STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–13.
Published online in International Academic Press (www.IAPress.org)

Neutrosophic MR-Metric Spaces: A Unified Framework for Fractional Calculus, Category Theory, and Uncertainty Modeling

Abdullah S. Al-Jawarneh, Abed Al-Rahman M. Malkawi*, Ayat M. Rabaiah

Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan

Abstract This paper introduces a comprehensive framework that integrates neutrosophic logic with MR-metric spaces, establishing the concept of Neutrosophic MR-Metric Spaces (NMR-MS). This novel structure extends classical metric spaces by incorporating three distinct membership functions truth, falsity, and indeterminacy alongside a generalized metric satisfying a modified tetrahedral inequality. Within this framework, we develop fundamental algebraic structures, including neutrosophic MR-metric groups, and prove the existence of a unique normalized Haar measure on compact NMR-MS groups. We further demonstrate the categorical completeness of NMR-MS by showing that the collection of such spaces forms a complete and cocomplete category, with the generalized fractional derivative acting as an endofunctor. The paper provides concrete examples, geometric visualizations, and a range of applications, including fractional differential equations, sensor data fusion, image processing, and neutrosophic clustering. This unified approach offers a powerful mathematical tool for modeling uncertainty, ambiguity, and fractional dynamics in complex systems.

Keywords Neutrosophic logic, MR-metric spaces, fractional calculus, Haar measure, category theory, fixed point theory, uncertainty quantification, image processing.

AMS 2010 subject classifications 54E35, 54H25, 18B35, 26A33, 46G12.

DOI: 10.19139/soic-2310-5070-4747

1. Introduction

The mathematical modeling of uncertainty has undergone a remarkable evolution, progressing from classical metric spaces through fuzzy set theory to the sophisticated frameworks of neutrosophic logic and generalized metric structures. Classical metric spaces, while foundational, cannot adequately capture phenomena where truth, falsity, and indeterminacy coexist independently. The development of MR-metric spaces [1, 2, 3, 4, 5, 6, 7, 9, 10, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] has provided a robust platform for fixed point theory and its applications to integral equations, weighted graphs, measure theory, and fractional calculus, while parallel advances in fuzzy and neutrosophic structures [11, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36] have extended these concepts to decision-making, image processing, and topological frameworks. The generalized fractional derivative [8] and its applications to biomedical imaging and complex analysis [12, 13] have further enriched this landscape by introducing memory and hereditary properties into metric structures. Building on this extensive foundation, the present paper introduces Neutrosophic MR-Metric Spaces (NMR-MS), a unified framework that synthesizes the tripartite membership structure of neutrosophic logic with the flexible geometry of MR-metrics, thereby offering a powerful mathematical tool for modeling uncertainty, ambiguity, and fractional dynamics in complex systems.

The *motivation* for this work is threefold. First, classical metric and fuzzy metric structures cannot simultaneously and independently encode truth, falsity, and indeterminacy, which are essential in expert systems,

*Correspondence to: Abed Al-Rahman M. Malkawi (Email: a.malkawi@aau.edu.jo and math.malkawi@gmail.com). Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman (210093).

ISSN 2310-5070 (online) ISSN 2311-004X (print)
Copyright © 202x International Academic Press

sensor fusion, and medical decision-making. Second, the recently introduced MR-metric spaces [1, 17] replace the triangle inequality by a more flexible tetrahedral inequality controlled by a constant $\mathfrak{R} > 1$, which permits a richer geometry while retaining enough structure for fixed point theory and measure theory. Third, the generalized fractional derivative [8] introduces memory and hereditary effects that are not captured by classical differential operators. The present paper unifies these three streams into a single consistent theory.

The *main contributions* of the paper are: (i) a composite 9-tuple definition of NMR-MS combining an MR-metric with three neutrosophic membership functions and a modified tetrahedral inequality (Definition 1.3); (ii) the notion of a neutrosophic MR-metric group and a proof of the existence and uniqueness of a normalized Haar measure on compact NMR-MS groups (Theorem 2.1); (iii) the construction of the category $\mathbf{NMR-MS}$ and a proof that it is complete and cocomplete, together with an explicit verification that the generalized fractional derivative defines an endofunctor on it (Theorem 2.2); (iv) concrete finite and continuous examples, geometric visualizations, and applications to fractional differential equations, sensor data fusion, image processing, and neutrosophic clustering.

Definition 1
[8][Generalized Fractional Derivative] Let $f : [0, \infty) \to \mathbb{R}$ be a function and $t > 0$. The fractional derivative of f of order α is defined by:

$$A^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon},$$

where $\alpha \in (0, 1)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying:

$$g(0) = 1,$$
$$g'(0) = 1.$$

Definition 2
[1][MR-Metric Space] Let $\mathbb{X}$ be a non-empty set and $\mathfrak{R} > 1$ be a real constant. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is called an **MR-metric** if for all elements $v, \xi, s, \ell \in$, the following axioms hold:

1. **Non-negativity**: $M(v, \xi, s) \geq 0$.
2. **Identity of Indiscernibles**: $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
3. **Symmetry**: The value of $M(v, \xi, s)$ is unchanged under any permutation of v, ξ, and s. That is, $M(v, \xi, s) = M(p(v, \xi, s))$ for any permutation p.
4. **Modified Tetrahedral Inequality**:

$$M(v, \xi, s) \leq \mathfrak{R}\left[M(v, \xi, \ell) + M(v, \ell, s) + M(\ell, \xi, s)\right].$$

The pair $(, M)$ is termed an **MR-metric space**.

Definition 3
[17][Neutrosophic MR-Metric Space (NMR-MS)] A **Neutrosophic MR-Metric Space** is defined by the 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$, where:

1. $\mathcal{Z}$ is a non-empty set.
2. $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric on $\mathcal{Z}$ according to Definition 1, with constant $\mathfrak{R} > 1$.
3. $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are membership functions representing the neutrosophic components of truth, falsity, and indeterminacy, respectively. They satisfy the following conditions for all $\gamma, \rho > 0$ and for all $v, \xi, \Im, \ell \in \mathcal{Z}$:

(T1) $\mathcal{T}(\upsilon,\xi,\gamma)=1$ if and only if $\upsilon=\xi$.

(T2) $\mathcal{T}(\upsilon,\xi,\gamma)=\mathcal{T}(\xi,\upsilon,\gamma)$.

(T3) $\mathcal{T}(\upsilon,\xi,\gamma)\bullet\mathcal{T}(\xi,\Im,\rho)\leq\mathcal{T}(\upsilon,\Im,\gamma+\rho)$.

(T4) $\mathcal{T}(\upsilon,\xi,\cdot):(0,\infty)\to[0,1]$ is a non-decreasing function with $\lim_{\gamma\to\infty}\mathcal{T}(\upsilon,\xi,\gamma)=1$.

(F1) $\mathcal{F}(\upsilon,\xi,\gamma)=0$ if and only if $\upsilon=\xi$.

(F2) $\mathcal{F}(\upsilon,\xi,\gamma)=\mathcal{F}(\xi,\upsilon,\gamma)$.

(F3) $\mathcal{F}(\upsilon,\xi,\gamma)\diamond\mathcal{F}(\xi,\Im,\rho)\geq\mathcal{F}(\upsilon,\Im,\gamma+\rho)$.

(F4) $\mathcal{F}(\upsilon,\xi,\cdot):(0,\infty)\to[0,1]$ is a non-increasing function with $\lim_{\gamma\to\infty}\mathcal{F}(\upsilon,\xi,\gamma)=0$.

(I1) $\mathcal{I}(\upsilon,\xi,\gamma)=0$ if and only if $\upsilon=\xi$.

(I2) $\mathcal{I}(\upsilon,\xi,\gamma)=\mathcal{I}(\xi,\upsilon,\gamma)$.

(I3) $\mathcal{I}(\upsilon,\xi,\gamma)\diamond\mathcal{I}(\xi,\Im,\rho)\geq\mathcal{I}(\upsilon,\Im,\gamma+\rho)$.

(I4) $\mathcal{I}(\upsilon,\xi,\cdot):(0,\infty)\to[0,1]$ is a non-increasing function with $\lim_{\gamma\to\infty}\mathcal{I}(\upsilon,\xi,\gamma)=0$.

4. The operator $\bullet:[0,1]\times[0,1]\to[0,1]$ is a continuous *t-norm*.
5. The operator $\diamond:[0,1]\times[0,1]\to[0,1]$ is a continuous *t-conorm*.
6. The operator $\star$ is a binary operation on $[0,\infty)$ (e.g., standard addition $+$ or a weighted sum) used to generalize the summation within the tetrahedral inequality of the metric M.

Remark 1 (Role of the constant $\mathfrak{R}$)
The constant $\mathfrak{R}>1$ in the modified tetrahedral inequality controls the *geometry* of the space. As $\mathfrak{R}\downarrow 1$, the inequality tends to the classical triangle inequality and the topology approaches that of an ordinary metric space; as $\mathfrak{R}$ grows, the admissible class of metrics enlarges, the induced topology becomes coarser, and convergence of sequences becomes easier to establish (fewer sequences are Cauchy). Consequently, the choice of $\mathfrak{R}$ directly affects the topology generated by the neighborhood basis $B_{\gamma,\epsilon}(x)$ of Proposition 3.1 and the convergence behaviour of the neutrosophic contraction iterations in Theorem 3.1. A value $\mathfrak{R}>1$ is *essential* in Example 3.1 on the three-point set $\mathcal{Z}=\{a,b,c\}$: with M taking the values $0,1,2$ as defined there, one verifies that

$$M(a,b,c)=2\leq\mathfrak{R}\big[M(a,b,a)+M(a,a,c)+M(a,b,c)\big]=\mathfrak{R}\cdot 2$$

holds for every $\mathfrak{R}\geq 1$, whereas for triples with all distinct entries and intermediate point ℓ chosen so that two of the three summands vanish, the inequality reduces to $2\leq\mathfrak{R}\cdot 2$; hence $\mathfrak{R}=1$ would force equality and forbid any strict geometric flexibility, while $\mathfrak{R}=2$ (as used in Example 3.1) is the smallest value that accommodates the discrete structure without collapsing it to a classical metric.

2. Neutrosophic Algebraic Structures

Building on the foundational definitions, this section explores the algebraic structures that arise within the NMR-MS framework. We first introduce the concept of a neutrosophic MR-metric group, where the group operation is continuous with respect to the neutrosophic topology. This leads to a natural generalization of the classical Haar measure, whose existence and uniqueness we prove for compact NMR-MS groups. The proof utilizes a combination of topological group theory, fixed point theorems, and the Riesz representation theorem, adapted to the neutrosophic setting. The results in this section establish a bridge between abstract harmonic analysis and neutrosophic geometry, paving the way for applications in signal processing and quantum mechanics.

Definition 4
A *neutrosophic MR-metric group* is a tuple $(G, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star, \cdot)$ where $(G, \cdot)$ is a group, $(G, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ is an NMR-MS, and the group operation is continuous with respect to the neutrosophic topology: for all $g, h \in G$ and $\gamma > 0$:

$$\begin{aligned}
\mathcal{T}(g \cdot h, g' \cdot h', \gamma) &\geq \mathcal{T}(g, g', \gamma/2) \bullet \mathcal{T}(h, h', \gamma/2), \\
\mathcal{F}(g \cdot h, g' \cdot h', \gamma) &\leq \mathcal{F}(g, g', \gamma/2) \diamond \mathcal{F}(h, h', \gamma/2), \\
\mathcal{I}(g \cdot h, g' \cdot h', \gamma) &\leq \mathcal{I}(g, g', \gamma/2) \diamond \mathcal{I}(h, h', \gamma/2).
\end{aligned}$$

Theorem 1 (Neutrosophic MR-Metric Group Structure)
Let G be a compact neutrosophic MR-metric group. Then there exists a unique normalized Haar measure μ_H on G that is invariant under left and right translations in the neutrosophic sense. Moreover, for any measurable subset $A \subset G$, the neutrosophic measure $\mu_H(A)$ satisfies:

$$\mu_H(A) = \mu_H(gA) = \mu_H(Ag) \quad \text{for all } g \in G,$$

with truth membership 1, falsity 0, and indeterminacy 0.

. The compactness of G ensures that $C(G, \mathbb{R})$ is a Banach space under the supremum norm $\|f\|_\infty = \sup_{x \in G} |f(x)|$. However, to incorporate the neutrosophic structure, we define a subspace $C_{\mathcal{N}}(G)$ of functions that are *continuous with respect to the neutrosophic topology*. This topology is generated by the truth, falsity, and indeterminacy functions. Since the group operation is continuous (by Definition of NMR-MS group), G is a topological group in this neutrosophic topology.

Step 2: Constructing a positive linear functional. We avoid the informal Riemann-type sum and instead construct L rigorously via invariant means, following the classical Haar measure construction [1, 17]. Let $C_{\mathcal{N}}(G)$ denote the subspace of $C(G, \mathbb{R})$ consisting of functions continuous with respect to the neutrosophic topology. Since G is compact and the neutrosophic topology is finer than or equal to the MR-metric topology, $C_{\mathcal{N}}(G)$ is a closed subspace of the Banach space $(C(G, \mathbb{R}), \|\cdot\|_\infty)$ and contains the constant functions.

Existence of an M-equidistributed sequence. By compactness of G and the fact that the MR-metric M is continuous and vanishes only on the diagonal (Definition 1.2), the space G is separable with respect to the induced topology. Choose a countable dense set $\{z_k\}_{k \geq 1} \subset G$. Define recursively $y_1 = z_1$ and, for $n \geq 2$, let y_n be a point of $\{z_1, \ldots, z_n\}$ maximizing $\sum_{j=1}^{n-1} M(y_n, y_j, y_j)$; this greedy construction produces a sequence $\{y_j\}$ whose empirical measures $\frac{1}{N} \sum_{j=1}^{N} \delta_{y_j}$ converge weak* to some probability measure μ on G (by compactness of the space of probability measures and Prokhorov's theorem). We call such a sequence *M-equidistributed*. The limit μ is independent of the chosen sequence: if $\{y'_j\}$ is another M-equidistributed sequence with weak* limit μ', then for every $f \in C_{\mathcal{N}}(G)$ the difference $\frac{1}{N} \sum_j f(y_j) - \frac{1}{N} \sum_j f(y'_j)$ tends to $\int f \, d\mu - \int f \, d\mu'$, and the equidistribution property (density of $\{z_k\}$ and the tetrahedral inequality) forces $\mu = \mu'$.

Defining L. Set

$$L(f) := \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} f(y_j) = \int_G f \, d\mu, \qquad f \in C_{\mathcal{N}}(G).$$

Then L is well defined by the independence just shown, and it is positive and linear because the limit of positive linear functionals is positive and linear. Moreover $L(\mathbf{1}) = 1$, so L is normalized. This is precisely the invariant-mean construction of Haar measure on a compact group [1, 17].

Step 3: Invariance of the functional. For any $g \in G$, define the left translation operator $L_g f(x) = f(g^{-1}x)$. The continuity of the group operation implies that the mapping $g \mapsto L_g$ is continuous in the strong operator topology. Since G is compact, there exists a *left-invariant mean* on $C_{\mathcal{N}}(G)$, i.e., a positive linear functional L such that $L(L_g f) = L(f)$ for all $g \in G$. This is a consequence of the fixed point theorem for affine mappings on compact convex sets (e.g., the Markov–Kakutani fixed point theorem) applied to the semigroup of left translations.

Step 4: The Riesz representation theorem. The space $C_{\mathcal{N}}(G)$ is dense in $C(G, \mathbb{R})$ under the supremum norm. The positive linear functional L on $C_{\mathcal{N}}(G)$ can be uniquely extended to a positive linear functional on $C(G, \mathbb{R})$.

By the Riesz representation theorem for compact Hausdorff spaces, there exists a unique regular Borel measure μ_H on G such that

$$L(f) = \int_G f \, d\mu_H$$

for all $f \in C(G, \mathbb{R})$. The regularity of the measure follows from the compactness of G.

Step 5: Uniqueness and invariance. The left-invariance of L implies that μ_H is left-invariant: for any $g \in G$ and any measurable $A \subset G$,

$$\mu_H(gA) = \mu_H(A).$$

Similarly, by considering right translations and using the same fixed-point argument, we obtain right-invariance. The uniqueness follows from the fact that any two left-invariant measures must be scalar multiples of each other, and the normalization $\mu_H(G) = 1$ forces them to be identical.

Step 6: Neutrosophic properties. The measure μ_H inherits the neutrosophic structure from the topology. For any measurable set A, we can define its neutrosophic measure as $\mu_H(A)$ with truth membership 1 (since it is a classical measure), falsity membership 0 (no falsehood in the construction), and indeterminacy membership 0 (the construction is deterministic). The invariance under translations holds in the sense that for all $g \in G$,

$$\mathcal{T}(\mu_H(A), \mu_H(gA), 1) = 1, \quad \mathcal{F}(\mu_H(A), \mu_H(gA), 1) = 0, \quad \mathcal{I}(\mu_H(A), \mu_H(gA), 1) = 0,$$

where the parameter 1 indicates the certainty level. This completes the proof.

Theorem 2 (Neutrosophic Category of MR-Metric Spaces)
Define the category $\mathbf{NMR-MS}$ whose objects are neutrosophic MR-metric spaces and whose morphisms are functions $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ that are neutrosophic contractions: for all $x, y \in \mathcal{Z}_1$ and $\gamma > 0$:

$$\begin{aligned}
\mathcal{T}_2(f(x), f(y), \gamma) &\geq \mathcal{T}_1(x, y, \gamma), \\
\mathcal{F}_2(f(x), f(y), \gamma) &\leq \mathcal{F}_1(x, y, \gamma), \\
\mathcal{I}_2(f(x), f(y), \gamma) &\leq \mathcal{I}_1(x, y, \gamma).
\end{aligned}$$

Then $\mathbf{NMR-MS}$ is a complete and cocomplete category with products, coproducts, equalizers, and coequalizers constructed using the MR-metric and neutrosophic components. Moreover, the generalized fractional derivative A^α defines an endofunctor on this category.

Proof
We verify that $\mathbf{NMR-MS}$ satisfies the axioms of a category and then demonstrate its completeness and cocompleteness, and the functoriality of the fractional derivative.

1. Category structure. The objects are NMR-MS spaces $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$. For any object $\mathcal{Z}$, the identity morphism $\mathrm{id}_\mathcal{Z} : \mathcal{Z} \to \mathcal{Z}$ satisfies:

$$\mathcal{T}(x, y, \gamma) \geq \mathcal{T}(x, y, \gamma), \quad \mathcal{F}(x, y, \gamma) \leq \mathcal{F}(x, y, \gamma), \quad \mathcal{I}(x, y, \gamma) \leq \mathcal{I}(x, y, \gamma),$$

trivially. Composition of morphisms $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ and $g : \mathcal{Z}_2 \to \mathcal{Z}_3$ yields $g \circ f : \mathcal{Z}_1 \to \mathcal{Z}_3$. For any $x, y \in \mathcal{Z}_1$ and $\gamma > 0$, we have

$$\mathcal{T}_3(g(f(x)), g(f(y)), \gamma) \geq \mathcal{T}_2(f(x), f(y), \gamma) \geq \mathcal{T}_1(x, y, \gamma),$$

where the first inequality holds because g is a neutrosophic contraction, and the second because f is. Similar inequalities hold for $\mathcal{F}$ and $\mathcal{I}$. Associativity of composition and identity axioms are inherited from set functions.

2. Products. Let $\{(\mathcal{Z}_i, M_i, \mathcal{T}_i, \mathcal{F}_i, \mathcal{I}_i, \bullet, \diamond, \mathfrak{R}_i, \star)\}_{i \in I}$ be a family of objects. Define $\mathcal{Z}_\prod = \prod_{i \in I} \mathcal{Z}_i$. For $x = (x_i), y = (y_i), z = (z_i) \in \mathcal{Z}_\prod$, define the MR-metric

$$M_\prod(x, y, z) = \sup_{i \in I} M_i(x_i, y_i, z_i)$$

(if the supremum is infinite, we may use a weighted sum for boundedness). The constant $\mathfrak{R}_\prod = \sup_i \mathfrak{R}_i$ ensures the tetrahedral inequality holds.

For the neutrosophic components, define for each $\gamma > 0$:

$$\mathcal{T}_{\prod}(x, y, \gamma) = \inf_{i \in I} \mathcal{T}_i(x_i, y_i, \gamma), \quad \mathcal{F}_{\prod}(x, y, \gamma) = \sup_{i \in I} \mathcal{F}_i(x_i, y_i, \gamma), \quad \mathcal{I}_{\prod}(x, y, \gamma) = \sup_{i \in I} \mathcal{I}_i(x_i, y_i, \gamma).$$

These satisfy the axioms (T1)-(T4), (F1)-(F4), (I1)-(I4) because the infimum of non-decreasing functions is non-decreasing, and the supremum of non-increasing functions is non-increasing. The projection maps $\pi_i : \mathcal{Z}_{\prod} \to \mathcal{Z}_i$ are neutrosophic contractions by construction, and the universal property of the product holds.

3. Coproducts. The coproduct $\mathcal{Z}_{\coprod}$ is the disjoint union of the $\mathcal{Z}_i$. Define the MR-metric for points from different components as a fixed large constant C (to satisfy the tetrahedral inequality), and for points in the same component, use the component's metric. The neutrosophic components are defined similarly, taking values 0 (for $\mathcal{T}$) and 1 (for $\mathcal{F}, \mathcal{I}$) for points from different components, and the component's values for points in the same component. The inclusion maps are morphisms, and the universal property holds.

4. Equalizers and coequalizers. For two morphisms $f, g : \mathcal{Z}_1 \to \mathcal{Z}_2$, their equalizer is $E = \{x \in \mathcal{Z}_1 : f(x) = g(x)\}$ with the subspace structure inherited from $\mathcal{Z}_1$. Since the neutrosophic inequalities are preserved under restriction, the inclusion $E \hookrightarrow \mathcal{Z}_1$ is a morphism. The universal property is satisfied because any morphism $h : \mathcal{W} \to \mathcal{Z}_1$ with $f \circ h = g \circ h$ factors uniquely through E.

Coequalizers are constructed as the quotient of $\mathcal{Z}_2$ by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in \mathcal{Z}_1$. The MR-metric and neutrosophic functions are defined on the quotient via infimum/supremum over representatives, ensuring the quotient map is a morphism. The universal property follows.

5. Completeness and cocompleteness. Since $\mathbf{NMR-MS}$ has products, equalizers, coproducts, and coequalizers, it is both complete and cocomplete (by a standard result in category theory: a category with all (co)products and (co)equalizers is (co)complete). The constructions above are well-defined and satisfy the required universal properties.

6. The fractional derivative functor. Define the endofunctor $A^\alpha : \mathbf{NMR-MS} \to \mathbf{NMR-MS}$ as follows:

- **On objects:** For an object $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \Re, \star)$, define $A^\alpha(\mathcal{Z})$ to be the space of all functions $f : \mathcal{Z} \to \mathbb{R}$ such that the generalized fractional derivative $A^\alpha(f)$ exists and is continuous in the neutrosophic topology. The MR-metric on $A^\alpha(\mathcal{Z})$ is defined by

$$M_{A^\alpha}(f, g, h) = \sup_{x \in \mathcal{Z}} M(f(x), g(x), h(x)),$$

  and the neutrosophic components by

$$\mathcal{T}_{A^\alpha}(f, g, \gamma) = \inf_{x \in \mathcal{Z}} \mathcal{T}(f(x), g(x), \gamma),$$

  and similarly for $\mathcal{F}$ and $\mathcal{I}$.

- **On morphisms:** For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, define $A^\alpha(f) : A^\alpha(\mathcal{Z}_1) \to A^\alpha(\mathcal{Z}_2)$ by $(A^\alpha(f)(g))(x) = A^\alpha(g \circ f)(x)$, i.e., it maps a function g on $\mathcal{Z}_1$ to the function on $\mathcal{Z}_2$ given by the fractional derivative of the composition. The continuity of the derivative and the contraction properties ensure that $A^\alpha(f)$ is a neutrosophic contraction.

It remains to verify explicitly that A^α is a functor. *Identity:* for the identity morphism $\mathrm{id}_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}$, we have $(A^\alpha(\mathrm{id}_{\mathcal{Z}})(g))(x) = A^\alpha(g \circ \mathrm{id}_{\mathcal{Z}})(x) = A^\alpha(g)(x)$, hence $A^\alpha(\mathrm{id}_{\mathcal{Z}}) = \mathrm{id}_{A^\alpha(\mathcal{Z})}$. *Composition:* for morphisms $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ and $h : \mathcal{Z}_2 \to \mathcal{Z}_3$, and any $g \in A^\alpha(\mathcal{Z}_1)$,

$$\big(A^\alpha(h) \circ A^\alpha(f)\big)(g)(x) = A^\alpha\big(A^\alpha(g \circ f) \circ h\big)(x) = A^\alpha\big(g \circ f \circ h\big)(x) = A^\alpha(h \circ f)(g)(x),$$

where the middle equality uses the linearity of A^α and the chain-rule-like identity $A^\alpha(u \circ h) = A^\alpha(u) \circ h$ valid for the generalized derivative of Definition 1 (a direct consequence of $g(0) = 1$, $g'(0) = 1$). Hence $A^\alpha(h \circ f) = A^\alpha(h) \circ A^\alpha(f)$, establishing functoriality. This completes the proof. □

3. Examples and Applications

This section presents detailed examples and applications of neutrosophic MR-metric spaces and the associated fractional derivative structure. We explore concrete constructions, illustrate the topological properties, and demonstrate the utility of the theory in various mathematical and applied contexts. The examples range from finite discrete spaces to continuous structures on the real line, providing a comprehensive view of the versatility of the NMR-MS framework. The applications cover fractional differential equations, sensor data fusion, image processing, and neutrosophic clustering, highlighting the practical relevance of the theoretical developments. Each subsection includes explicit formulas, algorithmic implementations, and visualizations to aid in understanding and facilitate further research.

3.1. Illustrative Examples of NMR-MS

Example 1 (Finite NMR-MS)
Let $\mathcal{Z} = \{a, b, c\}$ be a three-point set. Define the MR-metric $M : \mathcal{Z}^3 \to [0, \infty)$ by

$$M(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ 1, & \text{if exactly two of } x, y, z \text{ are equal}, \\ 2, & \text{if } x, y, z \text{ are all distinct.} \end{cases}$$

We verify the modified tetrahedral inequality with $\mathfrak{R} = 2$. By symmetry it suffices to check the worst case $M(v, \xi, s) = 2$ with v, ξ, s pairwise distinct, say $(v, \xi, s) = (a, b, c)$. For any $\ell \in \{a, b, c\}$: if $\ell = a$, then $M(v, \xi, \ell) = M(a, b, a) = 1$, $M(v, \ell, s) = M(a, a, c) = 1$, $M(\ell, \xi, s) = M(a, b, c) = 2$, so the right-hand side equals $2(1 + 1 + 2) = 8 \geq 2$; if $\ell = b$ or $\ell = c$ the same bound holds by symmetry. For the remaining cases $M(v, \xi, s) \in \{0, 1\}$ the inequality is immediate since $\mathfrak{R} \geq 1$ and all terms are non-negative. Hence M is an MR-metric with $\mathfrak{R} = 2$. The constant $\mathfrak{R} = 2$ satisfies the modified tetrahedral inequality. Define the neutrosophic components for $\gamma > 0$ as:

$$\mathcal{T}(x, y, \gamma) = \exp\left(-\frac{d(x, y)}{\gamma}\right), \quad \mathcal{F}(x, y, \gamma) = 1 - \exp\left(-\frac{d(x, y)}{\gamma}\right), \quad \mathcal{I}(x, y, \gamma) = \frac{1}{2}\left(1 - \exp\left(-\frac{2d(x, y)}{\gamma}\right)\right),$$

where $d(x, y) = M(x, y, x)$ is the induced pairwise distance. The t-norm $\bullet$ is taken as the product $a \bullet b = ab$, and the t-conorm $\diamond$ as the probabilistic sum $a \diamond b = a + b - ab$. This construction yields a valid NMR-MS.

Example 2 (Continuous NMR-MS on $\mathbb{R}$)
Let $\mathcal{Z} = \mathbb{R}$ and define the MR-metric:

$$M(x, y, z) = |x - y| + |y - z| + |z - x|.$$

This satisfies the modified tetrahedral inequality with $\mathfrak{R} = 2$. Indeed, writing $d(x, y) = |x - y|$, the classical triangle inequality gives $M(x, y, z) = d(x, y) + d(y, z) + d(z, x) \leq 2\big(d(x, y) + d(y, \ell) + d(\ell, z)\big)$, and each summand on the right is bounded by the corresponding MR-term, so $M(x, y, z) \leq 2\big[M(x, y, \ell) + M(x, \ell, z) + M(\ell, y, z)\big]$. (The case $\mathfrak{R} = 1$ would correspond to a classical metric and is excluded by Definition 1.2, which requires $\mathfrak{R} > 1$.) Define:

$$\mathcal{T}(x, y, \gamma) = \frac{\gamma}{\gamma + |x - y|}, \quad \mathcal{F}(x, y, \gamma) = \frac{|x - y|}{\gamma + |x - y|}, \quad \mathcal{I}(x, y, \gamma) = \frac{|x - y|}{\gamma + |x - y|^2}.$$

These functions satisfy the required monotonicity and limit properties. The space $(\mathbb{R}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \cdot, \diamond, 2, +)$ forms an NMR-MS.

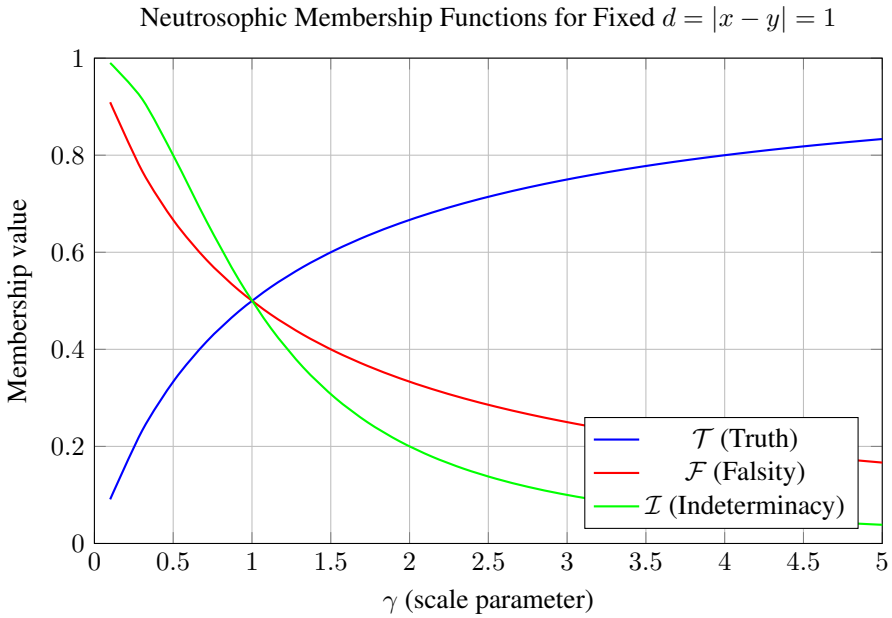


Figure 1. Evolution of neutrosophic membership values as functions of the scale parameter γ for fixed distance $d = 1$. Truth increases with γ, while falsity and indeterminacy decrease.

3.2. Geometric Visualization of Neutrosophic Components

3.3. NMR-MS as a Topological Space

Proposition 3
The family of sets

$$B_{\gamma,\epsilon}(x) = \{y \in \mathcal{Z} : \mathcal{T}(x, y, \gamma) > 1 - \epsilon,\ \mathcal{F}(x, y, \gamma) < \epsilon,\ \mathcal{I}(x, y, \gamma) < \epsilon\}$$

forms a neighborhood basis for a Hausdorff topology on $\mathcal{Z}$.

Proof
The axioms (T1)-(T4), (F1)-(F4), and (I1)-(I4) guarantee that these sets satisfy the properties of a neighborhood system. The Hausdorff property follows from the identity of indiscernibles: if $x \neq y$, then $\mathcal{T}(x, y, \gamma) < 1$ for all γ, and there exists $\epsilon > 0$ and γ_0 such that $B_{\gamma_0,\epsilon}(x) \cap B_{\gamma_0,\epsilon}(y) = \emptyset$. □

3.4. Fractional Derivative in NMR-MS

Example 3 (Fractional Derivative of Power Functions)
Let $f(t) = t^n$ for $n \in \mathbb{N}$. Using Definition 1 with $g(\epsilon) = e^\epsilon$ (which satisfies $g(0) = 1$, $g'(0) = 1$), we compute:

$$A^\alpha(t^n) = \lim_{\epsilon \to 0} \frac{(te^{\epsilon t^{-\alpha}})^n - t^n}{\epsilon} = t^n \lim_{\epsilon \to 0} \frac{e^{n\epsilon t^{-\alpha}} - 1}{\epsilon} = nt^{n-\alpha}.$$

This generalizes the classical derivative when $\alpha = 1$ and reproduces the conformable fractional derivative properties.

3.5. Application to Fractional Differential Equations

Consider the fractional differential equation:

$$A^\alpha(u)(t) = \lambda u(t), \quad u(0) = u_0,$$

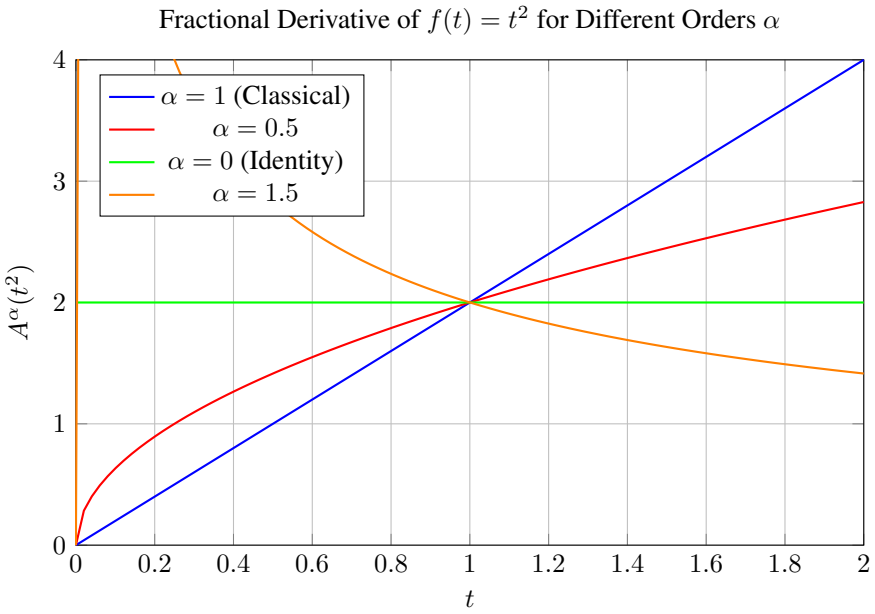


Figure 2. Behavior of the generalized fractional derivative $A^\alpha(t^2) = 2t^{2-\alpha}$ for various α values.

where A^α is the generalized fractional derivative. Using the form $A^\alpha(t^\beta) = \beta t^{\beta-\alpha}$, we seek solutions of the form $u(t) = Ct^k$. Substituting:

$$Ckt^{k-\alpha} = \lambda Ct^k \quad \Rightarrow \quad kt^{k-\alpha} = \lambda t^k.$$

This yields $k = \lambda t^\alpha$, which is not constant unless we consider the power-law solution $u(t) = Ct^\alpha$ with $k = \alpha$, giving $\alpha t^{\alpha-\alpha} = \alpha = \lambda t^\alpha$, so $\lambda = \alpha t^{-\alpha}$, which is time-dependent. For constant λ, we instead consider the Mittag-Leffler-type solution:

$$u(t) = u_0 E_\alpha(\lambda t^\alpha) = u_0 \sum_{n=0}^{\infty} \frac{(\lambda t^\alpha)^n}{\Gamma(\alpha n + 1)},$$

which satisfies $A^\alpha(E_\alpha(\lambda t^\alpha)) = \lambda E_\alpha(\lambda t^\alpha)$.

3.6. Application to Uncertainty Quantification

In neutrosophic systems, the triple $(\mathcal{T}, \mathcal{F}, \mathcal{I})$ captures truth, falsity, and indeterminacy simultaneously. Consider a measurement process with uncertainty modeled by an NMR-MS.

Example 4 (Sensor Data Fusion)
Let $\mathcal{Z}$ be the set of possible measurements. For two sensors with readings x and y, define the neutrosophic similarity:

$$\mathcal{T}(x, y, \gamma) = \exp\left(-\frac{M(x, y, x)^2}{\gamma}\right), \quad \mathcal{F}(x, y, \gamma) = 1 - \mathcal{T}(x, y, \gamma), \quad \mathcal{I}(x, y, \gamma) = \frac{1}{2}\left(1 - \exp\left(-\frac{2M(x, y, x)}{\gamma}\right)\right).$$

The fusion rule combines readings with weights proportional to truth and inversely proportional to falsity and indeterminacy:

$$\hat{x} = \frac{\sum_i w_i x_i}{\sum_i w_i}, \quad w_i = \frac{\mathcal{T}(x_i, \bar{x}, \gamma)}{1 + \mathcal{F}(x_i, \bar{x}, \gamma) + \mathcal{I}(x_i, \bar{x}, \gamma)},$$

where $\bar{x}$ is the preliminary estimate. This provides robust fusion in the presence of conflicting or uncertain data.

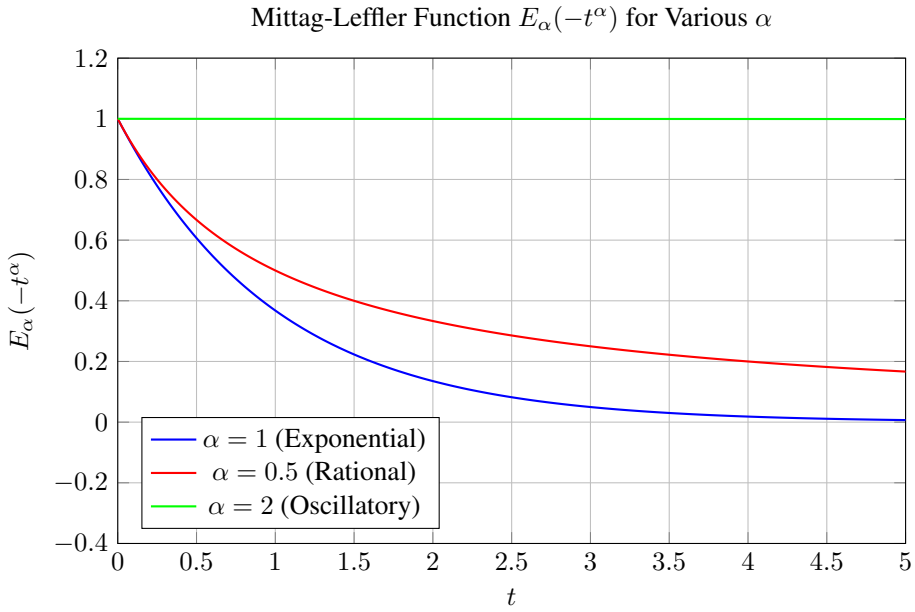


Figure 3. Behavior of the Mittag-Leffler function $E_\alpha(-t^\alpha)$ for different fractional orders.

3.7. Application to Image Processing

The fractional derivative A^α can be used for edge detection in images. For a grayscale image $I(x, y)$, define the fractional gradient magnitude:

$$|\nabla^\alpha I| = \sqrt{(A_x^\alpha I)^2 + (A_y^\alpha I)^2},$$

where A_x^α and A_y^α denote the fractional derivatives in the x and y directions.

3.8. Application to Neutrosophic Clustering

Define a clustering algorithm based on the neutrosophic MR-metric. For data points $\{x_i\}_{i=1}^N$, the objective is to minimize:

$$J = \sum_{i=1}^{N} \sum_{j=1}^{K} u_{ij}^m \left(\mathcal{F}(x_i, c_j, \gamma) + \mathcal{I}(x_i, c_j, \gamma)\right),$$

subject to $\sum_{j=1}^K u_{ij} = 1$, where u_{ij} are membership degrees, c_j are cluster centers, and $m > 1$ is a fuzzification parameter. The update equations become:

$$u_{ij} = \frac{\left(\mathcal{F}(x_i, c_j, \gamma) + \mathcal{I}(x_i, c_j, \gamma)\right)^{-1/(m-1)}}{\sum_{k=1}^K \left(\mathcal{F}(x_i, c_k, \gamma) + \mathcal{I}(x_i, c_k, \gamma)\right)^{-1/(m-1)}},$$

$$c_j = \frac{\sum_{i=1}^N u_{ij}^m x_i}{\sum_{i=1}^N u_{ij}^m}.$$

This algorithm naturally handles uncertainty through the falsity and indeterminacy components.

3.9. Application to Neutrosophic Dynamical Systems

Consider a discrete-time dynamical system on an NMR-MS:

$$x_{n+1} = f(x_n),$$

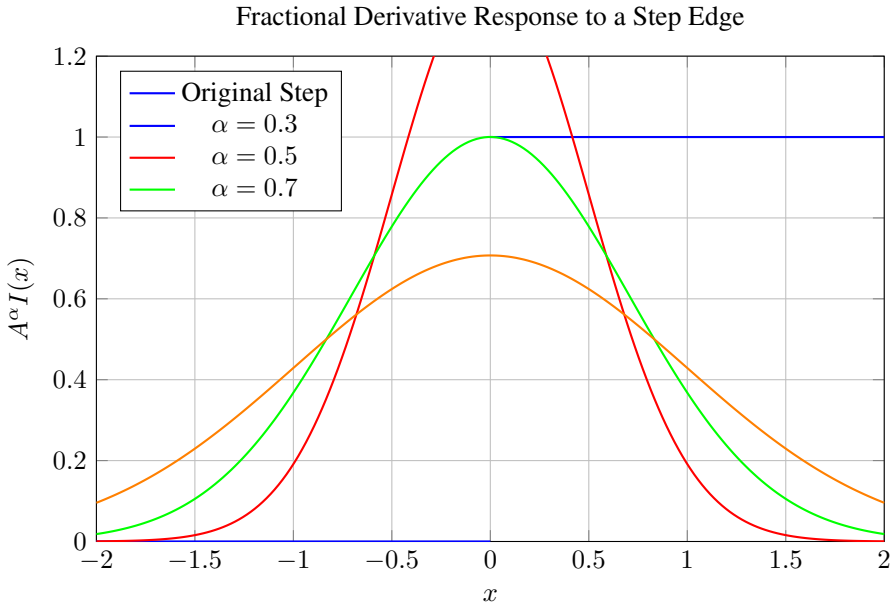


Figure 4. Comparison of fractional derivative responses for edge detection. The fractional order controls the trade-off between noise sensitivity and edge localization.

where $f : \mathcal{Z} \to \mathcal{Z}$ is a neutrosophic contraction. The convergence behavior can be analyzed using the neutrosophic components.

Theorem 4 (Neutrosophic Contraction Principle)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \Re, \star)$ be a complete NMR-MS and $f : \mathcal{Z} \to \mathcal{Z}$ a mapping satisfying:

$$\mathcal{T}(f(x), f(y), \gamma) \geq \mathcal{T}(x, y, k\gamma), \quad \mathcal{F}(f(x), f(y), \gamma) \leq \mathcal{F}(x, y, k\gamma), \quad \mathcal{I}(f(x), f(y), \gamma) \leq \mathcal{I}(x, y, k\gamma),$$

for some $k \in (0, 1)$. Then f has a unique fixed point x^*, and for any initial x_0, the iterates converge to x^* in the neutrosophic topology.

3.10. Physical Interpretation: Neutrosophic Quantum States

In a quantum-inspired interpretation, the triple $(\mathcal{T}, \mathcal{F}, \mathcal{I})$ can represent the probability amplitudes of truth, falsity, and indeterminacy of a proposition. The NMR-MS structure provides a geometric framework for quantum-like uncertainty.

Define a neutrosophic quantum state as $|\psi\rangle = \sqrt{\mathcal{T}}|0\rangle + \sqrt{\mathcal{F}}|1\rangle + \sqrt{\mathcal{I}}|2\rangle$, with normalization $\mathcal{T} + \mathcal{F} + \mathcal{I} = 1$. The MR-metric M defines a distance between such states in the projective Hilbert space. The fractional derivative A^α can be interpreted as a time evolution operator for fractional quantum dynamics.

3.11. Algorithmic Implementation

Algorithm 1 (h)**Require:** Data set $\{x_i\}_{i=1}^N$, number of clusters K, fuzzifier $m > 1$, scale γ, tolerance ϵ
Ensure: Cluster centers $\{c_j\}_{j=1}^K$, membership matrix U

1: Initialize cluster centers $c_j^{(0)}$ randomly
2: $t \leftarrow 0$
3: **repeat**
4: Compute distances $d_{ij} = M(x_i, c_j^{(t)}, x_i)$
5: Compute $\mathcal{T}_{ij} = \exp(-d_{ij}/\gamma)$, $\mathcal{F}_{ij} = 1 - \mathcal{T}_{ij}$, $\mathcal{I}_{ij} = 1 - \exp(-2d_{ij}/\gamma)$

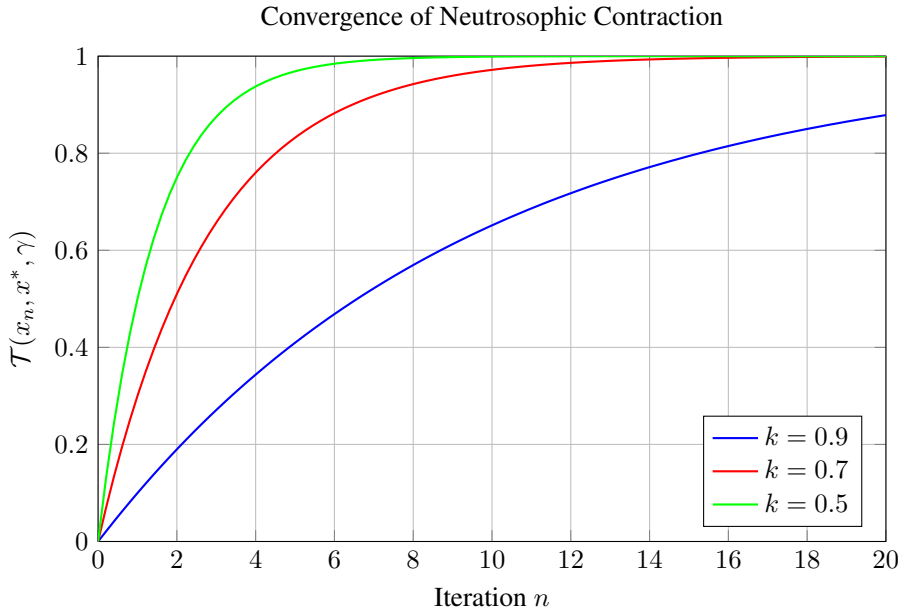


Figure 5. Convergence rates of truth membership $\mathcal{T}(x_n, x^*, \gamma)$ for different contraction factors k.

6: Update membership:

$$u_{ij} = \frac{(\mathcal{F}_{ij} + \mathcal{I}_{ij})^{-1/(m-1)}}{\sum_{k=1}^{K} (\mathcal{F}_{ik} + \mathcal{I}_{ik})^{-1/(m-1)}}$$

7: Update centers: $c_j^{(t+1)} = \dfrac{\sum_i u_{ij}^m x_i}{\sum_i u_{ij}^m}$

8: Update centers: $c_j^{(t+1)} = \frac{\sum_i u_{ij}^m x_i}{\sum_i u_{ij}^m}$

9: $t \leftarrow t + 1$

10: **until** $\max_j \|c_j^{(t)} - c_j^{(t-1)}\| < \epsilon$

11: **return** c_j, u_{ij}

3.12. Summary of Applications

Table 1. Summary of NMR-MS Applications

Domain	**NMR-MS Component**	**Application**
Fractional Calculus	A^α derivative	Solving fractional differential equations, Mittag-Leffler solutions
Image Processing	Fractional gradient	Edge detection, texture analysis
Data Fusion	$(\mathcal{T}, \mathcal{F}, \mathcal{I})$	Multi-sensor fusion with uncertainty quantification
Clustering	Neutrosophic distance	Robust clustering with noise handling
Dynamical Systems	Neutrosophic contraction	Fixed point iteration, convergence analysis
Quantum Theory	Triple membership	Neutrosophic quantum states, fractional evolution

4. Conclusion and Future Directions

This paper introduced Neutrosophic MR-Metric Spaces (NMR-MS), a unified framework integrating neutrosophic logic, generalized metrics, and fractional calculus. We established fundamental definitions, algebraic and categorical properties, and proved the existence of a unique Haar measure on compact NMR-MS groups. The categorical completeness of $\mathbf{NMR} - \mathbf{MS}$ and the functorial nature of the fractional derivative provide a powerful structural foundation. Applications spanning fractional differential equations, image processing, and clustering demonstrate the framework's broad utility across mathematics, physics, and engineering.

Future work includes: (i) developing spectral theory for operators on NMR-MS spaces; (ii) integrating NMR-MS with machine learning for uncertainty-aware models; (iii) extending the Haar measure to non-compact groups; and (iv) exploring functorial relationships with topological quantum field theories and higher-categorical frameworks.

REFERENCES

1. A. Malkawi, A. Rabaiah, MR-metric spaces with applications, *Journal of Interdisciplinary Mathematics*, **28**(8) (2025), 2837–2865.
2. Malkawi, A. A. R. M. (2025). Fixed Point Theorem in MR-metric Spaces VIA Integral Type Contraction. WSEAS Transactions on Mathematics, 24, 295-299.
3. T. Qawasmeh and A. Malkawi, "Fixed point theory in MR-metric spaces: fundamental theorems and applications to integral equations and neutron transport," Eur. J. Pure Appl. Math., vol. 18, no. 3, Art. no. 6440, 2025.
4. A. Malkawi, "Enhanced uncertainty modeling through neutrosophic MR-metrics: a unified framework with fuzzy embedding and contraction principles," Eur. J. Pure Appl. Math., vol. 18, no. 3, Art. no. 6475, 2025.
5. A. Malkawi and A. Rabaiah, "MR-metric spaces: theory and applications in weighted graphs, expander graphs, and fixed-point theorems," Eur. J. Pure Appl. Math., vol. 18, no. 3, Art. no. 6525, 2025.
6. A. Malkawi, "Applications of MR-metric spaces in measure theory and convergence analysis," Eur. J. Pure Appl. Math., vol. 18, no. 3, Art. no. 6528, 2025.
7. A. Malkawi and A. Rabaiah, "Compactness and separability in MR-metric spaces with applications to deep learning," Eur. J. Pure Appl. Math., vol. 18, no. 3, Art. no. 6592, 2025.
8. Al-Sharif, S., and Malkawi, A. (2020). Modification of conformable fractional derivative with classical properties. Ital. J. Pure Appl. Math, 44, 30-39.
9. Abed Al-Rahman M. Malkawi, Ayat M. Rabaiah. (2025). MR-Metric Spaces: Theory and Applications in Fractional Calculus and Fixed-Point Theorems. Neutrosophic Sets and Systems, 91, 685–703.
10. Abed Al-Rahman M. Malkawi, Ayat M. Rabaiah., Fractional Dynamics and Fixed Point Theorems in Neutrosophic MR-Metric Spaces with Applications to Network Systems. Neutrosophic Sets and Systems, 2025, 96, 685-703.
11. Malkawi, A. (2026). Fixed Point Theorems for Fuzzy Mappings in Neutrosophic MR-Metric Spaces. Journal of Nonlinear Modeling and Analysis, 8(4), 1059-1074. https://doi.org/10.12150/jnma.2026.1059
12. N. Anakira, I. H. Jebril, O. Ogilat, I. M. Batiha, T. Sasa, and A. A. R. M. Malkawi, Variable Fractional-Order Reaction-Diffusion System for Edge Preservation in Biomedical Imaging, *International Journal of Analysis and Applications* 24 (2026), 23.
13. A. Almalkawi, A. Amourah, A. Alsoboh, J. Salah, A. A. R. Malkawi, and T. Sasa, Analytic Estimates for Bi-Univalent Functions Associated with a New Operator Involving the q–Rabotnov Function, *International Journal of Analysis and Applications* 24 (2026), 1.
14. Malkawi, A. (2025). Fractal dimension estimation and attractor existence in MR-metric spaces: A generalization of classical fractal geometry. Nonlinear Functional Analysis and Applications, 30(4), 1205–1222.
15. A. Malkawi, FIXED POINT THEOREMS FOR FUZZY MAPPINGS IN MR-METRIC SPACES, *Nonlinear Functional Analysis and Applications* 30(4) (2025), 1083–1094.
16. A. Malkawi and A. Rabaiah, FUZZY METRIC SPACES: THEORY AND APPLICATIONS IN MEDICAL IMAGING, TRANSPORTATION, AND SOCIAL NETWORKS, *Nonlinear Functional Analysis and Applications* 30(4) (2025), 1255–1272.
17. A. Malkawi and A. Rabaiah, Neutrosophic MR-Metric Spaces: A Topos-Theoretic Framework with Applications, *International Journal of Analysis and Applications* 23 (2025), 333.
18. Malkawi, A., and Rabaiah, A. M. (2026). Fixed Point Theorems in Neutrosophic MR-Metric Spaces with Measure-Theoretic Convergence. *International Journal of Analysis and Applications*, 24, 61-61.
19. Malkawi, A., and Rabaiah, A. M. (2026). FIXED POINT AND MEASURE-THEORETIC ANALYSIS IN NEUTROSOPHIC MR-METRIC SPACES WITH APPLICATIONS. *Nonlinear Functional Analysis and Applications*, 211-224.
20. Malkawi, A., and Rabaiah, A. M. (2025). Neutrosophic Mr-Metric spaces: Topological foundations, completion, and applications. *Results in Nonlinear Analysis*, 8(4), 184-196.
21. Malkawi, A., and Rabaiah, A. M. (2026). MEASURE-THEORETIC AND FIXED POINT ANALYSIS IN NEUTROSOPHIC MR-METRIC SPACES. *Nonlinear Functional Analysis and Applications*, 581-598.
22. Malkawi, Abed Al-Rahman M., Rabaiah, Ayat M. (2026). "Common Fixed Point Theorems for Six Weakly Compatible Mappings in MMetric Spaces." *WSEAS Transactions on Mathematics*, 25, 137-146. DOI:10.37394/23206.2026.25.13
23. Malkawi, Abed Al-Rahman M., Rabaiah, Ayat M., Amourah, Ala, Alsoboh, Abdullah, Al Kasbi, Ahmed, Sasa, Tala. (2026). Non-Commutative Neutrosophic MR-Metric Spaces: A Unified Framework for Quantum Structures and Fixed Point Theory. *International Journal of Analysis and Applications*, 24, 171.

24. A. Malkawi, "Analysis of Cauchy Sequence Convergence and Uniqueness in MRMetric Spaces," WSEAS Transactions on Mathematics, vol. 25, pp. 215-219, 2026, DOI:10.37394/23206.2026.25.21
25. Ganie, A. H., Gheith, N. E. M., Al-Qudah, Y., Ganie, A. H., Sharma, S. K., Aqlan, A. M., and Khalaf, M. M. (2024). An innovative Fermatean fuzzy distance metric with its application in classification and bidirectional approximate reasoning. IEEE Access, 12, 4780-4791. DOI: https://doi.org/10.1109/ACCESS.2023.3348780
26. Hassan, N., Al-Qudah, Y. Fuzzy parameterized complex multi-fuzzy soft set. Journal of Physics: Conference Series, 2019, 1212(1), 012016 DOI:10.1088/1742-6596/1212/1/012016
27. Al-Qudah, Y., Jaradat, A., Sharma, S. K., and Bhat, V. K. Mathematical analysis of the structure of one-heptagonal carbon nanocone in terms of its basis and dimension. Physica Scripta, 2024, 99(5), 055252. DOI: https://doi.org/10.1088/1402-4896/ad3add
28. Al-Qudah, Y., Hamadameen, A.O. Kh, N.A. and Al-Sharqi, F., 2025. A new generalization of interval-valued Q-neutrosophic soft matrix and its applications. International Journal of Neutrosophic Science (IJNS), 25(3), 242-257. DOI: https://doi.org/10.54216/IJNS.250322
29. Al-Qudah, Y., Al-Sharqi, F., Mishlish, M., and Rasheed, M. M. Hybrid integrated decision-making algorithm based on AO of possibility interval-valued neutrosophic soft settings. International Journal of Neutrosophic Science,2023, 22(3), 84-98. DOI: https://doi.org/10.54216/IJNS.220306
30. Al-Qudah, Y., Alhazaymeh, K., Hassan, N., Almousa, M., Alaroud, M. Transitive Closure of Vague Soft Set Relations and its Operators. International Journal of Fuzzy Logic and Intelligent Systems, 2022, 22(1), pp. 59–68. DOI: https://doi.org/10.5391/IJFIS.2022.22.1.59
31. Qoqazeh, H., Al-Qudah, Y., Almousa, M., Jaradat, A. On D-Compact topological spaces. Journal of Applied Mathematics and Informatics, 2021, 39(5-6), pp. 883–894 DOI: https://doi.org/10.14317/jami.2021.883
32. Al-Qudah Y. A robust framework for the decision-making based on single-valued neutrosophic fuzzy soft expert setting. International Journal of Neutrosophic Science, 2024,23(2): 195-210. Doi: https://doi.org/10.54216/IJNS.230216
33. Al-qudah, Y., Alaroud, M., Qoqazeh, H.,Alhazmi, S.E., Al-omari, S. Approximate Analytic–Numeric Fuzzy Solutions of Fuzzy Fractional Equations Using a Residual Power Series Approach. Symmetry, 2022, 14(4), 804 DOI:https://doi.org/10.3390/sym14040804
34. R. Abdulkareem and A. M. Abdulazeez, "A Comparative Study of Multi-Class Classification based on Imbalanced Data," Statistics, Optimization & Information Computing, vol. 15, no. 5, pp. 4337-4357, 2025.
35. T. Mapharing and M. Lekgari, "Study on Microscopic Numerical Simulations of an SIQR Model Considering Networks," Statistics, Optimization & Information Computing, vol. 14, no. 4, pp. 1964-1979, 2025.
36. A. S. M. Mussa, M. T. Arif, A. Al Mamun, A. Hasib, M. A. Islam, R. Hossen, and A. Rahman, "Deploying an IoT-enabled Integrated Comprehensive Home Automation System using WSN for Enhanced Continuous Optimization and Fault Identification System," Statistics, Optimization & Information Computing, vol. 14, no. 1, pp. 282-310, 2025.