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A New Flexible Unit Haq Distribution with Entomology and Biomedical Applications

Mohammed Elgarhy [1,2], Rana H. Khashab [3,*], Muhammad Ahsan-u-Haq [4], Muhammad Ibrahim [5], Ghareeb A. Marei [1] and Neema M. Elharoun [6]

[1] *Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt; Mohammed.Elgarhy@ecu.edu.eg; ghareeb.adel@ecu.edu.eg*
[2] *Department of Computer Engineering, Biruni University, 34010, Istanbul, Turkey*
[3] *Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia; rhkhashab@uqu.edu.sa*
[4] *College of Statistical Sciences, University of the Punjab, Lahore Pakistan;*
[5] *Department of Statistics, Government College University (GCU) Lahore, Pakistan; lincoin786@yahoo.com*
[6] *Department of Statistics and Insurance, Faculty of Commerce, Arish University, El-Arish 45511, Egypt; na3omaalharoun@yahoo.com*

Abstract This paper proposes a new one-parameter unit Haq distribution to measure percentage data. Various statistical and reliability properties of the new distribution, such as moments and other measures connected with them, hazard function, and reversed hazard were examined. It was shown that the new model density function was suitable for modeling asymmetric data, and its failure rate function could effectively describe an increasing failure rate. Six estimation methods were used to estimate the parameters of the new distribution. A Monte Carlo simulation study was conducted on different sample sizes to evaluate the performance of these estimators. Lastly, the flexibility of the proposed model was exemplified by considering three datasets from entomology, COVID-19, and kidney dialysis patients to estimate the factors related to the unit interval.

Keywords Haq distribution; unit model; moments, reliability measures, Simulation, Biomedical data

AMS Mathematics Subject Classification: 62-XX, 62P10

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1. Introduction

Probability distributions are very important in statistical modeling; offer a structured approach to the analysis of predicting random phenomena. In several real-life situations, data are often restricted to unit intervals: in particular, when the data represents proportions, probabilities, or rates, then the data lies within unit interval. Approaches used to analyze standard statistics that assume infinite data frequently fails to capture the unique characteristics of unit interval data, requiring special unit distributions to be applied. For example, in public health, researchers investigate the rate of people recovering from infectious diseases such as COVID-19, the effectiveness of vaccines and success rates of medical treatments. Environmental science is often based on unit distribution to measure the proportion of land area affected by wildfires, the proportion of air samples that exceed pollution limits, or proportion of plastic waste that is successfully recycled. In engineering, unit models are used to model efficiency of renewable energy sources, such as proportion of sunlight converted into electricity by solar panels. In finance and business analytics, unit distributions are important when assessing the default rates on loans, customer retention in subscription services, and success rate of digital marketing campaigns. In sports analytics, these unit models help to quantify the success rate of free throws in basketball, completion rate of passes in football and how many

*Correspondence to: Corresponding author: rhkhashab@uqu.edu.sa

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games a team wins in a season. Even in the fields of psychology and behavioral science, researchers use these models in order to investigate response accuracy in cognitive tasks and percentage of correct responses among a response set in a behavioral assessment or the probability of a person adopting certain kind of behavior. Beyond these applications, unit distributions have important role in machine learning and artificial intelligence, where probabilities are the basis of predictive modeling, classification problems, and measuring uncertainty. Accurate modeling of unit datasets is essential for optimizing decision-making processes across disciplines. These several uses show how important it is to have probability distributions that work for unit intervals, which makes statistical modeling strong and useful in many areas.

With such widespread uses for unit models, there is a need to make probability distributions that are tailored to data that is bounded within the unit interval. Traditionally, statistical theory for distributions has mainly focused on models with unbounded support. Nevertheless, in recent years, more emphasis has been placed on building distributions that fall within a bounded interval. The most popular approach to doing this is to transform distributions initially defined in $(0,\ \infty)$ into unit intervals. This transformation maintains the most salient aspects of the target distribution without sacrificing much flexibility in modeling data with constraints to a particular range. To convert an unbounded model of probability into a bounded unit distribution of probability, some transformation methods are used frequently in the literature. Examples include $Y = e^{-X}$, $Y = \left(\frac{X}{1+X}\right)$ and $\left(\frac{1}{1+X}\right)$ where X is a positive real-valued random variable. Numerous unit probability distributions have been proposed using these modifications, such as unit Lindley [34], unit Gamma [37], unit Birnbaum-Saunders [35], unit Weibull [36], unit Teissier [32], unit exponentiated Fréchet [2], unit Xgamma [29], unit half-normal [31], unit exponentiated half-logistic [7], unit Muth [33], unit power Burr X [28], unit two parameter Bilal [38], unit Zeghdoudi [18], unit modified Burr III [4], unit XLindley [27], and unit moment exponential distribution [10]. For more details see [11, 12, 13, 14].

The unit interval models have proven to be more flexible and effective for analyzing data that fall within specific limits, such as percentages or proportions. Although numerous unit probability models exist in the literature, the development of new unit distributions remains essential to enhance data fitting, improve flexibility, and address the limitations of existing models. Novel distributions provide better tail behavior, increased computational efficiency, and adaptability to specific real-world applications. Therefore, in this study, we aim to contribute to this ongoing effort by proposing a new unit distribution based on the Haq distribution [3]. This new model is designed to offer better adaptability and accuracy for datasets constrained within the unit interval, providing a fresh tool for researchers and practitioners working with bounded data.

The Haq distribution is obtained by compounding the exponential distribution and the Xgamma distributions. The probability density function (pdf) and cumulative distribution (cdf) of the said distribution are given as follows.

$$f(x;\theta) = \left(\frac{\theta}{1+\theta}\right)^2 \left(2+\theta+\frac{\theta x^2}{2}\right) e^{-\theta x}, \quad x,\ \theta > 0,$$

and

$$F(x;\theta) = 1 - \left(\frac{1+2\theta+\theta^2+\theta x+\frac{(\theta x)^2}{2}}{(1+\theta)^2}\right) e^{-\theta x}.$$

Various researchers studied and extended Haq distribution using different generalization strategies. Some examples are size-biased Haq distribution [30], power Haq distribution [40], and unit Haq distribution [16]. The NUH distribution provides substantial shape flexibility on the unit interval. Its skewness coefficient varies from positive to negative values as the parameter changes, while the kurtosis coefficient also exhibits considerable variation. In addition, different parameter values produce distinct hazard-rate shapes. These properties make the NUH distribution suitable for modeling bounded data with diverse skewness, peakedness, and failure-rate patterns. To deal with this, we suggest New Unit Haq (NUH) distribution which is specially designed to fit skewed data. The motivation for the NUH distribution is twofold. First, it offers a parsimonious unit model that offers flexibility without introducing any additional parameters. Second, it serves as a practical alternative for modeling bounded data. The utility of the proposed distribution is proved by the fact that applying it to real datasets in epidemiology,

biomedical research, and physical sciences results in better performance than renowned considered well-established bounded models.

The utility of the proposed distribution is demonstrated through applications to real datasets in epidemiology, biomedical research, and physical sciences, where it outperforms several well-established models

The following are the objectives of the study.

- To introduce a new unit distribution capable of handling skewed datasets effectively.
- To derive and investigate its key statistical properties, including moments, reliability characteristics, order statistics, stochastic ordering, and record values.
- To develop methods for parameter estimation of the proposed distribution using maximum likelihood, Anderson-Darling, Cramer-von Mises, ordinary least squares, weighted least squares, and maximum product spacing approaches.
- To assess the goodness-of-fit of the proposed distribution through real-world data fitting, comparing it with existing unit interval distributions.

The rest of the paper is structured as follows. Section 2 is based on the derivation of the new distribution. Some reliability characteristics are discussed in Section 3. Key structural properties are derived and investigated in Section 4. An extensive statistical inference is discussed in Section 5. The adaptability and flexibility of the new unit model are assessed in Section 6, utilizing three datasets related to entomology, COVID-19, and Kidney dialysis patients. Section 7 is based on concluding remarks.

2. Derivation of new unit distribution

Let X be a continuous random variable following the Haq distribution with parameter $\theta > 0$. To construct a unit-interval version with a different density-shape structure from the existing UH distribution, we consider the nonlinear transformation $Y = \left(\frac{1}{X+1}\right)$ implies $X = \left(\frac{1}{Y} - 1\right)$ and we obtain a new unit distribution. The cdf and pdf of the NUH distribution are given in Equations (1) and (2), as follows.

$$F(y;\theta) = \left(\frac{\left(1 + 2\theta + \theta^2\right) + \theta\left(\frac{1}{y} - 1\right) + \frac{\theta^2}{2}\left(\frac{1}{y} - 1\right)^2}{(1+\theta)^2} \right) e^{-\theta\left(\frac{1}{y}-1\right)}, \quad 0 < y < 1, \tag{1}$$

and

$$f(y;\theta) = \left(\frac{\theta}{1+\theta}\right)^2 \left(2 + \theta + \frac{\theta}{2}\left(\frac{1}{y} - 1\right)^2\right) e^{-\theta\left(\frac{1}{y}-1\right)} \frac{1}{y^2}. \tag{2}$$

The limiting behavior of the NUH density function approaches 0 and 1 provides insights into how the probability distribution behaves at its boundaries.

$$\lim_{y\to 0} f(y;\theta) = 0 \;\&\; \lim_{y\to 1} f(y;\theta) = \frac{\theta^2(\theta+2)}{(1+\theta)^2}.$$

This expression shows that when y is close to 0, the density function decreases to zero, implying that higher values of y are increasingly unlikely. As $y \to 1$, the density function approaches a positive constant that depends on the parameter θ.

Figure 1 showcases the possible shapes of the PDF of the NUH distribution for selected parameter values. Notably, the proposed model exhibits positive skewness, making it suitable for modeling both positively skewed, extremely positively skewed, and exponentially increasing datasets.

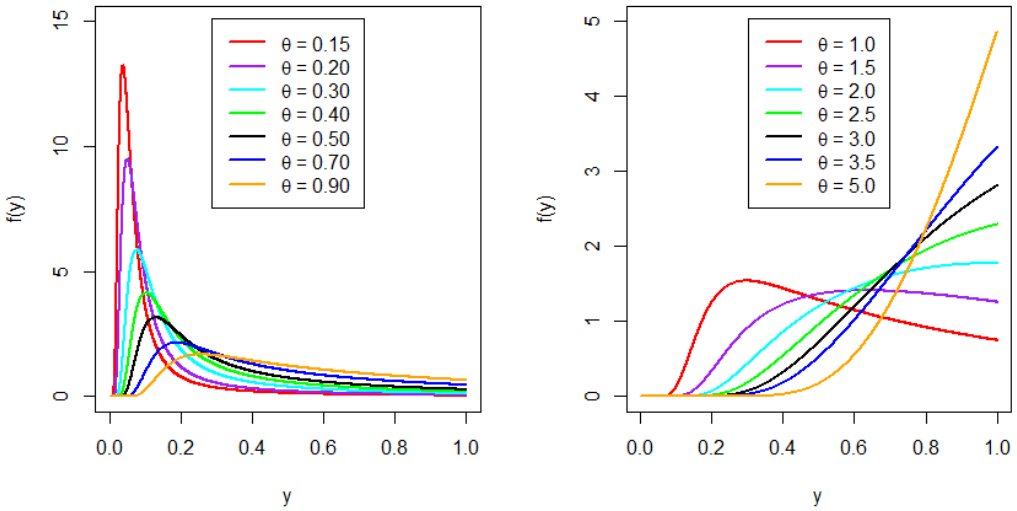


Figure 1. PDFs of the NUH distribution for selected values of the shape parameter θ.

3. Reliability characteristics

In this section of the study, we identify and analyze several key reliability characteristics of the NUH distribution. These characteristics are crucial in evaluating dependability, efficiency, and resilience under various conditions.

3.1. Reliability and hazard function

The reliability and hazard function of the NUH distribution can be determined through their relationship with Equations (1) and (2) and is expressed as follows

$$S(y;\theta) = 1 - \left(\frac{(1+2\theta+\theta^2) + \theta\left(\frac{1}{y}-1\right) + \frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2}{(1+\theta)^2} \right) e^{-\theta\left(\frac{1}{y}-1\right)}. \tag{3}$$

$$h(y;\theta) = \frac{\theta^2\left(2+\theta+\frac{\theta}{2}\left(\frac{1}{y}-1\right)^2\right)e^{-\theta\left(\frac{1}{y}-1\right)}}{y^2\left[(1+\theta)^2 - \left((1+\theta)^2 + \theta\left(\frac{1}{y}-1\right) + \frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2\right)e^{-\theta\left(\frac{1}{y}-1\right)}\right]}. \tag{4}$$

Figure 2 illustrates the effect of parameter θ on the failure rate function $h(y)$ of the NUH distribution. It can be observed that for values of θ ranges (0.15–0.9), $h(y)$ exhibits a bathtub-shaped pattern. It indicates an initial decrease followed by an eventual increase in failure rate. This implies that systems with early life and age-related failure phases may be modeled using the NUH distribution. Later, it also indicates that $h(y)$ becomes monotonically increasing for the values of θ that ranges (1.0–5.0, right panel), indicating a model appropriate for growing failure risk over time.

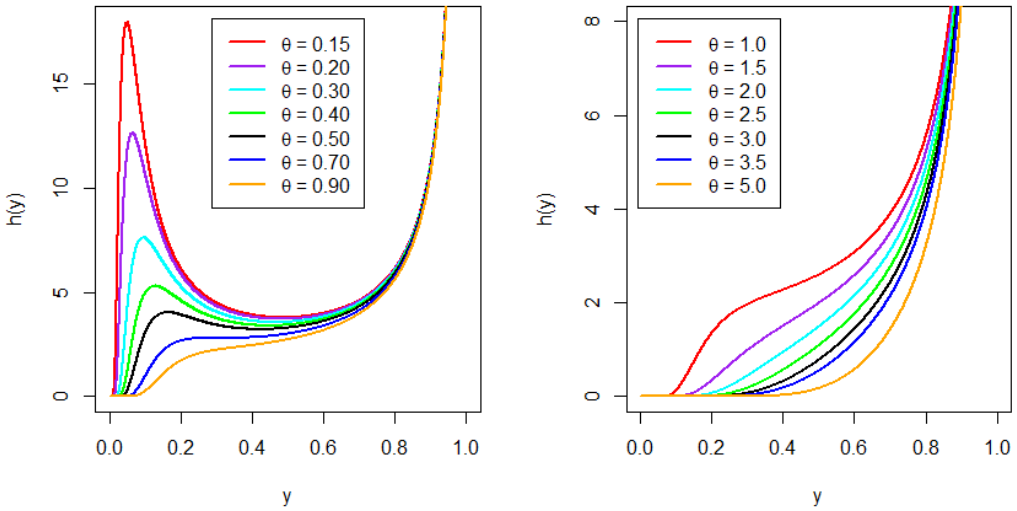


Figure 2. Hazard functions of the NUH distribution for selected values of the shape parameter θ.

3.2. Extensions of hazard rate

Equations (5), and (6) present various extensions of hazard rate, including reverse hazard rate $rh(y;\theta)$, and the cumulative hazard rate $H(y;\theta)$. These extensions provide a comprehensive description of the failure behavior of a system, offering insights into different aspects of reliability analysis and risk assessment.

$$rh(y;\theta) = \frac{\theta^2\left(2+\theta+\frac{\theta}{2}\left(\frac{1}{y}-1\right)^2\right)}{y^2\left[(1+\theta)^2+\theta\left(\frac{1}{y}-1\right)+\frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2\right]}. \tag{5}$$

and

$$H(y;\theta) = -\ln\left(1-\frac{\left((1+2\theta+\theta^2)+\theta\left(\frac{y}{1-y}\right)+\frac{\theta^2}{2}\left(\frac{y}{1-y}\right)^2\right)e^{-\theta\left(\frac{1}{y}-1\right)}}{(1+\theta)^2}\right). \tag{6}$$

3.3. Mills Ratio and Failure Odds function

Equations (7) and (8) introduce the Mills Ratio $M_r(y;\theta)$ and the Failure Odds Function $f_o(y;\theta)$ for the NUH distribution. These functions play a crucial role in characterizing the tail behavior and failure risk of the distribution.

$$M_r(y;\theta) = \frac{y^2\left[(1+\theta)^2-\left((1+\theta)^2+\theta\left(\frac{1}{y}-1\right)+\frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2\right)e^{-\theta\left(\frac{1}{y}-1\right)}\right]}{\theta^2\left(2+\theta+\frac{\theta}{2}\left(\frac{1}{y}-1\right)^2\right)e^{-\theta\left(\frac{1}{y}-1\right)}}. \tag{7}$$

$$f_o(y;\theta) = \frac{\left[(1+\theta)^2 + \theta\left(\frac{1}{y}-1\right) + \frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2\right] e^{-\theta(\frac{1}{y}-1)}}{(1+\theta)^2 - \left[(1+\theta)^2 + \theta\left(\frac{1}{y}-1\right) + \frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2\right] e^{-\theta(\frac{1}{y}-1)}}. \tag{8}$$

4. Statistical properties

This section explores various key statistical properties of the NUH distribution, including moments of origin, fractional positive and fractional negative moments, negative moments, and factorial moments. Furthermore, different generating functions and order statistics are also presented.

4.1. Moments

Theorem 1: The r^{th} moments of the origin of NUH distribution are given are follows

$$\mu'_r = E(Y^r) = \left(\frac{\theta}{1+\theta}\right)^2 \left[(2+\theta)U(1,2-r,\theta) + \theta U(3,4-r,\theta)\right],$$

where $U(a,b,z)$ denotes Tricomi's confluent hypergeometric function.

Proof: Let $Y \sim NUH(y;\theta)$ with the following r^{th} moments expression

$$\mu'_r = \left(\frac{\theta}{1+\theta}\right)^2 \int_0^1 y^{r-2}\left[2+\theta+\frac{\theta}{2}\left(\frac{1}{y}-1\right)^2\right] e^{-\theta(1/y-1)} dy.$$

Let $z = \left(\frac{1}{y}-1\right)$, then

$$\mu'_r = \left(\frac{\theta}{1+\theta}\right)^2 \int_0^\infty (1+z)^{-r}\left(2+\theta+\frac{\theta}{2}z^2\right) e^{-\theta z}\, dz.$$

Using the integral representation $U(a,b,\theta) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-\theta z} z^{a-1}(1+z)^{b-a-1}\, dz$, then

$$\mu'_r = \left(\frac{\theta}{1+\theta}\right)^2 \left[(2+\theta)U(1,2-r,\theta) + \theta U(3,4-r,\theta)\right].$$

The mean of the NUH distribution is expressed as

$$\mu'_1 = \left(\frac{\theta}{1+\theta}\right)^2 \left[(2+\theta)U(1,1,\theta) + \theta U(3,3,\theta)\right].$$

The fractional positive, fractional negative, negative, and factorial moments of the NUH distribution can be derived using their relationships with the rth moment of the origin. These moments provide a comprehensive understanding of the distribution's behavior in various contexts. They are expressed in Equations (9)–(12) respectively, as follows

,

$$E\left(Y^{\frac{p}{q}}\right) = \int_0^1 y^{\frac{p}{q}} f(y;\theta)dy = \left(\frac{\theta}{1+\theta}\right)^2 \left[(2+\theta)U(1,2-\frac{p}{q},\theta) + \theta U(3,4-\frac{p}{q},\theta)\right], \tag{9}$$

$$E\left(Y^{-\frac{p}{q}}\right) = \int_0^1 y^{-\frac{p}{q}} f(y;\theta)dy = \left(\frac{\theta}{1+\theta}\right)^2 \left[(2+\theta)U(1,2+\frac{p}{q},\theta) + \theta U(3,4+\frac{p}{q},\theta)\right], \tag{10}$$

$$E\left(Y^{-r}\right)=\int_0^1 y^{-r}f(y;\theta)dy=\left(\frac{\theta}{1+\theta}\right)^2\left[(2+\theta)U(1,2+r,\theta)+\theta U(3,4+r,\theta)\right], \tag{11}$$

$$E[Y]_n=\sum_{k=0}^{n}\psi_k\mu^{'}{}_r=\sum_{r=0}^{n}\psi_r\left(\frac{\theta}{1+\theta}\right)^2\left[(2+\theta)U(1,2-r,\theta)+\theta U(3,4-r,\theta)\right]. \tag{12}$$

where, $[Y]_n=Y(Y+1)(Y+2)\ldots(Y+i-1)$ and ψ_k is the Stirling number of the first kind. The harmonic mean of the NUH distribution is expressed as

$$HM=E\left(Y^{-1}\right)=\int_0^1 y^{-1}f(y;\theta)dy=\left(\frac{\theta}{1+\theta}\right)^2\left[(2+\theta)U(1,3,\theta)+\theta U(3,5,\theta)\right].$$

4.2. Generating functions

The moment-generating function $\phi(t)$ and the characteristic function $\phi(it)$ of the proposed model are presented in Equations (13) and (14), respectively. These functions can be derived directly by utilizing their relationship with the r*th* moment of the distribution.

$$\phi(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left\{\left(\frac{\theta}{1+\theta}\right)^2\left[(2+\theta)U(1,2-r,\theta)+\theta U(3,4-r,\theta)\right]\right\}, \tag{13}$$

$$\phi(it)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left\{\left(\frac{\theta}{1+\theta}\right)^2\left[(2+\theta)U(1,2-r,\theta)+\theta U(3,4-r,\theta)\right]\right\}. \tag{14}$$

4.3. Order Statistics

Order statistics play an important role in reliability analysis, lifetime modeling, record values, and inferential procedures. Let $Y_1,Y_2,\ldots,Y_n$ be a random sample from the NUH distribution with cdf function $F(y;\theta)$ and pdf $f(y;\theta)$. Denote the corresponding ordered sample by $Y_{1:n}\le Y_{2:n}\le\cdots\le Y_{n:n}$. The pdf of the ith order statistic, $Y_{i:n}$, is generally given by

$$f_{i:n}(y)=\frac{n!}{(i-1)!(n-i)!}[F(y;\theta)]^{i-1}[1-F(y;\theta)]^{n-i}f(y;\theta),\qquad 0<y<1, \tag{15}$$

for $i=1,2,\ldots,n$. Hence, substituting the cdf (1) and pdf (2) into Equation (15), the density function of $Y_{i:n}$ is obtained as

$$\begin{aligned}f_{i:n}(y)=&\frac{n!}{(i-1)!(n-i)!}\left[\frac{A(y;\theta)}{(1+\theta)^2}e^{-\theta\left(\frac{1}{y}-1\right)}\right]^{i-1}\\&\times\left[1-\frac{A(y;\theta)}{(1+\theta)^2}e^{-\theta\left(\frac{1}{y}-1\right)}\right]^{n-i}\\&\times\left(\frac{\theta}{1+\theta}\right)^2\left(2+\theta+\frac{\theta}{2}\left(\frac{1}{y}-1\right)^2\right)e^{-\theta\left(\frac{1}{y}-1\right)}\frac{1}{y^2}.\end{aligned} \tag{16}$$

where $0<y<1$ and $A(y;\theta)=(1+\theta)^2+\theta\left(\frac{1}{y}-1\right)+\frac{\theta^2}{2}\left(\frac{1}{y}-1\right)^2$.

The cumulative distribution function of the ith order statistic is given by

$$F_{i:n}(y)=\sum_{j=i}^{n}\binom{n}{j}[F(y;\theta)]^j[1-F(y;\theta)]^{n-j}, \tag{17}$$

or equivalently,

$$F_{i:n}(y) = \sum_{j=i}^{n} \binom{n}{j} \left[\frac{A(y;\theta)}{(1+\theta)^2} e^{-\theta\left(\frac{1}{y}-1\right)}\right]^j \left[1 - \frac{A(y;\theta)}{(1+\theta)^2} e^{-\theta\left(\frac{1}{y}-1\right)}\right]^{n-j}. \tag{18}$$

Two important special cases are the minimum and maximum order statistics. The density function of the minimum observation $Y_{1:n}$ is

$$f_{1:n}(y) = n[1 - F(y;\theta)]^{n-1} f(y;\theta), \tag{19}$$

which, for the NUH distribution, becomes

$$f_{1:n}(y) = \frac{n}{y^2}\left[1 - \frac{A(y;\theta)}{(1+\theta)^2} e^{-\theta\left(\frac{1}{y}-1\right)}\right]^{n-1} \left(\frac{\theta}{1+\theta}\right)^2 \left(2 + \theta + \frac{\theta}{2}\left(\frac{1}{y} - 1\right)^2\right) e^{-\theta\left(\frac{1}{y}-1\right)}. \tag{20}$$

Similarly, the density function of the maximum observation $Y_{n:n}$ is

$$f_{n:n}(y) = n[F(y;\theta)]^{n-1} f(y;\theta), \tag{21}$$

and hence

$$f_{n:n}(y) = \frac{n}{y^2}\left[\frac{A(y;\theta)}{(1+\theta)^2}\right]^{n-1} \left(\frac{\theta}{1+\theta}\right)^2 \left(2 + \theta + \frac{\theta}{2}\left(\frac{1}{y} - 1\right)^2\right) e^{-n\theta\left(\frac{1}{y}-1\right)}. \tag{22}$$

5. Estimation of parameter

In this section, six estimation methods, including maximum likelihood (ML), Anderson-Darling (AD), Cramer-von Mises (CvM), maximum product spacing (MPS), ordinary least squares (OLS), and weighted least squares (WLS), are presented [19, 20, 21, 22, 23, 24].

5.1. Maximum Likelihood (ML) Estimation

The following likelihood function is maximized to get the estimate of parameter θ.

$$\ell(\theta) = 2n\log(\theta) - 2n\log(1+\theta) + \sum_{i=1}^{n} \log\left[2 + \theta + \frac{\theta}{2}\left(\frac{1}{y_i} - 1\right)^2\right] - \theta\sum_{i=1}^{n}\left(\frac{1}{y_i} - 1\right) - 2\sum_{i=1}^{n}\log(y_i),$$

and its partial derivative is

$$\frac{\ell(\theta)}{\partial\theta} = \frac{2n}{\theta(1+\theta)} + \sum_{i=1}^{n} \frac{1 + \frac{1}{2}\left(\frac{1}{y_i} - 1\right)^2}{2 + \theta + \frac{\theta}{2}\left(\frac{1}{y_i} - 1\right)^2} - \sum_{i=1}^{n}\left(\frac{1}{y_i} - 1\right).$$

Equating the partial derivative of the likelihood function to zero and solving numerically gives an estimate.

5.2. Method of maximum product spacing (MPS) Estimation

A method named MPS was proposed by [25]. This method served as an alternative to ML estimation. In this method geometric mean of spacing is maximized to get an estimate.

$$MPS\,(y_i,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \ln \Delta_i\,(y_i),$$

where $\Delta_i(y_i) = F(y_{i:n},\theta) - F(y_{i-1:n},\theta)$, $F(y_{0:n},\theta) = 0$ and $F(y_{n+1:1},\theta) = 1$.

$$\frac{\partial MPS(y_i,\theta)}{\partial\theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{F_{\theta(i)} - F_{\theta(i-1)}}{\Delta_i(y_i)}$$

Numerical solution of this non-linear equation provides an estimate of θ.

5.3. *Method of Anderson-Darling (AD) Estimation*

The AD estimation is based on the Anderson-Darling goodness-of-fit statistic introduced by [17]. The AD estimates $\widehat{\theta}_{AD}$ of the parameters θ is obtained by minimizing

$$AD(y_i,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(y_{i:n};\theta) + \log S(y_{n+1-i:n};\theta)\right].$$

and

$$\frac{\partial AD(y_i,\theta)}{\partial\theta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\theta(i)}}{F(y_{i:n};\theta)} - \frac{F_{\theta(n+1-i)}}{1-F(y_{n+1-i:n};\theta)}\right].$$

The above partial derivative is minimized, and numerical estimation provides an estimate of the parameter.

5.4. *Method of Cramer von mises (CvM) Estimation*

A method introduced by [26], is based on minimizing the distance between the distribution CDF and the empirical distribution function.

$$CvM(y_i,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(y_{i:n};\theta) - \frac{2i-1}{2n}\right)^2,$$

and

$$\frac{\partial CvM(y_i,\theta)}{\partial\theta} = 2\sum_{i=1}^{n}F_{\theta(i)}\left(F(y_{i:n};\theta) - \frac{2i-1}{2n}\right).$$

Numerical solutions of the above equations provide an estimate.

5.5. *Method of ordinary least squares (OLS) Estimation*

The OLS estimate is obtained by minimizing the objective function specified below,
$OLS(y_i,\theta) = \sum_{i=1}^{n}\left[F(y_{i:n};\theta) - \frac{i}{n+1}\right]^2$, and $\frac{\partial OLS(y_i,\theta)}{\partial\theta} = 2\sum_{i=1}^{n}F_{\theta(i)}\left(F(y_{i:n};\theta) - \frac{i}{n+1}\right)$.
Numerical solutions serve the purpose of getting an estimate.

5.6. *Method of weighted least square (WLS) Estimation*

This subsection represents the WLS estimators for the parameters δ and τ of the NUH distribution, following the approach suggested by [39]. The WLS estimates $\widehat{\delta}$ and $\widehat{\tau}$ are obtained minimizing the distance given below.

$$WLS(y_i,\theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n+1-i)i}\left[F(y_{i:n};\theta) - \frac{i}{n+1}\right]^2,$$

and

$$\frac{\partial WLS\left(y_i,\theta\right)}{\partial\theta}=2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n+1-i)i}F_{\theta(i)}\left(F\left(y_{i:n};\theta\right)-\frac{i}{n+1}\right).$$

Numerical estimation is used to get the required estimate.

5.7. Numerical results of estimation and performance

Here, a detailed illustration of parameter estimation and performance-related measures is computed for the above-presented estimation methods in subsections 5.1–5.6. The estimated parameter (θ) of the NUH distribution is obtained by each method for each randomly selected sample (20, 50, 100, 150, 250), and then performance-related measures like absolute bias, mean relative error, and mean squared error are computed for 10,000 repetitions of each sample. The formulas for each goodness-of-fit method are given as:

$$AB=\frac{1}{10,000}\sum_{i=1}^{10,000}\left|\widehat{\theta}_i-\theta\right|,\ MRE=\frac{1}{10,000}\sum_{i=1}^{10,000}\frac{\left|\widehat{\theta}_i-\theta\right|}{\theta},\ and\ MSE=\frac{1}{10,000}\sum_{i=1}^{10,000}\left(\widehat{\theta}_i-\theta\right)^2$$

This simulation study was carried out for different parameters $\theta=(0.25,\ 0.50,\ 0.75,\ 1.00,\ 1.25,\ 1.50,\ 1.75,\ 2.00,\ 3.00,\ 5.00)$. The results of the simulation study are presented in tables numbered Tables 1–10.

Table 1. The parameter estimates based on different estimation approaches for parameter $\theta=0.25$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
20	$Bias\left(\widehat{\theta}\right)$	$0.00693^{(6)}$	$0.00422^{(3)}$	$0.00568^{(5)}$	$0.00443^{(4)}$	$0.00408^{(2)}$	$0.00357^{(1)}$
	$MRE\left(\widehat{\theta}\right)$	$0.02771^{(6)}$	$0.01689^{(3)}$	$0.02270^{(5)}$	$0.01771^{(4)}$	$0.01632^{(2)}$	$0.01426^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00155^{(2)}$	$0.00164^{(3)}$	$0.00186^{(6)}$	$0.00184^{(5)}$	$0.00173^{(4)}$	$0.00138^{(1)}$
	Sum Ranks	14	9	16	13	8	3
50	$Bias\left(\widehat{\theta}\right)$	$0.00254^{(5)}$	$0.00142^{(1.5)}$	$0.00204^{(4)}$	$0.00153^{(3)}$	$0.00142^{(1.5)}$	$0.00276^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.01014^{(5)}$	$0.00568^{(1)}$	$0.00815^{(4)}$	$0.00611^{(3)}$	$0.00570^{(2)}$	$0.01105^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00054^{(2)}$	$0.00059^{(3)}$	$0.00065^{(5.5)}$	$0.00065^{(5.5)}$	$0.00060^{(4)}$	$0.00052^{(1)}$
	Sum Ranks	12	5.5	13.5	11.5	7.5	13
100	$Bias\left(\widehat{\theta}\right)$	$0.00091^{(4)}$	$0.00057^{(1)}$	$0.00097^{(5)}$	$0.00072^{(3)}$	$0.00059^{(2)}$	$0.00217^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.00364^{(4)}$	$0.00227^{(1)}$	$0.00390^{(5)}$	$0.00288^{(3)}$	$0.00238^{(2)}$	$0.00869^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00027^{(2)}$	$0.00029^{(3)}$	$0.00033^{(5.5)}$	$0.00033^{(5.5)}$	$0.00030^{(4)}$	$0.00026^{(1)}$
	Sum Ranks	10	5	15.5	11.5	8	13
150	$Bias\left(\widehat{\theta}\right)$	$0.00076^{(4)}$	$0.00055^{(1)}$	$0.00079^{(5)}$	$0.00062^{(3)}$	$0.00060^{(2)}$	$0.00146^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.00304^{(4)}$	$0.00221^{(1)}$	$0.00316^{(5)}$	$0.00247^{(3)}$	$0.00242^{(2)}$	$0.00585^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00017^{(1.5)}$	$0.00020^{(3.5)}$	$0.00022^{(5.5)}$	$0.00022^{(5.5)}$	$0.00020^{(3.5)}$	$0.00017^{(1.5)}$
	Sum Ranks	9.5	5.5	15.5	11.5	7.5	13.5
250	$Bias\left(\widehat{\theta}\right)$	$0.00044^{(5)}$	$0.00023^{(1)}$	$0.00034^{(4)}$	$0.00024^{(2)}$	$0.00026^{(3)}$	$0.00101^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.00177^{(5)}$	$0.00090^{(1)}$	$0.00138^{(4)}$	$0.00097^{(2)}$	$0.00106^{(3)}$	$0.00406^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00010^{(1.5)}$	$0.00011^{(3)}$	$0.00013^{(5.5)}$	$0.00013^{(5.5)}$	$0.00012^{(4)}$	$0.00010^{(1.5)}$
	Sum Ranks	11.5	5	13.5	9.5	10	13.5

The simulation results are in Tables 1–11. These tables show the performance of different estimation methods for parameter θ of the NUH distribution at several true values under various sample sizes. It is evident that bias, MRE, and MSE decrease as sample size increases, which indicates that all estimators are consistent. On these grounds, the overall performance of ADE has been observed to be outstanding since it always ranks the minimum out of all the other estimators in Table 11. This is followed by WLSE and OLSE. From MPSE, good performance

Table 2. The parameter estimates based on different estimation approaches for parameter $\theta = 0.50$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.01449$^{(6)}$	0.00831$^{(1)}$	0.01216$^{(5)}$	0.00924$^{(3)}$	0.00836$^{(2)}$	0.00945$^{(4)}$
20	$MRE\left(\widehat{\theta}\right)$	0.02898$^{(6)}$	0.01663$^{(1)}$	0.02432$^{(5)}$	0.01847$^{(3)}$	0.01673$^{(2)}$	0.01891$^{(4)}$
	$MSE\left(\widehat{\theta}\right)$	0.00702$^{(2)}$	0.00740$^{(3)}$	0.00875$^{(6)}$	0.00861$^{(5)}$	0.00799$^{(4)}$	0.00618$^{(1)}$
	Sum Ranks	14	5	16	11	8	9
	$Bias\left(\widehat{\theta}\right)$	0.00657$^{(6)}$	0.00462$^{(1)}$	0.00633$^{(5)}$	0.00514$^{(3)}$	0.00485$^{(2)}$	0.00566$^{(4)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01315$^{(6)}$	0.00925$^{(1)}$	0.01266$^{(5)}$	0.01027$^{(3)}$	0.00969$^{(2)}$	0.01132$^{(4)}$
	$MSE\left(\widehat{\theta}\right)$	0.00250$^{(2)}$	0.00278$^{(3)}$	0.00317$^{(6)}$	0.00314$^{(5)}$	0.00287$^{(4)}$	0.00235$^{(1)}$
	Sum Ranks	14	5	16	11	8	9
	$Bias\left(\widehat{\theta}\right)$	0.00200$^{(5)}$	0.00080$^{(1)}$	0.00150$^{(4)}$	0.00090$^{(2)}$	0.00091$^{(3)}$	0.00506$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00400$^{(5)}$	0.00159$^{(1)}$	0.00300$^{(4)}$	0.00181$^{(2.5)}$	0.00181$^{(2.5)}$	0.01011$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00115$^{(2)}$	0.00131$^{(3)}$	0.00147$^{(5.5)}$	0.00147$^{(5.5)}$	0.00133$^{(4)}$	0.00113$^{(1)}$
	Sum Ranks	12	5	13.5	10	9.5	13
	$Bias\left(\widehat{\theta}\right)$	0.00162$^{(5)}$	0.00089$^{(1)}$	0.00139$^{(4)}$	0.00099$^{(2)}$	0.00101$^{(3)}$	0.00348$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00324$^{(5)}$	0.00178$^{(1)}$	0.00278$^{(4)}$	0.00198$^{(2)}$	0.00202$^{(3)}$	0.00696$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00078$^{(2)}$	0.00089$^{(3)}$	0.00098$^{(5.5)}$	0.00098$^{(5.5)}$	0.00090$^{(4)}$	0.00077$^{(1)}$
	Sum Ranks	12	5	13.5	9.5	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00126$^{(5)}$	0.00087$^{(1)}$	0.00115$^{(4)}$	0.00091$^{(2)}$	0.00096$^{(3)}$	0.00208$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00252$^{(5)}$	0.00173$^{(1)}$	0.00231$^{(4)}$	0.00183$^{(2)}$	0.00191$^{(3)}$	0.00417$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00047$^{(2)}$	0.00053$^{(3)}$	0.00060$^{(5.5)}$	0.00060$^{(5.5)}$	0.00054$^{(4)}$	0.00046$^{(1)}$
	Sum Ranks	12	5	13.5	9.5	10	13

Table 3. The parameter estimates based on different estimation approaches for parameter $\theta = 0.75$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.02551$^{(6)}$	0.01555$^{(2)}$	0.02307$^{(5)}$	0.01829$^{(4)}$	0.01677$^{(3)}$	0.01412$^{(1)}$
20	$MRE\left(\widehat{\theta}\right)$	0.03402$^{(6)}$	0.02073$^{(2)}$	0.03076$^{(5)}$	0.02438$^{(4)}$	0.02236$^{(3)}$	0.01882$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	0.01756$^{(2)}$	0.01914$^{(3)}$	0.02356$^{(6)}$	0.02318$^{(5)}$	0.02115$^{(4)}$	0.01508$^{(1)}$
	Sum Ranks	14	7	16	13	10	3
	$Bias\left(\widehat{\theta}\right)$	0.00887$^{(5)}$	0.00490$^{(1)}$	0.00751$^{(4)}$	0.00555$^{(3)}$	0.00536$^{(2)}$	0.01115$^{(6)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01183$^{(5)}$	0.00654$^{(1)}$	0.01001$^{(4)}$	0.00740$^{(3)}$	0.00714$^{(2)}$	0.01487$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00607$^{(2)}$	0.00689$^{(3)}$	0.00797$^{(6)}$	0.00791$^{(5)}$	0.00719$^{(4)}$	0.00574$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.00459$^{(5)}$	0.00237$^{(1)}$	0.00351$^{(4)}$	0.00252$^{(2)}$	0.00266$^{(3)}$	0.00707$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00611$^{(5)}$	0.00316$^{(1)}$	0.00468$^{(4)}$	0.00336$^{(2)}$	0.00355$^{(3)}$	0.00942$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00302$^{(2)}$	0.00348$^{(3)}$	0.00393$^{(6)}$	0.00391$^{(5)}$	0.00352$^{(4)}$	0.00294$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00353$^{(5)}$	0.00213$^{(1)}$	0.00297$^{(4)}$	0.00231$^{(2)}$	0.00240$^{(3)}$	0.00492$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00470$^{(5)}$	0.00284$^{(1)}$	0.00396$^{(4)}$	0.00309$^{(2)}$	0.00319$^{(3)}$	0.00656$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00198$^{(2)}$	0.00229$^{(3)}$	0.00260$^{(6)}$	0.00259$^{(5)}$	0.00232$^{(4)}$	0.00194$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00168$^{(5)}$	0.00087$^{(1)}$	0.00138$^{(4)}$	0.00098$^{(2)}$	0.00106$^{(3)}$	0.00386$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00224$^{(5)}$	0.00117$^{(1)}$	0.00184$^{(4)}$	0.00131$^{(2)}$	0.00141$^{(3)}$	0.00515$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00114$^{(2)}$	0.00130$^{(3)}$	0.00145$^{(5.5)}$	0.00145$^{(5.5)}$	0.00131$^{(4)}$	0.00113$^{(1)}$
	Sum Ranks	12	5	13.5	9.5	10	13

Table 4. The parameter estimates based on different estimation approaches for parameter $\theta = 1.00$

n	Measure	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.04036$^{(6)}$	0.02471$^{(2)}$	0.03521$^{(5)}$	0.02838$^{(4)}$	0.02585$^{(3)}$	0.01648$^{(1)}$
20	$MRE\left(\widehat{\theta}\right)$	0.04036$^{(6)}$	0.02471$^{(2)}$	0.03521$^{(5)}$	0.02838$^{(4)}$	0.02585$^{(3)}$	0.01648$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	0.03400$^{(2)}$	0.03678$^{(3)}$	0.04514$^{(6)}$	0.04427$^{(5)}$	0.04060$^{(4)}$	0.02853$^{(1)}$
	Sum Ranks	14	7	16	13	10	3
	$Bias\left(\widehat{\theta}\right)$	0.01378$^{(5)}$	0.00789$^{(1)}$	0.01180$^{(4)}$	0.00903$^{(3)}$	0.00836$^{(2)}$	0.01502$^{(6)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01378$^{(5)}$	0.00789$^{(1)}$	0.01180$^{(4)}$	0.00903$^{(3)}$	0.00836$^{(2)}$	0.01502$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.01208$^{(2)}$	0.01370$^{(3)}$	0.01586$^{(6)}$	0.01573$^{(5)}$	0.01422$^{(4)}$	0.01131$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.01087$^{(6)}$	0.00769$^{(2)}$	0.00948$^{(5)}$	0.00807$^{(3)}$	0.00821$^{(4)}$	0.00599$^{(1)}$
100	$MRE\left(\widehat{\theta}\right)$	0.01087$^{(6)}$	0.00769$^{(2)}$	0.00948$^{(5)}$	0.00807$^{(3)}$	0.00821$^{(4)}$	0.00599$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	0.00606$^{(2)}$	0.00686$^{(3)}$	0.00777$^{(6)}$	0.00773$^{(5)}$	0.00697$^{(4)}$	0.00574$^{(1)}$
	Sum Ranks	14	7	16	11	12	3
	$Bias\left(\widehat{\theta}\right)$	0.00488$^{(5)}$	0.00338$^{(1)}$	0.00469$^{(4)}$	0.00375$^{(3)}$	0.00370$^{(2)}$	0.00725$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00488$^{(5)}$	0.00338$^{(1)}$	0.00469$^{(4)}$	0.00375$^{(3)}$	0.00370$^{(2)}$	0.00725$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00382$^{(2)}$	0.00443$^{(3)}$	0.00503$^{(6)}$	0.00502$^{(5)}$	0.00449$^{(4)}$	0.00373$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.00229$^{(5)}$	0.00103$^{(1)}$	0.00164$^{(4)}$	0.00107$^{(2)}$	0.00126$^{(3)}$	0.00567$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00229$^{(5)}$	0.00103$^{(1)}$	0.00164$^{(4)}$	0.00107$^{(2)}$	0.00126$^{(3)}$	0.00567$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00217$^{(2)}$	0.00254$^{(3)}$	0.00286$^{(5.5)}$	0.00286$^{(5.5)}$	0.00256$^{(4)}$	0.00215$^{(1)}$
	Sum Ranks	12	5	13.5	9.5	10	13

Table 5. The parameter estimates based on different estimation approaches for parameter $\theta = 1.25$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.05118$^{(6)}$	0.03268$^{(2)}$	0.04619$^{(5)}$	0.03722$^{(4)}$	0.03341$^{(3)}$	0.02404$^{(1)}$
20	$MRE\left(\widehat{\theta}\right)$	0.04094$^{(6)}$	0.02614$^{(2)}$	0.03696$^{(5)}$	0.02978$^{(4)}$	0.02673$^{(3)}$	0.01923$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	0.06051$^{(2)}$	0.06629$^{(3)}$	0.08028$^{(6)}$	0.07867$^{(5)}$	0.07206$^{(4)}$	0.05058$^{(1)}$
	Sum Ranks	14	7	16	13	10	3
	$Bias\left(\widehat{\theta}\right)$	0.02045$^{(6)}$	0.01094$^{(1)}$	0.01514$^{(4)}$	0.01145$^{(2)}$	0.01160$^{(3)}$	0.01751$^{(5)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01636$^{(6)}$	0.00875$^{(1)}$	0.01212$^{(4)}$	0.00916$^{(2)}$	0.00928$^{(3)}$	0.01401$^{(5)}$
	$MSE\left(\widehat{\theta}\right)$	0.02113$^{(2)}$	0.02416$^{(3)}$	0.02783$^{(6)}$	0.02758$^{(5)}$	0.02503$^{(4)}$	0.01952$^{(1)}$
	Sum Ranks	14	5	14	9	10	11
	$Bias\left(\widehat{\theta}\right)$	0.00920$^{(4)}$	0.00675$^{(1)}$	0.00973$^{(5)}$	0.00788$^{(3)}$	0.00761$^{(2)}$	0.01307$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00736$^{(4)}$	0.00540$^{(1)}$	0.00778$^{(5)}$	0.00631$^{(3)}$	0.00609$^{(2)}$	0.01046$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.01004$^{(2)}$	0.01168$^{(3)}$	0.01329$^{(6)}$	0.01322$^{(5)}$	0.01197$^{(4)}$	0.00969$^{(1)}$
	Sum Ranks	10	5	16	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.00564$^{(5)}$	0.00333$^{(1)}$	0.00505$^{(4)}$	0.00382$^{(3)}$	0.00381$^{(2)}$	0.01039$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00451$^{(5)}$	0.00266$^{(1)}$	0.00404$^{(4)}$	0.00305$^{(2.5)}$	0.00305$^{(2.5)}$	0.00831$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00633$^{(2)}$	0.00734$^{(3)}$	0.00831$^{(6)}$	0.00828$^{(5)}$	0.00743$^{(4)}$	0.00621$^{(1)}$
	Sum Ranks	12	5	14	10.5	8.5	13
	$Bias\left(\widehat{\theta}\right)$	0.00299$^{(5)}$	0.00148$^{(1)}$	0.00246$^{(4)}$	0.00172$^{(2)}$	0.00183$^{(3)}$	0.00755$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00239$^{(5)}$	0.00118$^{(1)}$	0.00197$^{(4)}$	0.00138$^{(2)}$	0.00146$^{(3)}$	0.00604$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00380$^{(2)}$	0.00447$^{(3)}$	0.00507$^{(5.5)}$	0.00507$^{(5.5)}$	0.00452$^{(4)}$	0.00377$^{(1)}$
	Sum Ranks	12	5	13.5	9.5	10	13

Table 6. The parameter estimates based on different estimation approaches for parameter $\theta = 1.50$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
20	$Bias\left(\widehat{\theta}\right)$	$0.06441^{(6)}$	$0.04027^{(2)}$	$0.05881^{(5)}$	$0.04759^{(4)}$	$0.04366^{(3)}$	$0.02955^{(1)}$
	$MRE\left(\widehat{\theta}\right)$	$0.04294^{(6)}$	$0.02685^{(2)}$	$0.03921^{(5)}$	$0.03173^{(4)}$	$0.02911^{(3)}$	$0.01970^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	$0.08973^{(2)}$	$0.09717^{(3)}$	$0.12264^{(6)}$	$0.12024^{(5)}$	$0.10982^{(4)}$	$0.07472^{(1)}$
	Sum Ranks	14	7	16	13	10	3
50	$Bias\left(\widehat{\theta}\right)$	$0.02525^{(6)}$	$0.01385^{(1)}$	$0.01920^{(4)}$	$0.01462^{(2)}$	$0.01466^{(3)}$	$0.02240^{(5)}$
	$MRE\left(\widehat{\theta}\right)$	$0.01683^{(6)}$	$0.00923^{(1)}$	$0.01280_{(4)}$	$0.00975^{(2)}$	$0.00978^{(3)}$	$0.01493^{(5)}$
	$MSE\left(\widehat{\theta}\right)$	$0.03243^{(2)}$	$0.03704^{(3)}$	$0.04279^{(6)}$	$0.04240^{(5)}$	$0.03835^{(4)}$	$0.02992^{(1)}$
	Sum Ranks	14	5	14	9	10	11
100	$Bias\left(\widehat{\theta}\right)$	$0.01314^{(5)}$	$0.00812^{(1)}$	$0.01106^{(4)}$	$0.00875^{(2)}$	$0.00899^{(3)}$	$0.01472^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.00876^{(5)}$	$0.00541^{(1)}$	$0.00737^{(4)}$	$0.00584^{(2)}$	$0.00599^{(3)}$	$0.00981^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.01477^{(2)}$	$0.01708^{(3)}$	$0.01964^{(6)}$	$0.01954^{(5)}$	$0.01739^{(4)}$	$0.01418^{(1)}$
	Sum Ranks	12	5	14	9	10	13
150	$Bias\left(\widehat{\theta}\right)$	$0.00547^{(5)}$	$0.00221^{(1)}$	$0.00442^{(4)}$	$0.00288^{(3)}$	$0.00283^{(2)}$	$0.01459^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.00364^{(5)}$	$0.00147^{(1)}$	$0.00295^{(4)}$	$0.00192^{(3)}$	$0.00189^{(2)}$	$0.00972^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00986^{(2)}$	$0.01153^{(3)}$	$0.01311^{(6)}$	$0.01307^{(5)}$	$0.01169^{(4)}$	$0.00972^{(1)}$
	Sum Ranks	12	5	14	11	8	13
250	$Bias\left(\widehat{\theta}\right)$	$0.00480^{(5)}$	$0.00305^{(1)}$	$0.00413^{(4)}$	$0.00321^{(2)}$	$0.00347^{(3)}$	$0.00846^{(6)}$
	$MRE\left(\widehat{\theta}\right)$	$0.00320^{(5)}$	$0.00203^{(1)}$	$0.00276^{(4)}$	$0.00214^{(2)}$	$0.00231^{(3)}$	$0.00564^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	$0.00593^{(2)}$	$0.00707^{(3)}$	$0.00797^{(6)}$	$0.00796^{(5)}$	$0.00713^{(4)}$	$0.00585^{(1)}$
	Sum Ranks	12	5	14	9	10	13

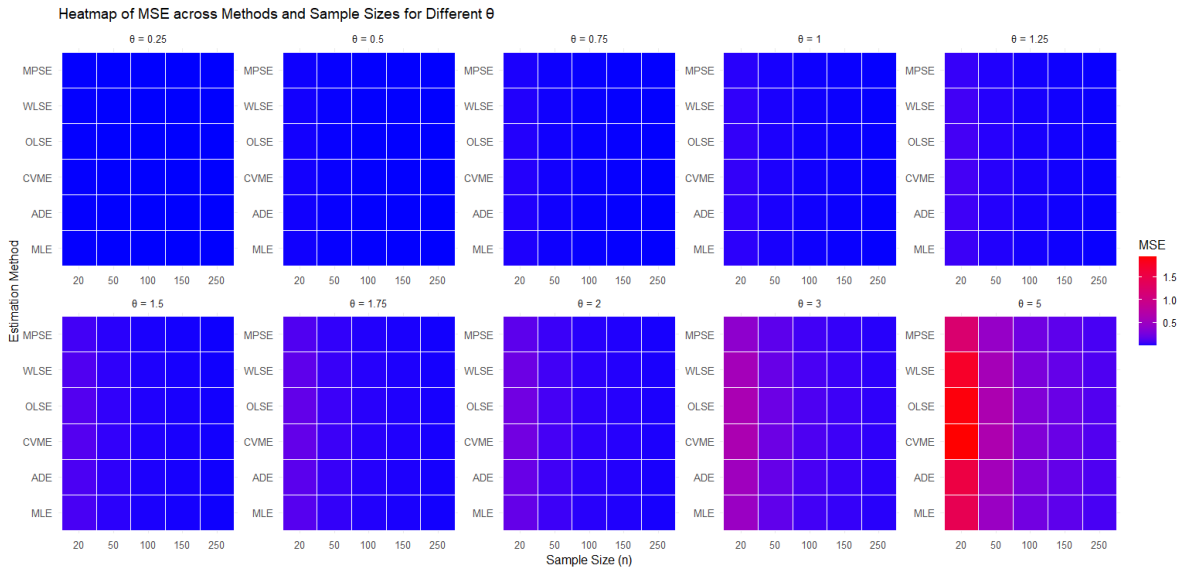


Figure 3. MSE heatmap plots for each choice of parameter θ.

is obtained for small θ; however, the estimator does not exhibit very reliable performance when this parameter has larger values. MLE and CVME generally provide poor performance for all sample sizes and parameters considered here. Heatmap analysis of MSEs in Figure 3 confirms that MLE provides the most accurate estimates with the least

Table 7. The parameter estimates based on different estimation approaches for parameter $\theta = 1.75$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.06529$^{(6)}$	0.03646$^{(1)}$	0.05876$^{(5)}$	0.04539$^{(3)}$	0.03956$^{(2)}$	0.04664$^{(4)}$
20	$MRE\left(\widehat{\theta}\right)$	0.03731$^{(6)}$	0.02083$^{(1)}$	0.03357$^{(5)}$	0.02594$^{(3)}$	0.02260$^{(2)}$	0.02665$^{(4)}$
	$MSE\left(\widehat{\theta}\right)$	0.12986$^{(2)}$	0.14031$^{(3)}$	0.17448$^{(6)}$	0.17126$^{(5)}$	0.15550$^{(4)}$	0.11068$^{(1)}$
	Sum Ranks	14	5	16	11	8	9
	$Bias\left(\widehat{\theta}\right)$	0.03270$^{(6)}$	0.02063$^{(1)}$	0.02781$^{(5)}$	0.02223$^{(3)}$	0.02137$^{(2)}$	0.02485$^{(4)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01869$^{(6)}$	0.01179$^{(1)}$	0.01589$^{(5)}$	0.01270$^{(3)}$	0.01221$^{(2)}$	0.01420$^{(4)}$
	$MSE\left(\widehat{\theta}\right)$	0.04573$^{(2)}$	0.05144$^{(3)}$	0.05961$^{(6)}$	0.05901$^{(5)}$	0.05314$^{(4)}$	0.04195$^{(1)}$
	Sum Ranks	14	5	16	11	8	9
	$Bias\left(\widehat{\theta}\right)$	0.01509$^{(5)}$	0.00800$^{(1)}$	0.01156$^{(4)}$	0.00878$^{(2)}$	0.00884$^{(3)}$	0.01852$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00863$^{(5)}$	0.00457$^{(1)}$	0.00661$^{(4)}$	0.00502$^{(2)}$	0.00505$^{(3)}$	0.01058$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.02221$^{(2)}$	0.02556$^{(3)}$	0.02915$^{(6)}$	0.02901$^{(5)}$	0.02607$^{(4)}$	0.02132$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00839$^{(5)}$	0.00404$^{(1)}$	0.00632$^{(4)}$	0.00447$^{(2)}$	0.00479$^{(3)}$	0.01593$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00479$^{(5)}$	0.00231$^{(1)}$	0.00361$^{(4)}$	0.00255$^{(2)}$	0.00274$^{(3)}$	0.00911$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.01432$^{(2)}$	0.01688$^{(3)}$	0.01911$^{(6)}$	0.01905$^{(5)}$	0.01704$^{(4)}$	0.01403$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00688$^{(5)}$	0.00469$^{(1)}$	0.00627$^{(4)}$	0.00515$^{(2)}$	0.00531$^{(3)}$	0.00919$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00393$^{(5)}$	0.00268$^{(1)}$	0.00358$^{(4)}$	0.00294$^{(2)}$	0.00303$^{(3)}$	0.00525$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.00833$^{(2)}$	0.00994$^{(3)}$	0.01131$^{(6)}$	0.01129$^{(5)}$	0.01002$^{(4)}$	0.00817$^{(1)}$
	Sum Ranks	12	5	14	9	10	13

Table 8. The parameter estimates based on different estimation approaches for parameter $\theta = 2.00$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.08732$^{(6)}$	0.05210$^{(2)}$	0.07708$^{(5)}$	0.06133$^{(4)}$	0.05472$^{(3)}$	0.04489$^{(1)}$
20	$MRE\left(\widehat{\theta}\right)$	0.04366$^{(6)}$	0.02605$^{(2)}$	0.03854$^{(5)}$	0.03067$^{(4)}$	0.02736$^{(3)}$	0.02244$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	0.17919$^{(2)}$	0.19379$^{(3)}$	0.23772$^{(6)}$	0.23285$^{(5)}$	0.21340$^{(4)}$	0.14942$^{(1)}$
	Sum Ranks	14	7	16	13	10	3
	$Bias\left(\widehat{\theta}\right)$	0.03029$^{(5)}$	0.01535$^{(1)}$	0.02324$^{(4)}$	0.01676$^{(3)}$	0.01604$^{(2)}$	0.03703$^{(6)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01515$^{(5)}$	0.00767$^{(1)}$	0.01162$^{(4)}$	0.00838$^{(3)}$	0.00802$^{(2)}$	0.01852$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.06212$^{(2)}$	0.07104$^{(3)}$	0.08133$^{(6)}$	0.08058$^{(5)}$	0.07333$^{(4)}$	0.05791$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.01674$^{(5)}$	0.01169$^{(1)}$	0.01669$^{(4)}$	0.01343$^{(3)}$	0.01285$^{(2)}$	0.02292$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00837$^{(5)}$	0.00585$^{(1)}$	0.00834$^{(4)}$	0.00672$^{(3)}$	0.00642$^{(2)}$	0.01146$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.02829$^{(2)}$	0.03314$^{(3)}$	0.03806$^{(6)}$	0.03785$^{(5)}$	0.03383$^{(4)}$	0.02723$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.01038$^{(5)}$	0.00541$^{(1)}$	0.00812$^{(4)}$	0.00595$^{(2)}$	0.00618$^{(3)}$	0.01831$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00519$^{(5)}$	0.00270$^{(1)}$	0.00406$^{(4)}$	0.00297$^{(2)}$	0.00309$^{(3)}$	0.00916$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.01989$^{(2)}$	0.02286$^{(3)}$	0.02590$^{(6)}$	0.02581$^{(5)}$	0.02311$^{(4)}$	0.01948$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00693$^{(5)}$	0.00467$^{(1)}$	0.00657$^{(4)}$	0.00527$^{(2)}$	0.00536$^{(3)}$	0.01204$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00346$^{(5)}$	0.00234$^{(1)}$	0.00329$^{(4)}$	0.00263$^{(2)}$	0.00268$^{(3)}$	0.00602$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.01150$^{(2)}$	0.01357$^{(3)}$	0.01541$^{(6)}$	0.01538$^{(5)}$	0.01370$^{(4)}$	0.01132$^{(1)}$
	Sum Ranks	12	5	14	9	10	13

Table 9. The parameter estimates based on different estimation approaches for parameter $\theta = 3.0$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.16342$^{(6)}$	0.10392$^{(2)}$	0.14027$^{(5)}$	0.11493$^{(4)}$	0.10656$^{(3)}$	0.04906$^{(1)}$
20	$MRE\left(\widehat{\theta}\right)$	0.05447$^{(6)}$	0.03464$^{(2)}$	0.04676$^{(5)}$	0.03831$^{(4)}$	0.03552$^{(3)}$	0.01635$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	0.48524$^{(2)}$	0.52054$^{(3)}$	0.63027$^{(6)}$	0.61610$^{(5)}$	0.56764$^{(4)}$	0.39648$^{(1)}$
	Sum Ranks	14	7	16	13	10	3
	$Bias\left(\widehat{\theta}\right)$	0.05437$^{(6)}$	0.03186$^{(1)}$	0.04495$^{(4)}$	0.03457$^{(3)}$	0.03336$^{(2)}$	0.05288$^{(5)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01812$^{(6)}$	0.01062$^{(1)}$	0.01498$^{(4)}$	0.01152$^{(3)}$	0.01112$^{(2)}$	0.01763$^{(5)}$
	$MSE\left(\widehat{\theta}\right)$	0.16161$^{(2)}$	0.18330$^{(3)}$	0.21196$^{(6)}$	0.20996$^{(5)}$	0.19058$^{(4)}$	0.14878$^{(1)}$
	Sum Ranks	14	5	14	11	8	11
	$Bias\left(\widehat{\theta}\right)$	0.01850$^{(5)}$	0.01040$^{(1)}$	0.01844$^{(4)}$	0.01329$^{(3)}$	0.01292$^{(2)}$	0.04393$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00617$^{(5)}$	0.00347$^{(1)}$	0.00615$^{(4)}$	0.00443$^{(3)}$	0.00431$^{(2)}$	0.01464$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.07481$^{(2)}$	0.08820$^{(3)}$	0.10156$^{(6)}$	0.10109$^{(5)}$	0.09043$^{(4)}$	0.07256$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.02149$^{(5)}$	0.01415$^{(1)}$	0.01895$^{(4)}$	0.01549$^{(2)}$	0.01569$^{(3)}$	0.02348$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00716$^{(5)}$	0.00472$^{(1)}$	0.00632$^{(4)}$	0.00516$^{(2)}$	0.00523$^{(3)}$	0.00783$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.04793$^{(2)}$	0.05661$^{(3)}$	0.06472$^{(6)}$	0.06447$^{(5)}$	0.05749$^{(4)}$	0.04609$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.00890$^{(4)}$	0.00596$^{(1)}$	0.00900$^{(5)}$	0.00692$^{(3)}$	0.00688$^{(2)}$	0.01849$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00297$^{(4)}$	0.00199$^{(1)}$	0.00300$^{(5)}$	0.00231$^{(3)}$	0.00229$^{(2)}$	0.00616$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.02923$^{(2)}$	0.03463$^{(3)}$	0.03897$^{(6)}$	0.03889$^{(5)}$	0.03485$^{(4)}$	0.02816$^{(1)}$
	Sum Ranks	10	5	16	11	8	13

Table 10. The parameter estimates based on different estimation approaches for parameter $\theta = 5.00$

n	Estimate	MLE	ADE	CVME	OLSE	WLSE	MPSE
	$Bias\left(\widehat{\theta}\right)$	0.24302$^{(6)}$	0.14888$^{(2)}$	0.22157$^{(5)}$	0.17816$^{(4)}$	0.15859$^{(3)}$	0.12730$^{(1)}$
20	$MRE\left(\widehat{\theta}\right)$	0.04860$^{(6)}$	0.02978$^{(2)}$	0.04431$^{(5)}$	0.03563$^{(4)}$	0.03172$^{(3)}$	0.02546$^{(1)}$
	$MSE\left(\widehat{\theta}\right)$	1.43588$^{(2)}$	1.55723$^{(3)}$	1.94242$^{(6)}$	1.90580$^{(5)}$	1.75331$^{(4)}$	1.20128$^{(1)}$
	Sum Ranks	14	7	16	13	10	3
	$Bias\left(\widehat{\theta}\right)$	0.09057$^{(5)}$	0.05549$^{(1)}$	0.08169$^{(4)}$	0.06375$^{(3)}$	0.05970$^{(2)}$	0.09982$^{(6)}$
50	$MRE\left(\widehat{\theta}\right)$	0.01811$^{(5)}$	0.01110$^{(1)}$	0.01634$^{(4)}$	0.01275$^{(3)}$	0.01194$^{(2)}$	0.01996$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.48410$^{(2)}$	0.55551$^{(3)}$	0.64420$^{(6)}$	0.63793$^{(5)}$	0.57767$^{(4)}$	0.44985$^{(1)}$
	Sum Ranks	12	5	14	11	8	13
	$Bias\left(\widehat{\theta}\right)$	0.04437$^{(5)}$	0.02447$^{(1)}$	0.03531$^{(4)}$	0.02627$^{(2)}$	0.02670$^{(3)}$	0.06740$^{(6)}$
100	$MRE\left(\widehat{\theta}\right)$	0.00887$^{(5)}$	0.00489$^{(1)}$	0.00706$^{(4)}$	0.00525$^{(2)}$	0.00534$^{(3)}$	0.01348$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.23816$^{(2)}$	0.28054$^{(3)}$	0.31831$^{(6)}$	0.31679$^{(5)}$	0.28548$^{(4)}$	0.22934$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.02517$^{(5)}$	0.01313$^{(1)}$	0.02066$^{(4)}$	0.01464$^{(2)}$	0.01515$^{(3)}$	0.05637$^{(6)}$
150	$MRE\left(\widehat{\theta}\right)$	0.00503$^{(5)}$	0.00263$^{(1)}$	0.00413$^{(4)}$	0.00293$^{(2)}$	0.00303$^{(3)}$	0.01127$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.15394$^{(2)}$	0.17590$^{(3)}$	0.19840$^{(6)}$	0.19780$^{(5)}$	0.17797$^{(4)}$	0.15114$^{(1)}$
	Sum Ranks	12	5	14	9	10	13
	$Bias\left(\widehat{\theta}\right)$	0.01733$^{(5)}$	0.00861$^{(1)}$	0.01285$^{(4)}$	0.00923$^{(2)}$	0.01006$^{(3)}$	0.03658$^{(6)}$
250	$MRE\left(\widehat{\theta}\right)$	0.00347$^{(5)}$	0.00172$^{(1)}$	0.00257$^{(4)}$	0.00185$^{(2)}$	0.00201$^{(3)}$	0.00732$^{(6)}$
	$MSE\left(\widehat{\theta}\right)$	0.08977$^{(2)}$	0.10456$^{(3)}$	0.11709$^{(6)}$	0.11687$^{(5)}$	0.10498$^{(4)}$	0.08821$^{(1)}$
	Sum Ranks	12	5	14	9	10	13

Table 11. Partial and overall ranks of all the methods of estimation

θ	n	MLE	ADE	CVME	OLSE	WLSE	MPSE
	20	14	9	16	13	8	3
	50	12	5.5	13.5	11.5	7.5	13
$\theta = 0.25$	100	10	5	15.5	11.5	8	13.5
	150	9.5	5.5	15.5	11.5	7.5	13.5
	250	11.5	5	13.5	9.5	10	13
	20	14	5	16	11	8	9
	50	14	5	16	11	8	9
$\theta = 0.50$	100	12	5	13.5	10	9.5	13
	150	12	5	13.5	9.5	10	13
	250	12	5	13.5	9.5	10	13
	20	14	7	16	13	10	3
	50	12	5	14	11	8	13
$\theta = 0.75$	100	12	5	14	9	10	13
	150	12	5	14	9	10	13
	250	12	5	13.5	9.5	10	13
	20	14	7	16	13	10	3
	50	12	5	14	11	8	13
$\theta = 1.0$	100	14	7	16	11	12	3
	150	12	5	14	11	8	13
	250	12	5	13.5	9.5	10	13
	20	14	7	16	13	10	3
	50	14	5	14	9	10	11
$\theta = 1.25$	100	10	5	16	11	8	13
	150	12	5	14	10.5	8.5	13
	250	12	5	13.5	9.5	10	13
	20	14	7	16	13	10	3
	50	14	5	14	9	10	11
$\theta = 1.50$	100	12	5	14	9	10	13
	150	12	5	14	11	8	13
	250	12	5	14	9	10	13
	20	14	5	16	11	8	9
	50	14	5	16	11	8	9
$\theta = 1.75$	100	12	5	14	9	10	13
	150	12	5	14	9	10	13
	250	12	5	14	9	10	13
	20	14	7	16	13	10	3
	50	12	5	14	11	8	13
$\theta = 2.00$	100	12	5	14	11	8	13
	150	12	5	14	9	10	13
	250	12	5	14	9	10	13
	20	14	7	16	13	10	3
	50	14	5	14	11	8	11
$\theta = 3.00$	100	12	5	14	11	8	13
	150	12	5	14	9	10	13
	250	10	5	16	11	8	13
	20	14	7	16	13	10	3
	50	12	5	14	11	8	13
$\theta = 5.00$	100	12	5	14	9	10	13
	150	12	5	14	9	10	13
	250	12	5	14	9	10	13
Sum Ranks		623	271	729	527	461	539
Overall ranks		5	1	6	3	2	4

bias and MSE, while ADE, WLSE, and OLSE produce comparable results. MPSE improves with larger sample sizes. We conclude hereby that ADE is the most robust and reliable estimator in NUH distribution, while WLSE follows as a good alternative. MLE provided the best absolute accuracy for larger sample sizes.

6. Applications

This section demonstrates the effectiveness of the proposed NUH distribution by applying it to three real-life data sets. The first data set represents COVID-19 death rates in France, whereas the second data set consists of time (in months) prior to infection in kidney dialysis patients, and the third dataset is related to entomology. Using these three datasets, the study evaluates the performance of the proposed NUH distribution by applying the MLE method. It then compares the results with several well-known statistical distributions. The considered competitive distributions are Kumaraswamy (Kw) distribution, Beta distribution, generalized exponential uniform (GEU) distribution [5], unit moment exponential (UMEx) distribution [10], Topp-Leone distribution, unit Haq (UH) distribution [16], unit Lindley (UL) distribution [34], unit Xgamma (UXg) distribution [29], log-Xgamma (LXg) distribution [15], and unit XLindley (UXL) distribution (Eliwa et al., 2022).

Now the next phase is to identify the statistical evaluation procedures to verify the most adaptive and efficient model for modeling three datasets. For this purpose, we rely on log-likelihood (l), Akaike information criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov-Smirnov (KS) test statistic and associated p-value.

6.1. COVID-19 Data

The first dataset reflects the COVID-19 mortality rates in France over a specific period from November 3 to November 11, 2021. This data captures the number of deaths attributed to COVID-19 during those nine days. The data set is: 0.4823, 0.6861, 0.1458, 0.3640, 0.1163, 0.7850, 0.6095, 0.4491, 0.2458, 0.6360, 0.2308, and 0.2448 [8]. Mortality rates from COVID-19 in France during the second week of November 2021 were mostly stable and ranged from 0.1163 to 0.7850. Since the mean (0.4163) and median (0.4066) are almost the same, this suggests a symmetric distribution of the data. The slight skew towards positivity (0.2041) shows that some values are higher than the rest, and the low kurtosis (1.6876) suggests the distribution is even without many extreme scores. We also illustrate the behavior of this dataset graphically and present it in Figure 4.

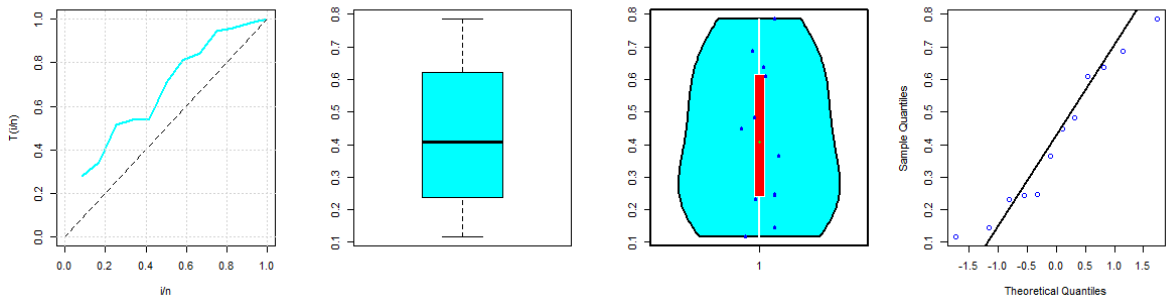


Figure 4. The non-parametric plots based on the COVID-19 data.

The model-fitting results in Table 12 indicate that the NUH distribution provides a comparatively good fit to the COVID-19 mortality data according to the reported information criteria and goodness-of-fit measures. The NUH model has AIC = -2.7965, BIC = -2.3116, KS statistic = 0.1202, and KS p-value = 0.9866. The relatively small AIC and BIC values, together with the relatively small KS statistic and large KS p-value, indicate good agreement between the fitted NUH distribution and the empirical data according to these criteria. The UXg and UMEx distributions also show relatively large KS p-values of 0.9036 and 0.8308, respectively, indicating that they provide acceptable fits under the KS criterion. In contrast, the UL model has a KS statistic of 1.2293 and p-value below 0.001, indicating substantial disagreement with the empirical distribution. These results suggest that NUH provides a competitive fit for the COVID-19 data among the models considered. We also plot the fitted PDF and CDF, PP plot, and profile log-likelihood plot for the first dataset, as presented in Figure 5.

Table 12. MLEs and model selection measures for the COVID-19 data

Models	MLE (SE) $\hat{\theta}$	$\hat{\beta}$	ℓ	AIC	BIC	K-S	p-value
Kw	1.7465 (0.5195)	2.9522 (1.3761)	2.4453	−0.8908	0.0791	0.1829	0.7532
Beta	1.9862 (0.7631)	2.7768 (1.1046)	2.4864	−0.9728	−0.0030	0.1824	0.7561
GEU	1.4817 (1.0283)	1.3896 (0.6147)	2.4321	−0.8642	0.1055	0.1987	0.6605
UMEx	0.5191 (0.1059)	–	2.2584	−2.5167	−2.0317	0.1685	0.8308
Topp	1.7000 (0.4907)	–	2.2838	−2.5677	−2.0828	0.1774	0.7841
UH	1.3223 (0.2997)	–	−0.3673	2.7346	3.2195	0.2389	0.4326
UL	1.3459 (0.3006)	–	2.1633	−2.3267	−1.8417	1.2293	0.0000
UXg	1.6421 (0.3489)	–	2.1979	−2.3959	−1.9111	0.1526	0.9036
LXg	1.1625 (0.2737)	–	0.1815	1.6368	2.1218	0.2295	0.4820
UXL	1.1625 (0.2737)	–	0.1815	1.6367	2.1218	0.2108	0.5886
NUH	0.7002 (0.1438)	–	2.3983	−2.7965	−2.3116	0.1202	0.9866

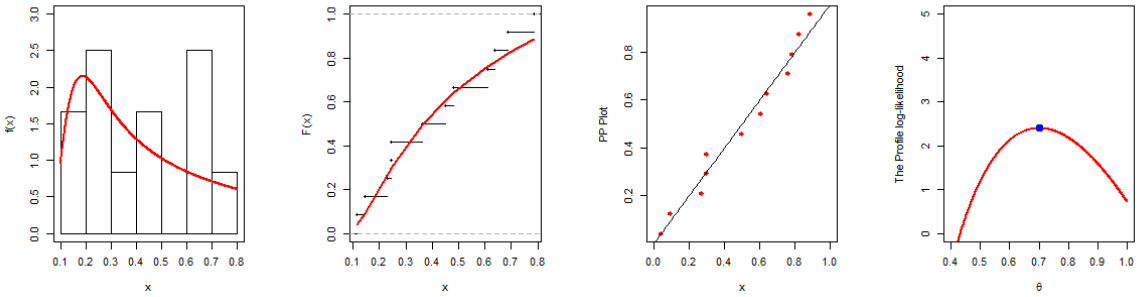


Figure 5. The fitted PDF, CDF, PP, and profile log-likelihood plot for the COVID-19 data.

6.2. Kidney Dialysis Data

This dataset represents the infection onset times (in months) for patients undergoing kidney dialysis, as reported by a specific study. The values include measurements such as 0.083333, 0.116667, 0.083333, 0.116667, 0.483333, 0.116667, 0.150000, 0.416667, 0.250000, 0.183333, 0.216667, 0.216667, 0.250000, 0.916667, 0.250000, 0.283333, 0.316667, 0.350000, 0.383333, 0.416667, 0.450000, 0.483333, 0.716667, 0.716667, 0.750000, 0.250000, 0.750000, and 0.850000 [6]. This transformation helps standardize the data for comparative distributional analysis. We also illustrate the behavior of this dataset graphically and present it in Figure 6.

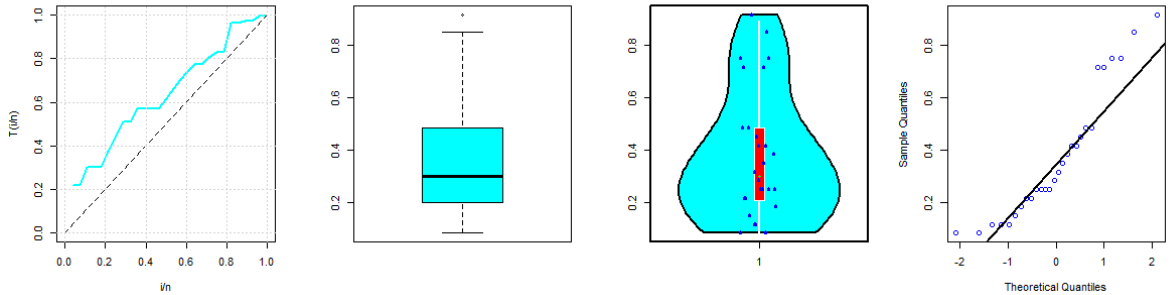


Figure 6. The non-parametric plots based on the kidney dialysis data.

Table 13. MLEs and model selection measures for the kidney dialysis data

Models	MLE (SE) $\hat{\theta}$	MLE (SE) $\hat{\beta}$	ℓ	AIC	BIC	K-S	p-value
Kw	1.2651 (0.2544)	2.0798 (0.5714)	3.6624	−3.3249	−0.6605	0.1377	0.6629
Beta	1.3566 (0.3332)	2.1057 (0.5496)	3.7776	−3.5552	−0.8908	0.1412	0.6321
GEU	0.5994 (0.3065)	0.8216 (0.3234)	−1.7434	7.4869	10.1514	0.2660	0.0380
UMEx	0.5986 (0.0800)	–	3.9957	−5.9915	−4.6593	0.1369	0.6695
Topp	1.3778 (0.2603)	–	3.8524	−5.7048	−4.3726	0.1442	0.6056
UH	1.1906 (0.1734)	–	−0.2870	2.5740	3.9062	0.2561	0.0507
UL	1.1217 (0.1616)	–	−5.2724	12.5448	13.8770	2.9202	0.0000
UXg	1.3062 (0.1809)	–	−5.1461	12.2922	13.6244	0.3318	0.0041
LXg	1.5505 (0.2116)	–	1.1873	−0.3745	0.9577	0.2379	0.0839
UXL	1.0315 (0.1566)	–	0.9843	0.0313	1.3635	0.2275	0.1102
NUH	0.5622 (0.0736)	–	6.3332	−10.6665	−9.3343	0.1010	0.9377

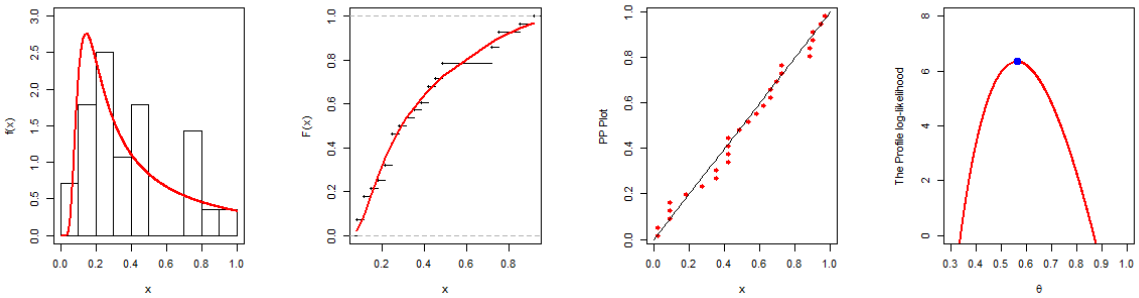


Figure 7. The fitted PDF, CDF, PP, and profile log-likelihood plot for the kidney dialysis data.

The statistical analysis of the time of infection onset amongst patients receiving kidney dialysis indicates that the NUH model fits the observed data the best. This model yielded the best log-likelihood value (6.3332), lowest AIC and BIC value (-10.6665 and -9.3343) and it provides a better model adequacy. The Kolmogorov-Smirnov value (0.1010) and the p-value (0.9377) help determine that the NUH model fits the empirical distribution very well and there is no significant difference between the model and the observed characteristics. The UMEx, Topp-Leone, Kw, and Beta models also fared reasonably well with p-values of over 0.60, which is an indication of the acceptable levels of fitting. Conversely, the UL and UXg models were performing poorly as indicated by low p-values and high values of AIC and BIC, which implies that it was deviating widely from the data. According to these findings, the NUH model is the most suitable model in a statistical sense to describe the way the patients under dialysis are distributed in the encouragement of the time of infection onset.

6.3. Stegobium paniceum Data

The third dataset was initially analyzing the number of new produced F_1 adults of *Stegobium paniceum* originating from parents that fed peppermint taken from both unirradiated and irradiated packets (non-choice non-packet test). The peppermint samples were subjected to gamma radiation at dose 6, 8, and 10 KGy, as well as microwave exposure for 1, 2, 3 minutes, resulting in the following observed observations 148, 42, 85, 31, 87, 145, 48, 33, 64, 152, 65, 46, 57, 87, 52, 67, 64, 36, 59, 69, and 64 [1]. Further, Alomani et al. [9] normalized this dataset to the unit interval $[0, 1]$, using the transformation $\left\{X = \frac{X_i}{\max(X_i)+1}\right\}$, to standardize the scaling of the datasets. The conforming bounded observations are: 0.9673, 0.2745, 0.5556, 0.2026, 0.5686, 0.9477, 0.3137, 0.2157, 0.4183, 0.9935, 0.4248, 0.3007, 0.3725, 0.5686, 0.3399, 0.4379, 0.4183, 0.2353, 0.3856, 0.4510, and 0.4183. We also illustrate the behavior of this dataset graphically and present it in Figure 8.

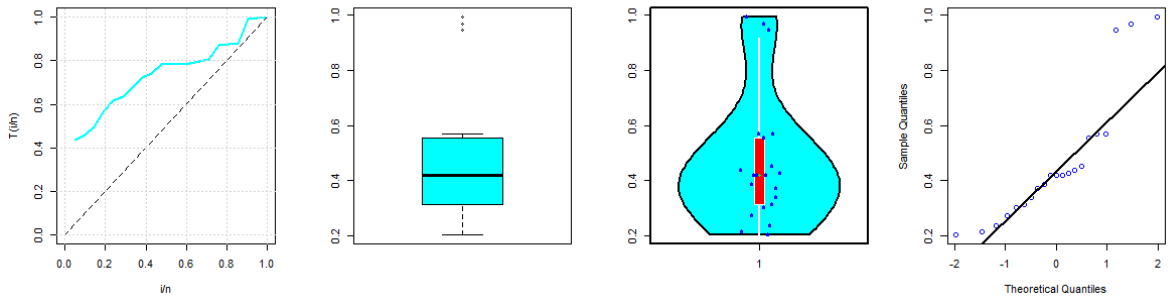


Figure 8. The non-parametric plots based on the entomology data.

Table 14. MLEs and model selection measures for the entomology data

	MLE (SE)						
Models	$\hat{\theta}$	$\hat{\beta}$	ℓ	AIC	BIC	K-S	p-value
Kw	1.3115 (0.3364)	1.2376 (0.3491)	0.4848	3.0302	5.1193	0.3055	0.0396
Beta	1.3654 (0.3968)	1.2691 (0.3643)	0.5344	2.9311	5.0201	0.3037	0.0415
GEU	0.0272 (0.0474)	0.2823 (0.4837)	−27.830	59.660	61.749	0.7115	0.0000
UMEx	0.4322 (0.0667)	–	−0.7660	3.5320	4.5765	0.2639	0.1072
Topp	2.2196 (0.4843)	–	−0.6490	3.2982	4.3427	0.2633	0.1089
UH	1.5201 (0.2667)	–	−0.5203	3.0407	4.0853	0.3513	0.0112
UL	0.1828 (0.0284)	–	−52.411	106.82	107.87	0.9838	0.0000
UXg	0.2000 (0.0300)	–	−56.271	114.54	115.59	0.8164	0.0000
LXg	2.0185 (0.3310)	–	0.4320	1.1359	2.1805	0.3475	0.0125
UXL	1.3576 (0.2465)	–	0.4532	1.0935	2.1380	0.3284	0.0215
NUH	0.9612 (0.1558)	–	5.1965	−8.3931	−7.3485	0.2543	0.1322

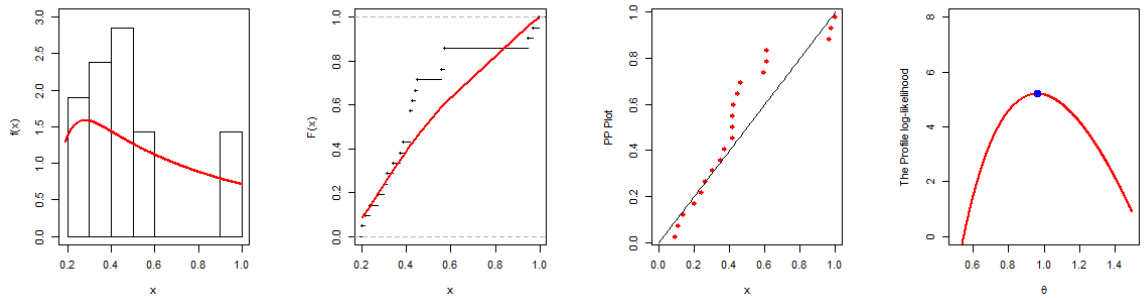


Figure 9. The fitted PDF, CDF, PP, and profile log-likelihood plot for the entomology data.

The analysis results presented in Table 14 and Figure 9 indicate that NUH distribution provides efficient results as compared to certain competitive distributions.

7. Conclusion

This paper proposed a one-parameter distribution as a flexible distribution that can be used in the analysis of data characterized by a unit interval, specifically data characterized by a percentage and proportion. The development of the NUH distribution was driven by the requirement of having a simple but potent model that can explain skewed data behaviors, such as asymmetry and monotonic hazard rate behavior.

The proposed model has been treated mathematically in detail, with extensive coverage of the statistical and reliability properties. The main attributes that are obtained are the moments, mean, the probability density and the distribution functions, and a few reliability functions, including the hazard rate, reversed hazard rate, cumulative hazard rate function. The analytical exploration revealed that the proposed distribution can model asymmetric and bounded data, with the failure rate function displaying an upward trend, which makes it appropriate for lifetime and reliability analyzes.

The estimation of the parameters of the NUH model was discussed in terms of six well-established methods of estimation. To assess and compare the effectiveness and stability of such estimation techniques, a Monte Carlo simulation experiment was conducted with different sample sizes. The simulation findings proved that each of the estimators worked well, but the ADE and WLSE approaches showed better accuracy and stability, especially with a larger sample size.

Lastly, the practical utility and flexibility of the proposed model were confirmed by applying it to three real-world datasets, one of which was COVID-19 patient data, another was entomology data, and the third was kidney dialysis patients. The findings indicated that the NUH distribution fits remarkably well for both datasets, being superior to rival models in goodness-of-fit measures and emphasizing its ability to serve as an effective model to characterize real life unit data.

In conclusion, the proposed distribution is a simple, efficient, and robust framework to use when analyzing bounded and percentage-based data. Its easy mathematical handling, generalized application, and dependable methods for estimating parameters have made it a valuable addition to the family of unit distributions.

8. Future Work

- Perform more extensive simulation studies using larger sample sizes and using different estimation methods, where performance is assessed based on empirical bias, variance, and mean squared error. Employ graphical diagnostics like box plots, density estimates, and Q–Q plots to summarize finite-sample performance of competing estimators.
- Develop interval estimation procedures based on asymptotic theory and resampling techniques, and discuss the evaluation of coverage probabilities and average confidence interval length.
- Study the performance of some bootstrap-based methods of variance estimation, bias correction, and confidence interval construction in some applications involving simulated and real data.
- Present the Bayesian inference of the proposed NUH distribution: prior specification, posterior analysis, and comparisons with the classical estimators.
- Extending the proposed model to more flexible frameworks, such as regression models for bounded responses, censored or truncated data, and other advanced modeling structures.

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REFERENCES

1. N. A. H. Abdelfattah, and R. M. Sayed, *Comparative efficacy of gamma and microwave radiation in protecting peppermint from infestation by drugstore beetle (Stegobium paniceum) L.*, International Journal of Tropical Insect Science, vol. 42, no. 2, pp. 1367–1372, 2022.
2. A. G. Abubakari, A. Luguterah, and S. Nasiru, *Unit exponentiated Fréchet distribution: Actuarial measures, quantile regression and applications*, Journal of the Indian Society for Probability and Statistics, vol. 23, no. 2, pp. 387–424, 2022.
3. M. Ahsan-ul-Haq, *Statistical Analysis of Haq Distribution: Estimation and Applications*, Pakistan Journal of Statistics, vol. 38, no. 4, pp. 473–490, 2022.
4. M. Ahsan-ul-Haq, S. Hashmi, K. Aidi, P. L. Ramos, and F. Louzada, *Unit Modified Burr-III Distribution: Estimation, Characterizations and Validation Test*, Annals of Data Science, 2020. https://doi.org/10.1007/s40745-020-00298-6
5. M. Ahsan-ul-Haq, A. Z. Memon, and S. Chand, *An Analysis of Generalized Exponential Uniform Distribution*, Mathematical Problems in Engineering, pp. 1–15, 2022. https://doi.org/10.1155/2022/6040943
6. A. I. Al-Omari, A. R. A. Alanzi, and S. S. Alshqaq, *The unit two parameters Mirra distribution: Reliability analysis, properties, estimation and applications*, Alexandria Engineering Journal, vol. 92, pp. 238–253, 2024. https://doi.org/10.1016/j.aej.2024.02.063
7. S. M. Alghamdi, M. Shrahili, A. S. Hassan, R. E. Mohamed, I. Elbatal, and M. Elgarhy, *Analysis of milk production and failure data: Using unit exponentiated half logistic power series class of distributions*, Symmetry, vol. 15, no. 3, p. 714, 2023.
8. E. M. Almetwally, T. M. Jawa, N. Sayed-Ahmed, C. Park, M. Zakarya, and S. Dey, *Analysis of unit-Weibull based on progressive type-II censored with optimal scheme*, Alexandria Engineering Journal, vol. 63, pp. 321–338, 2023.
9. G. Alomani, M. Kayid, and A. M. Elgazar, *Classical and Bayesian analysis with fitting radiation data: Using a new truncated distribution*, Journal of Radiation Research and Applied Sciences, vol. 18, no. 2, p. 101498, 2025. https://doi.org/10.1016/j.jrras.2025.101498
10. T. S. Alshammari, and K. A. Rashedi, *Analysis of kidney patients and pump failure data using a new unit interval distribution*, Alexandria Engineering Journal, vol. 116, pp. 374–384, 2025.
11. C. Chesneau and M. Elgarhy, Development and analysis of a new trigonometric unit distribution, *Innovation in Statistics and Probability*, vol. 2, no. 1, pp. 35–47, 2026.
12. A. Saboor, F. Jamal, S. Shafq, and R. Mumtaz, On the versatility of the unit logistic exponential distribution: Capturing bathtub, upside-down bathtub, and monotonic hazard rates, *Innovation in Statistics and Probability*, vol. 1, no. 1, pp. 28–46, 2025.
13. T. S. Helal, M. A. F. Elbarkawy, J. Zafar, and A. Hydara, The Unit Komal distribution: Bathtub risk function and applications, *Computational Journal of Mathematical and Statistical Sciences*, vol. 5, no. 2, pp. 731–760, 2026. doi:10.21608/cjmss.2026.491712.1425.
14. M. Elgarhy, G. Abdalla, A. Hassan, and E. Almetwally, Bayesian and non-Bayesian analysis of the novel unit inverse exponentiated Lomax distribution using progressive censoring schemes with optimal scheme and data application, *Computational Journal of Mathematical and Statistical Sciences*, vol. 5, no. 1, pp. 78–108, 2026. doi:10.21608/cjmss.2025.374277.1151.
15. E. Altun, *The log-xgamma distribution with inference and application*, vol. 159, no. 3, pp. 40–55, 2018.
16. M. R. Alzahrani, and M. Almohaimeed, *Analysis, inference, and application of Unit Haq distribution to engineering data*, Alexandria Engineering Journal, vol. 117, pp. 193–204, 2025.
17. T. W. Anderson, and D. A. Darling, *Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes*, The Annals of Mathematical Statistics, vol. 23, no. 2, pp. 193–212, 1952.
18. S. O. Bashiru, M. Kayid, R. Mahmoud, O. S. Balogun, M. M. Abd El-Raouf, and A. M. Gemeay, *Introducing the unit Zeghdoudi distribution as a novel statistical model for analyzing proportional data*, Journal of Radiation Research and Applied Sciences, vol. 18, no. 1, p. 101204, 2025.
19. A. S. Hassan, S. E. Hemeda, and S. G. Nassr, On the extension of exponentiated Pareto distribution, *Journal of Modern Applied Statistical Methods*, vol. 19, no. 1, Art. 21, 2021.
20. M. Kamal, M. M. Alsolmi, A. Al Mutairi, E. Hussam, M. S. Mustafa, and S. G. Nassr, A new distributional approach: Estimation, Monte Carlo simulation and applications to the biomedical data sets, *Networks and Heterogeneous Media*, vol. 18, no. 4, pp. 1575–1599, 2023.
21. A. A. H. Ahmadini, A. S. Hassan, M. Elgarhy, M. Elsehetry, S. S. Alshqaq, and S. G. Nassr, Inference of truncated Lomax inverse Lomax distribution with applications, *Intelligent Automation & Soft Computing*, vol. 29, no. 1, pp. 199–212, 2021.
22. Y. Zhang, Q. Mao, J. Xu, H. M. Aljohani, B. Elkalzah, and G. A. Marei, On the exponent power sine Lomax distribution with applications in the music education and radiation fields, *Journal of Radiation Research and Applied Sciences*, vol. 19, no. 1, Art. 102109, 2026.
23. M. Imran, M. Elbarkawy, F. Jamal, and G. A. Marei, An extended Lindley model with applications to entropy-based loss and actuarial risk measures, *Computational Journal of Mathematical and Statistical Sciences*, vol. 5, no. 2, pp. 819–851, 2026. doi:10.21608/cjmss.2026.486829.1418.
24. K. A. Rashedi, L. S. Diab, A. H. Alenezy, and G. A. Marei, A one-parameter decreasing-hazard lifetime model with exact inference on a bounded-below support, *AIMS Mathematics*, vol. 11, no. 8, pp. 24445–24489, 2026.
25. R. C. H. Cheng, and N. A. K. Amin, *Estimating parameters in continuous univariate distributions with a shifted origin*, Journal of the Royal Statistical Society: Series B (Methodological), vol. 45, no. 3, pp. 394–403, 1983. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x
26. K. Choi, and W. G. Bulgren, *An estimation procedure for mixtures of distributions*, Journal of the Royal Statistical Society Series B: Statistical Methodology, vol. 30, no. 3, pp. 444–460, 1968.
27. M. S. Eliwa, M. Ahsan-Ul-Haq, A. Al-Bossly, and M. El-Morshedy, *A Unit Probabilistic Model for Proportion and Asymmetric Data: Properties and Estimation Techniques with Application to Model Data from SC16 and P3 Algorithms*, Mathematical Problems in Engineering, vol. 2022, pp. 1–13, 2022. https://doi.org/10.1155/2022/9289721

28. A. Fayomi, A. S. Hassan, H. Baaqeel, and E. M. Almetwally, *Bayesian inference and data analysis of the unit–power Burr X distribution*, Axioms, vol. 12, no. 3, p. 297, 2023.
29. S. Hashmi, M. A. Ahsan-ul-Haq, J. Zafar, and M. A. Khaleel, *Unit Xgamma Distribution: Its Properties, Estimation and Application*, Proceedings of the Pakistan Academy of Sciences: Part A, vol. 59, no. 1, pp. 15–28, 2022. https://doi.org/10.53560/PPASA(59-1)636
30. H. A. Khogeer, *An improved one-parameter weighted distribution with mathematical properties, estimation, and applications*, Alexandria Engineering Journal, vol. 104, pp. 222–234, 2024. https://doi.org/10.1016/j.aej.2024.06.005
31. M. Ç. Korkmaz, *The unit generalized half normal distribution: A new bounded distribution with inference and application*, 2020.
32. A. Krishna, R. Maya, C. Chesneau, and M. R. Irshad, *The unit Teissier distribution and its applications*, Mathematical and Computational Applications, vol. 27, no. 1, p. 12, 2022.
33. R. Maya, P. Jodra, M. R. Irshad, and A. Krishna, *The unit Muth distribution: Statistical properties and applications*, Ricerche Di Matematica, vol. 73, no. 4, pp. 1843–1866, 2024.
34. J. Mazucheli, A. F. B. Menezes, and S. Chakraborty, *On the one parameter unit-Lindley distribution and its associated regression model for proportion data*, Journal of Applied Statistics, vol. 4763, 2018. https://doi.org/10.1080/02664763.2018.1511774
35. J. Mazucheli, A. F. B. Menezes, and S. Dey, *The unit-Birnbaum-Saunders distribution with applications*, Chilean Journal of Statistics, vol. 9, no. 1, pp. 47–57, 2018.
36. J. Mazucheli, A. F. B. Menezes, L. B. Fernandes, R. P. de Oliveira, and M. E. Ghitany, *The unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modeling of quantiles conditional on covariates*, Journal of Applied Statistics, vol. 47, no. 6, pp. 954–974, 2019. https://doi.org/10.1080/02664763.2019.1657813
37. M. H. B. Mustapha, and S. Nasiru, *Unit Gamma/Gompertz Quantile Regression with Applications to Skewed Data*, Sri Lankan Journal of Applied Statistics, vol. 23, no. 1, pp. 49–73, 2022. https://doi.org/10.4038/sljastats.v23i1.8066
38. T. N. Sindhu, A. Shafiq, M. B. Riaz, and T. A. Abushal, *A Statistical Framework for a New two-parameter Unit Bilal Distribution with Application to Model Asymmetric Data*, Heliyon, 2024.
39. J. J. Swain, S. Venkatraman, and J. R. Wilson, *Least-squares estimation of distribution functions in Johnson's translation system*, Journal of Statistical Computation and Simulation, vol. 29, no. 4, pp. 271–297, 1988.
40. Y. Tashkandy, M. E. Bakr, A. M. Gemeay, E. Hussam, M. M. Abd El-Raouf, and M. M. Hossain, *Modeling COVID-19 Real Data Set by a New Extension of Haq Distribution*, Axioms, vol. 12, no. 4, p. 327, 2023.