

A New Generalization of Double XGamma Distribution: Statistical Properties and Applications to Real Data

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Abstract In this article, we introduce and study a new extension of the double XGamma distribution, referred to as the power double XGamma distribution (PDXRD), by employing a power transformation strategy. The proposed PDXRD provides increased flexibility over the baseline distribution and is capable of accommodating a broader range of real-world data. The PDXRD has several attractive structural properties, including a flexible decrease, a unimodal shape, close to symmetry, and a right-skewed probability density function (PDF), as well as a versatile hazard rate function (HRF) that can exhibit either increasing or decreasing behavior. Several important mathematical and statistical properties are computed, covering ordinary and incomplete moments, mean, variance, skewness, kurtosis, coefficient of variation, moment generating function, conditional moments, mean residual life, Lorenz curve, mean past lifetime, extropy measures, and order statistics. The unknown two parameters of the PDXRD are estimated utilizing the maximum likelihood method, and the corresponding point and interval estimates are assessed through an extensive simulation study. Finally, the practical performance and importance of the proposed PDXRD are illustrated using two real-world datasets related to engineering and survival times data. The numerical results demonstrate that the PDXRD provides a superior fit compared with several well-known competing statistical distributions, highlighting its potential as an effective and flexible model for analyzing complex data arising in diverse applied fields such as engineering and survival times.

Keywords Double XGamma Distribution Power Transformation, Extropy Measures; Maximum Likelihood Estimation; Monte Carlo Simulation.

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1. Introduction

Probability statistical distributions serve as important tools for examining complex datasets, as classical distribution families such as; the exponential, normal, and gamma distributions have been widely utilized in several statistical areas; however, they frequently lack the adaptability required to capture complex behaviors adequately and are limited in their capacity to accurately represent actual data characterized by heavy tails, skewness, or hazard rates, which are commonly found in real-world datasets in different sciences. This foundation encouraged authors to develop novel generalized distributions that enhance traditional statistical distributions by incorporating additional parameters or employing new generating mechanisms to encompass a wider variety of data characteristics; various

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flexible lifetime distributions have also been developed to provide superior representations of real-world lifetime data, such as [1] studied the Truncated Weibull Rayleigh distribution under a progressive Type-II censoring scheme and applied it to lifetime data, describing the importance of flexible distributional models in reliability and survival analysis.

The improvement of new mixture probability distributions has become an established area of study, motivated by the need for more flexible models capable of capturing complicated real-world datasets. A main contribution in this field is the Lindley distribution, first investigated by [2] as a single-parameter alternative to the exponential model. This distribution has received a lot of attention and is now widely used in reliability studies and survival analysis. Expanding on this structure, a number of authors have developed numerous mixture distributions aimed at enhancing the performance of the original Lindley distribution. For examples the Shanker distribution by [3], the Akash distribution by [4], the Garhy distribution [5], the Ishita distribution by [6], Haq distribution [7], and the Shukla distribution by [8]. Furthermore, accomplishments include Rama distribution [9], Zeghdoudi distribution (ZD) [10], xgamma distribution [11], the Komal distribution [12], Chris-Jerry distribution [13], Hamza distribution [14], Obulezi distribution [15], an updated Lindley distribution [16], xlindley distribution [17], and xrama distribution [18].

Recently, [19] proposed that the double XRama distribution (DXRD), the idea is to combine the simplicity of the exponential model with the flexibility of the Rama-based model to produce a distribution that is more capable of representing age-to-failure data and reliability patterns. The probability density function (PDF) and cumulative distribution function (CDF) of the DXRD are given by:

$$f(y; \theta) = \frac{\theta^4}{(\theta^3 + 6)^3} [\theta^6 + 18\theta^3 + 108 + 36y^3] e^{-\theta y}, \quad y > 0, \theta > 0, \quad (1)$$

and

$$F(y; \theta) = 1 - \left[1 + \frac{36\theta y}{(\theta^3 + 6)^3} (6 + 3\theta y + \theta^2 y^2) \right] e^{-\theta y}, \quad (2)$$

where θ is the rate parameter.

The power transformation strategy is One of the most effective ways to increase the flexibility of both classical and recently developed probability distributions. Since [20] introduced the power Lindley distribution (PLD), power-generated distributions have gained considerable attention in the statistical literature. The main principle includes integrating an additional shape parameter through the transformation $X = Y^\beta$, where $\beta > 0$ and Y follows a baseline distribution. This simple yet effective transformation results in a more adaptable class of distributions, capable of exhibiting a wide range of PDF and hazard rate shapes. Several authors have shown the efficacy of this strategy, such as the power quasi Lindley distribution [21], power Zeghdoudi distribution (PZD) [22], power Topp-Leone distribution [23], power inverted Topp-Leone distribution [24], power modified XLindley distribution [25], power Shanker distribution [26], power inverse Lindley distribution [27], power Darna distribution [28], power Hamza distribution [29], power Lomax distribution [30], power Burr XII distribution [31], power xgamma distribution [32], power weighted Akash distribution [33], power Rama distribution [34], power weighted Sujatha distribution [35], new power xgamma distribution [36], power Ishita distribution [37], power new XLindley distribution [38], power quasi Lindley distribution [45], and various other recently proposed lifetime models.

Motivated by recent advances in power-generated distributions and the desirable characteristics of the DXRD, this paper introduces a new two-parameter lifetime model called power DXRD (PDXRD). The main contributions of this study are listed below:

- The PDXRD exhibits flexible PDF shapes, including decreasing, unimodal, and right-skewed, heavy tailed shapes, while the hazard rate function (HRF) can be either decreasing or increasing.
- The PDXRD gives simple and explicit analytical expressions for several important reliability functions, such as; the PDF, CDF, survival function, HRF, reversed HRF, and cumulative HRF.

- Several essential statistical and mathematical properties are computed in closed form, such as the quantile function, mode, moments, mean, variance, skewness, kurtosis, coefficient of variation, moment generating function, incomplete moments, conditional moments, inequality measures, mean residual life, mean past lifetime, extropy measures, and order statistics.
- Point and interval estimations for the unknown parameters of the PDXRD are investigated using the maximum likelihood strategy.
- The performance of the suggested estimators is assessed via extensive Monte Carlo simulation study using different set of parameters and sample sizes.
- Finally, the practical relevance, applicability, and superiority of the PDXRD are demonstrated using two real-world datasets.

The rest of the current article is structured as follows. Section 2 proposes and discusses the formulation of the PDXRD. Section 3 calculates some important statistical features of the PDXRD. Section 4 discusses the point and interval estimations for the unknown parameters of the PDXRD using the maximum likelihood estimation strategy. Section 5 presents a comprehensive simulation study of the unknown parameters of the suggested model. Section 6 illustrates the importance and flexibility of the proposed PDXRD using two actual datasets and comparative studies with some well-known statistical distributions. Finally, some concluding remarks are mentioned in Section 7.

2. Formulation of the New Suggested Distribution

This section presents the extension of the DXRD by adding a power shape parameter β , which provides greater flexibility to the distribution in representing data with various patterns, especially regarding degrees of skewness and tail behavior, making it more suitable for modeling data with greater diversity in terms of spread. By using the power transformation $X = Y^{\frac{1}{\beta}}$ in Equations (3) and (4), we get the PDF and CDF of the PDXRD as follows

$$f(x; \theta, \beta) = \frac{\beta \theta^4 x^{\beta-1}}{(\theta^3 + 6)^3} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) e^{-\theta x^\beta}, \quad x > 0, \theta > 0, \beta > 0. \quad (3)$$

and,

$$F(x; \theta, \beta) = 1 - \left[1 + \frac{36\theta x^\beta}{(\theta^3 + 6)^3} (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right] e^{-\theta x^\beta}, \quad x > 0, \theta > 0, \beta > 0, \quad (4)$$

where θ is the rate parameter and β is the shape parameter, respectively. The survival function is a key concept in survival analysis and reliability theory, used to indicate the probability that a system, individual, or component remains operational beyond a given time. The formula for the survival function of PDXRD is given by

$$S(x; \theta, \beta) = \left[1 + \frac{36\theta x^\beta}{(\theta^3 + 6)^3} (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right] e^{-\theta x^\beta}. \quad (5)$$

Additionally, the HRF reflects the immediate failure rate at a certain time, representing the probability of an event (such as system failure or patient death) occurring in the next infinitesimal time interval, provided that the system has survived up to that moment. The HRF of the PDXRD is given by

$$h(x; \theta, \beta) = \frac{\beta \theta^4 (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) x^{\beta-1}}{(\theta^3 + 6)^3 + 36\theta x^\beta (\theta^2 x^{2\beta} + 3\theta x^\beta + 6)}.$$

The reversed HRF and cumulative HRF of PDXRD are given by

$$\bar{h}(x; \theta, \beta) = \frac{\beta \theta^4 x^{\beta-1} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta})}{e^{\theta x^\beta} (\theta^3 + 6)^3 - \left[(\theta^3 + 6)^3 + 36\theta x^\beta (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right]},$$

and

$$H(x; \theta, \beta) = \theta x^\beta - \ln \left[1 + \frac{36\theta x^\beta}{(\theta^3 + 6)^3} (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right].$$

Figures 1 and 2 show, respectively, the plots of the two key functions relevant to modeling: the PDF and the HRF. From Figure 1, the PDF for numerous parameter values demonstrates that the proposed PDXRD is highly flexible. It can assume various shapes, such as right-skewed and nearly symmetric forms, and it also exhibits both unimodal and decreasing patterns depending on the parameter values. The variation in the parameters substantially affects the peak, spread, and tail behavior of the distribution, illustrating its flexibility in accommodating different types of data. Also in Figure 2, the HRF under varying parameter values exhibits several patterns, including increasing, decreasing, and J-shapes. These diverse behaviors indicate that the model is highly responsive to changes in the parameter values and further demonstrate its adaptability for modeling different hazard rate characteristics.

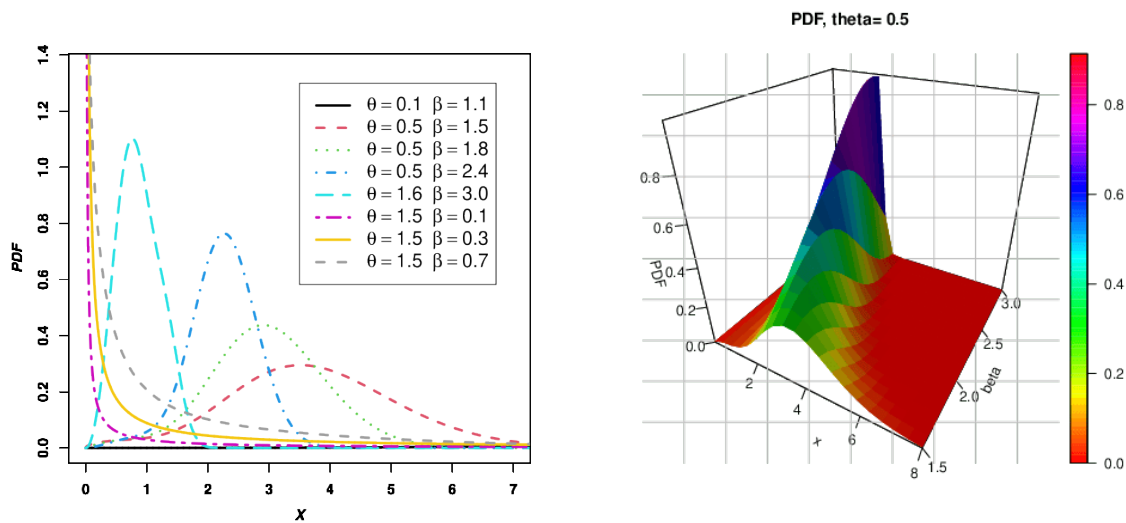


Figure 1. 2D and 3D Plots of PDF for the PDXRD.

3. Statistical Properties

This section presents essential statistical characteristics of the PDXRD, including ordinary moments (MO), descriptive statistical measures (skewness, kurtosis, coefficient of variation (CV)), moment generating function (MGF), incomplete moments (IMO), Lorenz curve (Lz), conditional moments, mean residual life (MRL), order statistics, and extropy.

3.1. Ordinary Moments

This subsection presents the r_{th} MO of the PDXRD. Ordinary moments play an essential role in describing the properties of a distribution and provide insight into the shape and behavior of the model. Let X be a random variable following the PDXRD. Then, the r_{th} ordinary moment about the origin, for any non-negative integer r , is obtained as:

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x; \theta, \beta) dx. \quad (6)$$

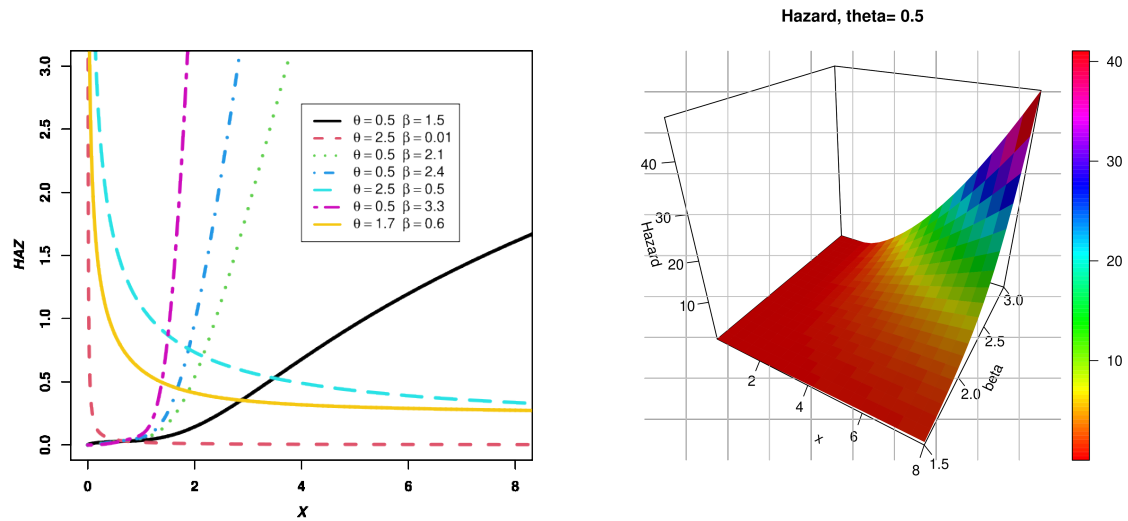


Figure 2. 2D and 3D Plots of HRF for the PDXRD.

By inserting the PDF (3) in (6), we get

$$\mu'_r = \frac{\beta\theta^4}{(\theta^3 + 6)^3} \int_0^\infty x^{r+\beta-1} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) e^{-\theta x^\beta} dx.$$

By rewriting the above equation, then

$$\mu'_r = \frac{\beta\theta^4}{(\theta^3 + 6)^3} \left[(\theta^6 + 18\theta^3 + 108) \int_0^\infty x^{r+\beta-1} e^{-\theta x^\beta} dx + 36 \int_0^\infty x^{r+4\beta-1} e^{-\theta x^\beta} dx \right].$$

Let $t = \theta x^\beta$, then

$$\mu'_r = \frac{\theta^4}{(\theta^3 + 6)^3} \left[(\theta^6 + 18\theta^3 + 108) \theta^{-\frac{r}{\beta}-1} \int_0^\infty t^{\frac{r}{\beta}} e^{-t} dt + 36 \theta^{-\frac{r}{\beta}-4} \int_0^\infty t^{\frac{r}{\beta}+3} e^{-t} dt \right].$$

By using the gamma function, we have

$$\mu'_r = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{r}{\beta} + 1\right) + 36\Gamma\left(\frac{r}{\beta} + 4\right)}{\theta^{\frac{r}{\beta}}(\theta^3 + 6)^3}. \quad (7)$$

Put $r = 1, 2, 3$ and 4 in Equation (7) to obtain the first four MOs about the origin as follows:

$$\begin{aligned} \mu'_1 &= \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{1}{\beta} + 1\right) + 36\Gamma\left(\frac{1}{\beta} + 4\right)}{\theta^{\frac{1}{\beta}}(\theta^3 + 6)^3}, \\ \mu'_2 &= \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{2}{\beta} + 1\right) + 36\Gamma\left(\frac{2}{\beta} + 4\right)}{\theta^{\frac{2}{\beta}}(\theta^3 + 6)^3}, \\ \mu'_3 &= \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{3}{\beta} + 1\right) + 36\Gamma\left(\frac{3}{\beta} + 4\right)}{\theta^{\frac{3}{\beta}}(\theta^3 + 6)^3}, \end{aligned}$$

and

$$\mu'_4 = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{4}{\beta} + 1\right) + 36\Gamma\left(\frac{4}{\beta} + 4\right)}{\theta^{\frac{4}{\beta}}(\theta^3 + 6)^3}.$$

The r_{th} MO equation provides a closed-form expression for all orders by using the gamma function, enabling precise characterization of PDXRD properties and simplifying the calculation of the essential metrics mentioned below. The mean and variance of the PDXRD, are given by

$$\mu_x = \mu'_1 = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{1}{\beta} + 1\right) + 36\Gamma\left(\frac{1}{\beta} + 4\right)}{\theta^{\frac{1}{\beta}}(\theta^3 + 6)^3},$$

and,

$$\sigma_x^2 = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{2}{\beta} + 1\right) + 36\Gamma\left(\frac{2}{\beta} + 4\right)}{\theta^{\frac{2}{\beta}}(\theta^3 + 6)^3} - \left[\frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{1}{\beta} + 1\right) + 36\Gamma\left(\frac{1}{\beta} + 4\right)}{\theta^{\frac{1}{\beta}}(\theta^3 + 6)^3} \right]^2.$$

Table 1. Numerical values of the moments, variance, skewness, kurtosis, coefficient of variation, and index of dispersion for different values of θ and β .

θ	β	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	Skewness	Kurtosis	CV	ID
10*1.5	0.6	2.090841	17.0973781	243.6412	5004.391	12.72576213	3.40724573	20.73455	1.7061645	6.08643237
	0.8	1.4303489	5.4542168	31.88043	243.6412	3.40831879	2.27719514	9.954361	1.2907083	2.3828583
	1.1	1.1255387	2.4460181	7.334487	26.817	1.17918086	1.50488978	5.446816	0.9647833	1.04765913
	1.5	0.9988373	1.5327151	2.986041	6.769118	0.53503918	0.98699916	3.589966	0.7323152	0.535662
	1.7	0.971638	1.3500202	2.311898	4.534687	0.40593977	0.81702491	3.171844	0.6557319	0.4177891
	2	0.9488554	1.1909547	1.778739	2.986041	0.29062804	0.6201754	2.802361	0.5681575	0.30629328
	2.3	0.9373992	1.1002869	1.495674	2.251248	0.22156967	0.46854447	2.600968	0.5021467	0.2363664
	2.5	0.9332077	1.0603063	1.374759	1.957094	0.18942968	0.38471229	2.520658	0.4663862	0.2029877
	3	0.9290432	0.9988373	1.190955	1.532715	0.13571608	0.2164017	2.426489	0.3965334	0.14608157
	3.3	0.9288965	0.9772455	1.125539	1.387853	0.11439677	0.13576951	2.4134	0.3641158	0.1231534
10*2	0.6	0.7203088	2.6107449	19.98542	238.6411	2.09190018	4.98785528	43.047585	2.0079449	2.90417151
	0.8	0.6331009	1.2089169	4.117466	19.98542	0.80810014	3.20589731	18.351036	1.4199062	1.27641604
	1.1	0.6197413	0.7731434	1.459979	3.621687	0.38906412	2.05453354	8.862932	1.0064685	0.62778477
	1.5	0.6455146	0.6442422	0.8542274	1.385965	0.22755321	1.33195427	5.216309	0.7389847	0.35251445
	1.7	0.6616131	0.6257971	0.7530657	1.080428	0.18806524	1.10569296	4.41737	0.655466	0.28425261
	2	0.6852349	0.6180758	0.6757859	0.8542274	0.14852895	0.85084701	3.708359	0.5624269	0.21675626
	2.3	0.7069874	0.6217396	0.6400214	0.7440252	0.12190837	0.65979183	3.308682	0.4938614	0.17243357
	2.5	0.7202699	0.6272423	0.6278764	0.7008722	0.10845358	0.55607534	3.139093	0.4572215	0.15057353
	3	0.7493965	0.6455146	0.6180758	0.6442422	0.08391938	0.35176805	2.903317	0.3865623	0.11198261
	3.3	0.7643913	0.6575728	0.6197413	0.6290424	0.07327864	0.25565119	2.838013	0.3541383	0.09586534
10*2.5	0.6	0.3728167	0.6548978	2.672342	18.83062	0.51590554	5.51469039	57.610958	1.926592	1.38380488
	0.8	0.3925939	0.4328223	0.8729168	2.672342	0.27869232	3.29085076	20.993226	1.34468	0.70987427
	1.1	0.4429529	0.3742806	0.4733739	0.803283	0.17807326	1.99387617	9.135296	0.9526685	0.40201397
	1.5	0.5080489	0.3855154	0.3815146	0.461357	0.12740169	1.23590163	5.127012	0.7025578	0.25076659
	1.7	0.5366897	0.4003539	0.3732825	0.412688	0.11231805	1.00570302	4.307785	0.6244555	0.2092793
	2	0.5747157	0.4256311	0.3762554	0.3815146	0.09533297	0.74944319	3.606284	0.5372402	0.1658785
	2.3	0.6075073	0.4515496	0.3877803	0.3729894	0.08248445	0.5589275	3.22649	0.4727532	0.13577523
	2.5	0.6268773	0.468434	0.3977082	0.3735353	0.07545892	0.45592594	3.071185	0.4382004	0.12037272
	3	0.6681893	0.5080489	0.4256311	0.3855154	0.06157191	0.25361796	2.868435	0.3713571	0.09214739
	3.3	0.6889989	0.5297859	0.4429529	0.3964059	0.05506641	0.15861579	2.82026	0.3405845	0.07992234

The descriptive statistical measures are utilized to effectively convey the primary traits of a dataset and aid in comprehending the fundamental characteristics of the data. Furthermore, Skewness = $\frac{\mu'_3}{\sigma^3}$, Kurtosis = $\frac{\mu'_4}{\sigma^4}$, coefficient of variation (CV) = $\frac{\sigma}{\mu}$, and index of dispersion (ID) = $\frac{\mu}{\sigma}$ serve to characterize the form of the distribution. Figures 3, 4, and 5 show 3D plots of mean, variance, skewness, kurtosis, CV and ID for the PDXRD.

Table 1 shows some numerical values of the first four moments, variance, skewness, kurtosis, CV, and ID for the PDXRD at selected values of θ and β . The numerical results show that increasing β generally reduces the variance, skewness, kurtosis, CV, and ID, while its effect on the mean depends on the value of θ . These numerical results confirm the substantial influence of the shape parameter β on the dispersion and shape features of the PDXRD.

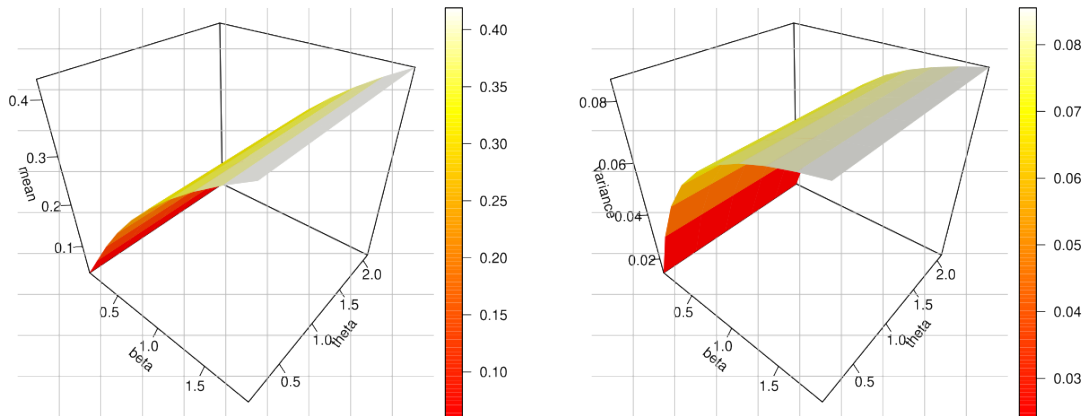


Figure 3. 3D Plots of mean and var for the PDXRD.

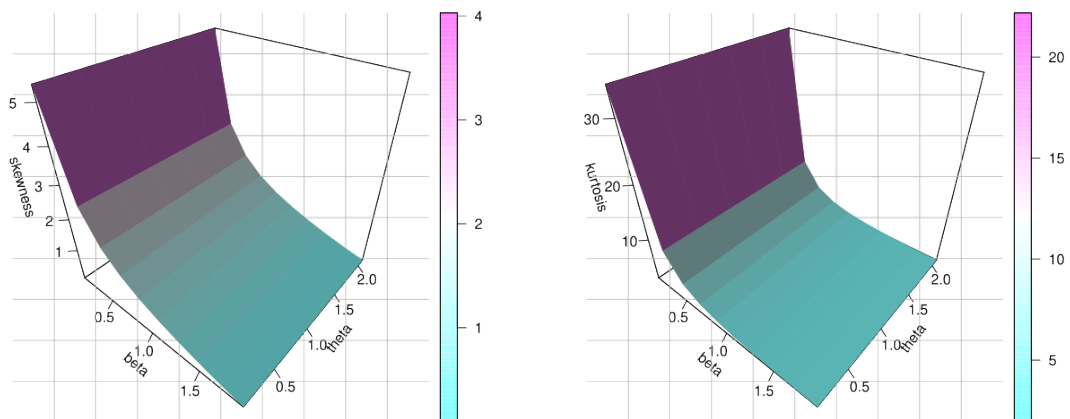


Figure 4. 3D Plots of skewness and kurtosis for the PDXRD.

The MGF is a crucial instrument for assessing the totals of random variables. It offers an alternative analytical method for obtaining statistical characteristics and distributional outcomes without directly relying on the probability density function or cumulative distribution function of a random variable. The MGF of the PDXRD is derived as:

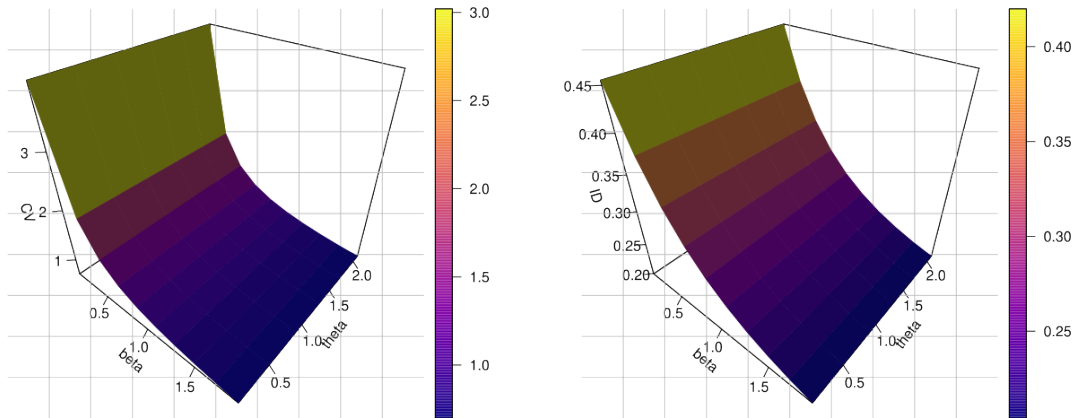


Figure 5. 3D Plots of CV and ID for the PDXRD.

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{r}{\beta} + 1\right) + 36\Gamma\left(\frac{r}{\beta} + 4\right)}{\theta^{\frac{r}{\beta}}(\theta^3 + 6)^3}, \quad r = 1, 2, 3, \dots \quad (8)$$

3.2. Incomplete Moments

The IMO serves as a valuable statistical measure that offers insightful details regarding the distributional characteristics of a random variable. They are extensively utilized in analyzing income inequality via the Lorenz curve and are additionally employed in actuarial science, economics, and medical statistics. Furthermore, incomplete moments are significant in determining mean deviations regarding the mean and median, which are key indicators of dispersion. Consequently, obtaining the incomplete moments of the suggested distribution is beneficial for understanding its mathematical properties and potential applications. The lower IMOs of a random variable that follows the PDXRD are specified by:

$$\begin{aligned} \mu_s(y) &= E[X^s; X \leq y] = \int_0^y x^s f(x; \theta, \beta) dx \\ &= \frac{\beta\theta^4}{(\theta^3 + 6)^3} \int_0^y x^{\beta+s-1} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) e^{-\theta x^\beta} dx. \end{aligned}$$

After some calculations, then

$$\mu_s(y) = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\gamma\left(\frac{s}{\beta} + 1, \theta y^\beta\right) + 36\gamma\left(\frac{s}{\beta} + 4, \theta y^\beta\right)}{\theta^{\frac{s}{\beta}}(\theta^3 + 6)^3}. \quad (9)$$

The first lower IMO can be computed by setting $s = 1$; in Equation (9) as follows

$$\mu_1(y) = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\gamma\left(\frac{1}{\beta} + 1, \theta y^\beta\right) + 36\gamma\left(\frac{1}{\beta} + 4, \theta y^\beta\right)}{\theta^{\frac{1}{\beta}}(\theta^3 + 6)^3}. \quad (10)$$

The upper IMO for the PDXRD is defined as:

$$\begin{aligned}\bar{\mu}_s(y) &= E[X^s; X > y] = \int_y^\infty x^s f(x; \theta, \beta) dx \\ &= \frac{\beta\theta^4}{(\theta^3 + 6)^3} \int_y^\infty x^{\beta+s-1} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) e^{-\theta x^\beta} dx.\end{aligned}$$

After some calculations, then

$$\bar{\mu}_s(y) = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{s}{\beta} + 1, \theta y^\beta\right) + 36\Gamma\left(\frac{s}{\beta} + 4, \theta y^\beta\right)}{\theta^{\frac{s}{\beta}} (\theta^3 + 6)^3}. \quad (11)$$

The first lower IMO can be computed by setting $s = 1$; in Equation (11) as follows

$$\bar{\mu}_1(y) = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{1}{\beta} + 1, \theta y^\beta\right) + 36\Gamma\left(\frac{1}{\beta} + 4, \theta y^\beta\right)}{\theta^{\frac{1}{\beta}} (\theta^3 + 6)^3}. \quad (12)$$

Equation (12) above will be employed to calculate the mean residual lifetime.

3.3. Lorenz Curve

In economics, the Lorenz curve is a visual representation of the level of inequality in the distribution of economic factors, especially income and wealth. To compute the Lz curve, we start with the lower IMO from Equation (9) and apply it to the next formula:

$$Lz(p) = \frac{\mu_1(y)}{\mu_y},$$

thus;

$$Lz(p) = \frac{\left[\gamma\left(\frac{1}{\beta} + 1, \theta y^\beta\right) (\theta^9 + 18\theta^6 + 108\theta^3) + 36\gamma\left(\frac{1}{\beta} + 4, \theta y^\beta\right)\right]}{\Gamma\left(\frac{1}{\beta} + 1\right) \left[\theta^9 + 18\theta^6 + 108\theta^3 + \left(216 + \frac{396}{\beta} + \frac{216}{\beta^2} + \frac{36}{\beta^3}\right)\right]}. \quad (13)$$

The ultimate form of the Lz curve, $Lz(p)$, offers a standardized assessment of cumulative distribution by connecting the lower incomplete gamma function with the complete gamma function, illustrating the extent of inequality determined by the model parameters.

3.4. Conditional moments

Conditional moments denote the expected values of a random variable based on specific conditions or occurring events. They find extensive applications in sectors such as machine learning, economics, and finance, as they offer a deeper understanding of a distribution's behavior beyond its unconditional properties. For a random variable X that adheres to the PDXR, the conditional moments of order s can be represented as follows

$$\eta_s(t) = E(X^s | X > t) = \frac{\bar{\mu}_s(t)}{S(t; \theta, \beta)} = \frac{1}{S(t; \theta, \beta)} \int_t^\infty x^s f(x; \theta, \beta) dx.$$

By using the Equations (11) and (5), we get

$$\eta_s(t) = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{s}{\beta} + 1, \theta t^\beta\right) + 36\Gamma\left(\frac{s}{\beta} + 4, \theta t^\beta\right)}{\theta^{\frac{s}{\beta}} \left[(\theta^3 + 6)^3 + 36\theta t^\beta (\theta^2 t^{2\beta} + 3\theta t^\beta + 6)\right] e^{-\theta t^\beta}}. \quad (14)$$

3.5. Mean Residual Life

This subsection presents the MRL, also known as expected remaining lifetime, which indicates the average life expectancy of a person or system that has already endured up to time t . The MRL for the PDXRD is given by:

$$\text{MRL}(t) = E(X - t | X > t) = \frac{\bar{\mu}_1(t)}{S(x; \theta, \beta)} - t. \quad (15)$$

Applying Equations (5) and (12) then, we get

$$\text{MRL}(t) = \frac{\theta^3(\theta^6 + 18\theta^3 + 108)\Gamma\left(\frac{1}{\beta} + 1, \theta t^\beta\right) + 36\Gamma\left(\frac{1}{\beta} + 4, \theta t^\beta\right)}{\theta^{\frac{1}{\beta}} \left[(\theta^3 + 6)^3 + 36\theta t^\beta (\theta^2 t^{2\beta} + 3\theta t^\beta + 6)\right] e^{-\theta t^\beta}} - t. \quad (16)$$

The MINT (mean past lifetime) gauges the average time an item (such as a component, system, or process) remains non-operational after a failure, specifically from the failure event to the subsequent event (e.g., repair, replacement, or maintenance). For , the MINT of is represented as:

$$\text{MINT}(t) = E(t - X | X \leq t) = t - E(X | X \leq t) = t - \frac{\mu_1(t)}{F(t; \theta, \beta)}. \quad (17)$$

Using Equations (4) and (10) yields:

$$\text{MINT}(t) = t - \frac{\theta^{-\frac{1}{\beta}} \left[\gamma\left(\frac{1}{\beta} + 1, \theta t^\beta\right) (\theta^9 + 18\theta^6 + 108\theta^3) + 36\gamma\left(\frac{1}{\beta} + 4, \theta t^\beta\right) \right]}{(\theta^3 + 6)^3 - [(\theta^3 + 6)^3 + 36\theta t^\beta (\theta^2 t^{2\beta} + 3\theta t^\beta + 6)] e^{-\theta t^\beta}}. \quad (18)$$

3.6. Order Statistics

Order statistics analyze the traits and behaviors of arranged observations, depicting the values of a data sample sorted in ascending or descending sequence. These statistics are crucial in areas like biology, economics, and engineering for examining data models and variability. The PDF of the r_{th} order statistics for the PDXRD is provided via

$$f_{r:n}(x; \theta, \beta) = C_r^n [f(x; \theta, \beta)] [F(x; \theta, \beta)]^{r-1} [1 - F(x; \theta, \beta)]^{n-r}, \quad (19)$$

where $C_r^n = \frac{n!}{(r-1)!(n-r)!}$. Inserting Equations (3) and (4) into (19), we then have:

$$\begin{aligned} f_{r:n}(x) &= C_r^n \left(\frac{\beta\theta^4}{(\theta^3 + 6)^3} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) x^{\beta-1} e^{-\theta x^\beta} \right) \\ &\times \left(1 - \left\{ 1 + \frac{36x^\beta}{(\theta^3 + 6)^3} (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right\} e^{-\theta x^\beta} \right)^{r-1} \\ &\times \left(\left\{ 1 + \frac{36x^\beta}{(\theta^3 + 6)^3} (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right\} e^{-\theta x^\beta} \right)^{n-r}. \end{aligned} \quad (20)$$

The PDF of the minimum order statistic is obtained by setting $r = 1$ as follows

$$\begin{aligned} f_{1:n}(x) &= \frac{n\beta\theta^4}{(\theta^3 + 6)^3} (\theta^6 + 18\theta^3 + 108 + 36x^{3\beta}) x^{\beta-1} e^{-\theta x^\beta} \\ &\times \left(\left\{ 1 + \frac{36x^\beta}{(\theta^3 + 6)^3} (\theta^2 x^{2\beta} + 3\theta x^\beta + 6) \right\} e^{-\theta x^\beta} \right)^{n-1}. \end{aligned} \quad (21)$$

The PDF of the maximum order statistic is obtained by setting $r = n$ as follows

$$f_{n:n}(x) = \frac{n\beta\theta^4}{(\theta^3+6)^3}(\theta^6+18\theta^3+108+36x^{3\beta})x^{\beta-1}e^{-\theta x^\beta} \times \left(1 - \left\{1 + \frac{36x^\beta}{(\theta^3+6)^3}(\theta^2x^{2\beta}+3\theta x^\beta+6)\right\}e^{-\theta x^\beta}\right)^{n-1}. \quad (22)$$

3.7. Extropy

Extropy illustrates the uncertainty resulting from outside influences, whereas entropy measures the internal information content and the intrinsic uncertainty within a system. To supplement the traditional notion of entropy, extropy was introduced as an alternative measure of uncertainty. This dual concept, which has applications in fields like risk assessment, heavy-tailed distributions, and information volatility, highlights how external influences shape uncertainty. Consequently, the extropy of PDXRD is defined as follows:

$$J(x) = -\frac{1}{2} \left[\int_0^\infty f^2(x; \theta, \beta) dx \right] = -\frac{1}{2} \left[\int_0^\infty \left(\frac{\beta\theta^4}{(\theta^3+6)^3} (A + 36x^{3\beta}) x^{\beta-1} e^{-\theta x^\beta} \right)^2 dx \right],$$

where $A = (\theta^6 + 18\theta^3 + 108)$. Then,

$$J(x) = -\frac{1}{2} \int_0^\infty \left[\frac{\beta^2\theta^8}{(\theta^3+6)^6} (A^2 + 72Ax^{3\beta} + 1296x^{6\beta}) x^{2\beta-2} e^{-2\theta x^\beta} dx \right].$$

By applying the substitution $u = 2\theta x^\beta$, and after simplifying the resulting expression, we obtain:

$$J(x) = \frac{-\beta\theta^8}{2(\theta^3+6)^6} \left[\frac{A^2}{(2\theta)^{2-\frac{1}{\beta}}} \Gamma\left(2 - \frac{1}{\beta}\right) + \frac{72A}{(2\theta)^{5-\frac{1}{\beta}}} \Gamma\left(5 - \frac{1}{\beta}\right) + \frac{1296}{(2\theta)^{8-\frac{1}{\beta}}} \Gamma\left(8 - \frac{1}{\beta}\right) \right], 2\beta > 1. \quad (23)$$

Weighted extropy is a supplemental measure of information to the traditional concept of extropy. It was created as an analog of weighted extropy. It is described as follows:

$$\begin{aligned} J^w(x) &= -\frac{1}{2} E[Xf(x; \theta, \beta)] \\ &= -\frac{1}{2} \left[\int_0^\infty x f^2(x; \theta, \beta) dx \right] = -\frac{1}{2} \left[\int_0^\infty x \left(\frac{\beta\theta^4 x^{\beta-1}}{(\theta^3+6)^3} (A + 36x^{3\beta}) e^{-\theta x^\beta} \right)^2 dx \right] \\ &= -\frac{1}{2} \int_0^\infty \left[\frac{\beta^2\theta^8}{(\theta^3+6)^6} (A^2 + 72Ax^{3\beta} + 1296x^{6\beta}) x^{2\beta-1} e^{-2\theta x^\beta} dx \right]. \end{aligned}$$

Then

$$J^w(x) = \frac{-\beta\theta^8}{2(\theta^3+6)^6} \left[\frac{A^2}{4\theta^2} + \frac{54A}{\theta^5} + \frac{25515}{\theta^8} \right] = \frac{-\beta}{8(\theta^3+6)^6} \left[A^2\theta^6 + 216A\theta^3 + 102060 \right]. \quad (24)$$

Table 2 presents the numerical values of $J(X)$ and $J^w(X)$ for different values of θ and β in PDXRD, both measures stays negative for all considered parameter sets, such as the increasing in θ leads to more negative values of both $J(X)$ and $J^w(X)$, additionally, for each fixed value of θ , both measures tend to decrease as β increases and $J^w(X)$ showing a clear and consistent downward trend across the considered values of β . Overall, the results indicate that both parameters impact the magnitude of the extropy measures and the weighted extropy exhibiting a more systematic variation with respect to β .

Table 2. Numerical values of extropy and weighted extropy measures.

$\theta = 1.5$										
β	0.6	0.8	1.1	1.5	1.7	2	2.3	2.5	3	3.3
$J(x)$	-0.69607137	-0.28281645	-0.22387131	-0.23448853	-0.24849913	-0.27368412	-0.30160171	-0.32111463	-0.37177778	-0.40300250
$J^w(x)$	-0.06661349	-0.08881799	-0.12212473	-0.16653372	-0.18873822	-0.22204496	-0.25535171	-0.27755620	-0.33306745	-0.36637419
$\theta = 2$										
β	0.6	0.8	1.1	1.5	1.7	2	2.3	2.5	3	3.3
$J(x)$	-1.72565206	-0.59661167	-0.40403654	-0.37043137	-0.37845183	-0.39618795	-0.42006553	-0.43796017	-0.48672366	-0.51771553
$J^w(x)$	-0.07011281	-0.09348374	-0.12854015	-0.17528202	-0.18965295	-0.23370935	-0.26876576	-0.29213669	-0.35056403	-0.38562044
$\theta = 2.5$										
β	0.6	0.8	1.1	1.5	1.7	2	2.3	2.5	3	3.3
$J(x)$	-2.81602939	-0.88043879	-0.54685415	-0.47345740	-0.47017118	-0.48034893	-0.50007916	-0.51628191	-0.56295818	-0.59360502
$J^w(x)$	-0.07347291	-0.09796388	-0.13470033	-0.18368227	-0.20817323	-0.24490969	-0.28164614	-0.30613711	-0.36736453	-0.40410099

4. Parameter Estimation

This section covers parameter estimation for the PDXRD using point and interval estimation [40, 41, 42, 43, 44].

4.1. Point Estimation

Maximum likelihood estimation (MLE) is an approach utilized to estimate the parameters of a statistical model based on observed data. Its aim is to find parameter values that optimize the likelihood function, which assesses how accurately the model aligns with the observed data. Under certain circumstances, MLE yields estimates that are consistent, efficient, and asymptotically normal. Assume that $x_1, x_2, \dots, x_n$ be a random sample of size n drawn from PDXRD with parameter vector $\underline{\Theta} = (\theta, \beta)^T$. The log-likelihood function (Ω) for the PDXRD is provided via:

$$\Omega = n \ln \beta + 4n \ln \theta - 3n \ln(\theta^3 + 6) + \sum_{i=1}^n \ln D_i + (\beta - 1) \sum_{i=1}^n \ln x_i - \theta \sum_{i=1}^n x_i^\beta, \quad (25)$$

where $D_i = \theta^6 + 18\theta^3 + 108 + 36x_i^{3\beta}$. Then, to optimize Ψ , The first partial derivative with respect to the parameters θ and β are provided via:

$$\frac{\partial \Omega}{\partial \theta} = \frac{4n}{\theta} - \frac{9n\theta^2}{(\theta^3 + 6)} + \sum_{i=1}^n \frac{6\theta^5 + 54\theta^2}{D_i} - \sum_{i=1}^n x_i^\beta, \quad (26)$$

and

$$\frac{\partial \Omega}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \frac{108x_i^{3\beta} \ln(x_i)}{D_i} + \sum_{i=1}^n \ln(x_i) - \theta \sum_{i=1}^n x_i^\beta \ln(x_i). \quad (27)$$

By setting Equations (26) and (27) to equal 0, we can estimate the two unknown θ and β parameters of the proposed PDXRD. Since the solution is not feasible analytically, R statistical software can be used.

4.2. Interval estimation

The interval estimation provides a wider perspective by showing a range within which the actual parameter value lies, accompanied by a defined confidence level. This provides greater information than just a single point estimate, as it captures the inherent uncertainty in the estimation process. In this context, we are discussing asymptotic confidence intervals (ACIs). The ACIs for the parameters will be developed using the asymptotic distribution of the MLE for the vector $\underline{\Theta} = (\theta, \beta)^T$, denoted as $(\Theta - \hat{\Theta}) \sim N(0, I^{-1}(\Theta))$ where, $\hat{\Theta}$ is the MLE of Θ , and $I^{-1}(\theta)$ is the inverse of the observed information matrix, which is given by:

$$I^{-1}(\Theta) = - \begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} \\ I_{\beta\theta} & I_{\beta\beta} \end{bmatrix}_{\hat{\Theta}=(\hat{\theta}, \hat{\beta})}^{-1} = \begin{bmatrix} \hat{\sigma}_{\theta\theta}^2 & \hat{\sigma}_{\theta\beta}^2 \\ \hat{\sigma}_{\beta\theta}^2 & \hat{\sigma}_{\beta\beta}^2 \end{bmatrix}.$$

The elements of the matrix $I^{-1}(\Theta)$ are provided via

$$I_{\theta\theta} = -E \left(\frac{\partial^2 \psi}{\partial \theta^2} \right) = -E \left(\frac{-4n}{\theta^2} - \frac{9n(12\theta - \theta^4)}{(\theta^3 + 6)^2} + \sum_{i=1}^n \frac{(30\theta^4 + 108\theta)D_i - (6\theta^5 + 54\theta^2)^2}{D_i^2} \right).$$

$$I_{\beta\beta} = -E \left(\frac{\partial^2 \psi}{\partial \beta^2} \right) = -E \left(\frac{-n}{\beta^2} + \sum_{i=1}^n \frac{(324x_i^{3\beta}(\ln x_i)^2) D_i - 11664x_i^{6\beta}(\ln x_i)^2}{D_i^2} - \theta \sum_{i=1}^n x_i^\beta (\ln x_i)^2 \right),$$

and,

$$I_{\theta\beta} = I_{\beta\theta} = -E \left(\frac{\partial^2 \psi}{\partial \beta \partial \theta} \right) = -E \left(\frac{\partial^2 \psi}{\partial \theta \partial \beta} \right) = E \left(\sum_{i=1}^n \frac{108x_i^{3\beta} \ln(x_i) (6\theta^5 + 54\theta^2)}{D_i^2} + \sum_{i=1}^n x_i^\beta \ln(x_i) \right).$$

The two-sided approximation $(1 - \nu)100\%$ ACIs for the parameters θ and β , respectively, are provided via: $\hat{\theta} \pm Z_{\nu/2} \hat{\sigma}_{\theta\theta}$ and $\hat{\beta} \pm Z_{\nu/2} \hat{\sigma}_{\beta\beta}$, where $Z_{\nu/2}$ is the upper $(\nu/2)$ percentile of the standard normal distribution.

5. Simulation Study

In this section, the Monte Carlo simulation study is a useful computational strategy utilized to analyze the behavior of statistical estimators by repeatedly generating random samples from specified probability distributions. It offers a useful framework for assessing estimation precision, bias, and variability, particularly in situations where precise mathematical analysis is challenging. The simulation procedure is carried out through a series of computational steps outlined as follows:

Step 1: Select $M = 1000$ random samples of sizes 30, 50, 100, 150, 200, 250, 300, 400, 500 from the PDXRD.

Step 2: Select various values for the model parameters (θ, β) are mentioned in Table 3 as below:

Table 3. Parameter settings considered in the simulation study.

Case	I	II	III	IV	V	VI	VII	VIII	IX	X
θ	0.25	0.50	0.75	1.50	1.00	2.00	2.50	3.00	3.75	4.00
β	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	7.50

Step 3: Compute the average of estimates derived from $M = 1000$ random samples utilizing the following formula:

$$Average(\hat{\theta}) = \frac{1}{M} \sum_{i=1}^M \hat{\theta}_i.$$

Step 4: Compute the root mean square error (RMSE) and the average bias (Bias) using the following formulas:

$$RMSE_{\theta}(n) = \sqrt{\frac{1}{M} \sum_{i=1}^M (\hat{\theta}_i - \Theta)^2}, \quad Bias_{\Theta}(n) = \frac{1}{M} \sum_{i=1}^M (\hat{\theta}_i - \Theta).$$

Step 5: Calculate the average length of confidence intervals (ACIs) at the 95% and 99% confidence levels using the following equations:

$$LCI = \hat{\Theta} - Z_{\nu/2} \sqrt{\text{Var}(\hat{\Theta})}, \quad UCI = \hat{\Theta} + Z_{\nu/2} \sqrt{\text{Var}(\hat{\Theta})}, \quad ACI = UCI - LCI.$$

Step 6: Calculate the coverage probability (CP), defined as the proportion of simulated confidence intervals that encompass the true parameter vector, $\underline{\Theta} = (\theta, \beta)^T$.

Step 7: The results of the simulation study were computed utilizing the R package.

From Tables 4 - 13 the findings lead to the following conclusions: With a larger sample size, both the RMSE and Bias diminish for every set of true parameters.

- As the sample size n increases, both the Root Mean Square Error (RMSE) and bias decrease for all sets of true parameter values, confirming the consistency of the estimators.
- The estimator of θ tends to underestimate the true value (negative bias), whereas the estimator of β exhibits a positive bias across the considered sample sizes.
- Increasing the sample size leads to narrower 95% and 99% confidence intervals, resulting in more precise parameter estimates and shorter Average Confidence Interval (ACI) lengths.
- The Coverage Probabilities (CPs) closely approach the nominal confidence levels of 95% and 99%, indicating enhanced estimation accuracy and reliability.

Furthermore, the simulated density functions of the PDXRD were illustrated for the selected cases. The reliability of the estimators was evaluated using RMSE and bias. Overall, the simulation results demonstrate that the maximum likelihood estimation (MLE) method provides accurate and reliable parameter estimation for the PDXRD.

Table 4. MLE simulation numerical results for Case (I).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	0.175361	0.298556	0.074639	0.009182	0.0958213	0.23566832	96.0	0.30973551	99.4
50	0.176889	0.292446	0.073111	0.007576	0.08704081	0.18524024	95.6	0.24345860	99.2
100	0.177389	0.290446	0.072611	0.006484	0.08052603	0.13653954	94.9	0.17945197	99.0
150	0.179374	0.282503	0.070626	0.005770	0.07595761	0.10964447	95.2	0.14410416	99.1
200	0.180950	0.276201	0.069050	0.005371	0.07328893	0.09633395	94.9	0.12661033	98.8
250	0.180151	0.279396	0.069849	0.005372	0.07329646	0.08712588	94.9	0.11450830	98.7
300	0.180051	0.279797	0.069949	0.005304	0.07282993	0.07954033	95.2	0.10453872	99.4
400	0.180628	0.277487	0.069372	0.005099	0.07140484	0.06635046	94.9	0.08720346	98.9
500	0.181251	0.274997	0.068749	0.004987	0.07062095	0.06334597	95.1	0.08325470	99.2

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	1.155302	0.155302	0.155302	0.041408	0.2034905	0.5157001	95.5	0.6777773	98.3
50	1.141970	0.141970	0.141970	0.029245	0.1710115	0.3739153	95.6	0.4914316	98.3
100	1.133998	0.133998	0.133998	0.022525	0.1500839	0.2651230	95.7	0.3484474	98.4
150	1.128085	0.128085	0.128085	0.019128	0.1383058	0.2046495	94.9	0.2689679	99.1
200	1.123301	0.123301	0.123301	0.017292	0.1314987	0.1792436	95.6	0.2355773	99.2
250	1.124456	0.124457	0.124457	0.017200	0.1311502	0.1622275	95.2	0.2132132	98.9
300	1.124005	0.124005	0.124005	0.016782	0.1295444	0.1469866	94.9	0.1931824	98.9
400	1.122183	0.122183	0.122183	0.015900	0.1260959	0.1222427	94.2	0.1606619	98.6
500	1.121273	0.121273	0.121273	0.015603	0.1249125	0.1173921	94.3	0.1542868	99.1

Table 5. MLE simulation numerical results for Case (II).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	0.413809	0.172382	0.086191	0.015825	0.12579796	0.35937332	95.3	0.4723192	98.8
50	0.419406	0.161187	0.080594	0.011431	0.10691668	0.27553938	95.0	0.3621375	99.3
100	0.421558	0.156884	0.078442	0.008754	0.09356439	0.20002718	94.8	0.2628929	98.9
150	0.424840	0.150320	0.075160	0.007306	0.08547398	0.15963838	95.2	0.2098104	98.8
200	0.427327	0.145346	0.072673	0.006521	0.08075518	0.13810680	95.1	0.1815118	99.0
250	0.425933	0.148135	0.074067	0.006505	0.08065162	0.12517790	95.0	0.1645195	99.1
300	0.425777	0.148446	0.074223	0.006364	0.07977360	0.11466414	94.9	0.1507014	99.2
400	0.426999	0.146002	0.073001	0.005936	0.07704476	0.09660437	94.8	0.1269657	99.0
500	0.427955	0.144091	0.072045	0.005710	0.07556292	0.08936767	95.2	0.1174547	99.2

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	1.675948	0.11729859	0.17594790	0.05812978	0.24110120	0.64649490	95.6	0.8496790	98.4
50	1.60513525	0.15770290	0.10513525	0.03867189	0.19665170	0.46075450	95.1	0.6055630	98.6
100	1.647974	0.09864954	0.14797430	0.02878250	0.16965410	0.32545430	95.8	0.4277400	98.4
150	1.641604	0.09440243	0.14160370	0.02405773	0.15510550	0.24823680	94.6	0.3262541	99.3
200	1.635133	0.09008891	0.13513340	0.02136834	0.14617910	0.21862300	95.6	0.2873331	99.2
250	1.637511	0.09167367	0.13751050	0.02142039	0.14635700	0.19653860	95.5	0.2583079	99.0
300	1.637080	0.09138684	0.13708030	0.02087314	0.14447540	0.17896090	95.1	0.2352057	98.8
400	1.634459	0.08963928	0.13445890	0.01950679	0.13966670	0.14818540	94.2	0.1947579	98.6
500	1.633569	0.08904633	0.13356950	0.01913406	0.13832590	0.14104050	94.8	0.1853675	98.9

Table 6. MLE simulation numerical results for Case (III).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	0.690143	0.079810	0.059857	0.012565	0.11209343	0.37169824	95.1	0.4885177	98.6
50	0.694994	0.073341	0.055006	0.008314	0.09117871	0.28519779	94.9	0.3748314	98.8
100	0.696090	0.071881	0.053910	0.005651	0.07517545	0.20548288	95.4	0.2700632	98.7
150	0.699876	0.066833	0.050124	0.004369	0.06609538	0.16896952	95.2	0.2220742	98.6
200	0.701054	0.065261	0.048946	0.003713	0.06093043	0.14232181	94.8	0.1870515	98.7
250	0.700451	0.066066	0.049549	0.003557	0.05964441	0.13021445	95.0	0.1711390	99.0
300	0.700192	0.066411	0.049808	0.003411	0.05840530	0.11962557	94.5	0.1572222	99.0
400	0.701338	0.064883	0.048662	0.003038	0.05512121	0.10154093	94.9	0.1334538	98.7
500	0.702246	0.063673	0.047754	0.002844	0.05332551	0.09306883	95.4	0.1223190	99.1

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	2.171174	0.085587	0.171174	0.059421	0.24376346	0.68066075	95.4	0.8945827	98.6
50	2.156030	0.078015	0.156030	0.040597	0.20148726	0.49998176	95.3	0.6571189	98.9
100	2.148641	0.074320	0.148641	0.030342	0.17419074	0.35619381	95.4	0.4681404	98.7
150	2.141854	0.070927	0.141854	0.025118	0.15848624	0.27719332	94.1	0.3643112	99.2
200	2.136116	0.068058	0.136116	0.022348	0.14949082	0.24240187	95.5	0.3185853	98.9
250	2.137924	0.068962	0.137924	0.022181	0.14893231	0.22039498	94.9	0.2896620	98.7
300	2.137509	0.068754	0.137509	0.021512	0.14666928	0.20010458	94.8	0.2629946	99.0
400	2.134717	0.067359	0.134717	0.019960	0.14127911	0.16690607	93.9	0.2193623	98.6
500	2.134379	0.067190	0.134379	0.019752	0.14054137	0.16142759	95.0	0.2121620	99.3

6. Applications to Real Data

This section illustrates the importance and flexibility of the PDXRD in modeling two real-world datasets from different areas related to Engineering and survival times data. To ensure a comprehensive evaluation, the performance of the PDXRD is compared with eleven competing statistical distributions such as; DXRD [19], quasi Lindley distribution (QLD) [39], power XLindley distribution (PXL) [46], power exponentiated Lindley

Table 7. MLE simulation numerical results for Case (IV).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	1.503284	0.002189	0.003284	0.017314	0.13158106	0.5158951	94.4	0.6780335	98.8
50	1.500866	0.000577	0.000866	0.010049	0.10024715	0.3931508	94.6	0.5167125	98.6
100	1.498023	0.001318	0.001977	0.005280	0.07266433	0.2848812	95.0	0.3744153	98.8
150	1.500619	0.000413	0.000619	0.003595	0.05995433	0.2351260	94.3	0.3090228	99.2
200	1.497871	0.001419	0.002129	0.002519	0.05019136	0.1966714	94.9	0.2584824	99.1
250	1.498921	0.000719	0.001079	0.002030	0.04505676	0.1766602	95.0	0.2321820	98.8
300	1.497793	0.001471	0.002207	0.001680	0.04098830	0.1605214	94.8	0.2109710	99.0
400	1.497394	0.001737	0.002606	0.001352	0.03677574	0.1438705	94.9	0.1890870	99.2
500	1.499574	0.000284	0.000426	0.000979	0.03128177	0.1226745	94.9	0.1612294	98.5

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	2.925901	0.170361	0.425901	0.474044	0.6885085	2.1216759	94.3	2.7884884	96.8
50	2.866464	0.146586	0.366465	0.275771	0.5251391	1.4751728	95.9	1.9387985	97.4
100	2.823445	0.129378	0.323445	0.167656	0.4094586	0.9847139	95.5	1.2941954	98.3
150	2.809833	0.129333	0.309833	0.134099	0.3661953	0.7655613	96.1	1.0061663	98.3
200	2.790767	0.116307	0.290767	0.110352	0.3321931	0.6300464	95.5	0.8280610	98.3
250	2.790961	0.116384	0.290961	0.105525	0.3248466	0.5665452	95.6	0.7446022	98.9
300	2.786547	0.114619	0.286547	0.098310	0.3135448	0.4992004	95.0	0.6560920	98.8
400	2.783209	0.113284	0.283209	0.093163	0.3052269	0.4464179	94.9	0.5867207	98.9
500	2.784660	0.113864	0.284660	0.091098	0.3018251	0.3935054	94.1	0.5171785	98.9

Table 8. MLE simulation numerical results for Case (V).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	0.9597984	0.04020161	0.04020161	0.01084020	0.10411628	0.37667231	94.9	0.4950550	98.3
50	0.9615586	0.03844140	0.03844140	0.006925623	0.08322033	0.28947905	94.5	0.3804582	98.7
100	0.9619544	0.03804559	0.03804559	0.004283605	0.06544925	0.20886563	96.0	0.2745091	98.9
150	0.9658783	0.03412171	0.03412171	0.003133802	0.05598037	0.17405339	94.6	0.2287559	98.5
200	0.9657864	0.03421362	0.03421362	0.002550229	0.05049979	0.14567632	95.4	0.1914603	98.8
250	0.9660460	0.03395399	0.03395399	0.002307567	0.04803714	0.13327134	95.2	0.1751566	99.1
300	0.9658720	0.03412805	0.03412805	0.002145557	0.04632016	0.12282903	95.1	0.1614324	98.8
400	0.9660151	0.03398491	0.03398491	0.001863040	0.04316294	0.10436145	94.9	0.1371608	98.8
500	0.9675345	0.03246547	0.03246547	0.001633955	0.04042221	0.09444908	94.7	0.1241331	99.1

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	3.260311	0.08677041	0.2603112	0.1327042	0.3642859	0.9994638	94.7	1.3135810	98.4
50	3.245793	0.08193113	0.2457934	0.09780737	0.3127417	0.7584001	95.2	0.9967544	98.5
100	3.234869	0.07828971	0.2348691	0.07356865	0.2712354	0.5320750	95.0	0.6992985	98.9
150	3.223768	0.07458931	0.2237679	0.06163939	0.2482728	0.4218125	94.4	0.5543821	99.0
200	3.216359	0.07211978	0.2163593	0.05549993	0.2355842	0.3655757	95.8	0.4804709	98.9
250	3.217320	0.07243984	0.2173195	0.05441226	0.2332644	0.3324307	94.8	0.4369089	98.8
300	3.216016	0.07200523	0.2160157	0.05248305	0.2290918	0.2992092	94.7	0.3932463	99.4
400	3.214358	0.07145276	0.2143583	0.04999181	0.2235885	0.2493558	95.0	0.3277248	99.1
500	3.212508	0.07083585	0.2125076	0.04908047	0.2215411	0.2455852	95.0	0.3227691	98.9

distribution (PELD) [47], ZD [10], exponentiated generalized Lindley distribution (EGLD) [48], alpha-power Burr-Hatke-exponential distribution (APBHED) [49], half logistic new-Weibull Pareto distribution (HLNWPD) [50], half logistic Weibull distribution (HLWD) [51], PZD [22], and PLD [20]. The parameters of all competing models are estimated using the ML estimation method. Also, the corresponding standard errors (S.E.) are computed. Model evaluation is conducted using log-likelihood values and selection criteria, including the log-likelihood (LnL), the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), Corrected Akaike Information

Table 9. MLE simulation numerical results for Case (VI).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	2.148707	0.074353	0.148707	0.095180	0.3085119	1.0601335	95.0	1.3933183	97.5
50	2.127729	0.063864	0.127729	0.053706	0.2317450	0.7583812	95.4	0.9967296	97.5
100	2.110281	0.055140	0.110281	0.029501	0.1717592	0.5164398	96.0	0.6787495	98.4
150	2.107588	0.053794	0.107588	0.022710	0.1506998	0.4138591	95.1	0.5439291	98.5
200	2.099378	0.049689	0.099378	0.017344	0.1316973	0.3389330	95.0	0.4454548	98.9
250	2.099430	0.049715	0.099430	0.015914	0.1261497	0.3044884	95.7	0.4001848	98.9
300	2.097255	0.048628	0.097255	0.014216	0.1192319	0.2705198	94.7	0.3555402	98.7
400	2.096451	0.048226	0.096451	0.013203	0.1149025	0.2449173	95.1	0.3218914	98.9
500	2.098080	0.049040	0.098080	0.012489	0.1117547	0.2100865	94.6	0.2761137	99.0

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	4.595284	0.148821	0.595284	0.695894	0.8342025	2.2920195	95.7	3.0123685	98.3
50	4.544387	0.136097	0.544387	0.485695	0.6969178	1.7065576	95.7	2.2429043	98.8
100	4.513614	0.128404	0.513614	0.359412	0.5995100	1.2127229	95.5	1.5938644	99.0
150	4.483414	0.120854	0.483414	0.294553	0.5427276	0.9675735	94.9	1.2716680	98.5
200	4.469725	0.117431	0.469725	0.265467	0.5152346	0.8303552	95.3	1.0913239	98.5
250	4.470512	0.117628	0.470512	0.258678	0.5086036	0.7574178	95.1	0.9954634	99.0
300	4.470112	0.117528	0.470112	0.252165	0.5021604	0.6923086	94.7	0.9098913	99.2
400	4.464866	0.116217	0.464866	0.238104	0.4879587	0.5817645	94.9	0.7646048	98.8
500	4.456385	0.114096	0.456385	0.227306	0.4767664	0.5408797	95.6	0.7108705	98.8

Table 10. MLE simulation numerical results for Case (VII).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	2.925901	0.170361	0.425901	0.474044	0.6885085	2.1216759	94.3	2.7884884	96.8
50	2.866464	0.146586	0.366465	0.275771	0.5251391	1.4751728	95.9	1.9387985	97.4
100	2.823445	0.129378	0.323445	0.167656	0.4094586	0.9847139	95.5	1.2941954	98.3
150	2.809833	0.123933	0.309833	0.134099	0.3661953	0.7655613	96.1	1.0061663	98.3
200	2.790767	0.116307	0.290767	0.110352	0.3321931	0.6300464	95.5	0.8280610	98.3
250	2.790961	0.116384	0.290961	0.105525	0.3248466	0.5665452	95.6	0.7446022	98.9
300	2.786547	0.114619	0.286547	0.098310	0.3135448	0.4992004	95.0	0.6560920	98.8
400	2.783209	0.113284	0.283209	0.093163	0.3052269	0.4464179	94.9	0.5867207	98.9
500	2.784660	0.113864	0.284660	0.091098	0.3018251	0.3935054	94.1	0.5171785	98.9

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	5.767611	0.153522	0.767611	1.137889	1.0667188	2.9050646	95.3	3.8180849	98.5
50	5.698023	0.139605	0.698023	0.783502	0.8851564	2.1347355	95.5	2.8056524	98.7
100	5.654711	0.130942	0.654711	0.576898	0.7595381	1.5100910	95.3	1.9846911	98.9
150	5.615289	0.123058	0.615289	0.471034	0.6863189	1.1925167	95.0	1.5673076	98.4
200	5.596286	0.119257	0.596286	0.424447	0.6514956	1.0293882	95.5	1.3529102	98.5
250	5.596406	0.119281	0.596406	0.412628	0.6423612	0.9357585	95.1	1.2298540	99.1
300	5.595599	0.119120	0.595599	0.402265	0.6342438	0.8550188	95.1	1.1237390	99.1
400	5.589133	0.117827	0.589133	0.380363	0.6167356	0.7155356	94.6	0.9404182	98.8
500	5.578466	0.115693	0.578466	0.363785	0.6031455	0.6697441	95.2	0.8802351	98.6

Criterion (CAIC), and the Hannan-Quinn Information Criterion (HQIC). Goodness-of-fit is assessed with the Kolmogorov-Smirnov (KS) and its p-value (PV), Anderson-Darling (AD) statistics, along with their corresponding

Table 11. MLE simulation numerical results for Case (VIII).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	3.712047	0.237349	0.712047	1.182703	1.0875213	3.2238692	94.5	4.2370853	96.9
50	3.618891	0.206297	0.618891	0.704189	0.8391596	2.2226220	95.5	2.9211604	97.8
100	3.553933	0.184644	0.553933	0.451633	0.6720366	1.4923641	95.6	1.9613929	98.6
150	3.529926	0.176642	0.529926	0.366934	0.6057504	1.1508939	95.9	1.5126035	98.6
200	3.500003	0.166668	0.500003	0.310166	0.5569254	0.9619814	95.5	1.2643184	98.5
250	3.501497	0.167166	0.501497	0.300345	0.5480375	0.8667941	95.6	1.1392151	99.1
300	3.494959	0.164986	0.494959	0.282911	0.5318936	0.7637948	95.4	1.0038446	98.8
400	3.488607	0.162869	0.488607	0.268269	0.5179467	0.6739801	95.0	0.8858024	98.8
500	3.490177	0.163392	0.490177	0.264357	0.5141564	0.6086368	94.6	0.7999227	98.7

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	6.959665	0.159944	0.959665	1.732688	1.3163159	3.5335349	95.2	4.6440740	98.4
50	6.877291	0.146215	0.877291	1.209710	1.0998683	2.6017468	95.4	3.4194390	98.7
100	6.825556	0.137593	0.825556	0.902350	0.9499211	1.8429351	95.3	2.4221430	98.7
150	6.778366	0.129728	0.778366	0.742915	0.8619252	1.4519848	94.8	1.9083230	98.3
200	6.754300	0.125717	0.754300	0.671561	0.8194881	1.2562080	95.4	1.6510160	98.6
250	6.755215	0.125869	0.755215	0.655102	0.8093835	1.1417666	95.1	1.5006080	99.1
300	6.754041	0.125674	0.754041	0.639095	0.7994343	1.0414807	95.2	1.3688030	99.1
400	6.745672	0.124279	0.745672	0.605302	0.7780117	0.8706041	94.6	1.1442230	98.6
500	6.733557	0.122260	0.733557	0.581742	0.7627203	0.8192711	94.8	1.0767560	98.8

Table 12. MLE simulation numerical results for Case (IX).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	4.876558	0.300416	1.126558	2.825227	1.680841	4.8923902	94.5	6.429999	97.1
50	4.731556	0.261748	0.981556	1.678348	1.295511	3.3160770	95.8	4.358273	98.0
100	4.633858	0.235695	0.883858	1.098481	1.048084	2.2091346	95.6	2.903434	98.4
150	4.593257	0.224869	0.843257	0.896085	0.946617	1.6869087	95.7	2.217080	98.6
200	4.548482	0.212929	0.798482	0.769468	0.877193	1.4243462	95.7	1.871998	98.6
250	4.551559	0.213749	0.801559	0.749675	0.865838	1.2839747	95.5	1.687510	99.2
300	4.542541	0.211344	0.792541	0.711366	0.843425	1.1315714	95.1	1.487208	99.0
400	4.532012	0.208537	0.782012	0.674784	0.821452	0.9862814	94.9	1.296256	99.1
500	4.532844	0.208759	0.782844	0.666522	0.816408	0.9086494	94.8	1.194225	98.8

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	8.134669	0.162096	1.134669	2.413156	1.553434	4.1611289	95.1	5.468912	98.4
50	8.041378	0.148768	1.041378	1.697115	1.302734	3.0697871	95.4	4.034577	98.7
100	7.982552	0.140365	0.982552	1.273333	1.128420	2.1763317	95.4	2.860322	98.7
150	7.928298	0.132614	0.928298	1.052747	1.026034	1.7140818	94.8	2.252793	98.3
200	7.899791	0.128542	0.899791	0.952952	0.976192	1.4848001	95.4	1.951452	98.7
250	7.901677	0.128811	0.901678	0.931317	0.965048	1.3489177	95.1	1.772863	99.1
300	7.900349	0.128621	0.900349	0.908973	0.953401	1.2299292	95.1	1.616478	99.2
400	7.890347	0.127193	0.890347	0.861437	0.928136	1.0281124	94.3	1.351233	98.5
500	7.876729	0.125247	0.876729	0.829778	0.910921	0.9696432	94.9	1.274388	98.8

p-values (AD-PV), and Cramér-von Mises (CVM), along with their corresponding p-values (CVM-PV). The best-fitting model is identified based on the smallest LnL, AIC, BIC, CAIC, HQIC, KS, AD and CVM, and the highest associated KS-PV, AD-PV, and CVM-PV.

Table 13. MLE simulation numerical results for Case (X).

$n \uparrow$	$\hat{\theta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	5.269071	0.317268	1.269071	3.561506	1.887195	5.478077	94.6	7.199758	97.0
50	5.104590	0.276148	1.104590	2.106749	1.451465	3.692958	96.0	4.853603	98.0
100	4.994860	0.248715	0.994860	1.380341	1.174879	2.451133	95.7	3.221489	98.3
150	4.948014	0.237003	0.948014	1.125245	1.060776	1.866605	95.4	2.453252	98.6
200	4.898190	0.224548	0.898190	0.968752	0.984252	1.578591	95.8	2.074720	98.7
250	4.901658	0.225415	0.901658	0.944705	0.971959	1.423395	95.6	1.870747	99.1
300	4.891749	0.222937	0.891749	0.897487	0.947358	1.254237	95.0	1.648425	99.0
400	4.879563	0.219891	0.879563	0.850724	0.922347	1.088960	94.7	1.431204	99.0
500	4.880095	0.220024	0.880095	0.840689	0.916891	1.008503	94.8	1.325461	98.8

$n \uparrow$	$\hat{\beta}$	$RBias \downarrow$	$Bias \downarrow$	$MSE \downarrow$	$RMSE \downarrow$	$95\%ACI \downarrow$	$95\%CPs$	$99\%ACI \downarrow$	$99\%CPs$
30	8.716694	0.162226	1.216694	2.775846	1.666087	4.463981	95.1	5.866946	98.4
50	8.617357	0.148981	1.117357	1.953816	1.397790	3.293817	95.5	4.329017	98.7
100	8.554700	0.140627	1.054700	1.466945	1.211175	2.335303	95.4	3.069255	98.7
150	8.496698	0.132893	0.996698	1.213339	1.101517	1.839278	94.8	2.417336	98.3
200	8.466112	0.128815	0.966112	1.098454	1.048072	1.593499	95.4	2.094313	98.7
250	8.468286	0.129105	0.968286	1.073811	1.036249	1.447586	95.2	1.902542	99.1
300	8.466866	0.128915	0.966866	1.048067	1.023751	1.319771	95.1	1.734556	99.2
400	8.455857	0.127448	0.955857	0.992785	0.996386	1.103196	94.3	1.449915	98.5
500	8.441542	0.125539	0.941542	0.956896	0.978211	1.040580	94.9	1.367619	98.8

6.1. Aircraft Windshield Data Application

The first dataset represents the lifetimes of aircraft windshields (in units of 1000 h) and was obtained from Tahir et al. [52]. A windshield is a multi-layered structure consisting of a strong outer skin and inner layers. If the heating system malfunctions or the outer ply delaminates, a failure will result, and replacement of the windshield is required. The failure times for 63 aircraft windshields are as follows: 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 2.600, 0.150, 1.580, 0.046, 1.436, 2.592, 0.140, 1.492, 2.670, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 1.003, 3.665, 1.092, 2.183, 3.695, 1.152, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Figure 6 presents the profile log-likelihood curve of the parameters θ and β for the first dataset, which displays a clear maximum, confirming identifiability of the parameter. Tables 14 and 15 show all estimates of parameters for competing distributions, their associated standard errors (SEs), and goodness-of-fit statistics for the fitted distributions for the first data. Figures 7, 8, and 9 illustrate the estimated PDF, CDF, and probability-probability (P-P) plots for all competing models using the first dataset.

6.2. Survival Times Data Application

The second dataset was obtained from Chesneau and Artault [53], who analyze the survival durations of patients who received chemotherapy as their sole treatment. The dataset consists of the survival times of 46 patients and was originally reported by Chesneau and Artault (2021). A summary of the dataset is presented below. 3.658, 3.743, 3.978, 0.260, 1.485, 1.553, 1.099, 1.219, 2.444, 2.825, 2.830, 3.578, 1.271, 0.047, 0.115, 0.197, 0.203, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.841, 0.863, 4.003, 0.282, 0.534, 0.540, 0.570, 0.641, 0.644, 1.326, 1.447, 1.581, 1.589, 2.178, 0.121, 0.132, 0.164, 2.343, 2.416, 0.696, 4.033.

Figure 10 presents the profile log-likelihood curve of the parameters θ and β for the second dataset, which displays a clear maximum, confirming identifiability of the parameter. Tables 16 and 17 show all estimates of parameters for competing distributions, their associated standard errors (SEs), and goodness-of-fit statistics for the

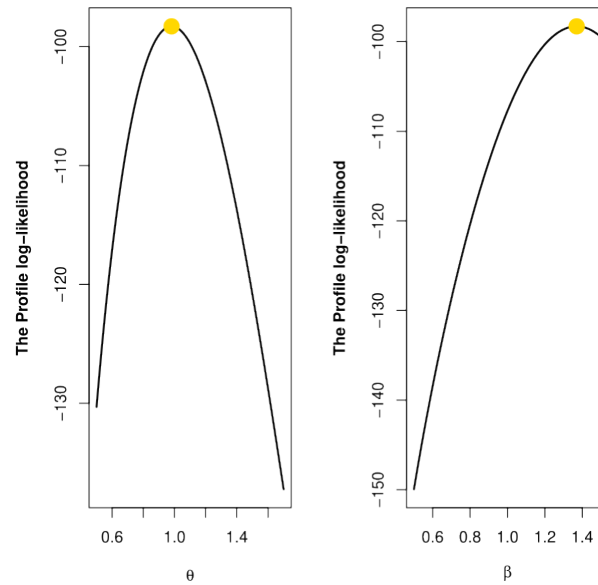
Figure 6. The PDXRD Profile Log-likelihood function plots for θ and β for the dataset 1.

Table 14. Parameter estimates and standard errors for the fitted models for the dataset 1.

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$SE(\hat{\beta})$	$SE(\hat{\theta})$	$SE(\hat{\lambda})$
PDXRD	1.369967	0.981208		0.0843028	0.06840704	
DXRD	1.165426			0.06006226		
QLD	0.115696	0.909438		0.13264968	0.09589357	
PXLD	0.438115	1.491731		0.065611	0.13670684	
PELD	0.970964	1.13443	15.83072	0.44085972	0.12451232	23.5979619
ZD	1.253924			0.09355436		
EGLD	0.023034	436.8513	1.159891	0.01570092	572.7633947	0.1961338
APBHE	53.88115	0.5829		47.67898296	0.0765	
HLNWD	0.374563	1.416488	0.110178	2.07355832	0.15144822	0.8635368
HLWD	1.41662	0.442652		0.15144766	0.08684352	
PZD	1.2984	0.958764		0.14041094	0.09326942	
PLD	1.383652	0.543903		0.12913065	0.08022224	

fitted distributions for the second data. Figures 11, 12, and 13 illustrate the estimated PDF, CDF, and P-P plots for all competing models using the second dataset.

The PDXRD compared with several well-established statistical models, including DXRD, QLD, PXLD, PELD, ZD, EGLD, APBHE, HLNWD, HLWD, PZD, and PLD. From Tables 15 and 17, the proposed PDXRD exhibits the best fit for both data sets according to the goodness-of-fit statistics and information criteria. This superior performance is indicated by the lowest values of information criteria such as LnL, AIC, BIC, CAIC, HQIC, KS, AD and CVM, as well as the highest KS-PV, AD-PV, and CVM-PV. Also, Figures 7, 8, 9, 11, 12, and 13 support the numerical values in Tables 15 and 17.

Table 15. Goodness-of-fit statistics for the fitted distributions for the dataset 1.

Models	LnL	AIC	BIC	CAIC	HQIC	KS	PV-KS	CVM	CVM-PV	AD	AD-PV
PDXRD	98.29024	200.5805	204.8668	200.7805	202.2663	0.06752051	0.91757927	0.03534562	0.956665	0.250537	0.970033
DXRD	107.7387	217.4773	219.6204	217.5429	218.3202	0.19593807	0.01355667	0.5450165	0.030335	3.034867	0.026419
QLD	102.1007	208.2015	212.4877	208.4015	209.8873	0.13850518	0.16218997	0.21126348	0.246958	1.110117	0.304095
PXLD	99.59505	203.1901	207.4764	203.3901	204.8759	0.09622967	0.5707759	0.06670827	0.773707	0.495824	0.750431
PELD	152.0823	310.1645	316.5939	310.5713	312.6932	0.09490423	0.58828319	0.06295069	0.797213	0.417123	0.831046
ZD	102.4542	206.9084	209.0515	206.974	207.7513	0.11858343	0.31321717	0.12085682	0.493124	0.961311	0.377868
EGLD	99.10242	204.2048	210.6342	204.6116	206.7336	0.10125858	0.50594194	0.07634077	0.715021	0.440463	0.807278
APBHE	101.8416	207.6831	211.9694	207.8831	209.3689	0.11816571	0.31719803	0.12248288	0.486604	0.790937	0.487086
HLNWD	99.26558	204.5312	210.9606	204.9379	207.0599	0.09209048	0.62581674	0.05916253	0.821051	0.431549	0.816391
HLWD	99.26558	202.5312	206.8174	202.7312	204.217	0.09200356	0.62698181	0.05903486	0.821854	0.431219	0.816727
PZD	102.3582	208.7164	213.0027	208.9164	210.4023	0.12547757	0.25248697	0.14166451	0.416909	0.971363	0.372306
PLD	99.58629	203.1726	207.4589	203.3726	204.8584	0.097049	0.56002765	0.06782785	0.766754	0.494044	0.752253

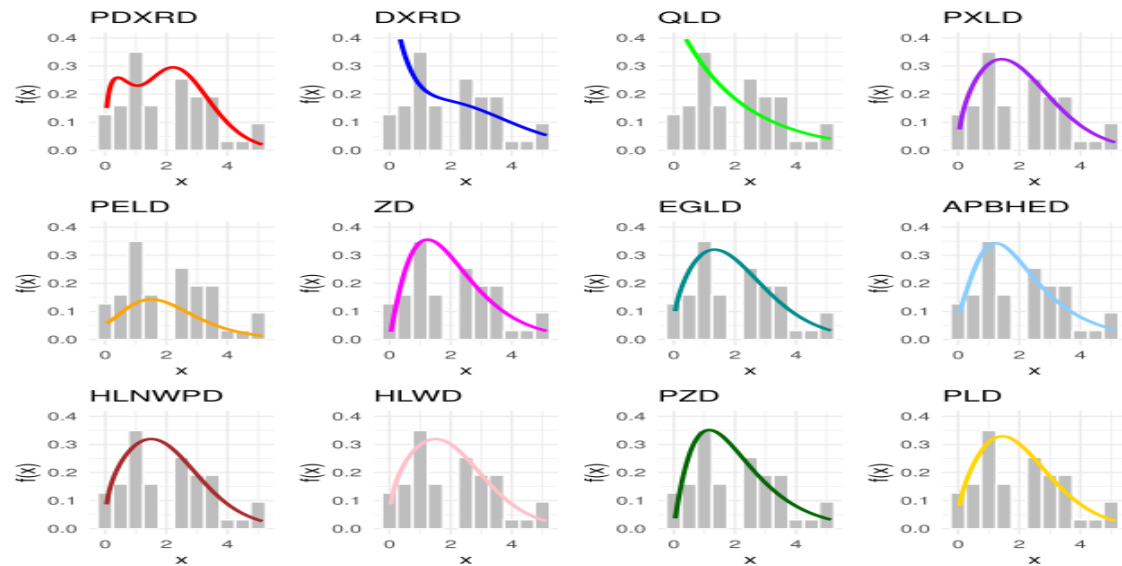


Figure 7. The estimated PDF for dataset 1.

Table 16. Maximum likelihood estimates and standard errors for the fitted models for dataset 2.

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$SE(\hat{\beta})$	$SE(\hat{\theta})$	$SE(\hat{\lambda})$
PDXRD	1.177478	1.3277035		0.10295532	0.09419637	
DXRD	1.397676			0.08959371		
QLD	1.4578358	0.9995203		2.1261171	0.27607573	
PXLD	0.8599061	1.0881391		0.1217715	0.12510927	
PELD	1.1696252	0.4174081	0.9764054	0.25489764	0.56827599	0.2626188
ZD	1.7954047			0.16432382		
EGLD	8.0632173	0.106691	1.2179875	0.02738575	0.02027246	0.2521458
APBHED	3.9462722	0.5873412		3.85983818	1.51E-01	
HLNWD	0.3639056	0.9588213	0.3798244	7.64369094	0.12159276	7.648489
HLWD	0.9586987	1.0013556		0.12158044	0.16467315	
PZD	2.1099339	0.7120353		0.21081237	0.0808939	
PLD	1.0105266	1.0493269		0.11598901	0.14198049	

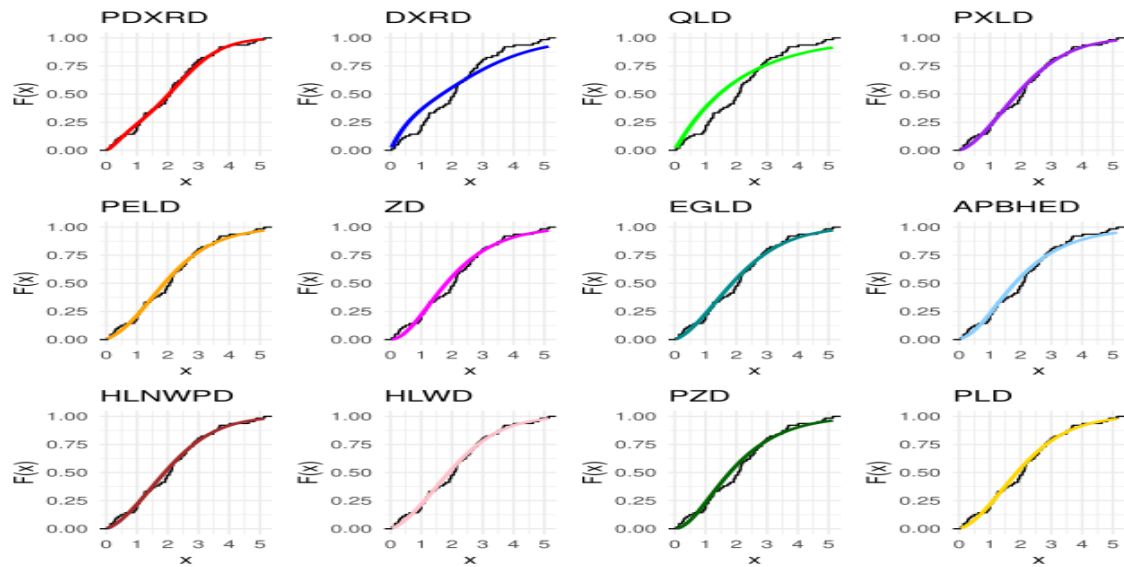


Figure 8. The estimated CDF for dataset 1.

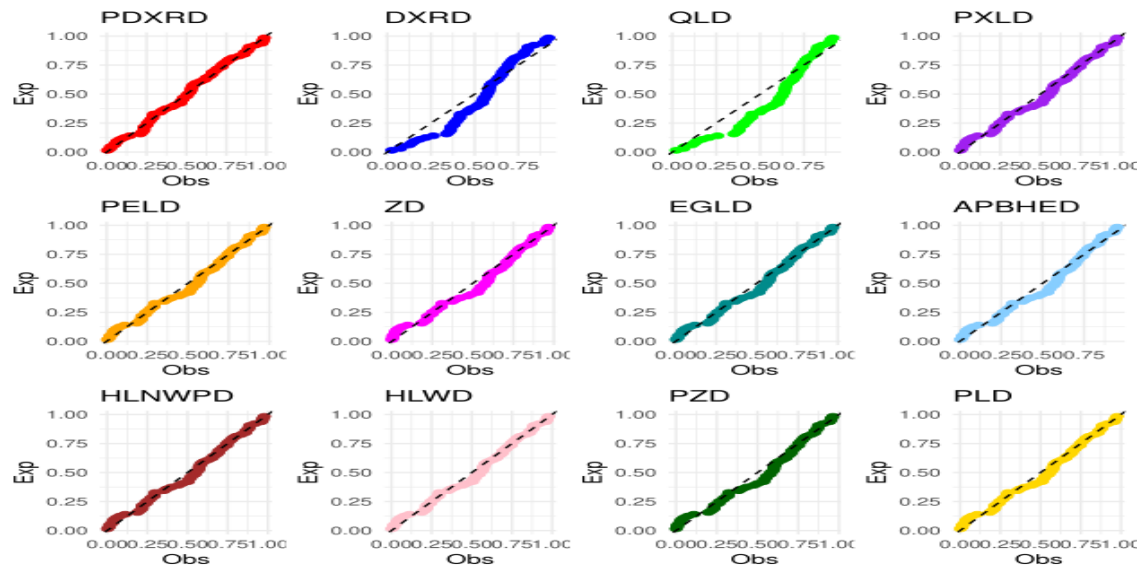


Figure 9. The P-P Plot for dataset 1.

7. Concluding remarks

The present study introduces a new flexible extension of the continuous family of distributions denoted by the PDXRD, which is a generalized mixed model formulated to yield greater flexibility for representing data with various hazard-rate shapes. The mathematical and statistical properties of the PDXRD are systematically obtained, such as the closed-form expressions for ordinary moments, the moment-generating function, and incomplete moments. Additionally, the study fundamentals include studies of reliability measures such as the mean residual life and order statistics. The estimation of parameters using the maximum likelihood estimation (MLE) method. A study of Monte Carlo simulations is conducted to determine the performance and asymptotic properties

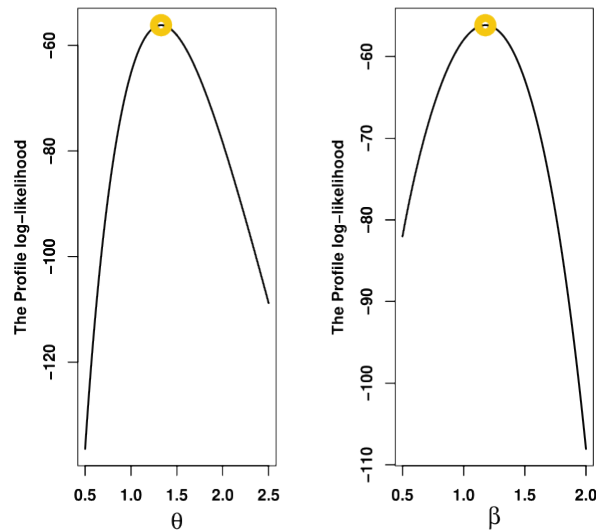
Figure 10. The PDXR D Profile Log-likelihood function plots for θ and β for dataset 2.

Table 17. Goodness-of-fit statistics for the fitted models for dataset 2.

Models	LnL	AIC	BIC	CAIC	HQIC	KS	PV-KS	CVM	CVM-PV	AD	AD-PV
PDXRD	56.15859	116.3172	119.8396	116.6172	117.6161	0.09688566	0.77850884	0.07437327	0.72758535	0.5285234	0.71691772
DXRD	57.63701	117.274	119.0352	117.3716	117.9235	0.11610966	0.56825216	0.10705358	0.55336935	0.807582	0.474834
QLD	57.437	118.874	122.3964	119.174	120.1729	0.10502242	0.69075676	0.079298	0.69842716	0.5467345	0.69870445
PXLD	57.23824	118.4765	121.9989	118.7765	119.7754	0.11926895	0.53424483	0.09681671	0.60289268	0.6094475	0.63813104
PELD	93.06559	192.1312	197.4148	192.7466	194.0796	0.11493446	0.58106701	0.0923784	0.62581495	0.5897219	0.65678357
ZD	62.5052	127.0104	128.7716	127.108	127.6599	0.23119624	0.01671006	0.5259516	0.03375107	3.796351	0.01108833
EGLD	57.14363	120.2873	125.5709	120.9026	122.2357	0.11736171	0.5546914	0.09340928	0.62041199	0.584244	0.66203211
APBHED	57.63704	119.2741	122.7965	119.5741	120.573	0.10428868	0.69885524	0.07816407	0.70505993	0.535415	0.70999768
HLNWD	57.79385	121.5877	126.8713	122.2031	123.5361	0.1255056	0.46959075	0.10801943	0.5489335	0.6695617	0.58385296
HLWD	57.79385	119.5877	123.1101	119.8877	120.8866	0.12543973	0.47025342	0.10791539	0.54940939	0.6691979	0.58416929
PZD	57.0607	118.1214	121.6438	118.4214	119.4203	0.10940005	0.64218496	0.08241227	0.68047584	0.5214937	0.72400811
PLD	57.46099	118.922	122.4444	119.222	120.2209	0.11973111	0.52933235	0.09870159	0.59342522	0.6274491	0.62146129

of the MLEs. The applied effectiveness of the proposed PDXR D is demonstrated on real-life two datasets, which produce a better fit than various competing known distributions. While this study analyzes parameter estimation for complete samples, future research may explore the performance of the PDXR D under different censoring techniques. The main limitation of this article is that the PDXR D is investigated only for complete data employing the maximum likelihood estimation strategy. Future directions may consider truncated and censored samples, alternative estimation approaches like Bayesian and least-squares approaches, and regression extensions incorporating covariates.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request. The data are not publicly available due to privacy or ethical restrictions.

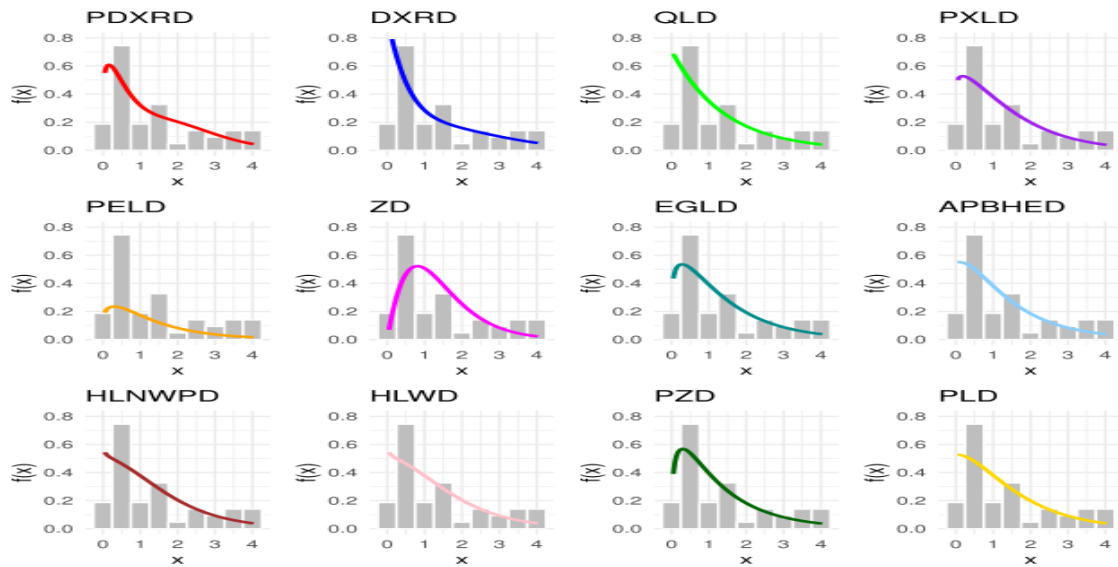


Figure 11. The estimated PDF for dataset 2.

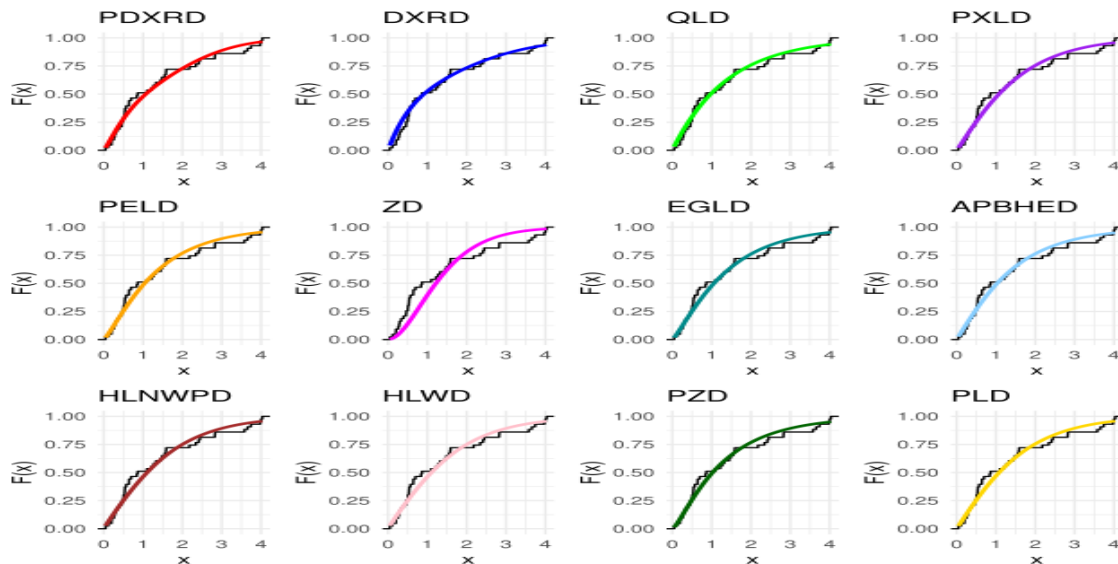


Figure 12. The estimated CDF for dataset 2.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

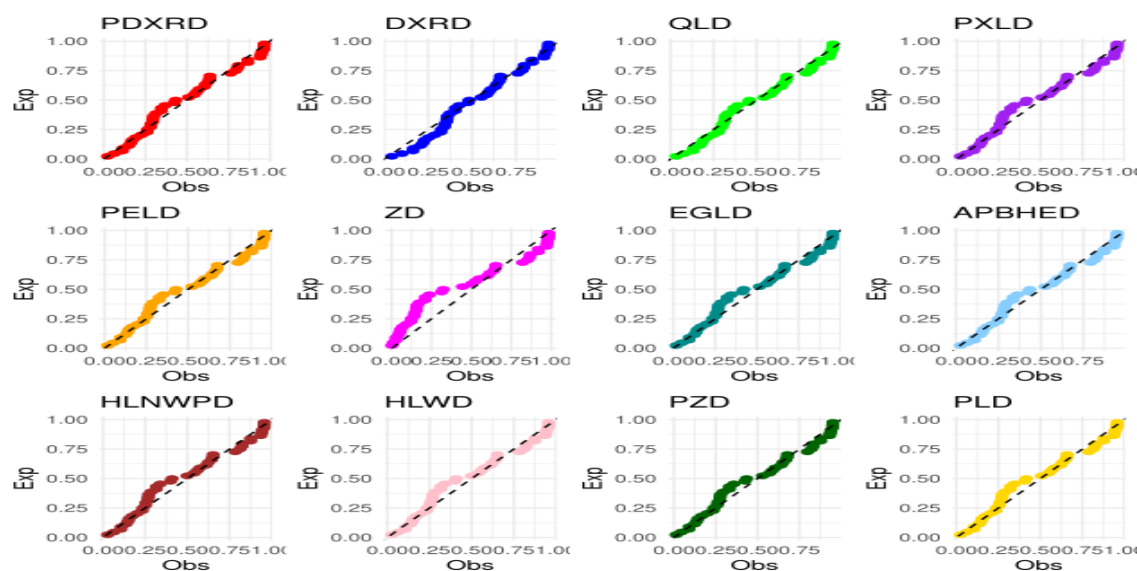


Figure 13. The P-P Plot for dataset 2.

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