.00097	-0.00227	0.00098	-0.00272	0.00097
		Lower	-0.00280	0.00103	-0.00317	0.00104	-0.00342	0.00102
		Mixed	-0.00166	0.00133	-0.00166	0.00134	-0.00170	0.00132
50	50	Upper	-0.00073	0.00075	-0.00109	0.00076	-0.00147	0.00074
		Lower	-0.00223	0.00085	-0.00259	0.00084	-0.00268	0.00083
		Mixed	0.00137	0.00120	0.00138	0.00121	0.00134	0.00121
70	70	Upper	-0.00057	0.00059	-0.00088	0.00060	-0.00107	0.00059
		Lower	-0.00219	0.00056	-0.00236	0.00055	-0.00239	0.00055
		Mixed	-0.00090	0.00082	-0.00088	0.00082	-0.00096	0.00081

Table 2. Bias and MSE of the MLE and Bayesian estimators of R under different record schemes for configuration $(\alpha_1 = 0.5,\ \alpha_2 = 1.2) \Rightarrow R_{true} = 0.7273$.

n	m	Scheme	MLE		Bayes (Jeffreys)		Bayes (Ext. Jeffreys)	
			Bias	MSE	Bias	MSE	Bias	MSE
10	10	Upper	-0.00584	0.00682	-0.00591	0.00680	-0.00624	0.00685
		Lower	-0.00587	0.00879	-0.00565	0.00874	-0.00591	0.00889
		Mixed	-0.01057	0.00844	-0.01876	0.00808	-0.01583	0.00805
15	15	Upper	-0.00729	0.00604	-0.00774	0.00606	-0.00836	0.00615
		Lower	-0.00836	0.00481	-0.00837	0.00487	-0.00818	0.00483
		Mixed	-0.00256	0.00549	-0.00977	0.00506	-0.00788	0.00511
30	30	Upper	-0.00108	0.00239	-0.00134	0.00241	-0.00151	0.00245
		Lower	-0.00198	0.00267	-0.00189	0.00265	-0.00185	0.00268
		Mixed	-0.00092	0.00320	-0.00315	0.00302	-0.00287	0.00304
40	40	Upper	-0.00075	0.00167	-0.00096	0.00169	-0.00111	0.00168
		Lower	-0.00178	0.00175	-0.00182	0.00176	-0.00164	0.00177
		Mixed	0.00232	0.00227	0.00088	0.00211	0.00107	0.00215
50	50	Upper	0.00045	0.00131	0.00033	0.00132	0.00021	0.00129
		Lower	-0.00156	0.00142	-0.00157	0.00141	-0.00138	0.00143
		Mixed	-0.00183	0.00199	-0.00281	0.00192	-0.00258	0.00194
70	70	Upper	0.00027	0.00102	0.00011	0.00104	0.00009	0.00103
		Lower	-0.00199	0.00094	-0.00201	0.00092	-0.00184	0.00093
		Mixed	-0.00046	0.00122	-0.00076	0.00119	-0.00080	0.00120

Table 3. Bias and MSE of the MLE and Bayesian estimators of R under different record schemes for configuration $(\alpha_1 = 0.8,\ \alpha_2 = 0.8) \Rightarrow R_{true} = 0.4999$.

n	m	Scheme	MLE		Bayes (Jeffreys)		Bayes (Ext. Jeffreys)	
			Bias	MSE	Bias	MSE	Bias	MSE
10	10	Upper	0.00943	0.01031	0.00918	0.01032	0.00907	0.01053
		Lower	0.00628	0.01217	0.00658	0.01223	0.00641	0.01220
		Mixed	-0.00009	0.01333	0.00019	0.01213	0.00008	0.01262
15	15	Upper	0.00488	0.00674	0.00432	0.00676	0.00411	0.00680
		Lower	-0.00167	0.00798	-0.00172	0.00789	-0.00175	0.00797
		Mixed	-0.00113	0.01024	-0.00093	0.00975	-0.00130	0.00991
30	30	Upper	0.00231	0.00391	0.00201	0.00393	0.00179	0.00392
		Lower	0.00148	0.00417	0.00157	0.00414	0.00166	0.00418
		Mixed	0.00187	0.00535	0.00168	0.00533	0.00203	0.00527
40	40	Upper	0.00162	0.00276	0.00138	0.00279	0.00119	0.00278
		Lower	0.00038	0.00277	0.00034	0.00278	0.00057	0.00280
		Mixed	-0.00203	0.00416	-0.00208	0.00417	-0.00204	0.00415
50	50	Upper	0.00261	0.00219	0.00248	0.00222	0.00232	0.00221
		Lower	0.00016	0.00225	0.00014	0.00224	0.00040	0.00226
		Mixed	-0.00064	0.00323	-0.00065	0.00325	-0.00107	0.00325
70	70	Upper	0.00194	0.00170	0.00175	0.00173	0.00172	0.00173
		Lower	-0.00113	0.00148	-0.00116	0.00147	-0.00093	0.00147
		Mixed	-0.00126	0.00199	-0.00109	0.00201	-0.00147	0.00203

Table 4. Bias and MSE of the MLE and Bayesian estimators of R under different record schemes for configuration $(\alpha_1 = 0.8,\ \alpha_2 = 1.5) \Rightarrow R_{true} = 0.6694$.

n	m	Scheme	MLE		Bayes (Jeffreys)		Bayes (Ext. Jeffreys)	
			Bias	MSE	Bias	MSE	Bias	MSE
10	10	Upper	-0.01020	0.00889	-0.01096	0.00894	-0.00979	0.00899
		Lower	-0.00879	0.00938	-0.00974	0.00941	-0.00873	0.00934
		Mixed	-0.02347	0.00976	-0.03954	0.00925	-0.03497	0.00949
15	15	Upper	-0.01156	0.00669	-0.01244	0.00672	-0.01198	0.00673
		Lower	-0.01090	0.00618	-0.01142	0.00632	-0.01088	0.00627
		Mixed	-0.01038	0.00597	-0.02448	0.00575	-0.02079	0.00573
30	30	Upper	-0.00225	0.00282	-0.00267	0.00281	-0.00254	0.00284
		Lower	0.00031	0.00354	0.00037	0.00353	0.00052	0.00354
		Mixed	-0.00404	0.00386	-0.01443	0.00358	-0.01277	0.00360
40	40	Upper	0.00171	0.00236	0.00152	0.00238	0.00163	0.00235
		Lower	-0.00259	0.00255	-0.00255	0.00257	-0.00239	0.00256
		Mixed	-0.00659	0.00277	-0.01464	0.00257	-0.01345	0.00260
50	50	Upper	0.00203	0.00167	0.00189	0.00168	0.00188	0.00168
		Lower	-0.00239	0.00194	-0.00247	0.00193	-0.00223	0.00196
		Mixed	-0.00113	0.00196	-0.00868	0.00180	-0.00764	0.00179
70	70	Upper	0.00072	0.00127	0.00055	0.00129	0.00041	0.00129
		Lower	-0.00263	0.00118	-0.00260	0.00119	-0.00261	0.00120
		Mixed	0.00092	0.00163	-0.00503	0.00143	-0.00432	0.00144

Table 5. Bias and MSE of the MLE and Bayesian estimators of R under different record schemes for configuration $(\alpha_1 = 0.9,\ \alpha_2 = 0.5) \Rightarrow R_{true} = 0.3407$.

n	m	Scheme	MLE		Bayes (Jeffreys)		Bayes (Ext. Jeffreys)	
			Bias	MSE	Bias	MSE	Bias	MSE
10	10	Upper	0.01195	0.00931	0.01190	0.00929	0.01117	0.00935
		Lower	0.00989	0.01025	0.01044	0.01035	0.00943	0.01020
		Mixed	0.01902	0.01197	0.02162	0.01167	0.02049	0.01168
15	15	Upper	0.00849	0.00572	0.00799	0.00573	0.00783	0.00575
		Lower	0.00351	0.00606	0.00361	0.00611	0.00346	0.00607
		Mixed	0.00288	0.00788	0.00431	0.00771	0.00364	0.00777
30	30	Upper	0.00443	0.00324	0.00417	0.00325	0.00397	0.00324
		Lower	0.00381	0.00336	0.00388	0.00335	0.00398	0.00338
		Mixed	0.00092	0.00343	0.00091	0.00339	0.00136	0.00341
40	40	Upper	0.00312	0.00229	0.00293	0.00231	0.00275	0.00230
		Lower	0.00201	0.00224	0.00197	0.00224	0.00219	0.00226
		Mixed	0.00482	0.00289	0.00481	0.00288	0.00466	0.00290
50	50	Upper	0.00366	0.00184	0.00357	0.00186	0.00342	0.00185
		Lower	0.00149	0.00181	0.00146	0.00179	0.00171	0.00182
		Mixed	0.00251	0.00258	0.00242	0.00260	0.00250	0.00259
70	70	Upper	0.00276	0.00141	0.00261	0.00143	0.00259	0.00143
		Lower	-0.00014	0.00117	-0.00016	0.00116	0.00004	0.00117
		Mixed	0.00312	0.00182	0.00314	0.00182	0.00298	0.00182

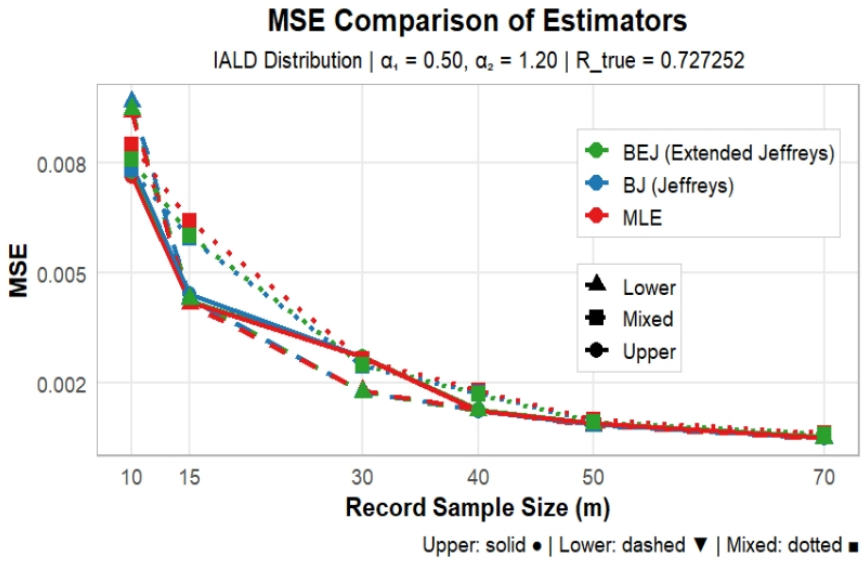


Figure 2. MSE comparison of MLE and Bayesian methods for the SSR based on the results of Table 2.

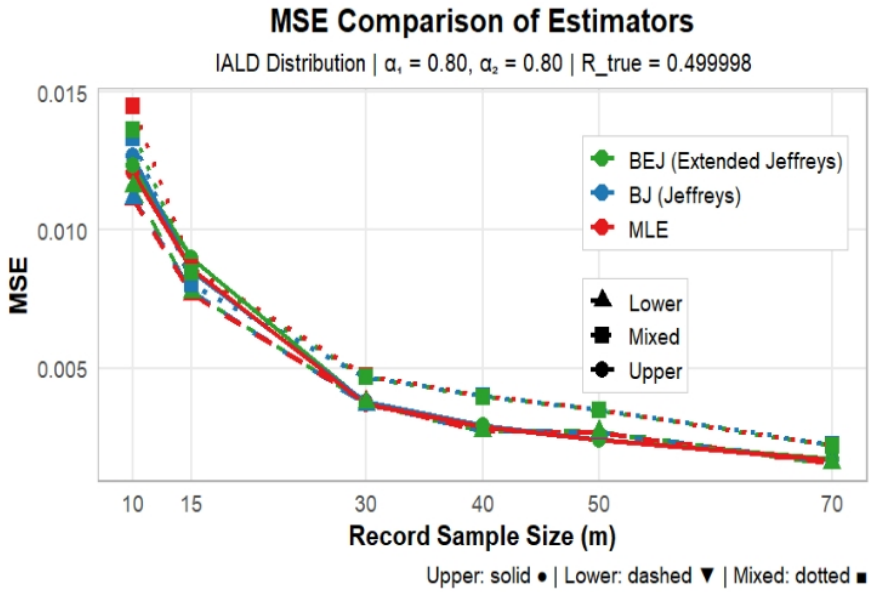


Figure 3. MSE comparison of MLE and Bayesian methods for the SSR based on the results of Table 3.

7. Applications to Real Datasets

In this section, we apply the suggested estimation techniques to two actual datasets to demonstrate the adaptability of the IAL distribution. We assessed the adequacy of the IAL distribution for each dataset. The assessment of the stress-strength reliability parameter $R = P(Y < X)$ is performed using upper, lower, and mixed record schemes through maximum likelihood and Bayesian approaches. A bootstrap method utilizing 500 resampling iterations is implemented to derive reliable estimates and standard errors. The purpose of the real-data applications is to assess the adequacy of the IAL distribution before conducting the stress-strength reliability analysis, rather than to compare competing distributions. This assessment is based on graphical methods and goodness-of-fit tests, including the Kolmogorov–Smirnov, Cramér–von Mises, and Anderson–Darling tests, to verify whether the IAL distribution provides an adequate fit to the observed data.

7.1. Failure time dataset

The first dataset is related to the failure times of 50 items given by Murthy et al. [33]. The elements of the data are: 0.008, 0.017, 0.058, 0.061, 0.084, 0.090, 0.134, 0.238, 0.245, 0.353, 0.374, 0.480, 0.495, 0.535, 0.564, 0.681, 0.686, 0.688, 0.921, 0.959, 1.022, 1.092, 1.260, 1.284, 1.295, 1.373, 1.395, 1.414, 1.760, 1.858, 1.892, 1.921, 1.926, 1.933, 2.135, 2.169, 2.301, 2.320, 2.405, 2.506, 2.598, 2.808, 2.971, 3.087, 3.492, 3.669, 3.926, 4.446, 5.119, 8.596. Figure 4 presents graphical representations, including the histogram, kernel density, violin plot, box plot, TTT plot, and QQ plot, which reveal a positively skewed, non-normal distribution characterized by a prolonged right tail and upper-tail outliers. The concave TTT plot lying above the diagonal line indicates an increasing hazard rate behavior, which aligns analytically with the bounded increasing HRF of the IAL model.

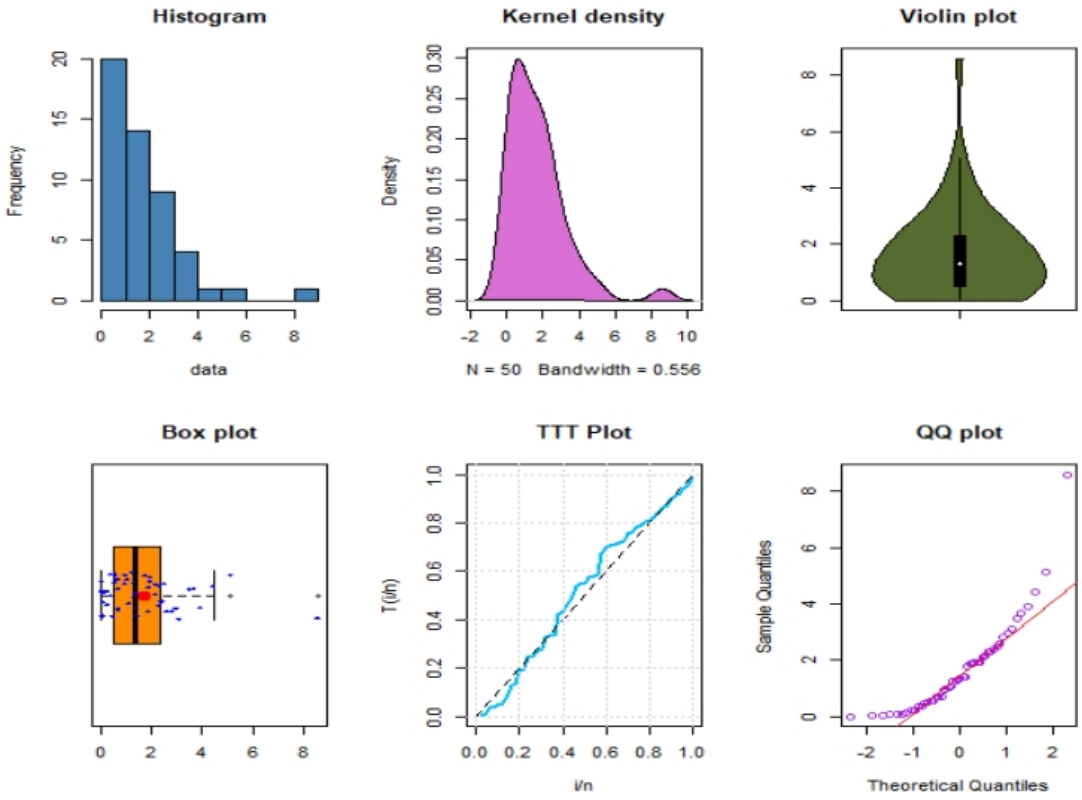


Figure 4. Graphical representation of the failure time dataset.

The proposed distribution demonstrates a satisfactory and stable fit to the observed data, effectively capturing its underlying characteristics. This adequacy is supported by goodness-of-fit measures and model selection results, which indicate strong model performance. The MLE of the parameter α is $\hat{\alpha} = 0.4434$, with standard error SE $= 0.0552$. The Kolmogorov–Smirnov (K-S) test statistic is 0.0807, with its related p-value $= 0.8745$. The Cramér–von Mises test statistic is 0.0054, with p-value $= 0.8524$. Also, the Anderson–Darling test supports the model fit, producing a statistic of 0.5758 with a p-value $= 0.6703$. Collectively, these results confirm that the proposed distribution provides an adequate and statistically consistent representation of the data. Figures 5 and 6 depict the fit via the estimated PDF and the empirical CDF, validating the goodness-of-fit metrics and statistical sufficiency of the suggested distribution for the observed data.

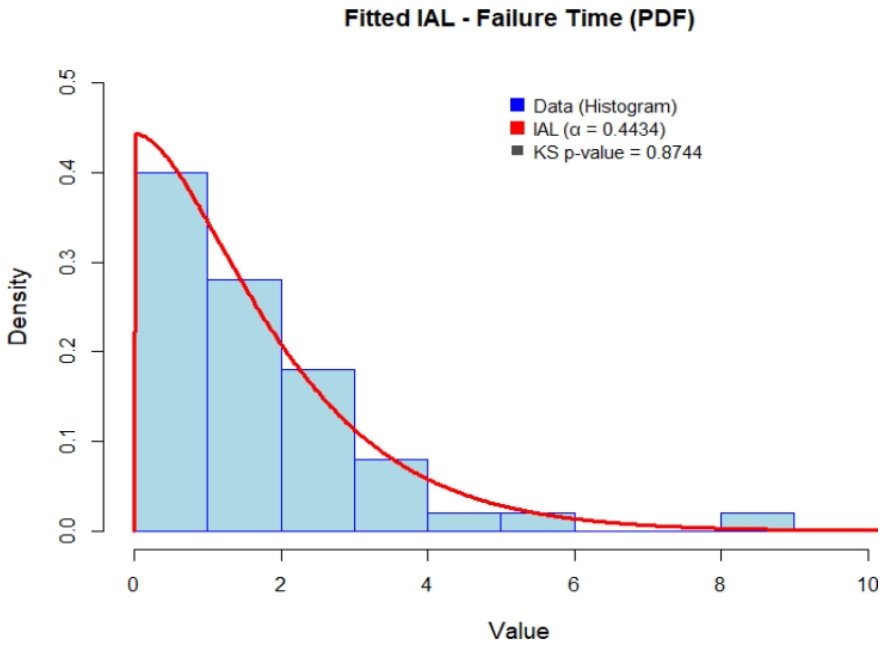


Figure 5. Histogram and estimated PDF for the IAL model with the failure time dataset.

The dataset was randomly divided into two equal segments (strength X and stress Y), resulting in the extraction of records. Table 6 presents the bootstrap estimates of $R = P(Y < X)$ for the failure time dataset across several recording strategies. For lower records, the algorithm failed to converge in all bootstrap replications for all three estimation methods, as indicated by the dashes (–) in Table 6. The convergence failure observed under the lower record scheme highlights an important practical limitation. Since the number of observed records may be much smaller than the original sample size, the available information may be insufficient for stable parameter estimation, particularly in bootstrap resampling. Therefore, the suitability of this scheme should be assessed based on the effective number of records rather than the original sample size alone. When repeated convergence failures occur, the estimates should be treated with caution, and alternative record schemes may be preferable. A universal minimum sample size or record count for reliable convergence remains to be investigated. This reflects the instability of pure record schemes when the effective sample size becomes extremely small due to the positively skewed nature of the data. In contrast, upper and mixed records produced viable estimates across all methods, with the extended Jeffreys prior yielding the highest R estimates for both schemes, followed by Jeffreys and then MLE.

The IAL distribution implementation on the failure time dataset aligns with theoretical predictions and simulation outcomes. Upper records yield the highest reliability estimates (0.664-0.789), mixed records produce moderately lower estimates (0.570–0.630), while lower records failed to converge due to the positively skewed distribution with an extended right tail. Regarding estimator performance, both Jeffreys and extended Jeffreys priors consistently generate higher estimates than MLE. For upper records, MLE yields 0.6641 (SE $= 0.0899$), Jeffreys 0.7194 (SE

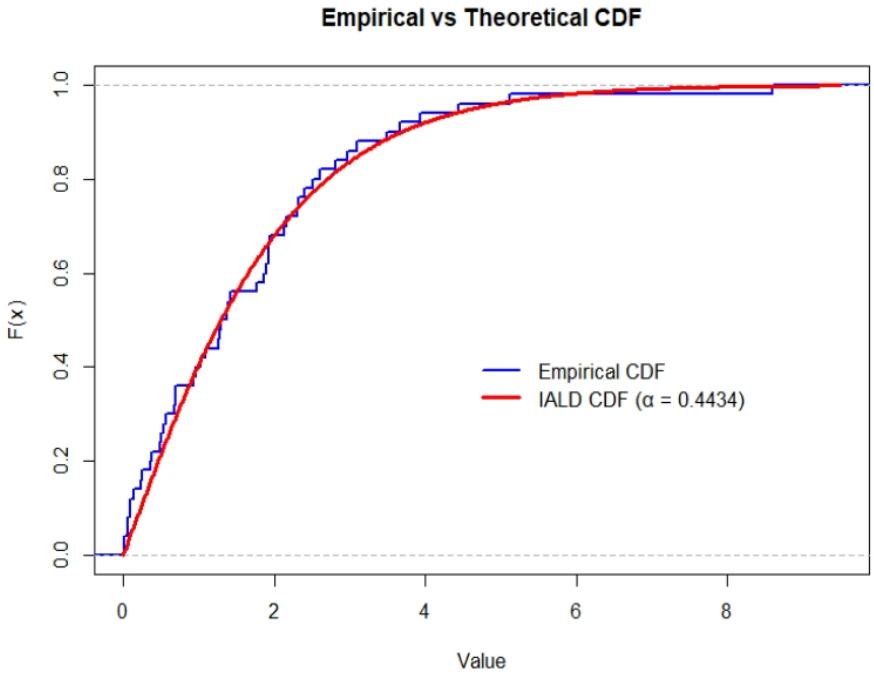


Figure 6. Plot of the empirical CDF for the IAL model with the failure time dataset.

$= 0.0949$), and extended Jeffreys 0.7888 (SE $= 0.1153$). For mixed records, the corresponding values are 0.5698 (SE $= 0.0895$), 0.5978 (SE $= 0.0913$), and 0.6302 (SE $= 0.0958$), respectively. The extended Jeffreys prior exhibits higher estimates than Jeffreys but with increased standard errors, reflecting a bias-variance trade-off. Upper records produce the strongest reliability signal ($\hat{R} = 0.7888$, SE $= 0.1153$), while mixed records indicate a positive but modest reliability framework ($\hat{R} = 0.6302$, SE $= 0.0958$).

Figures 7 and 8 present additional information about the efficacy and dependability of the MLE, Jeffreys, and extended Jeffreys estimators for the stress-strength parameter $R = P(Y < X)$ within upper and mixed record frameworks utilizing the IAL distribution. Figure 7 shows that the Bayesian extended Jeffreys prior consistently produces the highest R estimates under both upper and mixed records, followed by Jeffreys, then MLE. Figure 8 confirms that MCMC chains for both Bayesian priors converge rapidly and mix well after burn-in, with no evidence of non-stationarity. Collectively, these data indicate that the extended Jeffreys prior not only attains enhanced point estimates but also ensures dependable computational convergence, hence affirming its practical applicability for reliability estimation.

Table 6. Bootstrap estimates of $R = P(Y < X)$ for the failure time dataset (standard errors in parentheses).

Record scheme	MLE	Bayes (Jeffreys)	Bayes (Ext. Jeffreys)
Upper records	0.6641 (0.0899)	0.7194 (0.0949)	0.7888 (0.1153)
Lower records	–	–	–
Mixed records	0.5698 (0.0895)	0.5978 (0.0913)	0.6302 (0.0958)

7.2. Automobile fatalities dataset

The second dataset, analyzed by Mann [34], concerns the total number of automotive deaths in 39 counties of South Carolina for the year 2012. The data observations are as follows: 1, 8, 20, 12, 26, 13, 15, 17, 4, 48, 9, 2, 16, 9, 31, 27, 6, 5, 14, 50, 9, 16, 3, 48, 33, 9, 20, 68, 22, 13, 51, 13, 4, 17, 6, 52, 23, 12, 10. In accordance with the

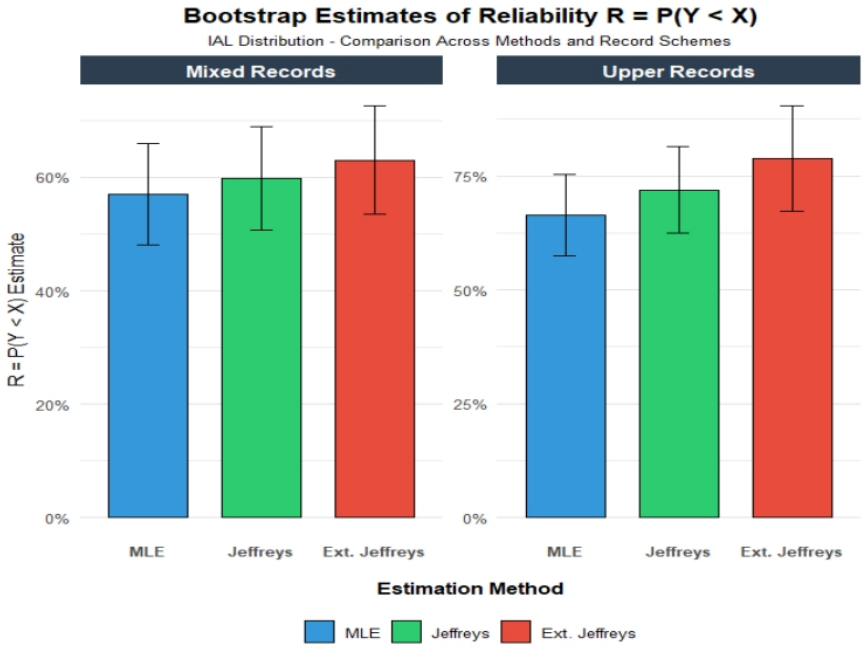


Figure 7. Comparison of R estimates for the failure time dataset.

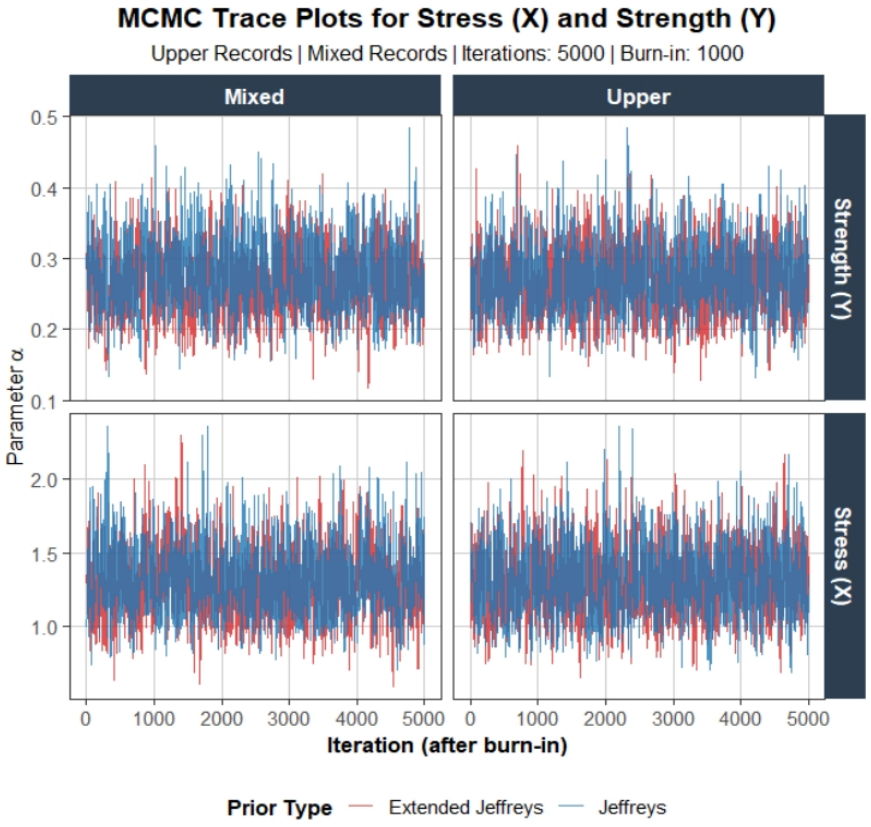


Figure 8. MCMC trace plots of α_1 and α_2 for the failure time dataset.

preprocessing outlined in the literature, the data are standardized by dividing by 69 to fall inside the range $(0, 1)$. The normalized observations are: 0.3188, 0.3768, 0.2464, 0.0580, 0.6957, 0.1304, 0.1304, 0.4493, 0.3913, 0.2899, 0.1739, 0.0870, 0.0725, 0.2030, 0.1304, 0.2319, 0.0435, 0.4783, 0.1304, 0.2899, 0.9855, 0.1884, 0.7391, 0.1884, 0.0290, 0.0580, 0.2464, 0.2319, 0.0870, 0.7536, 0.7246, 0.6957, 0.3333, 0.1739, 0.1884, 0.1449, 0.2174, 0.1159, 0.0145.

Figure 9 presents graphical representations, including the histogram, kernel density, violin plot, box plot, TTT plot, and QQ plot. There are multiple outliers and a concentration at lower levels in the right-skewed data. A bimodal tendency is suggested by the kernel density and histogram. Asymmetry is confirmed by box and violin plots; however, at larger values, the QQ plot deviates from normality.

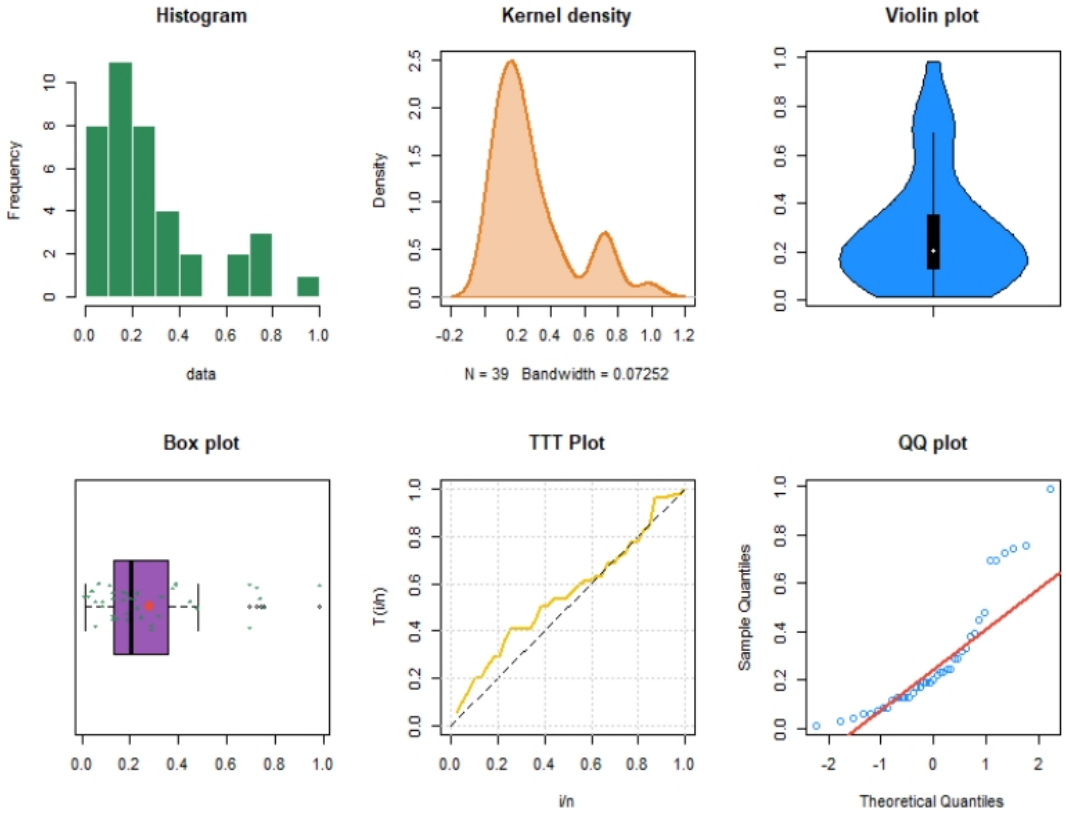


Figure 9. Graphical representation of the automobile fatalities dataset.

The proposed distribution exhibits a reasonable and consistent fit to the observed data, accurately reflecting its fundamental properties. This adequacy is corroborated by goodness-of-fit metrics and model selection outcomes, which demonstrate robust model performance. The MLE of the parameter α is $\hat{\alpha} = 2.6752$, with SE $= 0.3752$. The Kolmogorov–Smirnov (K-S) test statistic is 0.0979, with a corresponding p-value of 0.8491. The Cramér–von Mises test statistic is 0.0712, with a p-value of 0.7469. The Anderson–Darling test corroborates the model fit, yielding a statistic of 0.5065 with a p-value of 0.7391. These results collectively affirm that the proposed distribution offers a sufficient and statistically coherent representation of the data. Figures 10 and 11 illustrate the alignment between the estimated PDF and the empirical CDF, confirming the adequacy of the goodness-of-fit metrics and the statistical validity of the proposed distribution for the observed data.

The dataset was randomly partitioned into two equal pieces (strength X and stress Y), leading to the extraction of records. Table 7 displays the bootstrap estimates of $R = P(Y < X)$ for the automobile fatalities dataset under various recording strategies. All three recording schemes successfully converged, producing estimates centered around the neutral value $R = 0.5$, indicating an absence of strong reliability or unreliability signals in this dataset.

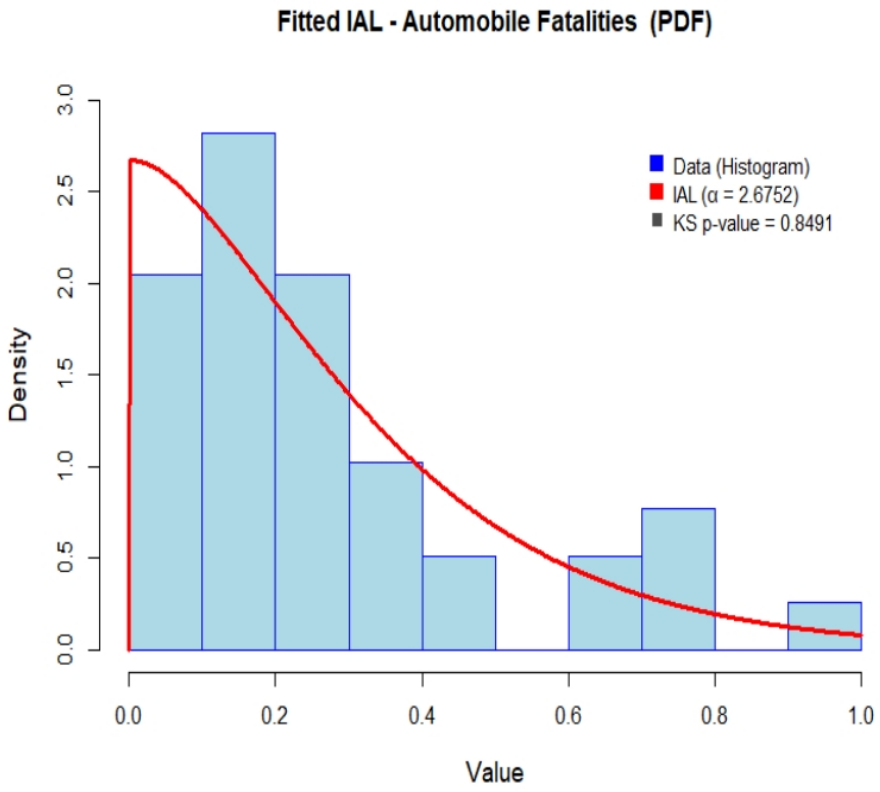


Figure 10. Histogram and estimated PDF for the IAL model with the automobile fatalities dataset.

The extended Jeffreys prior consistently yields higher estimates than both MLE and standard Jeffreys across all schemes, validating its theoretical bias-reduction characteristic. For mixed records, the extended Jeffreys prior produces $\hat{R} = 0.5301$ (SE $= 0.1038$), compared to 0.5096 (SE $= 0.0786$) for MLE and 0.5177 (SE $= 0.0871$) for the Jeffreys prior. Nonetheless, bias mitigation incurs elevated standard errors. The extended Jeffreys prior demonstrates substantially larger standard errors (0.2153 for upper, 0.2182 for lower, and 0.1038 for mixed records) compared to the standard Jeffreys prior (0.1028, 0.1655, and 0.0871, respectively). This precision-bias trade-off necessitates careful consideration in practical applications. Mixed records exhibit the most stable estimates with relatively smaller standard errors (0.0786–0.1038), indicating that mixed schemes yield the most dependable inferences when the underlying data structure does not strongly favor extreme values. The considerable standard errors for lower records and the extended Jeffreys estimate underscore substantial sampling variability, necessitating larger sample sizes to achieve more accurate reliability estimates.

Figure 12 demonstrates that the extended Jeffreys prior consistently yields the highest R estimates across all record schemes, followed by Jeffreys, then MLE, while Figure 13 confirms rapid convergence and good mixing of MCMC chains for both Bayesian priors with no signs of non-stationarity. Collectively, these findings establish that extended Jeffreys provides both statistical superiority and computational reliability for estimating $R = P(Y < X)$.

Table 7. Bootstrap estimates of $R = P(Y < X)$ for the automobile fatalities dataset (standard errors in parentheses).

Record scheme	MLE	Bayes (Jeffreys)	Bayes (Ext. Jeffreys)
Upper records	0.4865 (0.0773)	0.4867 (0.1028)	0.5104 (0.2153)
Lower records	0.4574 (0.1655)	0.4774 (0.1655)	0.5383 (0.2182)
Mixed records	0.5096 (0.0786)	0.5177 (0.0871)	0.5301 (0.1038)

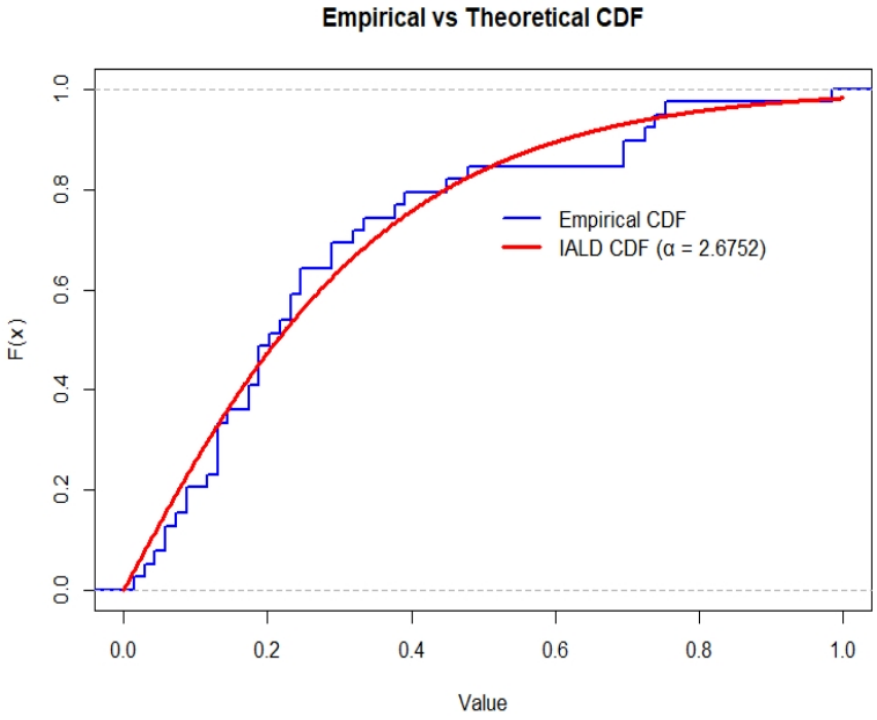


Figure 11. Plot of the empirical CDF for the IAL model with the automobile fatalities dataset.

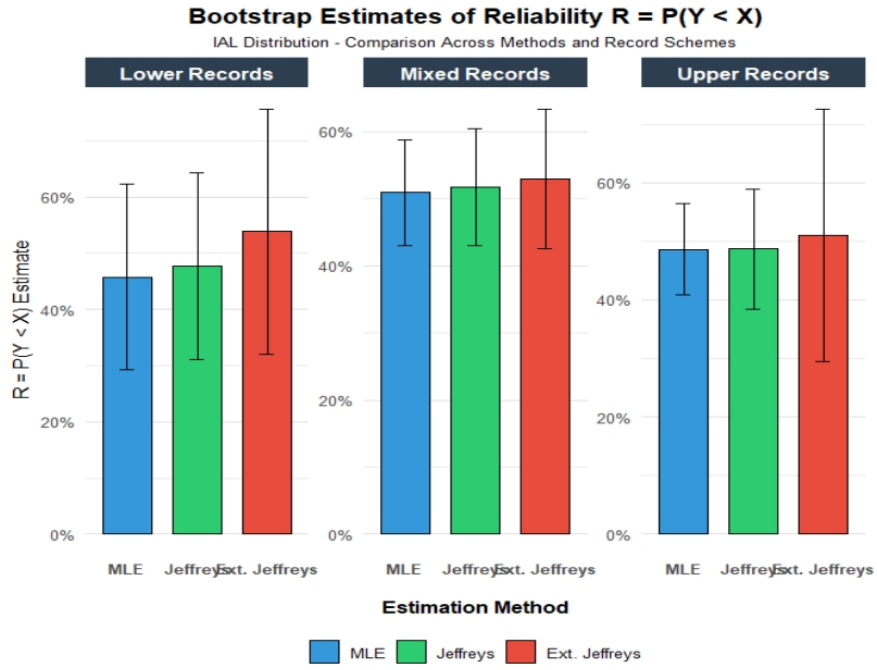


Figure 12. Comparison of R estimates for the automobile fatalities dataset.

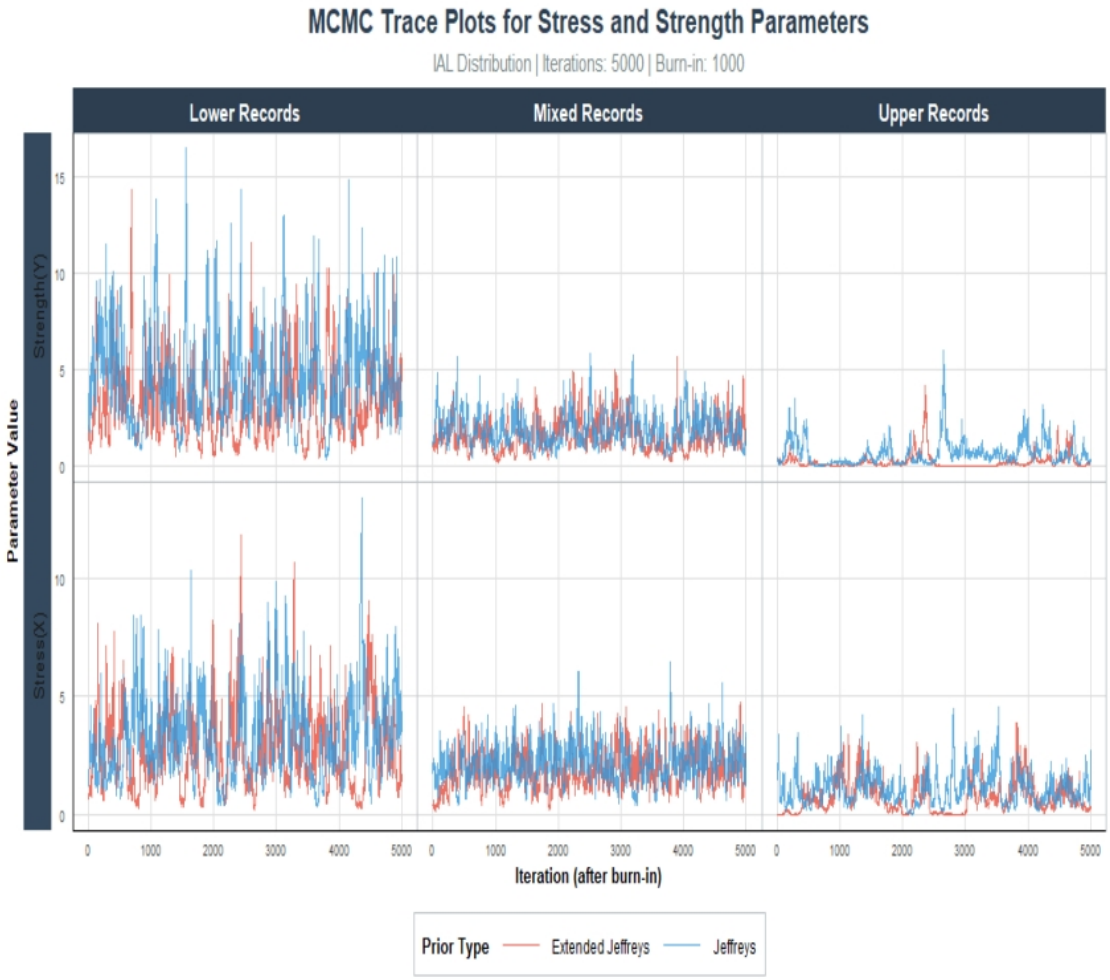


Figure 13. MCMC trace plots of α_1 and α_2 for the automobile fatalities dataset.

8. Conclusion

In this paper, the stress-strength reliability parameter $R = P(Y < X)$ for the induced Ailamujia (IAL) distribution under upper, lower, and mixed record sampling schemes is investigated, deriving a closed-form expression for R in (3.3) and estimating it using maximum likelihood, Jeffreys prior, and extended Jeffreys prior methods. The extensive simulation results and real-data applications involving industrial failure times and automobile fatalities indicate that mixed record schemes generally provide more balanced estimation performance, particularly in terms of bias reduction, since they incorporate information from both upper and lower tails of the distribution. In contrast, pure upper and lower record schemes tend to exhibit more pronounced directional bias, especially for small sample sizes. The comparative analysis further shows that the MLE, Bayesian Jeffreys, and Bayesian extended Jeffreys estimators exhibit broadly similar performance across most configurations. Although the Bayesian estimators occasionally achieve slightly smaller MSE values in small-sample settings, these improvements are generally marginal and diminish as the sample size increases. For moderate and large samples, the three methods become numerically very close in estimation accuracy. The MLE demonstrates stable and robust performance throughout the study and remains an attractive practical choice due to its simplicity and computational efficiency, particularly for moderate-to-large sample sizes. The real-data analysis further confirms the applicability of the proposed

methodology under different reliability structures, including cases where the estimated stress-strength reliability is close to $R = 0.5$. Overall, the proposed framework provides an effective and computationally efficient approach for stress-strength reliability estimation based on record-breaking observations from the IAL distribution, with potential applicability in reliability engineering and resource-constrained data environments.

A. Likelihood functions for α_2 under upper records

The likelihood and log-likelihood functions for the stress parameter α_2 based on $\mathbf{Y}_U$ are

$$L_U(\alpha_2) = \prod_{i=1}^{n} \alpha_2(1 + 2\alpha_2 y_{U(i)})e^{-2\alpha_2 y_{U(i)}} \prod_{i=1}^{n-1} \frac{1}{(1 + \alpha_2 y_{U(i)})e^{-2\alpha_2 y_{U(i)}}}, \tag{A.1}$$

and, by taking natural logarithms,

$$\ln L_U(\alpha_2) = n\log(\alpha_2) + \sum_{i=1}^{n} \log(1 + 2\alpha_2 y_{U(i)}) - 2\alpha_2 \sum_{i=1}^{n} y_{U(i)} - \sum_{i=1}^{n-1} \log(1 + \alpha_2 y_{U(i)}) + 2\alpha_2 \sum_{i=1}^{n-1} y_{U(i)}. \tag{A.2}$$

B. Likelihood functions for α_2 under lower records

The likelihood and log-likelihood functions for the stress parameter α_2 based on $\mathbf{Y}_L$ are

$$L_L(\alpha_2) = \prod_{i=1}^{n} \alpha_2(1 + 2\alpha_2 y_{L(i)})e^{-2\alpha_2 y_{L(i)}} \prod_{i=1}^{n-1} \frac{1}{1 - (1 + 2\alpha_2 y_{L(i)})e^{-2\alpha_2 y_{L(i)}}}, \tag{B.1}$$

and the associated log-likelihood function is

$$\ln L_L(\alpha_2) = n\log(\alpha_2) + \sum_{i=1}^{n} \log(1 + 2\alpha_2 y_{L(i)}) - 2\alpha_2 \sum_{i=1}^{n} y_{L(i)} - \sum_{i=1}^{n-1} \log\left[1 - (1 + \alpha_2 y_{L(i)})e^{-2\alpha_2 y_{L(i)}}\right]. \tag{B.2}$$

Disclosure Statement

The authors declare no competing interests.

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