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On the Composition of Left Jumarie and Right Caputo Fractional Derivatives of Power Functions

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Abstract The composition of fractional differential operators is a fundamental topic in fractional calculus because of its central role in the analysis of fractional differential equations, boundary value problems, and fractional variational models. Although Jumarie's modified fractional derivative has been extensively studied and successfully applied, explicit composition formulas involving the successive application of the left Jumarie and right Caputo fractional derivatives have received comparatively limited attention. Such formulas are important because they provide closed-form analytical expressions that simplify theoretical analysis and serve as reliable benchmarks for numerical computations. In this paper, a composition theorem for the left Jumarie and right Caputo fractional derivatives acting on power functions is established and proved. The proof is derived directly from the definitions of the two operators together with fundamental properties of the Gamma and Beta functions. Moreover, the compositions of operators of the same type reduced to pure power formulas. Illustrative examples are verified through direct numerical computation, comparison tables and graphical illustration. Furthermore, the limiting case as the fractional order of the derivative tends to integer order of one is investigated, showing that the obtained formula is consistent with the corresponding classical composition of first-order derivatives.

Keywords Fractional calculus, Jumarie's modified fractional derivative, Caputo derivative, Composition theorem, Power functions, Gamma function, Beta function.

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1. Introduction

The mathematical foundations of fractional calculus are closely related to the theory of special functions, particularly the Gamma and Beta functions, whose analytical properties are indispensable in defining fractional operators and deriving explicit solutions. In 1960, Rainville et al., established Classical references on special functions [1], while, in 1964 Artin gave a comprehensive introduction to the Gamma function, including its definition and basic properties [2]. The Gamma function extends the classical factorial function to non-integer arguments and satisfies the recurrence relation as

$$\Gamma(z+1) = z\,\Gamma(z), \tag{1}$$

In 1965 Abramowitz and Stegun compiled the most common formulas and identities for special functions, including the Gamma and Beta functions and the basic relationship between them as the following

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{2}$$

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These classical relationships play a central role in simplifying fractional integrals and derivatives and constitute indispensable tools in the analysis of fractional differential equations and variational problems [3]. while Podlubny in 1998 established one of the standard references on fractional differential equations and their analytical foundations [4]. These classical works were further complemented by Andrews et al., in 1999 [5], subsequent work in 2000 on discrete fractional calculus further enriched the operational framework of the subject. whose comprehensive treatment of special functions has become a standard reference for the properties of the Gamma and Beta functions employed throughout fractional calculus. An essential mathematical ingredient in fractional calculus is the theory of special functions, particularly the Gamma and Beta functions [6].

Building upon these mathematical foundations, the theory of fractional differential equations has undergone remarkable development over the past decades, Kilbas, et al., in 2006 subsequently developed a rigorous and systematic theory of fractional differential equations together with numerous analytical results and applications [7]. In the same period, Jumarie in 2006 introduced his modified Riemann–Liouville fractional derivative and derived several fractional Taylor expansions and differentiation formulas for continuous functions [8]. Shortly thereafter, Sabatier et al., in 2007 summarized the major theoretical advances and emerging research directions in fractional calculus [9], while Jumarie in 2009 further established several operational identities and explicit fractional calculus formulae based on his modified derivative [10]. Collectively, these contributions provided a solid mathematical foundation for the rapid development of modern fractional analysis in both theoretical and applied mathematics.

Fractional calculus has now evolved into a mature and interdisciplinary research field with numerous applications in science and engineering. Owing to the nonlocal nature of fractional operators, fractional models provide an effective mathematical framework for describing memory and hereditary phenomena that cannot be adequately captured by classical integer-order models. Consequently, fractional differential equations have been successfully applied to viscoelasticity, anomalous diffusion, heat and mass transfer, fluid mechanics, signal and image processing, control theory, mathematical biology, finance, and many other scientific disciplines. Comprehensive historical surveys and modern reviews by Machado et al., in 2011 [11], De Oliveira et al. in 2014 [12].

Among the numerous fractional operators proposed in the literature, Jumarie's modified Riemann–Liouville fractional derivative has received considerable attention because of its simple analytical formulation and its applicability to continuous functions that are not necessarily differentiable in the classical sense. By modifying the classical Riemann–Liouville definition through the subtraction of the initial function value before applying the fractional operator, Jumarie's formulation yields a fractional derivative of a constant function equal to zero while preserving several useful computational properties. These characteristics have motivated its application in various branches of fractional analysis, particularly in the study of fractional differential equations, generalized Taylor expansions, and analytical representations of non-differentiable functions [8, 10]. The applicability of Jumarie's fractional derivative has been further demonstrated in several subsequent studies. Ghosh et al. in 2015 obtained explicit analytical solutions of linear fractional differential equations in terms of the Mittag–Leffler function by employing Jumarie's derivative, illustrating its effectiveness in constructing closed-form solutions [13]. Tarasov in 2016 investigated fundamental analytical properties of fractional derivatives, including generalized chain rules, and emphasized that many classical identities cannot be transferred directly to fractional operators without appropriate mathematical assumptions [14]. Kumar in 2016 studied linear fractional non-homogeneous differential equations involving Jumarie-type derivatives and developed analytical solution procedures for several classes of problems [15]. Banerjee et al. in 2017 subsequently applied Jumarie's derivative to a fractional Schrödinger equation, demonstrating its applicability in mathematical physics and fractional dynamical systems [16]. More recently, the rapidly expanding range of applications of fractional calculus has continued to stimulate the development of new analytical and computational techniques. Bastos in 2018 derived integration-by-parts formulas and necessary optimality conditions for variational problems involving the Caputo–Fabrizio fractional derivative [17], while modern reviews by Teodoro et al. in 2019 have summarized the growing diversity of fractional operators and their mathematical properties. In addition [18], Abdeljawad et al., in 2020 investigated inequalities involving Riemann–Liouville fractional integrals, providing further analytical tools that have influenced subsequent developments in fractional analysis [19]. Despite these significant developments, comparatively limited attention has been devoted to deriving explicit composition identities for Jumarie-type fractional operators. Most existing contributions primarily focus on analytical solution techniques, explicit differentiation formulas, or applications of

Jumarie's derivative to specific classes of fractional differential equations. By contrast, rigorous analytical formulas describing the successive action of the left and right Jumarie fractional derivatives on elementary functions, particularly power functions, remain relatively scarce in the literature. Consequently, further investigation of such composition properties is both mathematically interesting and potentially valuable for future developments in fractional differential equations, fractional variational theory, and operational fractional calculus.

The mathematical analysis of fractional differential equations has also benefited from rigorous theoretical studies concerning the existence, uniqueness, stability, and qualitative behaviour of their solutions. In particular, Diethelm and Ford in 2010 established a comprehensive analytical framework for fractional differential equations, providing several fundamental results that have become standard references in the field [20]. More recently, Al-Refai and Luchko in 2023 introduced generalized fractional integrals and derivatives on finite intervals, thereby extending the classical theory and enriching the mathematical structure of fractional operators [21]. Also, Hilfer and Kleiner in 2024 further broadened the scope of fractional calculus by developing a distributional framework for fractional operators [22], while Kórus and Nápoles Valdés in 2024 presented a comprehensive review of recent advances in fractional calculus, highlighting several emerging directions for future research [23]. In addition, Mohammad and Saadaoui in 2024 proposed a new fractional derivative extending several classical concepts [24], and Khan et al., in 2024 developed efficient analytical techniques for nonlinear fractional partial differential equations involving Caputo derivatives [25]. Zhao in 2024 further demonstrated the effectiveness of fractional operators in nonlinear wave propagation through the construction of solitary and periodic solutions of generalized time-fractional Schrödinger equations [26].

Despite these developments, relatively little attention has been devoted to the explicit study of composition operators associated with Jumarie's modified derivative. In particular, composition formulas involving the successive action of the left Jumarie and right Caputo fractional derivatives on power functions remain scarcely discussed in the available literature. This is noteworthy because power functions constitute the natural basis of polynomial approximations and generalized fractional Taylor expansions, and their fractional derivatives can often be represented explicitly through Gamma functions. As recently emphasized by Ortigueira in 2025, power functions occupy a fundamental position in unified formulations of fractional differentiation [27]. Motivated by these observations, the present paper investigates the composition of the left Jumarie and right Caputo fractional derivatives acting successively on the power function $(x-a)^m$. More precisely, for $0 < \beta < 1$ and $m \in \mathbb{N}$, we establish an explicit closed-form expression for

$$ {}_{x}^{C}D_{b}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right), \tag{3}$$

under the assumption that $m > 2\beta - 1$.

The proof combines the explicit formula for the left Jumarie derivative of a power function, the definition of the right Caputo derivative, and the classical identities of the Beta and Gamma functions. To the best of the authors' knowledge, a systematic investigation of this composition has not been reported in the available literature, and a closed-form formula of this type is still lacking. Unlike many existing analytical approaches, the derivation is concise and relies only on elementary properties of special functions. Furthermore, the obtained formula is verified through representative numerical examples and shown to be consistent with the corresponding classical composition operator as the fractional order approaches one. These results provide a useful analytical tool for future investigations involving Jumarie-type fractional differential equations, boundary value problems, and fractional variational formulations.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries and basic definitions. Section 3 establishes the auxiliary results required for the proof of the main theorem. Section 4 contains the principal composition theorem together with its proof and two immediate 3 consequences. Section 5 provides illustrative examples and discusses the limiting classical case. Finally, Section 6 concludes the paper and outlines possible directions for future research.

2. Preliminaries

This section recalls the fundamental properties of the Gamma and Beta functions together with the definitions of the left Jumarie and right Caputo fractional derivatives. These results constitute the analytical foundation for the composition formula proved in Section 4. The Gamma function plays a central role in fractional calculus: it extends the factorial function from positive integers to positive real and complex numbers and appears naturally in the definitions of most fractional differential and integral operators [1–3].

Definition 1
For every real number $z > 0$, the Gamma function [2] [6], is defined by

$$\Gamma(z) = \int_0^\infty t^{\,z-1} e^{-t}\, dt.$$

The integral converges absolutely for all $z > 0$. Among its many properties, the following recurrence relation will be used repeatedly throughout this paper .

Proposition 1
For every $z > 0$, $\Gamma(z+1) = z\,\Gamma(z)$. Consequently, if $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

This identity establishes the connection between the Gamma function and the classical factorial function.

The Beta function is another classical special function that frequently appears in fractional calculus, particularly in the evaluation of fractional integrals and in deriving explicit formulas involving power functions [3, 5].

Definition 2
For $x > 0$ and $y > 0$, the Beta function is defined as

$$B(x,y) = \int_0^1 t^{\,x-1}(1-t)^{\,y-1}\, dt.$$

The integral is symmetric with respect to its two arguments, that is, $B(x,y) = B(x,y)$.

After defining the Gamma and Beta functions, it is natural to emphasize their interconnection. Relationship Between the Gamma and Beta Functions: One of the most important identities connecting them is the classical formula for all $x > 0$ and $y > 0$,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

This identity will be used repeatedly in the proof of the main theorem to simplify coefficients involving Gamma functions. Among the various definitions of fractional differentiation, the modified Riemann–Liouville derivative introduced by Jumarie in 2006 has attracted considerable attention because it extends fractional differentiation to continuous functions that are not necessarily differentiable in the classical sense, while preserving several computational properties analogous to those of ordinary calculus [8].

Definition 3
Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The left Jumarie fractional derivative of order $0 < \beta < 1$ is defined by

$${}^{J}_{a}D^{\beta}_{x} f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x (x-\tau)^{-\beta} \left[f(\tau) - f(a)\right] d\tau,$$

and the right Jumarie fractional derivative is defined by

$${}^{J}_{x}D^{\beta}_{b} f(x) = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^b (\tau-x)^{-\beta} \left[f(b) - f(\tau)\right] d\tau,$$

where $\Gamma(\cdot)$ denotes the Gamma function [16] [8]..

Both operators annihilate constants. As $\beta \to 1^-$ theyreduce, for sufficiently smooth f, to $+f(x)$, the sign convention of the right operator is discussed in Remark1.

Definition 4
Let y be a mapping defined over the closed interval $[a, b]$. The left Caputo fractional differentiation of order $\alpha > 0$ is given by:

$$ {}^{C}_{a}D^{\alpha}_{x}y(x) = {}^{R}_{a}I^{n-\alpha}_{x}\left(y^{(n)}(x)\right) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-s)^{n-\alpha-1}\left(y^{(n)}(s)\right)ds $$

whereas the right Caputo fractional derivative of order α is give by:

$$ {}^{C}_{x}D^{\alpha}_{b}y(x) = {}^{R}_{x}I^{n-\alpha}_{b}\left(y^{(n)}(x)\right) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{b}(s-x)^{n-\alpha-1}\left(y^{(n)}(s)\right)ds $$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$.

There is no Caputo fractional integral, the Caputo derivative is expressed through the Riemann–Liouville fractional integrals. Our work is given for the right operators, and the left case is analogous with straight forward modifications.

The Gamma function, the Betafunction, and the definitions of the left and right Jumarie and Caputo derivative constitute the basic tools required throughout the paper. In the next section an auxiliary result on the left Jumarie derivative of power functions is established, followed by a remark on the sign relation between the right Jumarie and right Caputo derivatives [28].

3. Auxiliary Results

This section presents the auxiliary result required in the proof of the main theorem. The formula gives an explicit expression for the left Jumarie derivative of a power function and serves as the principal tool in establishing the composition formula of Section4.

Lemma 1
(Left Jumarie derivative of a power function). Let $f : [a, b] \to \mathbb{R}$, $f(x) = (x-a)^m$, $m \in \mathbb{N}$, $0 < \beta < 1$. Then, the left Jumarie fractional derivatives of order β is given by:

$$ {}^{J}_{a}D^{\beta}_{x}(x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}(x-a)^{m-\beta} \qquad x \in [a, b]. \tag{4} $$

Proof
From Definition 3, we have

$$ {}^{J}_{a}D^{\beta}_{x}f(x) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_{a}^{x}(x-s)^{-\beta}\left[f(s)-f(a)\right]ds, \qquad 0 < \beta < 1, $$

where $(x-s)^{-\beta}$ is called the fractional kernel or power-law singular kernel in fractional calculus.

Let $f(x) = (x-a)^m$, then $f(s) - f(a) = (s-a)^m - (a-a)^{\beta} = (s-a)^m$ Then Eq. (4) will imply to:

$$ {}^{J}_{a}D^{\beta}_{x}(x-a)^m = \frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_{a}^{x}(x-s)^{-\beta}(s-a)^m ds, \qquad 0 < \beta < 1. \tag{5} $$

Now, we will consider variable transformation of Eq. (5):

Let $s - a = u(x-a)$ Then $s = a + u(x-a)$,

and hence $ds = (x-a)du$, $u \in [0, 1]$.

At $s = a$ Then $u = 0$, while at $s = x$, we have $u = 1$

Thus, the original integration I_1 given in Eq. (5) becomes,

$$\begin{aligned} I_1 &= \int_a^x (x-s)^{-\beta} \left[(s-a)^m\right] ds \\ &= \int_0^1 \left(x-a-u(x-a)\right)^{-\beta} \left[(a+u(x-a)-a)^m\right] (x-a) du \\ &= \int_0^1 \left((x-a)(1-u)\right)^{-\beta} u^m (x-a)^m (x-a) du \\ &= \int_0^1 (x-a)^{-\beta} (1-u)^{-\beta} u^m (x-a)^{m+1} du \\ &= (x-a)^{m+1-\beta} \int_0^1 (1-u)^{-\beta} u^m du\,. \end{aligned} \tag{6}$$

Now, recalling the Beta function the integral in Eq. (6) is exactly the Beta function:

$$B(m+1, 1-\beta) = \int_0^1 (1-u)^{-\beta} u^m du.$$

Also, from the by relationship between the beta and gamma functions, which is:

$$B(m+1, 1-\beta) = \int_0^1 (1-u)^{-\beta} u^m du = \frac{\Gamma(m+1)\Gamma(1-\beta)}{\Gamma(m-\beta+2)}\,,$$

Substitute back into the original integral presented in Eq. (6), getting:

$$I_1 = (x-a)^{m+1-\beta} \frac{\Gamma(m+1)\Gamma(1-\beta)}{\Gamma(m-\beta+2)}\,. \tag{7}$$

Substitute Eq. (7) back into Eq. (5), we get:

$${}_a^J D_x^\beta (x-a)^m = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \left((x-a)^{m+1-\beta} \frac{\Gamma(m+1)\Gamma(1-\beta)}{\Gamma(m-\beta+2)} \right), \tag{8}$$

Apply the definition of a regular derivative on the right-hand side of Eq. (8), implies to:

$${}_a^J D_x^\beta (x-a)^m = \frac{(m+1-\beta)}{\Gamma(1-\beta)} \ \frac{\Gamma(m+1)\Gamma(1-\beta)}{\Gamma(m-\beta+2)} (x-a)^{m-\beta}, \tag{9}$$

By using the gamma function property:

$$\Gamma(m-\beta+2) = (m-\beta+1)\Gamma(m-\beta+1)$$

So, Eq. (9) becomes:

$${}_a^J D_x^\beta (x-a)^m = \frac{(m+1-\beta)\Gamma(m+1)}{(m-\beta+1)\Gamma(m-\beta+1)} (x-a)^{m-\beta}$$

i.e.,

$${}_a^J D_x^\beta (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m-\beta+1)} (x-a)^{m-\beta}\,. \tag{10}$$

which is called the left Jumari's derivative for the power function $(x-a)^m$. □

Remark 1
(sign convention) [28]. Definition (4) is the right Caputo derivative applied to $f(b) - f(x)$. Both operators annihilate constants. After differentiation under the integral sign (see Step Second in the Proof of Theorem 1),

$$ {}_x^C D_b^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \int_x^b (s-x)^{-\beta} f'(s)\, ds, $$

which tends to $+f'(x)$ as $\beta \to 1$, whereas the classical right derivative tends to $-f'(x)$.

Consequently, a composition built with the right Caputo derivative differs by a factor (-1) from the same composition built with the right Jumarie derivative, whereas a composition of two right derivatives is the same for both.

4. Main Results

In this section, we establish the principal result of the paper, namely, an explicit closed-form formula for the composition of the left Jumarie and right Caputo fractional derivatives acting on power functions. The proof is based solely on the definitions of the two fractional operators together with the fundamental properties of the Gamma and Beta functions presented in Section 2 and the auxiliary result established in Section 3. The following theorem constitutes the main result of this paper.

Theorem 1
Suppose that $m \in \mathbb{N}$ satisfies $m > 2\beta - 1$, where $0 < \beta < 1$. Then, the composition of the left and right Jumarie fractional derivatives of order β, applied to the power function $(x-a)^m$ is given by:

$$ {}_x^C D_b^\beta \left({}_a^J D_x^\beta (x-a)^m \right) = (-1)^n \frac{\Gamma(m+1)}{\Gamma(m-\beta)\Gamma(1-\beta)} \int_x^b (s-x)^{-\beta} (s-a)^{m-\beta-1} ds, \qquad x \in [a,b]. $$

where $n-1 < \beta \leq n$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$

Proof
First, applying the Lemma 1, the left Jumarie fractional derivative of the power function is given explicitly by

$$ {}_a^J D_x^\beta (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} (x-a)^{m-\beta}. $$

Second, assume that,

$$ u(x) = {}_a^J D_x^\beta (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} (x-a)^{m-\beta}. $$

By applying the right Caputo fractional derivative for $x > a$, $u'(x)$ s continuous on $[x,b]$

$$ {}_x^C D_b^\beta \left({}_a^J D_x^\beta (x-a)^m \right) = {}_x^C D_b^\beta \left(\frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} (x-a)^{m-\beta} \right). \tag{11} $$

Since the Gamma quotient $\frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}$ is independent of the variable x, it may be taken outside the fractional derivative. Hence,

$$ {}_x^C D_b^\beta \left({}_a^J D_x^\beta (x-a)^m \right) = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} {}_x^C D_b^\beta (x-a)^{m-\beta}. \tag{12} $$

From Eq. (12) assume that, $u(x) = {}_a^J D_x^\beta (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} (x-a)^{m-\beta}$, we have

$$ {}_x^J D_b^\beta u(x) = (-1)^n \frac{1}{\Gamma(1-\beta)} \int_x^b (s-x)^{-\beta} u'(s)\, ds, $$

i.e.,

$$
\begin{aligned}
{}_{x}^{J}D_{b}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right) &= {}_{x}^{J}D_{b}^{\beta}\left(\frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}(x-a)^{m-\beta}\right) \\
&= (-1)^{n}\frac{1}{\Gamma(1-\beta)}\int_{x}^{b}(s-x)^{-\beta}\left(\frac{\Gamma(m+1)(m-\beta)}{\Gamma(m+1-\beta)}(s-a)^{m-\beta-1}\right)ds, \qquad (13)
\end{aligned}
$$

where $u'(s) = \frac{\Gamma(m+1)(m-\beta)}{\Gamma(m+1-\beta)}(s-a)^{m-\beta-1}$.

Since, $\Gamma(m+1-\beta) = (m-\beta)\Gamma(m-\beta)$, then Eq. (13) imply to

$$
{}_{x}^{C}D_{b}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right) = (-1)^{n}\frac{1}{\Gamma(1-\beta)}\int_{x}^{b}(s-x)^{-\beta}\left(\frac{\Gamma(m+1)(m-\beta)}{(m-\beta)\Gamma(m-\beta)}(s-a)^{m-\beta-1}\right)ds,
$$

Therefor,

$$
{}_{x}^{C}D_{b}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right) = (-1)^{n}\frac{\Gamma(m+1)}{\Gamma(m-\beta)\Gamma(1-\beta)}\int_{x}^{b}(s-x)^{-\beta}(s-a)^{m-\beta-1}ds, \qquad x\in[a,b].
$$

which is exactly the statement establishing the desired composition formula and completes the proof. □

Theorem 1 shows that composing a right derivative with a left one gives, in general, an incomplete Beta function, because the integral I_1 runs over $[a,b]$, while one of its singular points, a, lies outside this interval. When the two operators are of the same type, each integral runs between the singular points of its integrand, every step reduces to a complete Beta integral, and pure-power formulas follow. The two corollaries below record these cases. They use only the power-function computation of Theorem 1, and they hold for either the Jumarie or the Caputo right derivative, since the two differ by a sign that cancels in Corollary 2.

Corollary 1 (Composition of Left- Left Jumarie Fractional Derivatives of Power Functions)
For $m \geq 1$, $0 < \beta < 1$ and $x > a$,

$$
{}_{a}^{J}D_{x}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right) = \frac{\Gamma(m+1)}{\Gamma(m+1-2\beta)}(x-a)^{m-2\beta}.
$$

Proof
By Lemma 1, applying Left Jumarie Fractional Derivative of Power Function $(x-a)^m$, as following

$$
{}_{a}^{J}D_{x}^{\beta}(x-a)^{m} = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}\cdot(x-a)^{m-\beta},
$$

Also, applying Left Jumarie Fractional Derivative once again, of Power Function $(x-a)^{m-\beta}$, we getting

$$
\begin{aligned}
{}_{a}^{J}D_{x}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right) &= {}_{a}^{J}D_{x}^{\beta}\left(\frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}\cdot(x-a)^{m-\beta}\right) \\
&= \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m-\beta}\right) \\
&= \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}\left(\frac{\Gamma(m-\beta+1)}{\Gamma(m-\beta+1-\beta)}\cdot(x-a)^{m-2\beta}\right)
\end{aligned}
$$

Since, $\Gamma(m-\beta+1) = (m+1-\beta)$, then,

$$
{}_{a}^{J}D_{x}^{\beta}\left({}_{a}^{J}D_{x}^{\beta}(x-a)^{m}\right) = \frac{\Gamma(m+1)}{\Gamma(m+1-2\beta)}\cdot(x-a)^{m-2\beta}
$$

where $m+1-2\beta > 0$ because $m \geq 1 > 2\beta - 1$. □

Corollary 2 (Composition of right–right Jumarie Fractional Derivatives of Power Functions)
By using Definition 3. (the right Jumarie fractional derivative) of order $0 < \beta < 1$, getting

$$ {}_{x}^{J}D_{b}^{\beta} f(x) = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{x}^{b} (s-x)^{-\beta} \left[f(b) - f(s) \right] ds, $$

Let $f(x) = (b-x)^m$, then $f(b) = 0$, and $f(b) - f(s) = -(b-s)^m$,

Thus,

$$ {}_{x}^{J}D_{b}^{\beta} (b-x)^m = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{x}^{b} (s-x)^{-\beta} (b-s)^m ds, $$

Now introduce the change of variable $s = x + u(b-x)$, $0 \le u \le 1$. Then, $ds = (b-x)\,du$, $\quad s - x = u\,(b-x)$, $\quad b - s = (1-u)(b-x)$, and $s = x$ gives $u = 0$, while $s = b$ gives $u = 1$.

Hence,

$$ \begin{aligned} \int_{x}^{b} (s-x)^{-\beta} (b-s)^m \, ds &= \int_{0}^{1} \left(u(b-x) \right)^{-\beta} \left((1-u)(b-x) \right)^m (b-x)\, du \\ &= (b-x)^{m+1-\beta} \int_{0}^{1} u^{-\beta} (1-u)^m \, du. \end{aligned} $$

The last integral is the Beta function $B(1-\beta,\, m+1)$, so

$$ \int_{x}^{b} (s-x)^{-\beta} (b-s)^m \, ds = (b-x)^{m+1-\beta} \, \frac{\Gamma(1-\beta)\,\Gamma(m+1)}{\Gamma(m+2-\beta)}. $$

Differentiating with respect to x, and using

$$ \frac{d}{dx} (b-x)^{m+1-\beta} = -(m+1-\beta)(b-x)^{m-\beta}, $$

$$ {}_{x}^{J}D_{b}^{\beta} (b-x)^m = -\frac{(m+1-\beta)\,\Gamma(m+1)}{\Gamma(m+2-\beta)} \, (b-x)^{m-\beta}. $$

By the gamma function property $\Gamma(m+2-\beta) = (m+1-\beta)\,\Gamma(m+1-\beta)$, this becomes

$$ {}_{x}^{J}D_{b}^{\beta} (b-x)^m = -\frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} \, (b-x)^{m-\beta}. $$

Differentiating with respect to x, and using

$$ \frac{d}{dx} (b-x)^{m+1-\beta} = -(m+1-\beta)(b-x)^{m-\beta}, $$

$$ {}_{x}^{J}D_{b}^{\beta} (b-x)^m = -\frac{(m+1-\beta)\,\Gamma(m+1)}{\Gamma(m+2-\beta)} \, (b-x)^{m-\beta}. $$

By the gamma function property $\Gamma(m+2-\beta) = (m+1-\beta)\,\Gamma(m+1-\beta)$, this becomes

$$ {}_{x}^{J}D_{b}^{\beta} (b-x)^m = -\frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} \, (b-x)^{m-\beta}. $$

4.2 Classical limit and its role as a benchmark

As $\beta \to 1^-$ the operators reduce in the composition reduce to ordinary differentiation. By Lemma 1, ${}_a^J D_x^\beta (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}(x-a)^{m-\beta} \to m(x-a)^{m-1}$ as $\beta \to 1^-$, and, by Definition 4 with $n=1$, the right Caputo derivative reduces to $-d/dx$. Hence, at $\beta = 1$, the composition is

$$\begin{aligned} {}_x^C D_b^\beta \left({}_a^J D_x^\beta (x-a)^m \right) &= \left(-\frac{d}{dx} \right) \left(\frac{d}{dx} \right) (x-a)^m \\ &= -m(m-1)(x-a)^{m-2}, \end{aligned} \tag{14}$$

for $m \in \mathbb{N}$, $m \geq 1$. The minus sign is inherited from the right Caputo derivative; with the right Jumarie derivative it is reversed, by the sign relation of Remark 1. We refer to Eq. (14) as the classical law.

The classical law (14) gives a necessary condition for Theorem 1; its right-hand side must tend to (14) as $\beta \to 1^-$.

5. Illustrative Example and Numerical Verification

In the following section, we present an example that illustrate Theorem 1 for $\beta < 1$ and show how the composition approaches (14) as $\beta \to 1^-$..

Example 1
Consider the power function $f(x) = (x-a)^m$, with $a = 0$, $b = 1$, $m = 2$, and $\beta = 0.5$, since, $m > 2\beta - 1$, the assumptions of the main theorem are satisfied, and the composition:

$$ {}_x^C D_b^\beta \left({}_a^J D_x^\beta (x-a)^m \right), \qquad x \in [a,b], $$

Can be evaluated as follows. ByLemma1,

$$ {}_0^J D_x^{0.5} x^2 = \frac{\Gamma(3)}{\Gamma(2.5)} x^{1.5} = \frac{2}{0.75\sqrt{\pi}} x^{1.5} = \frac{8}{3\sqrt{\pi}} x^{1.5} $$

By Theorem 1, with the coefficient $\frac{\Gamma(3)}{\Gamma(1.5)\Gamma(0.5)} = \frac{4}{\pi}$, then

$$\begin{aligned} {}_x^C D_1^{0.5} \left({}_0^J D_x^{0.5} x^2 \right) &= -\frac{4}{\pi} \int_x^1 (s-x)^{-0.5} (s-0)^{0.5} ds \\ &= -\frac{4}{\pi} \left(\sqrt{1-x} + x \ln \frac{1+\sqrt{1-x}}{\sqrt{x}} \right), \qquad x \in (x,1]. \end{aligned}$$

At the end points, so the value at $x = 0$ is equal 1, so the value at $x = 1$ is equal $1 - \frac{4}{\pi} \approx -1.2732$, and the value at $x = 1$ is equal 0. The composition is negative, attains it $minimum \approx -1.5275$ $near$ $x \approx 0.305$, and is not a pure power of $(1-x)$.

To validate this result, the composition is also evaluated directly from the definitions, as described above. The values are compared with those of the main theorem in (Table 1) and (Figure 1).

Table 1. Direct computation versus Theorem 1 for Example1.

x	Analytical evaluation	Formula (Theorem	Absolute error
0.00	-1.273240	-1.273240	$< 10^{-24}$
0.20	-1.506439	-1.506439	$< 10^{-24}$
0.40	-1.511697	-1.511697	$< 10^{-24}$
0.60	-1.374786	-1.374786	$< 10^{-24}$
0.80	-1.059568	-1.059568	$< 10^{-24}$
1.00	0.000000	0.000000	0.00000

The graph illustrates (Figure 1) the be haviour of the composition operator over $[0, 1]$, while (Table 1) confirms the values predicted by the closed-form expression of Theorem 1. The agreement, to within the working precision, between the direct computation and the theoretical formula demonstrates the correctness of the derived composition formula for the selected fractional order and polynomial degree.

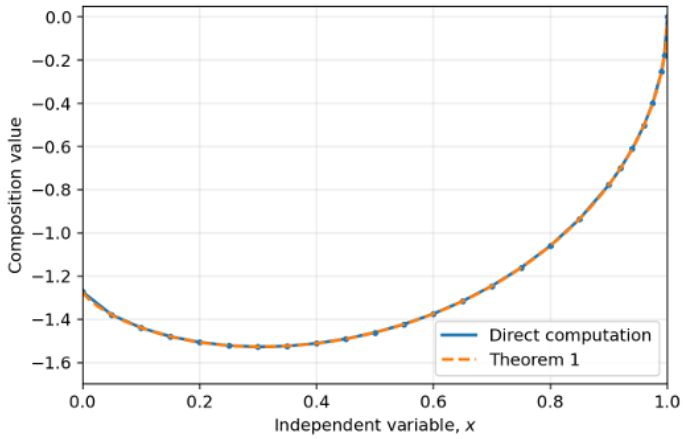


Figure 1. Comparison between the direct computation and the analytical solution obtained from Theorem 1 for Example 1.

Example 2
Consider $f(x) = (x-a)^m$ with $a = 0, b = 1, m = 4$ and $\beta = 0.75$, since $m \geq 1$ and $m > 2\beta - 1$, the assumptions of Theorem 1 are satisfied. Following the same procedure as in Example 1, Lemma 1 give

$$ {}_0^J D_x^{0.75} x^4 = \frac{\Gamma(5)}{\Gamma(4.25)} x^{3.25} $$

and Theorem 1, gives

$$ \frac{\Gamma(5)}{\Gamma(3.25)\Gamma(0.25)} $$

$$ \begin{aligned} {}_x^C D_1^{0.75} \left({}_0^J D_x^{0.75} x^4 \right) &= -\frac{\Gamma(5)}{\Gamma(3.25)\Gamma(0.25)} \int_x^1 (s-x)^{-0.75} (s-0)^{2.25} ds \\ &= -\frac{42}{\Gamma(3.25)\Gamma(0.25)} x^{2.5} B_{1-x}(0{,}25\ ,\ 2.5\ ) \end{aligned} $$

At the end points, so the value at $x = 0$ is $-\frac{\Gamma(5)}{\Gamma(3.25)\Gamma(0.25)} \approx -1.0387$, so the value at $x = 1$ is equal 0. In this case the incomplete Beta function has no elementary closed form, and the formula is evaluated numerically. The composition is again compared with the direct computation in (Table 2) and (Figure 2).

Table 2. Direct computation versus Theorem 1 for Example 2

x	Analytical evaluation	Formula (Theorem	Absolute error
0.00	-1.038668	-1.038668	$< 10^{-24}$
0.20	-1.499245	-1.499245	$< 10^{-24}$
0.40	-2.387154	-2.387154	$< 10^{-24}$
0.60	-3.590492	-3.590492	$< 10^{-24}$
0.80	-4.718836	-4.718836	$< 10^{-24}$
1.00	0.000000	0.000000	0.00000

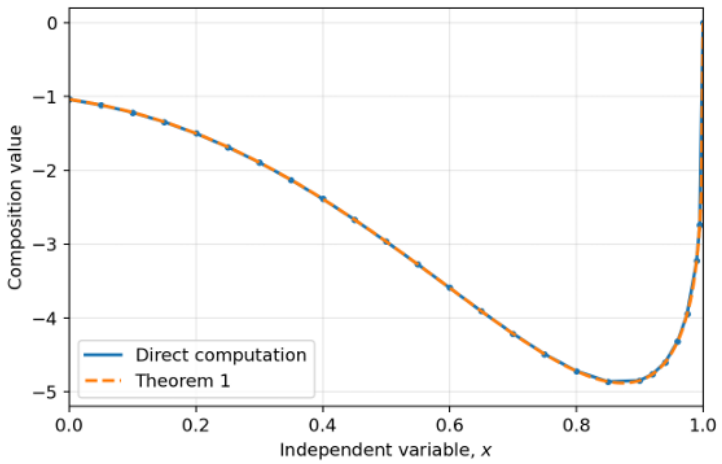


Figure 2. Comparison between the direct computation and the analytical formula of Theorem 1 for Example 2.

The values are negative and the composition rises steeply towards 0 as $x \to 1^-$ in agreement with the behaviour $\sim (1-x)^{1-\beta}$ predicted by the incomplete Beta representation. The examples illustrate the influence of a higher fractional order and polynomial degree.

6. Conclusion

In this paper we established an explicit formula for the composition of the left Jumarie and right Caputo fractional derivatives acting on power functions. More precisely, for $0 < \beta < 1, m \in N$ with $m \geq 1$ and $x \in (a, b)$, we derived the expression ${}_x^C D_b^\beta \left({}_a^J D_x^\beta (x-a)^m \right)$, which is negative and, unlike the compositions of operators of the same type (Corollaries 1 and 2), is not a pure power function: the mixed composition leads to an incomplete Beta function. The result was derived directly from the definitions of the two operators together with classical properties of the Gamma and Beta functions. The proposed formula provides a simple analytical tool for evaluating successive Jumarie–Caputo fractional differentiations; the coefficient is expressed explicitly in terms of Gamma functions, which simplifies theoretical calculations involving Jumarie operators. The validity of the result was illustrated through representative examples: numerical computations obtained directly from the definitions agree with the explicit formula to within the working precision, as confirmed by comparison tables, graphical illustrations and absolute-error analysis.

Moreover, the limiting case $\beta \to 1^-$ shows that the formula is consistent with the corresponding classical composition $(-d/dx)(d/dx)$. The results contribute to the analytical theory of Jumarie fractional calculus and provide benchmark formulas for future investigations. Possible directions for further research include composition

formulas for more general classes of functions, the analysis of higher-order fractional compositions, applications to fractional differential equations and boundary value problems, and extensions to other fractional operators beyond the Jumarie framework.Future research may explore the application of various approximation techniques, including established homotopy analysis method [29] and the Laplace variational method [30], to obtain accurate approximate solutions to the problem under consideration.,

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