STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, December 2026, pp 5650–5660.
Published online in International Academic Press (www.IAPress.org)

On N^*-Open Sets and N^*-Connectedness in Bitopological Spaces

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Abstract This study introduces the concept of N^*-open sets and their complements in bi-topological spaces. The fundamental properties of N^*-open sets are investigated, with particular emphasis on their relationships with conventional open and closed sets in bi-topological spaces. Furthermore, the concept of N^*-open sets is employed to introduce and characterize a new type of connectedness, called N^*-connectedness. The study establishes the basic properties of N^*-connected spaces and provides a framework for examining connectedness through the newly defined class of N^*-open sets.

Keywords N^*-open sets, N^*-closed sets, bi-topological spaces, connected spaces, N^*-connected spaces

AMS 2010 subject classifications 54E55, 54A10, 54A20, 54D20, 54D30, 54A05, 54D05, 54D35

DOI: 10.19139/soic-2310-5070-4554

1. Introduction

Novosad [1] introduced the topics of simple connectedness in bi-topological spaces, with examples and a number of important results. The idea of developing a general theory about the connectednesses and disconnectednesses of topological spaces first began in Preuss's doctoral thesis [2]. Pervin [3] introduced the concepts of connected bi-topological spaces, discussed a number of examples, and drew many results and conclusions. In 1971, Reilly [4] studied the idea of connectedness in bi-topological spaces (see also [5, 6]).

In 1975, the states of disconnectednesses and connectednesses in topological spaces were discussed, studied, and characterized by Arhangel'skiĭ and Wiegandt [7]. In the work given by Xiaodong [9], definitions, properties, and characterizations of connectedness on completely distributive topological lattices and fuzzy topological spaces were discussed and studied (see also [8, 10]).

In 1989, Padlewska [11] studied the concepts of connected spaces and generalized some important results. A general study introduced by Padlewska [12] defined separated sets, connected sets, connected spaces, and components of a topological space, proving several related theorems. Chaudhuri and Das [13] explained that connectedness in fuzzy topological spaces is a generalization of the concept in classical topological spaces, where a fuzzy set is connected if it cannot be expressed as the union of two disjoint, non-empty open fuzzy sets. In 2000, Li [14] defined connected spaces in fuzzy spaces for arbitrary fuzzy sets, establishing several facts and results. Ekici and Noiri [15] studied connectedness in ideal topological spaces.

Arhangel'skiĭ [16] explained that if an extremally disconnected space X has a homogeneous compactification, then X itself is finite. It also follows that if a totally bounded topological space has a dense extremally disconnected subspace, then it is finite. Several properties of connectedness and continuity in bi-topological spaces using the notion of β-open sets were introduced and studied by Abbas [17].

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ISSN 2310-5070 (online) ISSN 2311-004X (print)
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In 2013, Szczuka [18] discussed a connected topology τ on the set of natural numbers $\mathbb{N}$, whose basis consists of all connected arithmetic sequences in Golomb's topological spaces. El-Sheikh and Hosny [19] gave the notion of P^*-* connectedness in ideal bi-topological spaces. Shekutkovski [20] compared two important definitions of connected spaces: the standard one and another definition using coverings. Arunmaran and Kannan [21] introduced the concepts of semi-connectedness in bi-topological spaces.

Misajleski et al. [22] generalized the concepts of connectedness as a set in a topological space called a chain-connected set, introducing chain-disconnected sets and studying their properties. In 2021, Al-shami and Abo-Tabl [23] characterized new facts about connectedness and local connectedness on infra soft topological spaces. In 2023, Acikgoz and Esenbel [24] defined neutrosophic connected spaces and some related properties, introducing the neutrosophic product space.

Abu-Alkishik et al., [25], introduced C-compact spaces endowed with triple topologies, exploring their characteristics and interplay with T-topological spaces. To connect the topic with advanced subjects, view the studies [26] and [27].

In this paper, we will use the following standard symbols to denote the corresponding topologies, as established below.

$\mathcal{T}_{\text{ind}}$, $\mathcal{T}_{\text{dis}}$, $\mathcal{T}_{\text{coc}}$, $\mathcal{T}_{\text{cof}}$, $\mathcal{T}_u$, $\mathcal{T}_{l.r}$, $\mathcal{T}_{r.r}$, $\mathcal{T}_u$ and $\mathcal{T}_s$ to denote the indiscrete, discrete, co-countable, co-finite, usual, left ray, right ray and standard topologies. N^*-open set to denote any set belongs to the supremum topology $\mathcal{T}_H^1 \vee \mathcal{T}_H^2$ of any two topologies $\mathcal{T}_H^1$ and $\mathcal{T}_H^2$ on H.

2. Motivations and Main Contributions

Motivation

Connectedness is a fundamental concept in general topology, playing a significant and active role in the study of the structure of topological and bi-topological spaces. The study of generalized forms of connectedness clearly provides a more comprehensive framework for understanding topological structures. Driven by the scientific need to investigate and define new classes of generalized open sets and their associated topological connectedness properties, this study introduces the concept of N^*-open sets in bi-topological spaces. It discusses their key properties, related fundamental concepts, and general theorems, utilizing these elements to establish the notion of N^*-connectedness and to examine the topological structure of these new spaces.

Main Contributions

The main contributions of this study are summarized and clarified as follows:

1. Clearly introducing the concepts of N^*-open and N^*-closed sets in bitopological spaces, accompanied by concrete examples illustrating these concepts.
2. Most of the fundamental topological properties of N^*-open sets are investigated, with a particular focus on their relationships with traditional open and closed sets in bi-topological spaces.
3. A new type of connection, termed N^*-connected spaces, is introduced for the first time, based on the newly defined N^*-open sets.
4. The fundamental properties of N^*-connected spaces are established, constructed, and a suitable topological characterization for them is derived.
5. The primary relationship between N^*-connectedness and general connectedness concepts in bi-topological spaces is investigated.
6. The concepts proposed in the study provide a new and realistic framework for investigating connectedness in bi-topological spaces through N^*-open sets; they also contribute to the introduction and development of generalized concepts of connectedness in topology.

3. N^*-open sets

In this section we define N^*-open sets and its complement in bilopological spaces. We studied the basic properties of them and the relationship between them with open and closed sets in bitopological spaces.

Definition 1
Let H be any non-empty set and $\mathcal{T}_H^1, \mathcal{T}_H^2$ be two topologies defined on H. The set $O \subseteq H$ is open in H if $O \in \mathcal{T}_H^1 \cup \mathcal{T}_H^2$. The complement of O is closed in H. The space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is called pairwise topological space.

Result 2
Any topological space $(H, \mathcal{T}_H)$ is a bitopological space since $(H, \mathcal{T}_H) \equiv (H, \mathcal{T}_H, \mathcal{T}_H)$.

Result 3
Any bitopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is not a topological space in general but we can define a single topology from it. One of these methods is taking the least upper bound topology of $\mathcal{T}_H^1$ and $\mathcal{T}_H^2$.

Definition 4
Let $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ be a bitopological space. The smallest topology contains $\mathcal{T}_H^1$ and $\mathcal{T}_H^2$ is called the least upper bound topology of both $\mathcal{T}_H^1$ and $\mathcal{T}_H^2$ or it is called the supremum topology of $\mathcal{T}_H^1$ and $\mathcal{T}_H^2$ and denoted by $\mathcal{T}_H^1 \vee \mathcal{T}_H^2$ or $\mathcal{T}_H^\vee$.

Remark 5
$\mathcal{T}_H^1 \cup \mathcal{T}_H^2 \subseteq \mathcal{T}_H^1 \vee \mathcal{T}_H^2$.

Example 6
Let $H = \{1, 2, 3\}$, $\mathcal{T}_H^1 = \{\emptyset, H, \{1\}\}$, $\mathcal{T}_H^2 = \{\emptyset, H, \{1, 2\}\}$. Then $\mathcal{T}_H^\vee = \mathcal{T}_H^1 \vee \mathcal{T}_H^2 = \{\emptyset, H, \{1\}, \{1, 2\}\}$ is a topology on H. So $(H, \mathcal{T}_H^\vee)$ is a single topology defined on H. Let $H = \{a, b, c, d\}$, $\mathcal{T}_H^1 = \{\emptyset, H, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{T}_H^2 = \{\emptyset, H, \{a\}, \{a, b\}\}$. Then $\mathcal{T}_H^\vee = \{\emptyset, H, \{a\}, \{b\}, \{a, b\}\}$.

Example 7
Let $H = \mathbb{R}$, $\mathcal{T}_H^1 = \mathcal{T}_{\text{ind}}$, $\mathcal{T}_H^2 = \mathcal{T}_{\text{u}}$ then $\mathcal{T}_H^\vee = \mathcal{T}_u$.

Example 8
Let $H = \mathbb{R}$, $\mathcal{T}_H^1 = \mathcal{T}_{\text{l.r}}$, $\mathcal{T}_H^2 = \mathcal{T}_{\text{r.r}}$ then $\mathcal{T}_H^\vee = \mathcal{T}_u$.

Result: If $\mathcal{T}_H^1 \subseteq \mathcal{T}_H^2$ then $\mathcal{T}_H^\vee = \mathcal{T}_H^2$.

Definition 9
Let $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ be a bitopological space and $A \subseteq H$. Then A is called N^*-open set if $A \in \mathcal{T}_H^\vee = \mathcal{T}_H^1 \vee \mathcal{T}_H^2$. The complement of A is called N^*-closed set.

Example 10
Let $H = \{A, B, C\}$, $\mathcal{T}_H^1 = \{\emptyset, H, \{B\}\}$, and $\mathcal{T}_H^2 = \{\emptyset, H, \{C\}\}$. Then $\mathcal{T}_H^\vee = \mathcal{T}_H^1 \vee \mathcal{T}_H^2$=$\{\emptyset, H, \{C\}, \{B\}, \{C, B\}\}$. Now the sets $\emptyset, H, \{B\}, \{C\}$ and $\{B, C\}$ are called N^*-open sets. $\emptyset, H, \{A, C\}, \{A, B\}$ and $\{A\}$ are called N^*-closed sets.

Example 11
Let $H = \mathbb{R}$, $\mathcal{T}_H^1 = \mathcal{T}_{\text{cof}}$, $\mathcal{T}_H^2 = \mathcal{T}_u$, then $\mathcal{T}_H^\vee = \mathcal{T}_u$ so any open set in $(\mathbb{R}, \mathcal{T}_u)$ is N^*-open.

Remark 12

(1) Every open set in a bitopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-open set since $\mathcal{T}_H^1 \cup \mathcal{T}_H^2 \subseteq \mathcal{T}_H^1 \vee \mathcal{T}_H^2$, but the converse need not be true.

2 Every closed set in a bitopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-open set $\mathcal{T}_H^1 \cup \mathcal{T}_H^2 \subseteq \mathcal{T}_H^1 \vee \mathcal{T}_H^2$, but the converse need not be true.

The following example illustrated above remark:

Example 13
Let $H=\{n,m,l\}$,$\mathcal{T}_H^1=\{\emptyset,H,\{n\},\{n,m\},\{n,l\}\}$ $\mathcal{T}_H^2=\{\emptyset,H,\{m,l\}\}$. Then

$$\mathcal{T}_H^{\vee}=\mathcal{T}_H^1\vee\mathcal{T}_H^2=\{\emptyset,H,\{n\},\{n,m\},\{n,l\},\{m,l\},\{m\},\{l\}\}=\mathcal{T}_H^{\text{dis}}.$$

It is clear that $\{m\}$ is N^*-open set but not open in $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ since $\{m\}\notin\mathcal{T}_H^1\cup\mathcal{T}_H^2$ $\{x,y\}$ is N^*-closed set but not closed in $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ since $\{x,z\}$ not closed in both $\mathcal{T}_H^1$ and $\mathcal{T}_H^2$.

Remark 14

1 Any N^*-open set in $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ is not necessary $\mathcal{T}_H^i$-open set;$i=1,2$.

2 Any N^*-closed set in $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ is not necessary $\mathcal{T}_H^i$-closed set;$i=1,2$.

Example 15
$H=\{a,b,c\}$, $\mathcal{T}_H^1=\{H,\emptyset,\{a,c\}\}$, $\mathcal{T}_H^2=\{H,\emptyset,\{b,c\}\}$, $\mathcal{T}_H^{\vee}=\mathcal{T}_H^1\vee\mathcal{T}_H^2=\{H,\emptyset,\{c\},\{a,c\},\{b,c\}\}$. $\{c\}$ is N^*-open but it is not a $\mathcal{T}_H^i$-open set $i=1,2$. $\{a,b\}$ is N^*-closed but it is not a $\mathcal{T}_H^i$-closed set $i=1,2$.

Remark 16

1 Any N^*-open set in $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ is not necessarily open in it.

2 Any N^*-closed set in $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ is not necessarily closed in it.

The above example 15 illustrates the previous remark.

Remark 17
Let $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ be a bitopological space. Then:

1. $\mathcal{T}_H^1\cap\mathcal{T}_H^2\subseteq\mathcal{T}_H^1\vee\mathcal{T}_H^2$.
2. $\mathcal{T}_H^1\cup\mathcal{T}_H^2\subseteq\mathcal{T}_H^1\vee\mathcal{T}_H^2$.

Theorem 18
Let $(H,\mathcal{T}_H^1,\mathcal{T}_H^2)$ be a bitopological space, then:

1) The union of any collection of N^*-open sets on N^*-open.
2) The intersection of any finite number of N^*-closed sets is N^*-closed.

Proof:

1) Let $\{w_\alpha:\alpha\in\Delta\}$ be a family of N^*-open subsets of H. So $W_\alpha\in\mathcal{T}_H^1\vee\mathcal{T}_H^2$ $\forall\alpha\in\Delta$. But $\mathcal{T}_H^1\vee\mathcal{T}_H^2$ is a topological space, so $\bigcup_{\alpha\in\Delta}w_\alpha\in\mathcal{T}_H^1\vee\mathcal{T}_H^2$.
2) Let $\{F_{\gamma_1},F_{\gamma_2},\ldots,F_{\gamma_n}\}$ be a finite number of N^*-closed sets, then $(F_{\gamma_i})^c$ is N^*-open $\forall i=1,2,\ldots,n$. So

$$\begin{aligned}(F_{\gamma_i})^c\in\mathcal{T}_H^1\vee\mathcal{T}_H^2&\Rightarrow\bigcup_{i=1}^{n}(F_{\gamma_i})^c\in\mathcal{T}_H^1\vee\mathcal{T}_H^2.\\&\Rightarrow\left[\bigcup_{i=1}^{n}(F_{\gamma_i})^c\right]^c\text{ is an }N^*\text{-closed set.}\\&\Rightarrow\bigcap_{i=1}^{n}F_{\gamma_i}\text{ is an }N^*\text{-closed set.}\end{aligned}$$

Example 19

Consider the bitopological space $(\mathbb{N},\mathcal{T}_{\text{coc}},\mathcal{T}_{\text{cof}})$, then $\overset{\vee}{\mathcal{T}}_{\mathbb{N}}=\mathcal{T}_{\text{coc}}\vee\mathcal{T}_{\text{cof}}=\mathcal{T}_{\text{coc}}$. Hence, the set of all N^*-open sets are $\{u\subseteq\mathbb{N}:u\in\mathcal{T}_{\text{coc}}\}\equiv(\mathbb{N},\mathcal{T}_{\text{dis}})$. Since $\mathcal{T}_{\text{dis}}$ is a topology on $\mathbb{N}$, it is closed under the arbitrary union of open sets and also under the finite intersection of closed sets.

4. N^*-Connected Spaces

In this section, the concept of N^*-open sets is employed to introduce and characterize a new type of connectedness, called N^*-connectedness. The study establishes the basic properties of N^*-connected spaces and provides a framework for examining connectedness through the newly defined class of N^*-open sets.

Definition 20
A space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is called N^*-connected if we can't write H as a union of two non-empty disjoint N^*-open sets. Otherwise, H is called N^*-disconnected.

Result 21
A space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is called N^*-connected if we can't write H as a union of two non-empty disjoint N^*-closed sets. Otherwise, H is called N^*-disconnected.

Result 22
A space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is called N^*-connected if we can't write H as a union of two non-empty disjoint N^*-clopen sets. Otherwise, H is called N^*-disconnected.

Example 23
Let $H = \{1, 2, 3\}$, $\mathcal{T}_H^1 = \{\emptyset, H, \{1\}\}$, $\mathcal{T}_H^2 = \{\emptyset, H, \{2\}\}$, then $\mathcal{T}_H^\vee = \mathcal{T}_H^1 \vee \mathcal{T}_H^2 = \{\emptyset, H, \{1\}, \{2\}, \{1, 2\}\}$.

So we can't write H as a union of two non-empty disjoint N^*-open sets, Hence $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-connected.

Example 24
Let $H = \mathbb{R}$, $\mathcal{T}_H^1 = \mathcal{T}_{ind}$, $\mathcal{T}_H^2 = \mathcal{T}_{cof}$ then $\mathcal{T}_H^\vee = \mathcal{T}_H^1 \vee \mathcal{T}_H^2 = \mathcal{T}_{cof}$. Hence $(\mathbb{R}, \mathcal{T}_{ind}, \mathcal{T}_{cof})$ is N^*-connected. To show this: if $\mathbb{R} = U_1 \cup U_2$ where U_1, U_2 are non-empty N^*-open sets, then $\mathbb{R} - U_1 =$ finite and $\mathbb{R} - U_2 =$ finite. So, $\mathbb{R} = (\mathbb{R} -$ finite $) \cup (\mathbb{R} -$ finite $) = \mathbb{R} -$ finite which is a contradiction.

Example 25
The bitopological space $(\mathbb{R}, \mathcal{T}_{l.r}, \mathcal{T}_{r.r})$ is N^*-connected.

Example 26
If $|H| \geq 2$, $\mathcal{T}_H^1 = \mathcal{T}_{dis}$, $\mathcal{T}_H^2$ be any topology on H then $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected. Let $a \in H$, $U = \{a\}$, $V = H - \{a\}$ then U and V are non-empty N^*-open sets and $U \cup V = H$. So $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected.

Example 27
The bitopological space $(\mathbb{R}, \mathcal{T}_{cof}, \mathcal{T}_{coc})$ is N^*-connected.

Since $(\mathbb{R}, \mathcal{T}_{\text{cof}}) \subseteq (\mathbb{R}, \mathcal{T}_{\text{coc}})$, then $(\mathbb{R}, \mathcal{T}_{\text{cof}}, \mathcal{T}_{\text{coc}}) \equiv (\mathbb{R}, \mathcal{T}_{\text{coc}})$. Since $(\mathbb{R}, \mathcal{T}_{\text{coc}})$ is connected, then $(\mathbb{R}, \mathcal{T}_{\text{cof}}, \mathcal{T}_{\text{coc}})$ is N^*-connected.

Example 28
The bitopological space $(\mathbb{N}, \mathcal{T}_{cof}, \mathcal{T}_{coc})$ is N^*-disconnected.

Since $(\mathbb{N}, \mathcal{T}_{\text{cof}}, \mathcal{T}_{\text{coc}}) \equiv (\mathbb{N}, \mathcal{T}_{\text{coc}}) \equiv (\mathbb{N}, \mathcal{T}_{\text{dis}})$ and $(\mathbb{N}, \mathcal{T}_{\text{dis}})$ is disconnected, then $(\mathbb{N}, \mathcal{T}_{\text{cof}}, \mathcal{T}_{\text{coc}})$ is N^*-disconnected.

Example 29
The bitopological space $(\mathbb{R}, \mathcal{T}_{cof}, \mathcal{T}_{l.r})$ is N^*-connected.

Since we can not write $\mathbb{R}$ as the union of two non-empty disjoint open set.

Theorem 30
A space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-connected if H and $\emptyset$ are the only N^*-clopen sets.

Proof: $\Rightarrow$) Let H be an N^*-connected. Suppose that $\emptyset \neq A \neq H$ and A is N^*-clopen. Let $B = H - A$, then B is N^*-clopen and $\emptyset \neq B \neq H$. Now, $A \cap B = \emptyset$ and $A \cup B = H$. Hence H is disconnected, which leads to a contradiction. So H and $\emptyset$ are the only N^*-clopen sets.

$\Leftarrow$) Assume that $\emptyset$ and H are the only clopen sets. Suppose that H is disconnected, so $H = A \cup B$ where A and B are non-empty disjoint N^*-open sets. Since B is N^*-open and $A = H - B$, then A is N^*-closed set and $A \neq H$. So A is N^*-clopen, which is a contradiction.

Example 31
The bitopological space $(\mathbb{R}, \mathcal{T}_{l.r}, \mathcal{T}_{r.r})$ is N^*-connected since $\emptyset$ and $\mathbb{R}$ the only N^*-clopen sets.

Example 32
The bitopological space $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_{cof})$ is N^*-connected since $\emptyset$ and $\mathbb{R}$ the only N^*-clopen sets.

Example 33
The bitopological space $(\mathbb{R}, \mathcal{T}_s, \mathcal{T}_{coc})$ is N^*-disconnected since $\mathcal{T}_{\mathbb{R}}^{\vee} = \mathcal{T}_s$, let $a \in \mathbb{R}$ and $U = [a, \infty)$ and $V = \mathbb{R} - [a, \infty)$ then U and V are non-empty disjoint N^*-clopen sets and $\mathbb{R} = U \cup V$.

Theorem 34
A space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-connected if and only if every continuous function $f : (H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2) \to (\{a, b\}, \mathcal{T}_{dis})$ is constant.

Proof: $\Rightarrow$) Assume that $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ is N^*-connected and suppose that there exists a continuous function $f : (H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2) \to (\{a, b\}, \mathcal{T}_{dis})$ such that f is not constant. Then there exists $x, y \in H$ such that $f(x) = a$ and $f(y) = b$. Let $U = f^{-1}(\{a\})$ and $V = f^{-1}(\{b\})$, then U and V are open sets so N^*-open sets and

$$
\begin{aligned}
&U \neq \emptyset \text{ with } x \in U,\ V \neq \emptyset \text{ with } y \in V,\\
&U \cap V = f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = f^{-1}(\{a\} \cap \{b\}) = f^{-1}(\emptyset) = \emptyset\\
&U \cup V = f^{-1}(\{a\}) \cup f^{-1}(\{b\}) = f^{-1}(\{a, b\}) = H
\end{aligned}
$$

Hence $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ isN^*-disconnected which is a contradiction.

$\Leftarrow$) Suppose that every continuous function $f : (H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2) \to (\{a, b\}, \mathcal{T}_{dis})$ is constant. Assume that H is disconnected say $H = A \cup B$ where A and B are non-empty disjoint N^*-open sets.

Define $f : H \to (\{a, b\}, \mathcal{T}_{dis})$ by

$$
f(x) = \begin{cases} a & \text{if } x \in A,\\ b & \text{if } x \in B \end{cases}
$$

Let w be an open set in $\{a, b\}$, we have the following cases:

1) If $w = \emptyset$ then $f^{-1}(w) = \emptyset$ is open in H so it is N^*-open set.
2) If $w = \{a, b\}$ then $f^{-1}(w) = H$ is open in H so it is N^*-open set.
3) If $w = \{a\}$ then $f^{-1}(w) = A$ open in H so it is N^*-open set.
4) If $w = \{b\}$ then $f^{-1}(w) = B$ open in H so it is N^*-open set.

Hence f is continuous but not constant, which is a contradiction.
So $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ is connected.

Example 35
Consider $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_{cof})$, then $\overset{\vee}{\mathcal{T}}_{\mathbb{R}} = \mathcal{T}_u \vee \mathcal{T}_{\text{cof}} = \mathcal{T}_u$. Since $(\mathbb{R}, \mathcal{T}_u)$ is a connected space, then every continuous function $f : (\mathbb{R}, \mathcal{T}_u) \to (\{a, b\}, \mathcal{T}_{\text{dis}})$ is constant. Hence, every continuous function $f : (\mathbb{R}, \mathcal{T}_u \vee \mathcal{T}_{\text{cof}}) \to (\{a, b\}, \mathcal{T}_{\text{dis}})$ is constant, so by above theorem, $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_{\text{cof}})$ is N^*-connected space.

Example 36

$$\begin{aligned} X &= \{1,2,3\} \\ \mathcal{T}_1 &= \{X, \emptyset, \{1\}, \{2\}, \{1,2\}\}, \\ \mathcal{T}_2 &= \{X, \emptyset, \{3\}\}, \\ \mathcal{T}_1 \vee \mathcal{T}_2 &= \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\} = \mathcal{T}_{dis}. \end{aligned}$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ are connected spaces but $\mathcal{T}_1 \vee \mathcal{T}_2$ is N^*-disconnected spaces.

The following theorem provides the fundamental necessary and sufficient condition to ensure that $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected.

Theorem 37
If $(H, \mathcal{T}_H^i)$ is disconnected for some $i = 1, 2$, then $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected.

Proof: Assume that $(H, \mathcal{T}_H^i)$ is disconnected for some $i = 1, 2$. Then $H = A_1 \cup A_2$ where A_1, A_2 are two non-empty disjoint clopen sets $\mathcal{T}_H^i$. But $\mathcal{T}_H^i \subseteq \mathcal{T}_H^1 \vee \mathcal{T}_H^2$, so $A_1, A_2$are two non-empty disjoint clopen set in $\mathcal{T}_H^1 \vee \mathcal{T}_H^2$ so $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ is disconnected space. Hence $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected.

Corollary 38
If $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-connected, then $(H, \mathcal{T}_H^1)$ and $(H, \mathcal{T}_H^2)$ are connected spaces.

Example 39
Let

$$\begin{aligned} H &= \{1,2,3\} \text{ and} \\ \mathcal{T}_H^1 &= \{\emptyset, H, \{1\}, \{1,3\}\}, \\ \mathcal{T}_H^2 &= \{\emptyset, H, \{2\}, \{2,4\}, \{4\}\} \text{ then,} \\ \mathcal{T}_H^\vee &= \{\emptyset, H, \{1\}, \{2\}, \{1,3\}, \{2,4\}, \{1,2\}, \{1,2,4\}, \{1,2,3\}, \{4\}, \{1,4\}, \{1,3,4\}\}, \\ &(H, \mathcal{T}_H^1) \text{ and } (H, \mathcal{T}_H^2) \text{ are connected spaces, but } (H, \mathcal{T}_H^\vee) \text{ is disconnected,} \\ &\text{so } (H, \mathcal{T}_H^1, \mathcal{T}_H^2) \text{ is } N^*\text{-disconnected.} \end{aligned}$$

Example 40
Let

$$\begin{aligned} H &= \{1,2,3\} \\ \mathcal{T}_H^1 &= \{\emptyset, H, \{1\}, \{1,2\}\} \text{ connected,} \\ \mathcal{T}_H^2 &= \{\emptyset, H, \{1\}, \{2,3\}\} \text{ disconnected,} \\ \text{but } \mathcal{T}_H^\vee &= \{\emptyset, H, \{1\}, \{1,2\}, \{2,3\}, \{2\}\} \text{ is disconnected,} \\ &\text{so } (H, \mathcal{T}_H^1, \mathcal{T}_H^2) \text{ is } N^*\text{-disconnected.} \end{aligned}$$

Remark 41
The converse of Theorem 37 holds when the following extra conditions are satisfied:

a. The space $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ is a clopen space (that is, every open set in H is clopen).
b. $|H| \geq 2$,
c. $\mathcal{T}_H^1 \neq \mathcal{T}_{\text{dis}}$, $\mathcal{T}_H^2 \neq \mathcal{T}_{\text{dis}}$,
d. $|\mathcal{T}_H^1| \geq 2$ and $|\mathcal{T}_H^2| \geq 2$,

In this case, there exists a clopen set in each $(H, \mathcal{T}_H^1)$ and $(H, \mathcal{T}_H^2)$ other than $\emptyset$ and H. Thus, $(H, \mathcal{T}_H^1)$ and $(H, \mathcal{T}_H^2)$ are disconnected spaces.

Definition 42
A biotopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is connected if $(H, \mathcal{T}_H^1)$ and $(H, \mathcal{T}_H^2)$ are connected spaces.

Result 43
Every N^*-connected space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is connected.

Since $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-connected, then $(H, \mathcal{T}_H^1)$ and $(H, \mathcal{T}_H^2)$ are connected spaces. Thus, $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is connected space.

Definition 44
A biotopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is disconnected if both $(H, \mathcal{T}_H^1)$ and $(H, \mathcal{T}_H^2)$ are disconnected spaces.

Result 45
Every disconnected biotopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected.

The result is a direct deduction from Theorem 37 and Definition 44.

As an example of this result, consider $(\mathbb{R}, \mathcal{T}_s, \mathcal{T}_{\mathrm{u}})$ which is a disconnected space, so it is an N^*-disconnected, since ϕ and $\mathbb{R}$ are not the only N^*-clopen sets.

Definition 46
A subset Q of a biotopological space is N^*-connected if we can not write Q as a union of two non-empty disjoint N^*-open sets in Q.

Example 47
$H = \{1, 2, 3\}$, $\mathcal{T}_H^1 = \{\emptyset, H, \{1\}\}$, $\mathcal{T}_H^2 = \{\emptyset, H, \{2\}, \{2, 3\}\}$, $\mathcal{T}_H^\vee = \{\emptyset, H, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$
Let $Q = \{1, 2\} \subseteq H$ then $\mathcal{T}_Q^\vee = \{\emptyset, Q, \{1\}, \{2\}\}$. Q is disconnected since $Q = \{1\} \cup \{2\}$ where $\{1\}, \{2\}$ are non-empty disjoint N^*-open sets in Q.

Definition 48
Let $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ be a biotopological space and $Q \subseteq H$ then

$$\mathcal{T}_Q^\vee = \mathcal{T}_Q^1 \vee \mathcal{T}_Q^2 = \{Q \cap A : \text{ where } A \text{ is } N^* \text{ - open set in } H\}.$$

Definition 49
A subset Q of a biotopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is connected if $(Q, \mathcal{T}_Q^1)$ and $(Q, \mathcal{T}_Q^2)$ are connected spaces where

$$\mathcal{T}_Q^1 = \{Q \cap U : U \in \mathcal{T}_H^1\}$$
$$\mathcal{T}_Q^2 = \{Q \cap V : V \in \mathcal{T}_H^2\}$$

Example 50
Let $H = \mathbb{R}$, $\mathcal{T}_H^1 = \mathcal{T}_{ind}$, $\mathcal{T}_H^2 = \mathcal{T}_{l.r}$, $Q = \{1, 2\}$

$$\begin{aligned}\mathcal{T}_Q^\vee = \mathcal{T}_Q^1 \vee \mathcal{T}_Q^2 &= \{Q \cap A : A \text{ is } N^* \text{- open set}\} \\ &= \{\emptyset, Q, \{1\}\}.\end{aligned}$$

then Q is N^*-connected. Also we can see that Q is connected.

Definition 51
(Willard, 1970): Let $(H, \mathcal{T}_H)$ be a topological space and $S,\ T \subseteq H$, then $S,\ T$ are separated in H if $S \neq \emptyset \neq T$ and $\overline{S} \cap T = S \cap \overline{T} = \emptyset$.

Example 52
Consider $(\mathbb{R}, \mathcal{T}_{dis})$ then $S = \{1\}$ and $T = \{2\}$ are separated sets $\mathbb{R}$.

Theorem 53
A subset Q of a bitopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-connected if and only if Q can not be written as a union of two separated sets in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$.

Proof: $\Rightarrow$ Let Q be an N^*-connected subset of H. Suppose that $Q = S \cup T$ where S and T are separated in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$.
Now;

$$\begin{aligned} S \cap T = \emptyset,\ (\overline{S})^Q &= (\overline{S})^H \cap Q \\ &= (\overline{S})^H \cap (S \cup T) \\ &= \left((\overline{S})^H \cap S\right) \cup \left((\overline{S})^H \cap T\right) \\ &= S \cup \emptyset = S \end{aligned}$$

Therefor, $(\overline{S})^{\,Q} = S \Rightarrow S$ is closed in Q.
Similarly, we show that T is closed in Q. But $S \cup T = Q$ and $S \cap T = \emptyset$ then S and T are open in Q and $S \neq \emptyset \neq T$. Therefore, Q is N^*-disconnected which is a contradiction.

$\Longleftarrow$) Suppose that Q is N^*-disconnectd. Say $Q = A \cup B$ where A and B are non empty disjoint N^*-closed sets in Q. Now,

$$\begin{aligned} A &= (N^* - cl(A))^{\mathrm{Q}} \\ &= (N^* - cl(A))^H \cap Q \\ &= (N^* - cl(A))^H \cap (A \cup B) \\ &= \left(\left(N^* - cl(A)^H\right) \cap A\right) \cup \left((N^* - cl(A))^H \cap B\right) \\ &= A \cup ((N^* - cl(A))^H \cap B) \end{aligned}$$

So, $(N^* - cl(A))^H \cap B \subseteq A$ but $(N^* - cl(A))^H \cap B \subseteq B$. Which is a contradiction, since $A \cap B = \phi$.

Theorem 54
If Q is a non–empty N^*-connected set and $Q \subseteq S \cup T$ where S and T are separated sets in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ then $Q \subseteq S$ or $Q \subseteq T$.

Proof: Let $S_1 = S \cap Q$ and $T_1 = T \cap Q$. Suppose that $S_1 \neq \emptyset$ and $T_1 \neq \emptyset$.

$$\begin{aligned} N^* - cl(S_1) \cap T_1 \subseteq N^* - cl(S) \cap T = \emptyset \\ S_1 \cap N^* - cl(T_1) \subseteq S \cap N^* - cl(T) = \emptyset \end{aligned}$$

Hence S_1 and T_1 are separated sets in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ but

$$S_1 \cup T_1 = (Q \cap S) \cup (T \cap Q) = Q \cap (S \cup T) = Q.$$

So Q is N^*-disconnected which is a contradiction.

Theorem 55
If Q is N^*-connected in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$ then N^*-$cl(Q)$ is also N^*-connected in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$.

Proof. Suppose that Q is N^*-connected but N^*-$cl(Q)$ is N^*-disconnected then N^*-$cl(Q) = S \cup T$ where S and T are separated in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$. Now, $Q \subseteq N^*$-$cl(Q) = S \cup T$ then $Q \subseteq S$ or $Q \subseteq T$. Suppose that $Q \subseteq S$ then N^*-$cl(Q) \subseteq N^*$-$cl(S) \subseteq H$-T but N^*-$cl(Q) = S \cup T$ so $T = \emptyset$ which is a contradiction.

Theorem 56
Let $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ be a bitopological space and $Q \subseteq H$ is an N^*-connected. If $Q \subseteq D \subseteq N^*$-cl$(Q)$ then D is N^*-connected.

Proof. Suppose that D is not N^*-connected, then $D = A \cup B$ where A and B are separated sets in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$. Now, $Q \subseteq D = A \cup B$ then $Q \subseteq A$ or $Q \subseteq B$. Suppose $Q \subseteq A$, then N^*-cl$(Q) \subseteq N^*$-cl$(A) \subseteq H - B$. Then $D \subseteq N^*$-cl$(Q) \subseteq H - B$, so $B = \phi$, which is a contradiction since A and B are separated sets.

Theorem 57
In any bitopological space $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$, the empty set is N^*-connected and connected.

Proof. Because the only way for writing an empty set as the union of two sets is $\phi = \phi \cup \phi$. We can't write it as the union of two non empty disjoint open sets.

Theorem 58
Let $\{Q_\alpha : \alpha \in \Delta\}$ be a family of N^*-connected sets in $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ with $Q_\alpha \cap Q_\beta \neq \phi$ for all $\alpha, \beta \in \Delta$. Then $\bigcup_{\alpha\in\Delta} Q_\alpha$ is N^*-connected.

Proof. Suppose that $\bigcup_{\alpha\in\Delta} Q_\alpha$ is N^*-disconnected. Then $\bigcup_{\alpha\in\Delta} Q_\alpha = A \cup B$ where A and B are separated sets in $(H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2)$. Choose $\alpha_0 \in \Delta$, then $Q_{\alpha_0} \subseteq \bigcup_{\alpha\in\Delta} Q_\alpha = A \cup B$. Since Q_{α_0} is N^*-connected, then $Q_{\alpha_0} \subseteq A$ or $Q_{\alpha_0} \subseteq B$. (Assume that $Q_{\alpha_0} \subseteq A$.) If $\beta_0 \in \Delta$, then $Q_{\beta_0} \subseteq \bigcup_{\alpha\in\Delta} Q_\alpha \subseteq A \cup B$, since Q_{β_0} is N^*-connected, then $Q_{\beta_0} \subseteq A$ or $Q_{\beta_0} \subseteq B$, but $Q_{\alpha_0} \cap Q_{\beta_0} \neq \phi$, so $Q_{\beta_0} \subseteq A$. Hence, $B \subseteq \bigcup_{\alpha\in\Delta} Q_\alpha \subseteq A \subseteq X - B$, so $B = \phi$, which is a contradiction.

Definition 59
A function $f : (H, \mathcal{T}_H^1, \mathcal{T}_H^2) \longrightarrow (W, \mathcal{T}_W^1, \mathcal{T}_W^2)$ is *continuous* if $f : (H, \mathcal{T}_H^1) \to (W, \mathcal{T}_W^1)$ and $f : (H, \mathcal{T}_H^2) \to (W, \mathcal{T}_W^2)$ are continuous functions.

Definition 60
A function $f : (H, \mathcal{T}_H^1, \mathcal{T}_H^2) \longrightarrow (W, \mathcal{T}_W^1, \mathcal{T}_W^2)$ is N^*-*continuous* if the inverse image of any N^*-open set in W is N^*-open set in H. This means that $f : (H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2) \longrightarrow (W, \mathcal{T}_W^1 \vee \mathcal{T}_W^2)$ is continuous.

Theorem 61
The N^*-continuous image of an N^*-connected space is an N^*-connected space.

Proof. Let $f : (H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2) \longrightarrow (W, \mathcal{T}_W^1 \vee \mathcal{T}_W^2)$ be an N^*-continuous on to function. Suppose that W is N^*-disconnected, then there exists a continuous function $g : (W, \mathcal{T}_W^1 \vee \mathcal{T}_W^2) \longrightarrow (\{a, b\}, \mathcal{T}_{dis})$ such that g is not constant. Since f is continuous then $g \circ f : (H, \mathcal{T}_H^1 \vee \mathcal{T}_H^2) \longrightarrow (\{a, b\}, \mathcal{T}_{dis})$ is continuous and not constant. Hence, $(H, \mathcal{T}_H^1, \mathcal{T}_H^2)$ is N^*-disconnected, which is a contradiction.

Corollary 62
The N^*-continuous image of an N^*-connected space is an N^*-connected.

Conclusion

This study introduces and investigates the concept of N^*-open sets and their complements in bitopological spaces. The fundamental properties of these sets are established, with particular emphasis on their relationships with the conventional notions of open and closed sets in bi-topological spaces. Furthermore, the newly introduced class of N^*-open sets is utilized to define and characterize a new form of connectedness, referred to as N^*-connectedness. Several basic properties of the resulting N^*-connected spaces are established, providing a framework for studying connectedness through this newly defined class of N^*-open sets. The results obtained contribute to the broader development of connectedness in bitopological spaces and provide a basis for further investigation of related topological properties.

Acknowledgments

The researcher extends sincere thanks, gratitude, and appreciation to all researchers without exception who have contributed to the development of general topology and to the enrichment of scientific knowledge and research in the field of general and generalized topological spaces. The researcher also expresses deep gratitude

and appreciation to everyone who offered advice and guidance, or provided valuable information and relevant references that contributed to the completion of this research in its final form.

Funding Disclosure

The author acknowledges and declares that this study was entirely self-funded and that no financial support or grants were received from any public, private, or external entity.

Future Recommendations:

1. Extending the N^*-open set framework to generalized multi-topological structures.
2. Investigating new topological invariants, such as N^*-compactness and N^*-separation axioms.
3. Exploring potential applications of N^*-connectedness in related mathematical disciplines.

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