

Solutions to the Identification Elliptic Problem with Elzaki Transform and Sumudu Transform

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Abstract In this article, the Sumudu and Elzaki transforms for solving the source identification problem for partial differential equations are presented. Several properties of these transforms are reviewed and examined. Application of the Elzaki transform to obtain the solution for the identification elliptic problem is provided. In addition, the solution of the identification elliptic problem with Sumudu transform is given.

Keywords Source identification problem, Elzaki transform, Sumudu transform, elliptic Equation

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1. Introduction

Integral transforms have long been a cornerstone in the analytical and semi-analytical solution of differential equations. Among these, the Elzaki transform (ET), proposed by T.M. Elzaki [1], has emerged as a powerful alternative technique due to its flexibility in handling a wide range of differential equations with various initial and boundary conditions. In addition, the Sumudu transform (ST), introduced by Watugala in the 1990s [2], has gained growing attention for its simplicity and dimensional preservation properties. Unlike the Laplace transform, the ST maintains the original units of the function, which is particularly advantageous in engineering and physical applications.

In recent decades, several works have been conducted by many researchers on studying novel methods for solving ordinary differential equations with several integral transforms (see [3], [4], [5], [6], [7], [8], [20], [21]). Additionally, many authors are solving the partial differential equations among these are the applications on Laplace Transforms, ETs, and STs [9], solving the PDEs using the Sumudu and Elzaki transforms [10].

In [11], the solution of PDEs using the ST and ET is provided. The Elzaki-Laplace transform and the modified ST solutions to the telegraph problem are provided in [12] and [13].

On the other side, numerous authors have examined the source identification problems (SIPs) of PDEs as can be seen in [14], [15], [16], [17], [18], [19]).

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Let us illustrate the identification Elliptic equation

$$\begin{cases} -u_{tt}(t, x) - u_{xx}(t, x) = p(x) + f(t, x), \\ x \in (0, \infty), -T < t < 0, \\ u(0, x) = \varphi_1(x), u(-T, x) = \varphi_2(x), x \in (0, \infty), \\ u(t, 0) = \alpha_1(t, x), u_x(t, 0) = \alpha_2(t, x), -T < t < 0. \end{cases} \quad (1)$$

where $f(t, x)$, $\varphi_1(x)$, $\varphi_2(x)$, $\alpha_1(x)$, and $\alpha_2(x)$ are smooth given functions. $p(x)$ is unknown parameter.

This paper aims to establish a comprehensive methodology for applying the ET and ST to (1), emphasizing both theoretical development and practical computation. Finally, five sections are presented in this article: The fundamental concepts of these integral transformations are introduced in Section 2. The identification of the elliptic equation with ET and ST solutions are shown in Sections 3 and 4. We provide our findings in Section 5.

2. Motivations and Main Contributions

2.1. Motivations

Numerous branches of applied mathematics, physics, engineering, and mathematical modeling deal with source identification problems (SIPs) for partial differential equations (PDEs). An inverse SIP aims to recover an unknown source function using extra measurements or observations of the solution, in contrast to direct problems where the source term is known and the state variable is computed. Such problems are generally more challenging because the available information is indirect. In applications where the source cannot be directly measured, it is crucial to identify unknown source functions. Diffusion processes, material characterization, heat-source identification, pollutant detection, and the identification of internal driving mechanisms in physical systems are a few examples. Integral transform methods provide a powerful framework for solving differential equations because they can convert differentiation operators into algebraic expressions or simpler differential relations. Because of their useful operating features and ease of handling initial and boundary conditions, the Sumudu and Elzaki transforms, in particular, have attracted a lot of attention.

The motivation of this work is therefore to investigate the applicability of the Sumudu and Elzaki transforms to source identification problems for partial differential equations. Special attention is given to the identification elliptic equation, for which the unknown source function must be determined simultaneously with the solution of the governing PDE. The study provides a unified analytical treatment based on integral transforms and demonstrates how the two transform techniques can be employed to obtain representations of both the state solution and the unknown source.

Another motivation is to examine the similarities and differences between the Elzaki and Sumudu transform approaches. Although both transforms possess analogous operational properties, their definitions and transformed representations are different. Establishing their applicability to the same identification problem provides a useful basis for assessing the effectiveness and flexibility of these methods.

2.2. Main Contributions

The main contributions of this study are summarized as follows:

1. A unified analytical framework is developed for source identification problems in partial differential equations using the Sumudu and Elzaki transforms.
2. The essential properties of both transforms are established and employed to transform the governing elliptic equation while incorporating the prescribed conditions.
3. Elzaki- and Sumudu-transform-based approaches are developed to obtain the solution and recover the unknown source function from additional information.
4. The proposed framework demonstrates the applicability and effectiveness of integral transforms for inverse source problems and provides a basis for extending the approach to more general parabolic and hyperbolic equations.

Overall, the study provides a systematic and analytical methodology for identifying unknown source functions in PDEs through the combined use of the Elzaki and Sumudu transforms.

3. Basic Concepts of Integral Transforms

In this section, we present some of the basic concepts for ET and ST.

3.1. The Elzaki Transform

The Elzaki Transform is a relatively integral transform, and it has been used successfully for solving differential equations, integral equations, and partial differential equations (PDEs) in applied mathematics and engineering.

The Elzaki transform of $f(t)$ is defined as

$$\varepsilon[f(t)](v) = W(v) = v \int_0^\infty f(t) e^{-t/v} dt, \quad v \in (k_1, k_2).$$

Equivalently $\varepsilon[f(t)](v) = v^2 \int_0^\infty f(vt) e^{-t} dt$. The Elzaki transform shares many properties with Sumudu and Laplace transforms but has distinctions that make it convenient for certain variable-coefficient problems.

Let $W(v, x) = \varepsilon\{f(t, x)\}$. Then [10]

1. $\varepsilon\{f_t(t, x)\}(v) = \frac{W(v, x)}{v} - v f(0, x),$
2. $\varepsilon\{f_{tt}(t, x)\}(v) = \frac{1}{v^2} W(v, x) - \frac{f(0, x)}{v} - f_t(0, x).$

ET of some standard function are obtained in the following [10]

1. $\varepsilon\{1\}(v) = v^2.$
2. $\varepsilon\{e^{a\tau}\}(v) = \frac{v^2}{1-av}.$
3. $\varepsilon\{e^{-a\tau}\}(v) = \frac{v^2}{1+av}.$

3.2. The Sumudu Transform

The Sumudu Transform is an integral transform. It has been used to solve differential, integral, and partial differential equations because of its simplicity and preservation of the function's dimensionality.

The Sumudu transform of a suitable function $f(t)$ is defined by

$$\mathcal{S}[f(t)](u) = F(z) = \int_0^\infty f(zt) e^{-t} dt, \quad -t_1 < z < t_2.$$

Important properties include linearity, duality with the Laplace transform, transforms of derivatives and integrals, and shift theorems.

Let $F(X, z) = \mathcal{S}\{f(x, t)\}$. Then [10]

1. $\mathcal{S}\{f_t(x, t)\}(z) = \frac{F(X, z) - f(X, 0)}{z},$
2. $\mathcal{S}\{f_{tt}(x, t)\}(z) = \frac{F(x, z) - f(x, 0) - z f_t(x, 0)}{z^2}.$

The STs of some standard functions are given as follows [10]

1. $\mathcal{S}\{1\}(z) = 1,$
2. $\mathcal{S}\{e^s\}(z) = \frac{1}{1-z}.$
3. $\mathcal{S}\{e^{-s}\}(z) = \frac{1}{1+z}.$

4. Application of the Elzaki Transform

In this section, we solve the problem (1) using ET in the following steps:

1. Rewrite the PDE in a way that makes it appropriate for the ET, which is often specified for non-negative time.
2. Using the ET and the necessary derivative rules.
3. Solve the (algebraic/ODE) problem that results in (1) and (2) after applying the transform in the time variable.
4. Apply the inverse of ET, to obtain the exact solution $u(t, x)$ and $p(x)$.

Let us illustrate the identification Elliptic equation (1) in the following examples showing the solution by using ET. Solve the following identification elliptic equation with the conditions by using ET.

$$\begin{cases} -u_{tt}(t, x) - u_{xx}(t, x) = p(x) - 2e^{-t-x} - e^{-x}, \\ 0 < x < 1, -1 < t < 0, \\ u(0, x) = e^{-x}, u(-1, x) = e^{1-x}, \\ u_t(0, x) = -e^{-x}, 0 < x < 1, \\ u(t, 0) = e^{-t}, u_x(t, 0) = -e^{-t}, -1 < t < 0. \end{cases} \quad (2)$$

Solution. First, we take

$$-u_{tt}(t, x) - u_{xx}(t, x) = p(x) - 2e^{-t-x} - e^{-x}.$$

To acquire a conventional sign for the time derivative, multiply by (-1). This yields

$$u_{tt}(t, x) + u_{xx}(t, x) = -p(x) + 2e^{-t-x} + e^{-x}. \quad (3)$$

As we know, the ET is defined for non-negative time $t \geq 0$. But in this problem, the interval is $-1 < t < 0$. Therefore, to apply the ET straightforwardly, we will use $\tau = -t, \tau \geq 0$.

Now and in the future, define

$$\omega(\tau, x) = u(-\tau, x).$$

Then

$$u_t = -\omega_\tau, u_{tt} = \omega_{\tau\tau}, \text{ and } u_{xx} = \omega_{xx}.$$

Hence, the formula (2) becomes

$$\begin{cases} -\omega_{\tau\tau}(\tau, x) - \omega_{xx}(\tau, x) = p(x) - 2e^{\tau-x} - e^{-x}, \\ 0 < x < 1, 0 < \tau < 1, \\ \omega(0, x) = u(0, x) = e^{-x}, \\ \omega(1, x) = u(-1, x) = e^{1-x}, \\ \omega_\tau(0, x) = e^{-x}, 0 < x < 1, \\ \omega(\tau, 0) = u(-\tau, 0) = e^\tau, \\ \omega_x(\tau, 0) = u_x(-\tau, 0) = -e^\tau, 0 < \tau < 1. \end{cases}$$

Second, let us denote

$$\begin{aligned} \varepsilon_\tau \{ \omega(\tau, x) \} (v) &= \varepsilon \{ \omega(v, x) \} = W(v, x), \\ \varepsilon \{ \omega_{xx}(v, x) \} &= W_{xx}(v, x), \end{aligned}$$

and

$$\varepsilon \{ \omega_{\tau\tau}(v, x) \} = \frac{1}{v^2} W(v, x) - \frac{\omega(0, x)}{v} - \omega_\tau(0, x).$$

Then, applying the ET in $\tau, \tau \geq 0$, we obtain

$$\varepsilon \{ \omega_{\tau\tau} \} (v, x) - \varepsilon \{ \omega_{xx} \} (v, x) = \varepsilon \{ p(x) \} - 2\varepsilon \{ e^{\tau-x} \} - \varepsilon \{ e^{-x} \}$$

Use the derivative rules for the ET, we obtain

$$\begin{aligned} & \left(\frac{1}{v^2} W(v, x) - \frac{\omega(0, x)}{v} - \omega_\tau(0, x) \right) - W_{xx}(v, x) \\ &= p(x) \varepsilon \{1\} - 2e^{-x} \varepsilon \{e^\tau\} - e^{-x} \varepsilon \{1\} \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & W_{xx}(v, x) - \frac{1}{v^2} W(v, x) \\ &= -p(x) \varepsilon \{1\} + 2e^{-x} \varepsilon \{e^\tau\} + e^{-x} \varepsilon \{1\} + \frac{\omega(0, x)}{v} + \omega_\tau(0, x). \end{aligned} \quad (4)$$

Applying the definition of ET, we obtain

$$\varepsilon \{1\}(v) = v \int_0^\infty e^{-\tau/v} d\tau = v \cdot v = v^2, \quad (5)$$

and

$$\begin{aligned} \varepsilon \{e^\tau\}(v) &= v \int_0^\infty e^\tau e^{-\tau/v} d\tau = v \int_0^\infty e^{-(\frac{1}{v}-1)\tau} d\tau \\ &= \frac{1}{1/v - 1} = \frac{v^2}{1 - v}. \end{aligned} \quad (6)$$

Since $\omega(0, x) = e^{-x}$, and applying the formulas (5) and (6), the equation (4) becomes

$$\begin{aligned} & W_{xx}(v, x) - \frac{1}{v^2} W(v, x) \\ &= -p(x)v^2 + 2e^{-x} \frac{v^2}{1 - v} + e^{-x}v^2 + \frac{e^{-x}}{v} + \omega_\tau(0, x). \end{aligned}$$

Then

$$\begin{aligned} & W_{xx}(v, x) - \frac{1}{v^2} W(v, x) \\ &= -p(x)v^2 + e^{-x} \left(\frac{2v^2}{1 - v} + v^2 + \frac{1}{v} \right) + \omega_\tau(0, x). \end{aligned} \quad (7)$$

Third, to obtain the exact solution $(u(t, x), p(x))$, we will consider $\omega(\tau, x) = A(\tau)e^{-x}$, for some scalar function $A(\tau)$. Then, the ET will be

$$W(v, x) = \tilde{A}(v)e^{-x}, \quad (8)$$

where,

$$\tilde{A}(v) = \varepsilon \{A(\tau)\}(v).$$

Therefore, substitute (8) in (7), we will write the LHS as

$$\begin{aligned} & W_{xx}(v, x) - \frac{1}{v^2} W(v, x) = \tilde{A}(v)e^{-x} - \frac{1}{v^2} \tilde{A}(v)e^{-x} \\ &= \tilde{A}(v)e^{-x} \left(1 - \frac{1}{v^2} \right). \end{aligned}$$

Then, (7) becomes

$$\tilde{A}(v)e^{-x} \left(1 - \frac{1}{v^2} \right) = -p(x)v^2 + e^{-x} \left(\frac{2v^2}{1 - v} + v^2 + \frac{1}{v} \right) + \omega_\tau(0, x). \quad (9)$$

Thus, LHS is proportional to e^{-x} . This means, the entire RHS must also be proportional to e^{-x} . Therefore, we claim

$$P(x) = \alpha e^{-x}, \quad \omega_\tau(0, x) = \beta e^{-x},$$

where α , and β are unknown constant. Therefore, (9) becomes

$$\tilde{A}(v)e^{-x} \left(1 - \frac{1}{v^2}\right) = -\alpha e^{-x}v^2 + e^{-x} \left(\frac{2v^2}{1-v} + v^2 + \frac{1}{v}\right) + \beta e^{-x}.$$

Now, dividing by e^{-x} , yields

$$\tilde{A}(v) \left(1 - \frac{1}{v^2}\right) = -\alpha v^2 + \left(\frac{2v^2}{1-v} + v^2 + \frac{1}{v}\right) + \beta. \quad (10)$$

Fourth, since $\omega(\tau, 0) = e^\tau$, and $\omega(\tau, 0) = A(\tau)e^0$, then $A(\tau) = e^\tau$, therefore

$$\omega(\tau, x) = e^{\tau-x},$$

then, applying derivative of τ and since $\tau = 0$, then we obtain

$$\omega_\tau(\tau, x)|_{\tau=0} = \omega_\tau(0, x) = e^{-x},$$

therefore $\beta = 1$.

Also, since $A(\tau) = e^\tau$, then

$$\tilde{A}(v) = \varepsilon \{A(\tau)\}(v) = \varepsilon \{e^\tau\}(v) = \frac{v^2}{1-v}. \quad (11)$$

Therefore, using (11) and since $\beta = 1$, (10) becomes

$$\frac{v^2}{1-v} \left(1 - \frac{1}{v^2}\right) = -\alpha v^2 + \frac{2v^2}{1-v} + v^2 + \frac{1}{v} + 1,$$

then,

$$-(v+1) = -\alpha v^2 + \frac{2v^2}{1-v} + v^2 + \frac{1}{v} + 1.$$

In this case, we could match rational functions in v , but there is a simpler and decisive method that we already have $A(\tau) = e^\tau$, such that

$$\omega(\tau, x) = A(\tau)e^{-x} = e^{\tau-x}.$$

Since $\tau = -t$, then

$$u(t, x) = \omega(\tau, x) = \omega(-t, x) = e^{-t-x}.$$

Also, from (3), we obtain $p(x) = e^{-x}$, where $\alpha = 1$. Therefore, the pair of exact solutions is

$$(u(t, x), p(x)) = (e^{-t-x}, e^{-x}).$$

5. Application of the Sumudu Transform

In this section, we solve the identification elliptic problem using the ST. We will:

1. Rewrite the PDE in a way that makes it appropriate for the ET, which is often non-negative.
2. Choose the ST properties we need.

3. Apply the ST to the PDE, solve the resulting ODE in x (with the transformed boundary).
4. invert the ST to obtain $u(t, x)$ and identify $p(x)$.

Let us illustrate the identification Elliptic equation (1) in the following example, which shows the solution by using ST.

Solve the identification elliptic equation with the conditions (2) by using ST.

Solution. First, we take

$$\begin{cases} -u_{tt}(t, x) - u_{xx}(t, x) = p(x) - 2e^{-t-x} - e^{-x}, \\ 0 < x < 1, -1 < t < 0, \\ u(0, x) = e^{-x}, u(-1, x) = e^{1-x}, \\ u_t(0, x) = -e^{-x}, 0 < x < 1, \\ u(t, 0) = e^{-t}, u_x(t, 0) = -e^{-t}, -1 < t < 0. \end{cases} \quad (12)$$

Here, the ST is defined for non-negative time $t \geq 0$. But in this problem, the interval is $-1 < t < 0$. Therefore, to apply the ET straightforwardly, we will use $s = -t, s \geq 0$.

Define

$$v(s, x) = u(t, x) = u(-s, x).$$

Derivatives transform as

$$\begin{aligned} u_t &= \frac{\partial}{\partial t} u(t, x) = -v_s(s, x), \\ u_{tt} &= v_{ss}(s, x). \end{aligned}$$

Hence, formula (12) becomes

$$\begin{cases} -v_{ss}(s, x) - v_{xx}(s, x) = p(x) - 2e^{s-x} - e^{-x}, \\ 0 < x < 1, 0 < s < 1, \\ v(0, x) = u(0, x) = e^{-x}, \\ v(1, x) = u(-1, x) = e^{1-x}, \\ v_s(0, x) = -u_t(0, x) = -e^{-x}, 0 < x < 1, \\ v(s, 0) = e^{-s}, v_x(s, 0) = -e^{-s}, 0 < s < 1. \end{cases} \quad (13)$$

Second, we apply ST to s . As we know, the definition of ST defined by

$$\begin{aligned} \mathcal{S}\{f(s)\}(z) &= F(z) = \int_0^\infty f(zs) e^{-s} ds \\ &= \frac{1}{z} \mathcal{L}\{f\}\left(\frac{1}{z}\right) \end{aligned}$$

Since $F(z) = \mathcal{S}\{f(s)\}(z)$. That is leads to

$$\mathcal{S}\{f'(s)\}(z) = \frac{F(z) - f(0)}{z},$$

$$\mathcal{S}\{f''(s)\}(z) = \frac{F(z) - f(0) - zf'(0)}{z^2}.$$

$$\mathcal{S}\{e^{as}\}(z) = \int_0^\infty e^{azs} e^{-s} ds = \frac{1}{1 - az},$$

and

$$\mathcal{S}\{1\}(z) = 1, \quad \mathcal{S}\{e^s\}(z) = \frac{1}{1 - z}.$$

Third, since $V(z, x) = \mathcal{S}\{v(s, x)\}(z)$. Taking the ST to the equation which is given in (12) with respect to s , we obtain

$$\begin{aligned} & \mathcal{S}\{-v_{ss}\}(z, x) - \mathcal{S}\{v_{xx}\}(z, x) \\ &= \mathcal{S}\{p(x)\}(z) - \mathcal{S}\{-2e^{s-x}\}(z) - \mathcal{S}\{-e^{-x}\}(z), \end{aligned}$$

then

$$\begin{aligned} & -\frac{V - v(0, x) - zv_s(0, x)}{z^2} - V_{xx}(z, x) \\ &= p(x) \cdot \mathcal{S}\{1\}(z) - 2e^{-x} \mathcal{S}\{e^s\}(z) - \mathcal{S}\{-e^{-x}\}(z). \end{aligned}$$

Thus the transformed PDE is

$$-\frac{V - v(0, x) - zv_s(0, x)}{z^2} - V_{xx}(z, x) = p(x) - 2e^{-x} \frac{1}{1-z} - e^{-x},$$

Fourth, inserting $v(0, x) = e^{-x}$ and $v_s(0, x) = e^{-x}$ to obtain

$$-\frac{V - e^{-x} - ze^{-x}}{z^2} - V_{xx} = p(x) - 2\frac{e^{-x}}{1-z} - e^{-x}.$$

This yields

$$V_{xx} + \frac{1}{z^2}V = -p(x) + e^{-x} + 2\frac{e^{-x}}{1-z} + \frac{e^{-x}}{z^2} + \frac{e^{-x}}{z}. \quad (14)$$

Fifthly, applying the boundary conditions to determine V , we obtain

$$v(s, 0) = e^s \quad \Rightarrow \quad V(z, 0) = \mathcal{S}\{e^s\}(z) = \frac{1}{1-z},$$

$$v_x(s, 0) = -e^s \quad \Rightarrow \quad V_x(z, 0) = -\frac{1}{1-z}.$$

Then, let us assume

$$V(z, x) = \frac{e^{-x}}{1-z}. \quad (15)$$

Then

$$V_x(z, x) = -\frac{e^{-x}}{1-z}, \quad V_{xx}(z, x) = \frac{e^{-x}}{1-z}. \quad (16)$$

Then, applying (15), and (16) in (14) becomes

$$\frac{e^{-x}}{1-z} + \frac{1}{z^2} \cdot \frac{e^{-x}}{1-z} = -p(x) + e^{-x} \left(1 + \frac{2}{1-z} + \frac{1}{z^2} + \frac{1}{z} \right).$$

Hence,

$$e^{-x} \frac{1 + \frac{1}{z^2}}{1-z} = -p(x) + e^{-x} \left(1 + \frac{2}{1-z} + \frac{1}{z^2} + \frac{1}{z} \right).$$

Therefore,

$$\begin{aligned} p(x) &= \left(-\frac{1 + \frac{1}{z^2}}{1-z} + 1 + \frac{2}{1-z} + \frac{1}{z^2} + \frac{1}{z} \right) e^{-x} \\ &= \left(\frac{1 - \frac{1}{z^2}}{1-z} + 1 + \frac{1}{z^2} + \frac{1}{z} \right) e^{-x} \\ &= \left(\frac{z^2 - 1}{z^2(1-z)} + 1 + \frac{1}{z^2} + \frac{1}{z} \right) e^{-x} \\ &= \left(-\frac{z+1}{z^2} + 1 + \frac{1}{z^2} + \frac{1}{z} \right) e^{-x} \\ &= \left(-\frac{1}{z} - \frac{1}{z^2} + 1 + \frac{1}{z^2} + \frac{1}{z} \right) e^{-x} \end{aligned}$$

Hence

$$p(x) = 1 \cdot e^{-x} = e^{-x}.$$

Finally, since $V(z, x) = \frac{e^{-x}}{1-z}$ and $\mathcal{S}\{e^s\}(z) = \frac{1}{1-z}$, taking the inverse of ST, we obtain

$$v(s, x) = e^{s-x}$$

Returning to the original variable $t = -s$, we obtain

$$u(t, x) = v(-t, x) = e^{-t-x}.$$

Thus the pair of exact solution is

$$(u(t, x), p(x)) = (e^{-t-x}, e^{-x}).$$

6. Conclusion

In this study, we investigated the solution of the elliptic identification problem using two integral transform techniques, the ET and the ST. Both transforms proved to be powerful analytical tools for simplifying and solving the governing partial differential equations associated with the identification problem. The main steps were to change $t \rightarrow s = -t$, then the ET and ST act on $s \geq 0$, apply them, and use derivative identities to get an ODE, use boundary data and the structure of the forcing to guess a simple transformed solution, simplify algebraically to obtain the exact solution for identifying $p(x)$ and $u(t, x)$.

These methods are effective here because the data and forcing are exponential in t and x , and the spatial ODE is elementary. Unlike other transforms, both transforms preserve the structure of differential operators while simplifying the inversion process, which makes them suitable for the considered problem for time-dependent problems in mathematical physics.

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