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Randic, Sum-Connectivity, and Their Product Energies of Non-Commuting Graph of Dihedral Groups

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Abstract One of the most extensively studied topics in chemical graph theory is graph energy. In this paper, we examine the non-commuting graph associated with the dihedral groups of order $2n$, for $n \geq 3$. We determine the spectral radius and the corresponding energies arising from the Randic, sum-connectivity, and the product of the Randic and sum-connectivity matrices of this graph. Our results show that all the obtained energies are strongly hypoenergetic. Moreover, it is shown that, in all cases, each energy is exactly equal to twice its corresponding spectral radius.

Keywords Non-commuting Graph, The Energy of a Graph, Dihedral Group, Randic Matrix, Sum-Connectivity Matrix, Product of Randic and sum-Connectivity Matrix

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1. Introduction

The energy of a graph in chemistry is utilized to estimate molecules' total π-electron energy. Chemical graph theory is a subfield of mathematical chemistry that employs graph theory to mathematically represent chemical substances, as indicated by [21]. The molecules are represented as graphs, with carbon atoms showing as vertices and hydrogen bonds connecting them showing as edges. We direct the reader to [12] for supplementary material and foundational ideas about graph energies. Diverse graph energies find applications in crystallography [23], macromolecular theory [15], protein sequencing [22], biology [7], air travel issues [10], and spacecraft design [16].

The non-commuting graph on a finite group G is defined by Abdollahi in 2006 [1], denoted by Γ_G, where the vertex set comprises non-central members of G and two distinct vertices are considered neighboring whenever they do not commute in G. Graphs Γ_G have been discussed by many authors, for instance, investigating the isomorphic properties of Γ_G of the respective groups. The isomorphism of two non-commuting graphs implies the groups are also isomorphic, as well [5]. More specifically, if Γ_G for a group G is always isomorphic to Γ_G for A_n, then $G \cong A_n$ [2].

The association between Γ_G and the adjacency matrix implies Γ_G has eigenvalues based on the corresponding matrix. The sum of the absolute values of its eigenvalues is the energy of Γ_G. Gutman [8] introduced this definition in 1978. In addition, the energy value is never an odd integer [4, 13].

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In this research, for n greater than or equal to 3, the dihedral group of order $2n$ is the vertex set of Γ_G. It is denoted by $D_{2n} = \langle t, u : t^n = u^2 = e, utu = t^{-1} \rangle$ [3]. The further discussion, we denote $\Gamma_{D_{2n}}$ as the non-commuting graph of D_{2n}. Since the center of D_{2n} is $\{e\}$ when n is odd, then $\Gamma_{D_{2n}}$ has $2n-1$ vertices. Meanwhile, for even n, the center of D_{2n} is $\{e, t^{\frac{n}{2}}\}$ which means $\Gamma_{D_{2n}}$ has $2n-2$ vertices. Moreover, the centralizer of t^i is $C_{D_{2n}}(t^i) = \{t^j : 1 \leq j \leq n\}$ and for $t^i u$ is either $C_{D_{2n}}(t^i u) = \{e, t^i u\}$, if n is odd or $C_{D_{2n}}(t^i u) = \{e, t^{\frac{n}{2}}, t^i u, t^{\frac{n}{2}+i} u\}$, if n is even.

The energy of Γ_G for D_{2n} has been studied by several authors, which involves degree-based matrices, including Wiener-Hosoya matrix [18], closeness matrix [19], and Seidel Laplacian matrix [20]. Gutman et al. introduced the Randic (R) matrix of a graph [9]. In addition, the definition of the sum-connectivity (SC) matrix can be seen in [24] and the product of Randic and sum-connectivity (RS) matrix definition [14]. Then this research is focused on the energy of Γ_G for D_{2n}, $n \geq 3$, based on R, SC, and RS-matrices.

2. Preliminaries

In this section, we begin with the common definition of the corresponding matrices. The definition is based on the degree of vertex v_p in D_{2n}, or simply denoted by d_{v_p}.

2.1. *Adjacency Criteria of* $\Gamma_{D_{2n}}$

Recall that $D_{2n} = \langle t, u : t^n = u^2 = e, utu = t^{-1} \rangle$ is the dihedral group of order $2n$, consisting of the set of rotations $\{e, t, t^2, \ldots, t^{n-1}\}$ and reflections $\{u, tu, t^2 u, \ldots, t^{n-1} u\}$. Let $\Gamma_{D_{2n}}$ denote the non-commuting graph of D_{2n}. The vertex set is defined by $V(\Gamma_{D_{2n}}) = D_{2n} \backslash Z(D_{2n})$, where $Z(D_{2n})$ denotes the center of the group. Two distinct vertices $x, y \in V(\Gamma_{D_{2n}})$ are said to be adjacent if and only if $xy \neq yx$ in $\Gamma_{D_{2n}}$.

The adjacency relations in $\Gamma_{D_{2n}}$ follow directly from the algebraic structure of the dihedral group. For any two rotations t^i and t^j, we have $t^i t^j = t^j t^i$, and hence no two rotations are adjacent in $\Gamma_{D_{2n}}$. On the other hand, for a rotation t^i and a reflection $t^j u$, we obtain $t^i (t^j u) \neq (t^j u) t^i$. Therefore, every non-central rotation is adjacent to every reflection in $\Gamma_{D_{2n}}$.

The adjacency between reflections depends on the parity of n. If n is odd, the centralizer of a reflection element $t^i u$ in D_{2n} is given by $\{e, t^i u\}$, which implies that no two distinct reflections commute. Hence, every pair of distinct reflections is adjacent. In contrast, if n is even, the centralizer of $t^i u$ is $\{e, t^{\frac{n}{2}}, t^i u, t^{\frac{n}{2}+i} u\}$. This shows that $t^i u$ commutes with $t^{\frac{n}{2}+i} u$, and therefore these two vertices are not adjacent in $\Gamma_{D_{2n}}$, while all other distinct pairs of reflections remain adjacent.

2.2. *Graph Matrices*

Definition 2.1. [9] The Randic matrix of $\Gamma_{D_{2n}}$ is given by $R(\Gamma_{D_{2n}}) = (r_{pq})$ where

$$r_{pq} = \begin{cases} \frac{1}{\sqrt{d_{v_p} d_{v_q}}}, & \text{for adjacent } v_p \text{ and } v_q \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. [24] The SC-matrix of $\Gamma_{D_{2n}}$ is given by $SC(\Gamma_{D_{2n}}) = (sc_{pq})$ where

$$sc_{pq} = \begin{cases} \frac{1}{\sqrt{d_{v_p} + d_{v_q}}}, & \text{for adjacent } v_p \text{ and } v_q \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. [14] The RS-matrix of $\Gamma_{D_{2n}}$ is $RS(\Gamma_{D_{2n}}) = (r_{pq})$ where

$$rs_{pq} = \begin{cases} \frac{1}{\sqrt{d_{v_p}^2 d_{v_q} + d_{v_p} d_{v_q}^2}}, & \text{for adjacent } v_p \text{ and } v_q \\ 0, & \text{otherwise.} \end{cases}$$

In addition, the characteristic polynomial of $\Gamma_{D_{2n}}$ can be determined by formulating the determinant of $R(\Gamma_{D_{2n}}) - \alpha I_{2n-1}$ for odd n (or $R(\Gamma_{D_{2n}}) - \alpha I_{2n-2}$ for even n) and is denoted by $P_{R(\Gamma_{D_{2n}})}(\alpha)$. The solutions of $P_{R(\Gamma_{D_{2n}})}(\alpha) = 0$ gives the eigenvalues of $R(\Gamma_{D_{2n}})$, we write α_i, for $i = 1, 2, ..., 2n-1$ and n is odd (or $i = 1, 2, ..., 2n-2$ for even n). The spectrum of $\Gamma_{D_{2n}}$ is $Spec(\Gamma_{D_{2n}}) = \left\{\alpha_1^{k_1}, \alpha_2^{k_2}, ..., \alpha_{2n-1}^{k_{2n-1}}\right\}$, with respective multiplicities k_i, for $i = 1, 2, ..., 2n-1$. The Randic spectral radius of $\Gamma_{D_{2n}}$ is

$$\rho_R(\Gamma_{D_{2n}}) = max\{|\alpha| : \alpha \in Spec_D(\Gamma_{D_{2n}})\}.$$

The Randic energy of $\Gamma_{D_{2n}}$ is

$$E_R(\Gamma_{D_{2n}}) = \sum_{i=1}^{n} |\alpha_i| \,.$$

The above notations also apply to the sum-connectivity and product of Randic and sum-connectivity matrices.

Furthermore, $\Gamma_{D_{2n}}$ can be stated as strongly hypoenergetic graph [12] whenever the energy meets the requirements as follows:

$$E(\Gamma_{D_{2n}}) < \begin{cases} 2(n-1), & \text{for odd } n \\ 2n-3, & \text{when } n \text{ is even.} \end{cases}$$

To construct the matrix, we need the information from the following theorem.

Theorem 2.4
[11] In $\Gamma_{D_{2n}}$,

1. $d_{t^i} = n$, and
2. $d_{t^i u} = \begin{cases} 2(n-1), & \text{for odd } n \\ 2(n-2), & \text{when } n \text{ is even} \end{cases}$

We now supply some previous results in support of the theorems derived in Section 3. Obtaining the graph energy requires formulating the characteristic formula of $\Gamma_{D_{2n}}$. An $n \times n$ matrix J_n is defined as a matrix with exactly all entries 1. Here are two essential results that assist in determining the characteristic formula of $\Gamma_{D_{2n}}$.

Lemma 2.5
[17] For real values a, b, c and d, and an $(n_1 + n_2) \times (n_1 + n_2)$ matrix

$$M = \begin{pmatrix} aJ_{n_1} - aI_{n_1} & cJ_{n_1 \times n_2} \\ dJ_{n_2 \times n_1} & bJ_{n_2} - bI_{n_2} \end{pmatrix},$$

the characteristic polynomial of M can be expressed as

$$\begin{aligned} P_M(\alpha) = & \begin{vmatrix} (\alpha + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\alpha + b)I_{n_2} - bJ_{n_2} \end{vmatrix} \\ = & (\alpha + a)^{n_1 - 1}(\alpha + b)^{n_2 - 1} \left((\alpha - (n_1 - 1)\, a)\, (\alpha - (n_2 - 1)\, b) - n_1 n_2 cd \right), \end{aligned}$$

where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 2.6
[6] If $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the partition into A, B, C, D, hence

$$|P| = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = |A| \left|D - CA^{-1}B\right|,$$

where P and A are square matrices and A is non-singular.

3. Main Results

This part delineates the energy of $\Gamma_{D_{2n}}$ for $n \geq 3$ utilizing the relevant matrices, including Randic, sum-connectivity, and the product of Randic and sum-connectivity matrices. We know that if $n = 1$ and $n = 2$, then D_{2n} is abelian, consequently, strictly it is for $n \geq 3$.

Theorem 3.1
For real numbers r, s and a $(2n-2) \times (2n-2)$ matrix

$$M = \begin{pmatrix} 0_{n-2} & sJ_{(n-2)\times\frac{n}{2}} & sJ_{(n-2)\times\frac{n}{2}} \\ sJ_{\frac{n}{2}\times(n-2)} & r(J-I)_{\frac{n}{2}} & r(J-I)_{\frac{n}{2}} \\ sJ_{\frac{n}{2}\times(n-2)} & r(J-I)_{\frac{n}{2}} & r(J-I)_{\frac{n}{2}} \end{pmatrix},$$

subsequently the characteristic formula of M can be expressed as

$$P_M(\alpha) = (\alpha)^{\frac{3n-6}{2}} (\alpha + 2r)^{\frac{n}{2}-1} \left(\alpha^2 - (n-2)r\alpha - n(n-2)s^2\right).$$

Proof
For the real numbers r, s, suppose now M as a square matrix of size $2n-2$. The characteristic formula of M is formulated by $P_M(\alpha) = |\alpha I_{2n-2} - M|$ as follows

$$P_M(\alpha) = \begin{vmatrix} \alpha I_{n-2} & -sJ_{(n-2)\times\frac{n}{2}} & -sJ_{(n-2)\times\frac{n}{2}} \\ -sJ_{\frac{n}{2}\times(n-2)} & (\alpha+r)I_{\frac{n}{2}} - rJ_{\frac{n}{2}} & -r(J-I)_{\frac{n}{2}} \\ -sJ_{\frac{n}{2}\times(n-2)} & -r(J-I)_{\frac{n}{2}} & (\alpha+r)I_{\frac{n}{2}} - rJ_{\frac{n}{2}} \end{vmatrix}.$$

Step 1: We perform the following row operation on the matrix. For each $1 \leq i \leq \frac{n}{2}$, replace the row $R_{n-2+\frac{n}{2}+i}$ with a new row defined by subtracting row R_{n-2+i}. In other words, $R_{n-2+\frac{n}{2}+i}$ by $R'_{n-2+\frac{n}{2}+i} = R_{n-2+\frac{n}{2}+i} - R_{n-2+i}$. Thus,

$$P_M(\alpha) = \begin{vmatrix} \alpha I_{n-2} & -sJ_{(n-2)\times\frac{n}{2}} & -sJ_{(n-2)\times\frac{n}{2}} \\ -sJ_{\frac{n}{2}\times(n-2)} & (\alpha+r)I_{\frac{n}{2}} - rJ_{\frac{n}{2}} & -r(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -\alpha I_{\frac{n}{2}} & \alpha I_{\frac{n}{2}} \end{vmatrix}.$$

Step 2: For each $1 \leq i \leq \frac{n}{2}$, replace the column C_{n-2+i} with a new column obtained by adding the column $C_{n-2+\frac{n}{2}+i}$, that is, $C'_{n-2+i} = C_{n-2+i} + C_{n-2+\frac{n}{2}+i}$, then

$$P_M(\alpha) = \begin{vmatrix} \alpha I_{n-2} & -2sJ_{(n-2)\times\frac{n}{2}} & -sJ_{(n-2)\times\frac{n}{2}} \\ -sJ_{\frac{n}{2}\times(n-2)} & (\alpha+2r)I_{\frac{n}{2}} - 2rJ_{\frac{n}{2}} & -r(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & \alpha I_{\frac{n}{2}} \end{vmatrix} = \begin{vmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{vmatrix}.$$

After performing steps 1 and 2, the matrix can be transformed into a block triangular form, where the lower-left block become zero. Using Theorem 2.6 with $A = \begin{pmatrix} M_1 & M_2 \\ M_4 & M_5 \end{pmatrix}$, $B = \begin{pmatrix} M3 \\ M_6 \end{pmatrix}$, $C = \begin{pmatrix} M_7 & M_8 \end{pmatrix}$, $D = M_9$, then the determinant reduces to the product of determinants of the diagonal blocks, that is, $P_M = |A|\,|D|$. The determinant $|A|$ is evaluated via Lemma 2.5 by setting $n_1 = n-2$, $n_2 = \frac{n}{2}$ and $a = 0$, $b = 2r$, $c = 2s$, $d = s$, leading to

$$\begin{aligned} |A| =& (\alpha+a)^{n_1-1}(\alpha+b)^{n_2-1}\left((\alpha-(n_1-1)a)(\alpha-(n_2-1)b) - n_1n_2cd\right) \\ =& (\alpha+0)^{n-2-1}(\alpha+2r)^{\frac{n}{2}-1}\left((\alpha-(n-2-1)0)\left(\alpha - \left(\frac{n}{2}-1\right)2r\right) - (n-2)\frac{n}{2}2s^2\right) \\ =& (\alpha)^{n-3}(\alpha+2r)^{\frac{n}{2}-1}\left(\alpha^2 - (n-2)r\alpha - n(n-2)s^2\right). \end{aligned}$$

We have the form of D as a diagonal matrix, its determinant equals the product of its diagonal entries, and hence

$$|D| = (\alpha)^{\frac{n}{2}}.$$

Therefore,

$$P_M(\alpha) = (\alpha)^{\frac{3n-6}{2}} (\alpha + 2r)^{\frac{n}{2}-1} \left(\alpha^2 - (n-2)r\alpha - n(n-2)s^2\right).$$

$\square$

3.1. Randic Energy

This part determines the Randic energy of $\Gamma_{D_{2n}}$ for $n \geq 3$.

Theorem 3.2
In $\Gamma_{D_{2n}}$, the characteristic formula for $R(\Gamma_{D_{2n}})$ is

$$P_{R(\Gamma_{D_{2n}})}(\alpha) = \begin{cases} \alpha^{n-2}\left(\alpha + \frac{1}{2(n-1)}\right)^{n-1}(\alpha-1)\left(\alpha+\frac{1}{2}\right), & \text{for odd } n \\ \alpha^{\frac{3n-6}{2}}\left(\alpha + \frac{1}{n-2}\right)^{\frac{n}{2}-1}(\alpha-1)\left(\alpha+\frac{1}{2}\right), & \text{for even } n. \end{cases}$$

Proof

1. Taking n be odd number, then using Theorem 2.4 (1), we have $d_{t^i} = n$ and $d_{t^iu} = 2n-2$, for $1 \leq i \leq n$. The set of rotations is $H_1 = \{t, t^2, \ldots, t^{n-1}\}$ and reflections $H_2 = \{u, tu, t^2u, \ldots, t^{n-1}u\}$. Since the centralizer of t^i consists of elements $\{e, t, t^2, \ldots, t^{n-1}\}$, the vertex corresponding to t^i, where $1 \leq i \leq n-1$, does not share edges with all vertices in H_1. However, it maintains adjacency with every vertex in H_2. Given that the centralizer of t^iu in D_{2n} is $\{e, t^iu\}$, one can conclude that for all $1 \leq i \leq n$, the vertex corresponding to t^iu is connected to all other vertices in $H_1 \cup H_2$. Now using Definition 2.1, then the Randic matrix of $\Gamma_{D_{2n}}$ of size $2n-1$,

$$R(\Gamma_{D_{2n}}) = \begin{array}{c} \\ t \\ \vdots \\ t^{n-1} \\ u \\ \vdots \\ t^{n-1}u \end{array} \begin{array}{c} \begin{array}{cccccc} t & \cdots & t^{n-1} & u & \cdots & t^{n-1}u \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-1)}} & \cdots & \frac{1}{\sqrt{2n(n-1)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-1)}} & \cdots & \frac{1}{\sqrt{2n(n-1)}} \\ \frac{1}{\sqrt{2n(n-1)}} & \cdots & \frac{1}{\sqrt{2n(n-1)}} & 0 & \cdots & \frac{1}{2(n-1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2n(n-1)}} & \cdots & \frac{1}{\sqrt{2n(n-1)}} & \frac{1}{2(n-1)} & \cdots & 0 \end{pmatrix} \end{array}.$$

Now $R(\Gamma_{D_{2n}})$ can be segmented into four block matrices as outlined below:

$$R(\Gamma_{D_{2n}}) = \begin{pmatrix} 0_{n-1} & \frac{1}{\sqrt{2n(n-1)}}J_{(n-1)\times n} \\ \frac{1}{\sqrt{2n(n-1)}}J_{n\times(n-1)} & \frac{1}{2(n-1)}(J_n - I_n) \end{pmatrix}.$$

Then the following determinant is the characteristic polynomial of $R(\Gamma_{D_{2n}})$ is $P_{R(\Gamma_{D_{2n}})}(\alpha) = |\alpha I_{2n-1} - R(\Gamma_{D_{2n}})|$ as follows.

$$P_{R(\Gamma_{D_{2n}})}(\alpha) = \begin{vmatrix} \alpha I_{n-1} & -\frac{1}{\sqrt{2n(n-1)}}J_{(n-1)\times n} \\ -\frac{1}{\sqrt{2n(n-1)}}J_{n\times(n-1)} & (\alpha + \frac{1}{2(n-1)})I_n - \frac{1}{2(n-1)}J_n) \end{vmatrix}.$$

Using Lemma 2.5 and taking $a=0$, $b=\frac{1}{2(n-1)}$, $c=d=\frac{1}{\sqrt{2n(n-1)}}$, $n_1=n-1$ and $n_2=n$, we derive the following expression.

$$\begin{aligned}
P_{R(\Gamma_{D_{2n}})}(\alpha) =& (\alpha+a)^{n_1-1}(\alpha+b)^{n_2-1}\left((\alpha-(n_1-1)\,a)\,(\alpha-(n_2-1)\,b)-n_1n_2cd\right)\\
=&(\alpha+0)^{n-1-1}\left(\alpha+\frac{1}{2(n-1)}\right)^{n-1}\left((\alpha-(n-1-1)\,0)\left(\alpha-(n-1)\,\frac{1}{2(n-1)}\right)-\right.\\
&\left.(n-1)n\left(\frac{1}{\sqrt{2n(n-1)}}\right)^2\right)\\
=&\alpha^{n-2}\left(\alpha+\frac{1}{2(n-1)}\right)^{n-1}\left(\alpha\left(\alpha-\frac{1}{2}\right)-\frac{1}{2}\right)\\
=&\alpha^{n-2}\left(\alpha+\frac{1}{2(n-1)}\right)^{n-1}(\alpha-1)\left(\alpha+\frac{1}{2}\right),
\end{aligned}$$

and the the proof has been finalized.

2. Let n is even. Then following Theorem 2.4 (1), we derive $d_{t^i}=n$ and $d_{t^iu}=2n-4$, for all $1\le i\le n$. Define $H_1=\left\{t,t^2,\dots,t^{\frac{n}{2}-1},t^{\frac{n}{2}+1},\dots,t^{n-1}\right\}$ and $H_2=\left\{u,tu,t^2u,\dots,t^{n-1}u\right\}$. From the structure of the centralizer of t^i in D_{2n}, it follows that vertices in H_1 are adjacent exclusively to vertices in H_2. Moreover, because the centralizer of t^iu is $\left\{e,t^{\frac{n}{2}},t^iu,t^{\frac{n}{2}+i}u\right\}$, the vertices t^iu and $t^{\frac{n}{2}+i}u$ are non-adjacent in the graph $\Gamma_{D_{2n}}$. Hence, following Definition 2.1, matrix $R(\Gamma_{D_{2n}})$ of size $(2n-2)$ is

$$R(\Gamma_{D_{2n}})=\begin{array}{c|cccccccccc}
 & t & \cdots & t^{n-1} & u & \cdots & t^{\frac{n}{2}-1}u & t^{\frac{n}{2}}u & \cdots & t^{n-1}u\\
\hline
t & 0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}} & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
t^{n-1} & 0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}} & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}}\\
u & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}} & 0 & \cdots & \frac{1}{2(n-2)} & 0 & \cdots & \frac{1}{2(n-2)}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
t^{\frac{n}{2}-1}u & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}} & \frac{1}{2(n-2)} & \cdots & 0 & \frac{1}{2(n-2)} & \cdots & 0\\
t^{\frac{n}{2}}u & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}} & 0 & \cdots & \frac{1}{2(n-2)} & 0 & \cdots & \frac{1}{2(n-2)}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
t^{n-1}u & \frac{1}{\sqrt{2n(n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-2)}} & \frac{1}{2(n-2)} & \cdots & 0 & \frac{1}{2(n-2)} & \cdots & 0
\end{array}.$$

Consequently, $R(\Gamma_{D_{2n}})$ can be represented in block matrix form as

$$R(\Gamma_{D_{2n}})=\begin{pmatrix}
0_{n-2} & \frac{1}{\sqrt{2n(n-2)}}J_{(n-2)\times\frac{n}{2}} & \frac{1}{\sqrt{2n(n-2)}}J_{(n-2)\times\frac{n}{2}}\\
\frac{1}{\sqrt{2n(n-2)}}J_{\frac{n}{2}\times(n-2)} & \frac{1}{2(n-2)}(J-I)_{\frac{n}{2}} & \frac{1}{2(n-2)}(J-I)_{\frac{n}{2}}\\
\frac{1}{\sqrt{2n(n-2)}}J_{\frac{n}{2}\times(n-2)} & \frac{1}{2(n-2)}(J-I)_{\frac{n}{2}} & \frac{1}{2(n-2)}(J-I)_{\frac{n}{2}}
\end{pmatrix}.$$

Substituting $s = \frac{1}{\sqrt{2n(n-2)}}$ and $r = \frac{1}{2(n-2)}$ into Theorem 3.1 yields

$$\begin{aligned}
P_{R(\Gamma_{D_{2n}})}(\alpha) =&\alpha^{\frac{3n-6}{2}} (\alpha+2r)^{\frac{n}{2}-1}\left(\alpha^2-(n-2)r\alpha-n(n-2)s^2\right)\\
=&\alpha^{\frac{3n-6}{2}}\left(\alpha+2\cdot\frac{1}{2(n-2)}\right)^{\frac{n}{2}-1}\left(\alpha^2-(n-2)\frac{1}{2(n-2)}\alpha-n(n-2)\left(\frac{1}{\sqrt{2n(n-2)}}\right)^2\right)\\
=&\alpha^{\frac{3n-6}{2}}\left(\alpha+\frac{1}{n-2}\right)^{\frac{n}{2}-1}\left(\alpha^2-\frac{1}{2}\alpha-\frac{1}{2}\right)\\
=&\alpha^{\frac{3n-6}{2}}\left(\alpha+\frac{1}{n-2}\right)^{\frac{n}{2}-1}(\alpha-1)\left(\alpha+\frac{1}{2}\right).
\end{aligned}$$

□

The following result yields the spectrum and spectral radius of $\Gamma_{D_{2n}}$, corresponding to the Randic matrix.

Theorem 3.3
In $\Gamma_{D_{2n}}$, the Randic spectral radius for $\Gamma_{D_{2n}}$ is $\rho_R(\Gamma_{D_{2n}}) = 1$.

Proof

1. Suppose n is odd. According to Theorem 3.2 (1), $P_{R(\Gamma_{D_{2n}}}(\alpha)$ has four eigenvalues, with multiplicity $(n-2)$, $\alpha_1 = 0$, $\alpha_2 = -\frac{1}{2(n-1)}$ of multiplicity $(n-1)$ and are a single $\alpha_3 = 1$ and $\alpha_4 = \frac{1}{2}$. Hence, R-spectrum for $\Gamma_{D_{2n}}$ is
$$Spec(\Gamma_{D_{2n}}) = \left\{(1)^1, (0)^{n-2}, \left(-\frac{1}{2}\right)^1, \left(-\frac{1}{2(n-1)}\right)^{n-1}\right\}.$$
It is possible to infer that, $\rho_R(\Gamma_{D_{2n}}) = 1$.
2. The information from Theorem 3.2 (2) for even n states that $R(\Gamma_{D_{2n}})$ has four eigenvalues $\alpha_1 = 0$ of multiplicity $\frac{3n-6}{2}$, $\alpha_2 = -\frac{1}{n-2}$ of multiplicity $(\frac{n}{2}-1)$, and a single $\alpha_3 = 1$ and $\alpha_4 = -\frac{1}{2}$. So that we can provide
$$Spec(\Gamma_{D_{2n}}) = \left\{(1)^1, (0)^{\frac{3n-6}{2}}, \left(-\frac{1}{2}\right)^1, \left(-\frac{1}{n-2}\right)^{\frac{n}{2}-1}\right\}.$$
Thus, the maximum of the absolute eigenvalues of $R(\Gamma_{D_{2n}})$ is $\rho_R(\Gamma_{D_{2n}}) = 1$.

□

Considering the result in the preceding theorem, we are now able to have a result on the Randic energy of $\Gamma_{D_{2n}}$.

Theorem 3.4
In$\Gamma_{D_{2n}}$, the Randic energy for $\Gamma_{D_{2n}}$ is $E_R(\Gamma_{D_{2n}}) = 2$.

Proof
By observing that spectrum in the proof of Theorem 3.3, the direct computation gives
$E_R(\Gamma_{D_{2n}}) = 2$. □

3.2. Sum-Connectivity Energy

This section presents the sum-connectivity energy of $\Gamma_{D_{2n}}$. Firstly, we show the characteristic polynomial of $\Gamma_{D_{2n}}$ corresponding to the sum-connectivity matrix.

Theorem 3.5
In $\Gamma_{D_{2n}}$, the characteristic formula of sum-connectivity matrix of $\Gamma_{D_{2n}}$ is

1. for odd n, $P_{SC(\Gamma_{D_{2n}})}(\alpha) = \alpha^{n-2}\left(\alpha + \frac{1}{2\sqrt{n-1}}\right)^{n-1}\left(\alpha^2 - \frac{\sqrt{n-1}}{2}\alpha - \frac{n(n-1)}{3n-2}\right)$,
2. for n is even, $P_{SC(\Gamma_{D_{2n}})}(\alpha) = \alpha^{\frac{3(n-2)}{2}}\left(\alpha + \frac{1}{\sqrt{n-2}}\right)^{\frac{n}{2}-1}\left(\alpha^2 - \frac{\sqrt{n-2}}{2}\alpha - \frac{n(n-2)}{3n-4}\right)$.

Proof

1. Let n is odd. By applying the same vertex partition into elements of type t^i and $t^i u$ as used in the proof of Theorem 3.2 (part 1), and accordance with Theorem 2.4 (1) and Definition 2.2, it follows that the matrix $SC(\Gamma_{D_{2n}})$ is of order $2n-1$ as follows:

$$SC(\Gamma_{D_{2n}}) = \begin{array}{c} \\ t \\ \vdots \\ t^{n-1} \\ u \\ \vdots \\ t^{n-1}u \end{array} \begin{array}{c} \begin{array}{cccccc} t & \cdots & t^{n-1} & u & \cdots & t^{n-1}u \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{\sqrt{3n-2}} & \cdots & \frac{1}{\sqrt{3n-2}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\sqrt{3n-2}} & \cdots & \frac{1}{\sqrt{3n-2}} \\ \frac{1}{\sqrt{3n-2}} & \cdots & \frac{1}{\sqrt{3n-2}} & 0 & \cdots & \frac{1}{2\sqrt{n-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{3n-2}} & \cdots & \frac{1}{\sqrt{3n-2}} & \frac{1}{2\sqrt{n-1}} & \cdots & 0 \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 0_{n-1} & \frac{1}{\sqrt{3n-2}}J_{(n-1)\times n} \\ \frac{1}{\sqrt{3n-2}}J_{n\times(n-1)} & \frac{1}{2\sqrt{n-1}}(J-I)_n \end{pmatrix}.$$

Now we can get

$$P_{SC(\Gamma_{D_{2n}})}(\alpha) = \begin{vmatrix} \alpha I_{n-1} & -\frac{1}{\sqrt{3n-2}}J_{(n-1)\times n} \\ -\frac{1}{\sqrt{3n-2}}J_{n\times(n-1)} & \left(\alpha + \frac{1}{2\sqrt{n-1}}\right)I_n - \frac{1}{2\sqrt{n-1}}J_n \end{vmatrix}.$$

Substituting $a = 0$, $b = \frac{1}{2\sqrt{n-1}}$, $c = d = \frac{1}{\sqrt{3n-2}}$, $n_1 = n-1$ and $n_2 = n$, into Lemma 2.5, we derive the desired expression for

$$\begin{aligned} P_{SC(\Gamma_{D_{2n}})}(\alpha) =& (\alpha + a)^{n_1-1}(\alpha + b)^{n_2-1}\left((\alpha - (n_1-1)a)(\alpha - (n_2-1)b) - n_1 n_2 cd\right) \\ =& (\alpha + 0)^{n-1-1}\left(\alpha + \frac{1}{2\sqrt{n-1}}\right)^{n-1}\left((\alpha - (n-1-1)0)\left(\alpha - (n-1)\frac{1}{2\sqrt{n-1}}\right) - \right. \\ & \left. (n-1)n\left(\frac{1}{\sqrt{3n-2}}\right)^2\right) \\ =& \alpha^{n-2}\left(\alpha + \frac{1}{2\sqrt{n-1}}\right)^{n-1}\left(\alpha\left(\alpha - \frac{\sqrt{n-1}}{2}\right) - \frac{n(n-1)}{3n-2}\right) \\ =& \alpha^{n-2}\left(\alpha + \frac{1}{2\sqrt{n-1}}\right)^{n-1}\left(\alpha^2 - \frac{\sqrt{n-1}}{2}\alpha - \frac{n(n-1)}{3n-2}\right). \end{aligned}$$

2. For even n, by employing the vertex partition into elements of type t^i and $t^i u$, as established in the proof of Theorem 3.2 (part 2), and in conjunction with Theorem 2.4 (2) and Definition 2.2, it follows that $SC(\Gamma_{D_{2n}})$ is a matrix of order $2n-2$, as shown below:

$$SC(\Gamma_{D_{2n}}) = \begin{array}{c} \\ t \\ \vdots \\ t^{n-1} \\ u \\ \vdots \\ t^{\frac{n}{2}-1}u \\ t^{\frac{n}{2}}u \\ \vdots \\ t^{n-1}u \end{array} \begin{array}{c} \begin{array}{ccccccccc} t & \dots & t^{n-1} & u & \dots & t^{\frac{n}{2}-1}u & t^{\frac{n}{2}}u & \dots & t^{n-1}u \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} & \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} & \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} \\ \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} & 0 & \dots & \frac{1}{2\sqrt{n-2}} & 0 & \dots & \frac{1}{2\sqrt{n-2}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} & \frac{1}{2\sqrt{n-2}} & \dots & 0 & \frac{1}{2\sqrt{n-2}} & \dots & 0 \\ \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} & 0 & \dots & \frac{1}{2\sqrt{n-2}} & 0 & \dots & \frac{1}{2\sqrt{n-2}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{3n-4}} & \dots & \frac{1}{\sqrt{3n-4}} & \frac{1}{2\sqrt{n-2}} & \dots & 0 & \frac{1}{2\sqrt{n-2}} & \dots & 0 \end{pmatrix} \end{array}.$$

In other words,

$$SC(\Gamma_{D_{2n}}) = \begin{pmatrix} 0_{n-2} & \frac{1}{\sqrt{3n-4}} J_{(n-2)\times\frac{n}{2}} & \frac{1}{\sqrt{3n-4}} J_{(n-2)\times\frac{n}{2}} \\ \frac{1}{\sqrt{3n-4}} J_{\frac{n}{2}\times(n-2)} & \frac{1}{2\sqrt{n-2}} (J-I)_{\frac{n}{2}} & \frac{1}{2\sqrt{n-2}} (J-I)_{\frac{n}{2}} \\ \frac{1}{\sqrt{3n-4}} J_{\frac{n}{2}\times(n-2)} & \frac{1}{2\sqrt{n-2}} (J-I)_{\frac{n}{2}} & \frac{1}{2\sqrt{n-2}} (J-I)_{\frac{n}{2}} \end{pmatrix}.$$

In accordance with Theorem 3.1 with $s = \frac{1}{\sqrt{3n-4}}$ and $r = \frac{1}{2\sqrt{n-2}}$, immediately we can provide

$$\begin{aligned} & P_{SC(\Gamma_{D_{2n}})}(\alpha) \\ &= \alpha^{\frac{3n-6}{2}} (\alpha + 2r)^{\frac{n}{2}-1} \left(\alpha^2 - (n-2)r\alpha - n(n-2)s^2\right) \\ &= \alpha^{\frac{3n-6}{2}} \left(\alpha + 2\left(\frac{1}{2\sqrt{n-2}}\right)\right)^{\frac{n}{2}-1} \left(\alpha^2 - (n-2)\left(\frac{1}{2\sqrt{n-2}}\right)\alpha - n(n-2)\left(\frac{1}{\sqrt{3n-4}}\right)^2\right) \\ &= \alpha^{\frac{3(n-2)}{2}} \left(\alpha + \frac{1}{\sqrt{n-2}}\right)^{\frac{n}{2}-1} \left(\alpha^2 - \frac{\sqrt{n-2}}{2}\alpha - \frac{n(n-2)}{3n-4}\right). \end{aligned}$$

□

To find the energy, we first need to determine the spectrum and spectral radius as follows.

Theorem 3.6
In $\Gamma_{D_{2n}}$, the SC-spectral radius for $\Gamma_{D_{2n}}$ is

$$\rho_{SC}(\Gamma_{D_{2n}}) = \begin{cases} \frac{1}{4}\left(\sqrt{n-1} + \sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right), & \text{for odd } n \\ \frac{1}{4}\left(\sqrt{n-2} + \sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right), & \text{for even } n. \end{cases}$$

Proof

1. According to Theorem 3.5 (1) where n is odd, $SC(\Gamma_{D_{2n}})$ has four eigenvalues, $\alpha_1 = 0$, $\alpha_2 = -\frac{1}{2\sqrt{n-1}}$ of multiplicity $(n-2)$, $(n-1)$ respectively, and $\alpha_{3,4} = \frac{1}{4}\left(\sqrt{n-1} \pm \sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right)$. Hence, SC-spectrum

for $\Gamma_{D_{2n}}$ is

$$Spec(\Gamma_{D_{2n}}) = \left\{ \left(\frac{1}{4}\left(\sqrt{n-1}+\sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right)\right)^1, (0)^{n-2}, \right.$$
$$\left. \left(\frac{1}{4}\left(\sqrt{n-1}-\sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right)\right)^1, \left(-\frac{1}{2\sqrt{n-1}}\right)^{n-1} \right\}.$$

Consequently, the maximum of the absolute eigenvalues for $\Gamma_{D_{2n}}$ is

$$\rho_{SC}(\Gamma_{D_{2n}}) = \frac{1}{4}\left(\sqrt{n-1}+\sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right).$$

2. Based on Theorem 3.5 (2) for even n, $SC(\Gamma_{D_{2n}})$ has four eigenvalues, $\alpha_1 = 0$ of multiplicity $\frac{3n-6}{2}$, $\alpha_2 = -\frac{1}{\sqrt{n-2}}$ of multiplicity $(\frac{n}{2}-1)$, and $\alpha_{3,4} = \frac{1}{4}\left(\sqrt{n-2} \pm \sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right)$. Thus, $SC-$spectrum for $\Gamma_{D_{2n}}$ is

$$Spec(\Gamma_{D_{2n}}) = \left\{ \left(\frac{1}{4}\left(\sqrt{n-2}+\sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right)\right)^1, (0)^{\frac{3n-6}{2}}, \right.$$
$$\left. \left(-\frac{1}{\sqrt{n-2}}\right)^{\frac{n}{2}-1}, \left(\frac{1}{4}\left(\sqrt{n-2}-\sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right)\right)^1 \right\}.$$

We then obtain SC-spectral radius of $\Gamma_{D_{2n}}$,

$$\rho_{SC}(\Gamma_{D_{2n}}) = \frac{1}{4}\left(\sqrt{n-2}+\sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right).$$

□

Theorem 3.7
In $\Gamma_{D_{2n}}$, the sum-connectivity energy for $\Gamma_{D_{2n}}$ is

$$E_{SC}(\Gamma_{D_{2n}}) = \begin{cases} \frac{1}{2}\left(\sqrt{n-1}+\sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right), & \text{for odd } n \\ \frac{1}{2}\left(\sqrt{n-2}+\sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right), & \text{when } n \text{ is even.} \end{cases}$$

Proof

1. Using $Spec(\Gamma_{D_{2n}})$ from Theorem 3.6 (1) for odd n, we then obtain the sum-connectivity energy of $\Gamma_{D_{2n}}$,

$$\begin{aligned} E_{SC}(\Gamma_{D_{2n}}) =& (n-2)|0| + (n-1)\left|-\frac{1}{2\sqrt{n-1}}\right| + \left|\frac{1}{4}\left(\sqrt{n-2}\pm\sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right)\right| \\ =& \frac{1}{2}\left(\sqrt{n-1}+\sqrt{\frac{(n-1)(19n-2)}{3n-2}}\right). \end{aligned}$$

2. For the even n, we know $SC-$spectrum of $\Gamma_{D_{2n}}$ from Theorem 3.6 (2). Therefore, $SC-$energy of $\Gamma_{D_{2n}}$ is given below

$$\begin{aligned} E_{SC}(\Gamma_{D_{2n}}) &= \left(\frac{3n-6}{2}\right)|0| + \left(\frac{n}{2}-1\right)\left|-\frac{1}{\sqrt{n-2}}\right| + \left|\frac{1}{4}\left(\sqrt{n-2} \pm \sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right)\right| \\ &= \frac{1}{2}\left(\sqrt{n-2} + \sqrt{\frac{(n-2)(19n-4)}{3n-4}}\right). \end{aligned}$$

□

3.3. *Product of Randic and Sum-Connectivity Energy*

In this part, we show the energy of $\Gamma_{D_{2n}}$ associated with RS-matrix. First, we need to formulate the characteristic polynomial $\Gamma_{D_{2n}}$ as shown as follows.

Theorem 3.8
In $\Gamma_{D_{2n}}$, the characteristic formula of RS-matrix of $\Gamma_{D_{2n}}$ is

1. for n is odd,

$$P_{RS(\Gamma_{D_{2n}})}(\alpha) = \alpha^{n-2}\left(\alpha + \frac{1}{4(n-1)\sqrt{n-1}}\right)^{n-1}\left(\alpha^2 - \frac{1}{4\sqrt{n-1}}\alpha + \frac{1}{2(3n-2)}\right),$$

2. for n is even,

$$P_{RS(\Gamma_{D_{2n}})}(\alpha) = \alpha^{\frac{3(n-2)}{2}}\left(\alpha + \frac{1}{4(n-2)\sqrt{n-2}}\right)^{\frac{n}{2}-1}\left(\alpha^2 - \frac{1}{2\sqrt{n-2}}\alpha - \frac{1}{2(3n-4)}\right).$$

Proof

1. Suppose n is odd. By applying the same vertex partition into elements of type t^i and t^iu as used in the proof of Theorem 3.2 (part 1), and in view of Theorem 2.4 (1) together with Definition 2.3, it follows that $RS(\Gamma_{D_{2n}})$ is a matrix of size $2n-1$,

$$RS(\Gamma_{D_{2n}}) = \begin{array}{c|cccccc} & t & \cdots & t^{n-1} & u & \cdots & t^{n-1}u \\ \hline t & 0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-1)(3n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-1)(3n-2)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t^{n-1} & 0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-1)(3n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-1)(3n-2)}} \\ u & \frac{1}{\sqrt{2n(n-1)(3n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-1)(3n-2)}} & 0 & \cdots & \frac{1}{4(n-1)\sqrt{n-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t^{n-1}u & \frac{1}{\sqrt{2n(n-1)(3n-2)}} & \cdots & \frac{1}{\sqrt{2n(n-1)(3n-2)}} & \frac{1}{4(n-1)\sqrt{n-1}} & \cdots & 0 \end{array}$$

$$= \begin{pmatrix} 0_{n-1} & \frac{1}{\sqrt{2n(n-1)(3n-2)}}J_{(n-1)\times n} \\ \frac{1}{\sqrt{2n(n-1)(3n-2)}}J_{n\times(n-1)} & \frac{1}{4(n-1)\sqrt{n-1}}(J-I)_n \end{pmatrix}.$$

Our analysis leads us to the characteristic formula of $RS(\Gamma_{D_{2n}})$ as follows:

$$P_{RS(\Gamma_{D_{2n}})}(\alpha) = \begin{vmatrix} \alpha I_{n-1} & -\frac{1}{\sqrt{2n(n-1)(3n-2)}}J_{(n-1)\times n} \\ -\frac{1}{\sqrt{2n(n-1)(3n-2)}}J_{n\times(n-1)} & \left(\alpha + \frac{1}{4(n-1)\sqrt{n-1}}\right)I_n - \frac{1}{4(n-1)\sqrt{n-1}}J_n \end{vmatrix}.$$

By applying Lemma 2.5 with $a = 0$, $b = \frac{1}{4(n-1)\sqrt{n-1}}$, $c = d = \frac{1}{\sqrt{2n(n-1)(3n-2)}}$, where $n_1 = n - 1$ and $n_2 = n$, we obtain the desired result:

$$\begin{aligned}
&P_{RS(\Gamma_{D_{2n}})}(\alpha) \\
&= (\alpha+0)^{n-1-1}\left(\alpha+\frac{1}{4(n-1)\sqrt{n-1}}\right)^{n-1}\left((\alpha-(n-1-1)\,0)\left(\alpha-(n-1)\,\frac{1}{4(n-1)\sqrt{n-1}}\right)-\right. \\
&\left.(n-1)n\left(\frac{1}{\sqrt{2n(n-1)(3n-2)}}\right)^2\right) \\
&= \alpha^{n-2}\left(\alpha+\frac{1}{4(n-1)\sqrt{n-1}}\right)^{n-1}\left(\alpha\left(\alpha-\frac{1}{4\sqrt{n-1}}\right)-(n-1)n\left(\frac{1}{2n(n-1)(3n-2)}\right)\right) \\
&= \alpha^{n-2}\left(\alpha+\frac{1}{4(n-1)\sqrt{n-1}}\right)^{n-1}\left(\alpha^2-\frac{1}{4\sqrt{n-1}}\alpha+\frac{1}{2(3n-2)}\right).
\end{aligned}$$

2. By adopting the vertex partition into elements of type t^i and t^iu, as developed in the proof of Theorem 3.2 (part 2), and using Theorem 2.4 (2) and Definition 2.3 for even n, matrix $RS(\Gamma_{D_{2n}})$ of size $2n-2$ is as given below:

$$\begin{array}{c}
\\ t \\ \vdots \\ t^{n-1} \\ u \\ \vdots \\ t^{\frac{n}{2}-1}u \\ t^{\frac{n}{2}}u \\ \vdots \\ t^{n-1}u
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
t & \cdots & t^{n-1} & u & \cdots & t^{\frac{n}{2}-1}u & t^{\frac{n}{2}}u & \cdots & t^{n-1}u
\end{array} \\
\begin{pmatrix}
0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} \\
\frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & 0 & \cdots & \frac{1}{4(n-2)\sqrt{n-2}} & 0 & \cdots & \frac{1}{4(n-2)\sqrt{n-2}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \frac{1}{4(n-2)\sqrt{n-2}} & \cdots & 0 & \frac{1}{4(n-2)\sqrt{n-2}} & \cdots & 0 \\
\frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & 0 & \cdots & \frac{1}{4(n-2)\sqrt{n-2}} & 0 & \cdots & \frac{1}{4(n-2)\sqrt{n-2}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{2n(n-2)(3n-4)}} & \cdots & \frac{1}{\sqrt{2n(n-2)(3n-4)}} & \frac{1}{4(n-2)\sqrt{n-2}} & \cdots & 0 & \frac{1}{4(n-2)\sqrt{n-2}} & \cdots & 0
\end{pmatrix}
\end{array}.$$

In other words,

$$RS(\Gamma_{D_{2n}}) = \begin{pmatrix}
0_{n-2} & \frac{1}{\sqrt{2n(n-2)(3n-4)}}J_{(n-2)\times\frac{n}{2}} & \frac{1}{\sqrt{2n(n-2)(3n-4)}}J_{(n-2)\times\frac{n}{2}} \\
\frac{1}{\sqrt{2n(n-2)(3n-4)}}J_{\frac{n}{2}\times(n-2)} & \frac{1}{4(n-2)\sqrt{n-2}}(J-I)_{\frac{n}{2}} & \frac{1}{4(n-2)\sqrt{n-2}}(J-I)_{\frac{n}{2}} \\
\frac{1}{\sqrt{2n(n-2)(3n-4)}}J_{\frac{n}{2}\times(n-2)} & \frac{1}{4(n-2)\sqrt{n-2}}(J-I)_{\frac{n}{2}} & \frac{1}{4(n-2)\sqrt{n-2}}(J-I)_{\frac{n}{2}}
\end{pmatrix}.$$

According to Theorem 3.1 where $s = \frac{1}{\sqrt{2n(n-2)(3n-4)}}$ and $r = \frac{1}{4(n-2)\sqrt{n-2}}$, immediately we can provide

$$\begin{aligned}
P_{RS(\Gamma_{D_{2n}})}(\alpha) &= (\alpha)^{\frac{3n-6}{2}}(\alpha+2r)^{\frac{n}{2}-1}\left(\alpha^2-(n-2)r\alpha-n(n-2)s^2\right) \\
&= (\alpha)^{\frac{3n-6}{2}}\left(\alpha+2\frac{1}{4(n-2)\sqrt{n-2}}\right)^{\frac{n}{2}-1} \\
&\quad\left(\alpha^2-(n-2)\left(\frac{1}{4(n-2)\sqrt{n-2}}\right)\alpha-n(n-2)\left(\frac{1}{\sqrt{2n(n-2)(3n-4)}}\right)^2\right) \\
&= \alpha^{\frac{3(n-2)}{2}}\left(\alpha+\frac{1}{2(n-2)\sqrt{n-2}}\right)^{\frac{n}{2}-1}\left(\alpha^2-\frac{1}{4\sqrt{n-2}}\alpha-\frac{1}{2(3n-4)}\right).
\end{aligned}$$

□

By considering the previous theorem, we present the spectrum and spectral radius of $\Gamma_{D_{2n}}$ based on RS-matrix.

Theorem 3.9
In $\Gamma_{D_{2n}}$, then the spectral radius for $\Gamma_{D_{2n}}$ corresponding to RS-matrix is

$$\rho_{RS}(\Gamma_{D_{2n}}) = \begin{cases} \frac{1}{8}\left(\frac{1}{\sqrt{n-1}} + \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right), & \text{when } n \text{ is odd} \\ \frac{1}{8}\left(\frac{1}{\sqrt{n-2}} + \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right), & \text{when } n \text{ is even.} \end{cases}$$

Proof

1. Let us prove the first case by Theorem 3.8 (1) where n is odd. $RS(\Gamma_{D_{2n}})$ has four eigenvalues, $\alpha_1 = 0$ of multiplicity $(n-2)$ and $\alpha_2 = -\frac{1}{4(n-1)\sqrt{n-1}}$ of multiplicity $(n-1)$. The last two eigenvalues are $\alpha_{3,4} = \frac{1}{8}\left(\frac{1}{\sqrt{n-1}} \pm \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right)$. Hence, the spectrum for Γ_G is

$$\begin{aligned} Spec(\Gamma_{D_{2n}}) = \Bigg\{ &\left(\frac{1}{8}\left(\frac{1}{\sqrt{n-1}} + \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right)\right)^1, (0)^{n-2}, \\ &\left(\frac{1}{8}\left(\frac{1}{\sqrt{n-1}} - \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right)\right)^1, \left(-\frac{1}{4(n-1)\sqrt{n-1}}\right)^{n-1} \Bigg\}. \end{aligned}$$

   Consequently, the maximum of the absolute eigenvalues for $\Gamma_{D_{2n}}$ is the spectral radius.
2. Using Theorem 3.8 (2) when n is even, $RS(\Gamma_{D_{2n}})$ has four eigenvalues, $\alpha_1 = 0$ of multiplicity $\frac{3n-6}{2}$, $\alpha_2 = -\frac{1}{2(n-2)\sqrt{n-2}}$ of multiplicity $(\frac{n}{2}-1)$, and $\alpha_{3,4} = \frac{1}{8}\left(\frac{1}{\sqrt{n-2}} \pm \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right)$. Then the spectrum for $\Gamma_{D_{2n}}$ is

$$\begin{aligned} Spec(\Gamma_{D_{2n}}) = \Bigg\{ &\left(\frac{1}{8}\left(\frac{1}{\sqrt{n-2}} + \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right)\right)^1, (0)^{\frac{3n-6}{2}}, \\ &\left(-\frac{1}{2(n-2)\sqrt{n-2}}\right)^{\frac{n}{2}-1}, \left(\frac{1}{8}\left(\frac{1}{\sqrt{n-2}} - \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right)\right)^1 \Bigg\}. \end{aligned}$$

   We then complete the proof.

□

Theorem 3.10
In $\Gamma_{D_{2n}}$, then RS-energy for $\Gamma_{D_{2n}}$ is

$$E_{RS}(\Gamma_{D_{2n}}) = \begin{cases} \frac{1}{4}\left(\frac{1}{\sqrt{n-1}} + \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right), & \text{for odd } n \\ \frac{1}{4}\left(\frac{1}{\sqrt{n-2}} + \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right), & \text{for even } n. \end{cases}$$

Proof

1. Using $Spec(\Gamma_{D_{2n}})$ given in Theorem 3.9 (1) for odd value n, we get RS-energy of $\Gamma_{D_{2n}}$,

$$\begin{aligned} E_{RS}(\Gamma_{D_{2n}}) &= (n-2)|0| + (n-1)\left|-\frac{1}{4(n-1)\sqrt{n-1}}\right| + \left|\frac{1}{8}\left(\frac{1}{\sqrt{n-1}} \pm \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right)\right| \\ &= \frac{1}{4}\left(\frac{1}{\sqrt{n-1}} + \sqrt{\frac{35n-34}{(n-1)(3n-2)}}\right). \end{aligned}$$

2. When n is even number, we know the spectrum of $\Gamma_{D_{2n}}$ from Theorem 3.9 (2). Therefore, RS-energy of $\Gamma_{D_{2n}}$ is given below

$$E_{RS}(\Gamma_{D_{2n}}) = \left(\frac{3n-6}{2}\right)|0| + \left(\frac{n}{2}-1\right)\left|-\frac{1}{2(n-2)\sqrt{n-2}}\right| + \left|\frac{1}{8}\left(\frac{1}{\sqrt{n-2}} \pm \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right)\right|$$
$$= \frac{1}{4}\left(\frac{1}{\sqrt{n-2}} + \sqrt{\frac{35n-68}{(n-2)(3n-4)}}\right).$$

□

4. Discussions

Now we are ready to give the conclusion regarding the obtained energies in Theorems 3.4, 3.7, and 3.10.

Corollary 4.1
$\Gamma_{D_{2n}}$ associated with the Randic, sum-connectivity, and product of Randic and sum-connectivity matrices is strongly hypoenergetic.

Corollary 4.2
The Randic energy for $\Gamma_{D_{2n}}$ is always an even integer.

Corollary 4.3
The sum-connectivity and product of Randic and sum-connectivity energies for $\Gamma_{D_{2n}}$ are never an odd integer.

The facts of Corollary 4.2 and 4.3 correspond to the well-known facts from [4] and [13].

Corollary 4.4
$E_{SC}(\Gamma_{D_{2n}}) > E_R(\Gamma_{D_{2n}})$ and $E_{SC}(\Gamma_{D_{2n}}) > E_{RS}(\Gamma_{D_{2n}})$.

In this research, the sum-connectivity energy of $\Gamma_{D_{2n}}$, where $n \geq 3$ is always greater than the Randic and product of Randic and sum-connectivity energies. Corollary 4.4 shows that E_{SC} is consistently larger than E_R and E_{RS}. This is mainly due to the differences in their matrix elements. The SC matrix uses $\frac{1}{\sqrt{d_u+d_v}}$, while the Randic matrix uses $\frac{1}{\sqrt{d_u \cdot d_v}}$. For high-degree vertices, the product $d_u \cdot d_v$ grows faster than the sum $d_u + d_v$, resulting in smaller entries in the R-matrix.

As a result, the SC matrix has larger edge weights, leading to larger eigenvalues and higher energy. Meanwhile the product of Randic and sum-connectivity matrix further reduces weights, making E_{RS} the smallest. This indicates that E_{SC} is more sensitive to high-degree vertices in the graph.

Moreover, the correlation between the energy and its spectral radius is stated in the following corollary.

Corollary 4.5
1. $E_R(\Gamma_{D_{2n}}) = 2 \cdot \rho_R(\Gamma_{D_{2n}})$.
2. $E_{SC}(\Gamma_{D_{2n}}) = 2 \cdot \rho_{SC}(\Gamma_{D_{2n}})$.
3. $E_{RS}(\Gamma_{D_{2n}}) = 2 \cdot \rho_{RS}(\Gamma_{D_{2n}})$.

For odd n, the spectrum consists of four types of eigenvalues: $n-2$ zeros, one eigenvalue $a+\sqrt{b}$, one eigenvalue $a-\sqrt{b}$, and $n-1$ repeated eigenvalues $-\frac{2}{n-1}a$. The energy is given by the sum of absolute values:

$$E = (n-1)\left|-\frac{2}{\sqrt{n-1}}a\right| + \left|a+\sqrt{b}\right| + \left|a-\sqrt{b}\right|.$$

Since $\left|-\frac{2}{n-1}a\right| = \frac{2}{n-1}|a|$, the first term simplify to $2|a|$. Assuming $a+\sqrt{b} \geq 0$ and $a-\sqrt{b} \leq 0$, we obtain $\left|a+\sqrt{b}\right| = a+\sqrt{b}$ and $\left|a-\sqrt{b}\right| = -(a-\sqrt{b}) = \sqrt{b}-a$. Thus,

$$E = 2a + a + \sqrt{b} + \sqrt{b} - a = 2(a+\sqrt{b}).$$

This shows that $E = 2\rho$, where $\rho = a + \sqrt{b}$ is the spectral radius.

For even n, the spectrum consists of four types of eigenvalues: $\frac{3n-6}{2}$ zeros, one eigenvalue $a + \sqrt{b}$, one eigenvalue $a - \sqrt{b}$, and $\frac{n}{2} - 1$ repeated eigenvalues $-\frac{4}{n-2}a$. The energy is given by the sum of absolute values:

$$E = \left(\frac{n}{2} - 1\right)\left|-\frac{4}{n-2}a\right| + \left|a + \sqrt{b}\right| + \left|a - \sqrt{b}\right|.$$

Since $\left|-\frac{4}{n-2}a\right| = \frac{4}{n-2}|a|$, the first term simplify to $\left(\frac{n}{2} - 1\right)\frac{4}{n-2}|a| = 2|a|$. Assuming $a + \sqrt{b} \geq 0$ and $a - \sqrt{b} \leq 0$, we have $\left|a + \sqrt{b}\right| = a + \sqrt{b}$ and $\left|a - \sqrt{b}\right| = -(a - \sqrt{b}) = \sqrt{b} - a$. Thus,

$$E = 2a + a + \sqrt{b} + \sqrt{b} - a = 2(a + \sqrt{b}),$$

which again shows that $E = 2\rho$, where $\rho = a + \sqrt{b}$.

5. Conclusions

In this paper, we investigated the non-commuting graph of the dihedral group by analyzing the spectrum, spectral radius, and energy associated with the Randic, sum-connectivity, and product of Randic and sum-connectivity matrices. Explicit expressions for these spectral quantities were obtained, revealing that the corresponding energies are strongly hypoenergetic and never attain odd integer values. Moreover, it was shown that, in all cases, each of these energies is exactly equal to twice its corresponding spectral radius. The approach developed in this work may be extended to other non-abelian groups, such as symmetric, alternating, and dicyclic groups, to provide deeper insight into the interplay between algebraic structures and spectral graph invariants. However, such extensions may present additional challenges, including the construction of the non-commuting graphs, the formulation of the associated matrices, and the increased complexity of solving the characteristic polynomials to determine their spectrum and energy.

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