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Ratio-cum-log ratio type exponential estimators of finite population mean in simple random sampling with two auxiliary variables

Wuttichai Srisodaphol*, Kritsadakorn Phuyodploy, Chayaphon Weluwanarak

Department of Statistics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Abstract This study proposes an estimator for the population mean in simple random sampling incorporating two auxiliary variables and a logarithmic transformation to enhance the efficiency of the estimation. The study explains the properties of the estimator, such as its bias and mean squared error, by using a second-order Taylor series expansion. The conditions under which the proposed estimator outperforms existing ones are analytically established. A comparative performance analysis, using both simulated and real-world data, demonstrates that the proposed estimator consistently achieves a higher relative efficiency than several commonly used alternatives. These findings suggest that the proposed approach offers a more practical and accurate solution to mean estimation in survey sampling contexts.

Keywords Estimator, Mean Squared Error, Percentage Relative Efficiency, Population Mean, Simple Random Sampling

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1. Introduction

Sampling is a method that selects a subset of individuals from a population to draw conclusions about that population. The advantage of sampling is that it saves time, costs, and labor and can be summarized more quickly than through a census. Simple random sampling, stratified sampling, and cluster sampling are frequently used as sampling methods. In simple random sampling, every individual has an equal chance of selection. Stratified sampling involves dividing the population into strata and randomly selecting samples from each stratum. Cluster sampling randomly selects entire groups of individuals, a technique often employed in large and dispersed populations [10].

In simple random sampling, the sample mean is an unbiased estimator of the population mean. In general, the estimator's performance tends to improve as the number of auxiliary variables increases, and the efficacy of the estimator will be enhanced by the collection of additional data. The study variables y are correlated with the auxiliary variables x and z, which the estimators consider. When there is a positive correlation between the study variable and auxiliary variables, a variety of methods are employed, including ratio, regression, and exponential estimators, to estimate the mean of the finite population. In contrast, when the correlation is negative, a product estimator is used [30]. Sampling offers the benefit of reducing costs and time. Auxiliary variables improve the efficiency of the estimator, but their frequent use can increase costs and time [10].

Many researchers have used auxiliary variables in developing estimators and compared them using mean squared error and percentage relative efficiency. Cochran [9] proposed a ratio estimator that used an auxiliary variable, and it performed better than the traditional estimator of the mean (sample mean), Lu and Yan [21] proposed a ratio

*Correspondence to: Wuttichai Srisodaphol (Email: wuttsr@kku.ac.th). Department of Statistics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand.

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type estimator using two auxiliary variables with improvement over the estimator proposed by [38]. The proposed estimator was more efficient than the estimators proposed by [38, 1]. Noor-ul-Amin et al. [25] proposed a product-cum-product exponential type estimator using two auxiliary variables improved from the estimator proposed by [5]. The proposed estimator was more efficient than the traditional mean estimator (sample mean), the estimator proposed by [9, 28, 37, 5]. Lu [20] proposed a regression estimator using two auxiliary variables, motivated by [32], This proposed estimator was more efficient than the estimator proposed by [1, 18, 22, 26]. Riyaz et al. [27] proposed an exponential-type estimator using two auxiliary variables. The proposed estimator was more efficient than the traditional mean estimator (sample mean), ratio estimator, product estimator [28], the estimator proposed by [37, 34]. Sher et al. [30] suggested exponential ratio-cum-product type estimators using two auxiliary variables, they changed the estimator of [8] and added the idea of [29], This suggested estimator was better than the traditional estimator of the mean (sample mean), as well as estimators using two auxiliary variables, including those proposed by [37, 31, 24, 13, 35].

In addition, some authors proposed estimators that used the logarithm function. Izunobi and Onyeka [15] proposed a logarithmic ratio type estimator using the natural logarithm in the auxiliary variables, and the proposed estimator was more efficient than the traditional mean estimator (sample mean), ratio estimator, and the estimator proposed by [5]. Yadav et al. [47] proposed a ratio-cum-exponential-log ratio estimator that added exponential and natural logarithm to the ratio estimator. It was more efficient than the traditional mean estimator (sample mean), the ratio estimator, the estimator proposed by [11, 6, 38, 5, 44, 17, 33, 36, 19, 3, 40, 41, 42, 43, 16, 39, 14, 48, 4, 46, 2].

The authors have presented an estimator that incorporates both the logarithm term and the ratio estimator, along with two auxiliary variables. More specifically, the proposed estimator integrates ratio, log-ratio, and exponential components within a unified estimation framework using two auxiliary variables. This suggests that the estimator has the potential to improve estimation efficiency, which is evaluated through comparisons with existing estimators. Yadav et al. [47] added logarithmic terms to the ratio estimator, while Sher et al. [30] proposed exponential ratio-cum-product type estimators in two auxiliary variables. Choudhury and Singh [7] modified these estimators, incorporating the idea of [29], enhancing their efficiency. Although these studies have contributed to the development of ratio-based and exponential-type estimators, further investigation into the combined use of ratio, log-ratio, and exponential components may provide additional opportunities to improve the efficiency of population mean estimation using auxiliary information. So, in this study, we propose a ratio-cum-log ratio type exponential estimator that provides a more efficient estimator by considering the mean squared error and percentage relative efficiency.

Therefore, this study develops an estimator for the population mean using two auxiliary variables under simple random sampling, called a ratio-cum-log ratio type exponential estimator, and compares the efficiency of the proposed estimator with the sample mean ($\overline{y}$), the estimators of [21] ($\overline{y}_{pmr}$), [25] (t_6), [20] ($\overline{y}_{pr}$), [27] ($\overline{y}_{SR}$), and [30] ($ESTp2$). This estimator is a valuable alternative for researchers and practitioners as it improves the accuracy and efficiency of estimating the population mean.

2. Estimator Development

This research focuses on developing and evaluating a proposed estimator of the population mean using two auxiliary variables and a logarithmic term to improve the efficiency of estimation under simple random sampling without replacement. We develop a ratio-cum-log ratio type exponential estimator based on the concepts of Yadav et al. [47] and Sher et al. [30], under simple random sampling without replacement. Yadav et al. [47] proposed the estimator as follows:

$$t_p = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) + k_1\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + k_2\bar{y}\left(1+\log\left(\frac{\bar{X}}{\bar{x}}\right)\right).$$

Sher et al. [30] proposed the estimator as follows:

$$ESTp2 = (t_3\bar{y}+t_4)\left(\left(\frac{\bar{X}}{\bar{x}}\right)+(1-l)\left(\frac{\bar{x}}{\bar{X}}\right)\right)\exp\left(\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right),$$

By improving the second term of the exponential estimator from the estimator t_p and using the last term of the estimator $ESTp2$. So, we propose a ratio-cum-log ratio-type exponential estimator $\hat{\bar{Y}}_{KC}$ that is given by

$$\hat{\bar{Y}}_{KC} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left(\frac{u(\bar{Z} - \bar{z})}{u(\bar{z} + \bar{Z}) + 2v}\right)$$

where k_1 and k_2 are the constants, u and v are the constants from the information of auxiliary variables including 1, 0, C_z and ρ_{yz} .

We substitute constants u and v to obtain possible estimators when u equals 0, only one term of the auxiliary variable remains. Therefore, we propose estimators that retain the remaining terms of two auxiliary variables. The proposed estimators are presented in Table 1.

Table 1. List of proposed estimators employing two auxiliary variables

No.	u	v	The possible estimators
1	1	0	$\hat{\bar{Y}}_{KC1} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\bar{Z}-\bar{z}}{\bar{z}+\bar{Z}}\right]$
2	1	1	$\hat{\bar{Y}}_{KC2} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\bar{Z}-\bar{z}}{(\bar{z}+\bar{Z})+2}\right]$
3	1	C_z	$\hat{\bar{Y}}_{KC3} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\bar{Z}-\bar{z}}{(\bar{z}+\bar{Z})+2C_z}\right]$
4	1	ρ_{yz}	$\hat{\bar{Y}}_{KC4} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\bar{Z}-\bar{z}}{(\bar{z}+\bar{Z})+2\rho_{yz}}\right]$
5	C_z	0	$\hat{\bar{Y}}_{KC5} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{C_z(\bar{Z}-\bar{z})}{C_z(\bar{z}+\bar{Z})}\right]$
6	C_z	1	$\hat{\bar{Y}}_{KC6} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{C_z(\bar{Z}-\bar{z})}{C_z(\bar{z}+\bar{Z})+2}\right]$
7	C_z	ρ_{yz}	$\hat{\bar{Y}}_{KC7} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{C_z(\bar{Z}-\bar{z})}{C_z(\bar{z}+\bar{Z})+2\rho_{yz}}\right]$
8	C_z	C_z	$\hat{\bar{Y}}_{KC8} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{C_z(\bar{Z}-\bar{z})}{C_z(\bar{z}+\bar{Z})+2C_z}\right]$
9	ρ_{yz}	0	$\hat{\bar{Y}}_{KC9} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\rho_{yz}(\bar{Z}-\bar{z})}{\rho_{yz}(\bar{z}+\bar{Z})}\right]$
10	ρ_{yz}	1	$\hat{\bar{Y}}_{KC10} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\rho_{yz}(\bar{Z}-\bar{z})}{\rho_{yz}(\bar{z}+\bar{Z})+2}\right]$
11	ρ_{yz}	C_z	$\hat{\bar{Y}}_{KC11} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\rho_{yz}(\bar{Z}-\bar{z})}{\rho_{yz}(\bar{z}+\bar{Z})+2C_z}\right]$
12	ρ_{yz}	ρ_{yz}	$\hat{\bar{Y}}_{KC12} = (k_1\bar{y} + k_2)\left[\left(\frac{\bar{X}}{\bar{x}}\right) + \left(1 + \log\left(\frac{\bar{X}}{\bar{x}}\right)\right)\right] \exp\left[\frac{\rho_{yz}(\bar{Z}-\bar{z})}{\rho_{yz}(\bar{z}+\bar{Z})+2\rho_{yz}}\right]$

3. Performance of the Estimators

The performance of the estimator is evaluated in terms of its bias and mean squared error (MSE). To determine the bias of the proposed estimator, the second-order Taylor series expansion is utilized, as illustrated in equation (1),

$$
\begin{aligned}
\hat{\bar{Y}}_{KC} =&(k_1\bar{y}+k_2)\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&+(k_1\bar{y}+k_2)\left[-\frac{\bar{X}}{\bar{x}^2}-\frac{1}{\bar{x}}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})\\
&+k_1\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{y}-\bar{Y})\\
&-(k_1\bar{y}+k_2)\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{2u^2\bar{Z}+2uv}{(u(\bar{z}+\bar{Z})+2v)^2}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{z}-\bar{Z})\\
&+\frac{1}{2}(k_1\bar{y}+k_2)\left[\frac{2\bar{X}}{\bar{x}^3}+\frac{1}{\bar{x}^2}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})^2\\
&+\frac{1}{2}(k_1\bar{y}+k_2)\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{(2u^2\bar{Z}+2uv)^2+(2u^2\bar{Z}+2uv)(2u^2(\bar{z}+\bar{Z}))+4uv}{(u(\bar{z}+\bar{Z})+2v)^4}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{z}-\bar{Z})^2\\
&+k_1\left[-\frac{\bar{X}}{\bar{x}^2}-\frac{1}{\bar{x}}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})(\bar{y}-\bar{Y})\\
&-(k_1\bar{y}+k_2)\left[-\frac{\bar{X}}{\bar{x}^2}-\frac{1}{\bar{x}}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{2u^2\bar{Z}+2uv}{(u(\bar{z}+\bar{Z})+2v)^2}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})(\bar{z}-\bar{Z})\\
&-k_1\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{2u^2\bar{Z}+2uv}{(u(\bar{z}+\bar{Z})+2v)^2}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{y}-\bar{Y})(\bar{z}-\bar{Z}). \qquad (1)
\end{aligned}
$$

From $e_0=\frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $e_1=\frac{\bar{x}-\bar{X}}{\bar{X}}$ and $e_2=\frac{\bar{z}-\bar{Z}}{\bar{Z}}$, we obtain $\bar{y}-\bar{Y}=\bar{Y}e_0$, $\bar{x}-\bar{X}=\bar{X}e_1$ and $\bar{z}-\bar{Z}=\bar{Z}e_2$, which are then substituted in Equation (1), and the result is shown in equation (2),

$$
\begin{aligned}
\hat{\bar{Y}}_{KC} =& 2k_1\bar{Y}+2k_2-2e_1(k_1\bar{Y}+k_2)+2e_0k_1\bar{Y}-e_2(k_1\bar{Y}+k_2)\left(\frac{u\bar{Z}}{u\bar{Z}+v}\right)+\frac{3}{2}e_1^2(k_1\bar{Y}+k_2)\\
&+e_2^2(k_1\bar{Y}+k_2)\left(\frac{3u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}\right)-2e_0e_1k_1\bar{Y}+e_1e_2(k_1\bar{Y}+k_2)\left(\frac{u\bar{Z}}{u\bar{Z}+v}\right)\\
&-e_0e_2k_1\bar{Y}\left(\frac{u\bar{Z}}{u\bar{Z}+v}\right). \qquad (2)
\end{aligned}
$$

From equation (2), we subtract $\bar{Y}$ from both sides of equation (2), resulting in equation (3),

$$\begin{aligned}\hat{\bar{Y}}_{KC} - \bar{Y} &= 2k_1\bar{Y} + 2k_2 - 2e_1(k_1\bar{Y} + k_2) + 2e_0k_1\bar{Y} - e_2(k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right) + \frac{3}{2}e_1^2(k_1\bar{Y} + k_2) \\ &+ e_2^2(k_1\bar{Y} + k_2)\left(\frac{3u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\right) - 2e_0e_1k_1\bar{Y} + e_1e_2(k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right) \\ &- e_0e_2k_1\bar{Y}\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right) - \bar{Y}. \end{aligned} \tag{3}$$

The expectation is taken in equation (3), resulting in the bias of $\hat{\bar{Y}}_{KC}$ as shown below,

$$\begin{aligned}Bias\left(\hat{\bar{Y}}_{KC}\right) &= 2k_1\bar{Y} + 2k_2 - 2(k_1\bar{Y} + k_2)E[e_1] + 2k_1\bar{Y}E[e_0] - (k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right)E[e_2] \\ &+ \frac{3}{2}(k_1\bar{Y} + k_2)E[e_1^2] + (k_1\bar{Y} + k_2)\left(\frac{3u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\right)E[e_2^2] - 2k_1\bar{Y}E[e_0e_1] \\ &+ (k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right)E[e_1e_2] - k_1\bar{Y}\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right)E[e_0e_2] - \bar{Y}.\end{aligned}$$

From $e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}$, $e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$ and $e_2 = \frac{\bar{z} - \bar{Z}}{\bar{Z}}$, we obtain $E[e_0] = 0$, $E[e_1] = 0$, $E[e_2] = 0$, $E[e_0^2] = \lambda C_y^2$, $E[e_1^2] = \lambda C_x^2$, $E[e_2^2] = \lambda C_z^2$, $E[e_0e_1] = \lambda C_{yx}$, $E[e_0e_2] = \lambda C_{yz}$, $E[e_1e_2] = \lambda C_{xz}$, $\lambda = \frac{1-f}{n}$, $f = \frac{n}{N}$, $C_{yx} = \rho_{yx}C_yC_x$, $C_{yz} = \rho_{yz}C_yC_z$ and $C_{xz} = \rho_{xz}C_xC_z$.

Therefore, the bias of the proposed estimator is given as equation (4),

$$\begin{aligned}Bias(\hat{\bar{Y}}_{KC}) &= (k_1\bar{Y} + k_2)\left(2 + \frac{3}{2}\lambda C_x^2 + \frac{3u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\lambda C_z^2 + \frac{u\bar{Z}}{u\bar{Z} + v}\lambda C_{xz}\right) \\ &- k_1\bar{Y}\left(2\lambda C_{yx} + \frac{u\bar{Z}}{u\bar{Z} + v}\lambda C_{yz}\right) - \bar{Y}.\end{aligned} \tag{4}$$

To find the mean squared error of the proposed estimator, we square equation (3), as shown in equation (5),

$$\begin{aligned}(\hat{\bar{Y}}_{KC} - \bar{Y})^2 &= (4k_1^2\bar{Y}^2 + 8k_1k_2\bar{Y} + 4k_2^2)\left(1 + e_1^2 + \frac{u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}e_2^2 - 2e_1 - \frac{u\bar{Z}}{u\bar{Z} + v}e_2\right. \\ &\left. + \frac{u\bar{Z}}{u\bar{Z} + v}e_1e_2\right) + \bar{Y}^2 + (k_1\bar{Y}^2 + k_2\bar{Y})\left(-4 + 4e_1 + \frac{2u\bar{Z}}{u\bar{Z} + v}e_2\right) \\ &+ (k_1^2\bar{Y}^2 + k_1k_2\bar{Y})\left(8e_0 - 8e_0e_1 - \frac{4u\bar{Z}}{u\bar{Z} + v}e_0e_2\right) + 4e_0^2k_1^2\bar{Y}^2 \\ &- 4e_0k_1\bar{Y}^2.\end{aligned} \tag{5}$$

Taking the expectation on both sides of equation (5), we obtain the mean squared error of $\hat{\bar{Y}}_{KC}$ as

$$\begin{aligned}MSE(\hat{\bar{Y}}_{KC}) &= E\left[(\hat{\bar{Y}}_{KC} - \bar{Y})^2\right] \\ &= \bar{Y}^2 + (4k_1^2\bar{Y}^2 + 8k_1k_2\bar{Y} + 4k_2^2)\left(1 + \lambda C_x^2 + \frac{u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\lambda C_z^2 + \frac{u\bar{Z}}{u\bar{Z} + v}\lambda C_{xz}\right) \\ &- 4k_1\bar{Y}^2 - 4k_2\bar{Y} + 4k_1^2\bar{Y}^2\lambda C_y^2 - (k_1^2\bar{Y}^2 + k_1k_2\bar{Y})\left(8\lambda C_{yx} + \frac{4u\bar{Z}}{u\bar{Z} + v}\lambda C_{yz}\right).\end{aligned}$$

Therefore, the mean squared error of the proposed estimator is given as

$$MSE(\hat{\bar{Y}}_{KC}) = \bar{Y}^2 + (4k_1^2\bar{Y}^2 + 8k_1k_2\bar{Y} + 4k_2^2)A - 4k_1\bar{Y}^2 - 4k_2\bar{Y} + 4k_1^2\bar{Y}^2\lambda C_y^2 - (k_1^2\bar{Y}^2 + k_1k_2\bar{Y})B \quad (6)$$

where $A = 1 + \lambda C_x^2 + \frac{u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}\lambda C_z^2 + \frac{u\bar{Z}}{u\bar{Z}+v}\lambda C_{xz}$ and $B = 8\lambda C_{yx} + \frac{4u\bar{Z}}{u\bar{Z}+v}\lambda C_{yz}$.

To find the constants k_1 and k_2 that make the minimum mean squared error of the proposed estimator, we find the first-order partial derivatives of $MSE(\hat{\bar{Y}}_{KC})$ with respect to k_1 and k_2, respectively, as shown in equations (7) and (8),

$$\frac{\partial MSE(\hat{\bar{Y}}_{KC})}{\partial k_1} = (8k_1\bar{Y}^2 + 8k_2\bar{Y})A - 4\bar{Y}^2 + 8k_1\bar{Y}^2\lambda C_y^2 - (2k_1\bar{Y}^2 + k_2\bar{Y})B, \quad (7)$$

and

$$\frac{\partial MSE(\hat{\bar{Y}}_{KC})}{\partial k_2} = (8k_1\bar{Y} + 8k_2)A - 4\bar{Y} - k_1\bar{Y}B. \quad (8)$$

Then, we set equations (7) and (8) equal to zero and simplify them, as shown in equation (9) and (10),

$$(8k_1\bar{Y} + 8k_2)A = 4\bar{Y} - 8k_1\bar{Y}\lambda C_y^2 + 2k_1\bar{Y}B + k_2B, \quad (9)$$

and

$$(8k_1\bar{Y} + 8k_2)A = 4\bar{Y} + k_1\bar{Y}B. \quad (10)$$

Now, we equate equation (9) to equation (10) to obtain

$$k_2 = \frac{8k_1\bar{Y}\lambda C_y^2 - k_1\bar{Y}B}{B} \quad (11)$$

Next, we substitute k_2 from equation (11) into equation (10) to obtain

$$k_1 = \frac{4B}{64\lambda C_y^2 A - B^2} \quad (12)$$

Finally, we substitute k_1 from equation (12) into equation (11) to obtain

$$k_2 = \frac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}.$$

Therefore, we get the k_1 and k_2 as

$$k_1 = \tfrac{4B}{64\lambda C_y^2 A - B^2} \text{ and } k_2 = \tfrac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}.$$

Substituting the k_1 and k_2 into equation (6), we get the minimum of $MSE(\hat{\bar{Y}}_{KC})$,

$$MSE_{\min}(\hat{\bar{Y}}_{KC}) = \bar{Y}^2 - \frac{64\bar{Y}^2\lambda C_y^2}{64\lambda C_y^2 A - B^2}.$$

The complete derivations for the bias and mean squared error are detailed in the Appendix.

4. Mean Squared Error Superiority Conditions

We find conditions for the proposed estimator to achieve a lower mean squared error than existing estimators, showing results as follows:

Condition 1, the proposed estimator $\hat{\bar{Y}}_{KC}$ has a lower mean squared error than the sample mean $\bar{y}$, $MSE(\hat{\bar{Y}}_{KC}) < Var(\bar{y})$ if

$$\lambda(C_y^2 + C_x^2 - 2C_{yx}) - 1 + \frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2} > 0.$$

Condition 2, the proposed estimator $\hat{\bar{Y}}_{KC}$ has a lower mean squared error than the estimator proposed by Lu and Yan [21] $\bar{y}_{pmr}$, $MSE(\hat{\bar{Y}}_{KC}) < MSE(\bar{y}_{pmr})$ if

$$\lambda(C_y^2 + k_1^2\omega_1^2 C_x^2 + k_2^2\omega_2^2 C_z^2 - 2k_1\omega_1 C_{yx} - 2k_2\omega_2 C_{yz} + 2k_1k_2\omega_1\omega_2 C_{xz}) - 1 + \frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2} > 0.$$

Condition 3, the proposed estimator $\hat{\bar{Y}}_{KC}$ has a lower mean squared error than the estimator proposed by Noor-ul-Amin et al. [25] t_6, $MSE(\hat{\bar{Y}}_{KC}) < MSE(t_6)$ if

$$\lambda\left[\left(C_y^2 + \frac{C_x^2}{4}(1 - 4H_{yx})\right) + \frac{C_z^2}{4}(1 - 4H_{yz} + 2H_{xz})\right] - 1 + \frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2} > 0.$$

Condition 4, the proposed estimator $\hat{\bar{Y}}_{KC}$ has a lower mean squared error than the estimator proposed by Lu [20] $\bar{y}_{pr}$, $MSE(\hat{\bar{Y}}_{KC}) < MSE(\bar{y}_{pr})$ if

$$\frac{\lambda\left[4C_y^2 L - U\lambda(C_x^2 + C_z^2)\right]}{4\left[U + C_y^2\lambda L + U\lambda(C_x^2 + C_z^2)\right]} - 1 + \frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2} > 0.$$

Condition 5, the proposed estimator $\hat{\bar{Y}}_{KC}$ has a lower mean squared error than the estimator proposed by Riyaz et al. [27] $\bar{y}_{SR}$, $MSE(\hat{\bar{Y}}_{KC}) < MSE(\bar{y}_{SR})$ if

$$C_y^2 + C_z^2 + \frac{\tau_1 C_x^2}{4} - \tau_1\rho_{yx}C_yC_x - 2\rho_{yz}C_yC_z + \tau_1\rho_{xz}C_xC_z - 1 + \frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2} > 0.$$

Condition 6, the proposed estimator $\hat{\bar{Y}}_{KC}$ has a lower mean squared error than the estimator proposed by Sher et al. [30] $ESTp2$, $MSE(\hat{\bar{Y}}_{KC}) < MSE(ESTp2)$ if

$$\frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2} - \frac{A_{S1}D_{S1}^2 + B_{S1}C_{S1}^2 - 2C_{S1}D_{S1}E_{S1} + B_{S1} + 2B_{S1}C_{S1} + D_{S1}^2 - 2D_{S1}D_{S1}}{A_{S1}B_{S1} - E_{S1}^2} > 0.$$

In practice, before an estimator is chosen, the conditions above can be checked using the known population information. If the condition for a given existing estimator is satisfied, the proposed estimator is expected to have a lower MSE than that estimator and can be used.

5. Percentage Relative Efficiency on Real Data

Percentage relative efficiency (PRE) is used to compare the performance of the proposed estimators with existing estimators using three real datasets, including (i) body measurements of 507 physically active individuals [12], (ii)

Table 2. Summary of various statistical values of variables from real data

Data set	N	n	$\bar{Y}$	$\bar{X}$	$\bar{Z}$	C_y	C_x	C_z	ρ_{yx}	ρ_{yz}	ρ_{xz}
1	507	52	69.148	96.681	76.979	0.193	0.069	0.143	0.763	0.904	0.692
2	178	90	68.751	69.564	69.139	0.143	0.132	0.139	0.977	0.991	0.988
3	514	300	15.086	13.201	12.857	0.684	0.671	0.659	0.922	0.965	0.923

life expectancy of people in 178 different countries [23], and (iii) height, depth, and width of 514 wild fruits in Uganda and Madagascar combined [45]. The statistics and information of the variables calculated from the three sets of real data are shown in Table 2.

PRE is a value used to determine how efficient one estimator is compared to another, based on their respective MSE values. The PRE [31] is given by

$$PRE(\bar{y}, \hat{\bar{Y}}_*) = \frac{MSE(\bar{y})}{MSE(\hat{\bar{Y}}_*)}.$$

where $MSE(\bar{y})$ is the mean squared error of $\bar{y}$,

$MSE(\hat{\bar{Y}}_*)$ is the mean squared error of the existing estimators and the proposed estimators.

The MSE of the estimators with three real data are presented in Table 3.

Table 3. Mean squared error of the estimators on three real data

Estimator	Data 1	Data 2	Data 3
$\bar{y}$	3.074	0.531	0.148
$\bar{y}_{pmr}$	5.717	4.632	0.152
t_6	2.657	0.975	0.288
$\bar{y}_{pr}$	2.200	0.995	0.113
$\bar{y}_{SR}$	27.057	20.914	29.500
$ESTp2$	4785.280	4727.369	227.782
$\hat{\bar{Y}}_{KC1}$	0.244	0.032	0.029
$\hat{\bar{Y}}_{KC2}$	0.242	0.032	0.029
$\hat{\bar{Y}}_{KC3}$	0.244	0.032	0.029
$\hat{\bar{Y}}_{KC4}$	0.242	0.032	0.029
$\hat{\bar{Y}}_{KC5}$	0.244	0.032	0.029
$\hat{\bar{Y}}_{KC6}$	**0.231**	**0.031**	**0.028**
$\hat{\bar{Y}}_{KC7}$	0.233	0.031	0.028
$\hat{\bar{Y}}_{KC8}$	0.242	0.032	0.029
$\hat{\bar{Y}}_{KC9}$	0.244	0.032	0.029
$\hat{\bar{Y}}_{KC10}$	0.242	0.032	0.029
$\hat{\bar{Y}}_{KC11}$	0.244	0.032	0.029
$\hat{\bar{Y}}_{KC12}$	0.242	0.032	0.029

The PRE of both the proposed and existing estimators is presented in Table 4.

From Table 4, which presents the percentage relative efficiency of the twelve proposed estimators and the existing estimators for three real datasets, it is found that all twelve proposed estimators exhibit a higher percentage relative

Table 4. Percentage relative efficiency of the estimators on three real data

Estimator	Data 1	Data 2	Data 3
$\bar{y}$	100	100	100
$\bar{y}_{pmr}$	53.772	11.470	97.018
t_6	115.692	54.466	51.410
$\bar{y}_{pr}$	139.692	53.407	131.116
$\bar{y}_{SR}$	11.361	2.540	0.501
$ESTp2$	0.064	0.011	0.049
$\hat{\bar{Y}}_{KC1}$	1259.972	1656.092	505.307
$\hat{\bar{Y}}_{KC2}$	1270.356	1666.122	517.406
$\hat{\bar{Y}}_{KC3}$	1261.471	1657.499	513.466
$\hat{\bar{Y}}_{KC4}$	1269.368	1666.035	517.012
$\hat{\bar{Y}}_{KC5}$	1259.972	1656.092	505.307
$\hat{\bar{Y}}_{KC6}$	**1327.973**	**1723.801**	**523.048**
$\hat{\bar{Y}}_{KC7}$	1321.890	1723.253	522.491
$\hat{\bar{Y}}_{KC8}$	1270.356	1666.122	517.046
$\hat{\bar{Y}}_{KC9}$	1259.972	1656.092	505.307
$\hat{\bar{Y}}_{KC10}$	1271.445	1666.209	517.811
$\hat{\bar{Y}}_{KC11}$	1261.630	1657.512	513.746
$\hat{\bar{Y}}_{KC12}$	1270.356	1666.122	517.406

efficiency than the existing estimators. Among them, $\hat{\bar{Y}}_{KC6}$ achieve the highest percentage relative efficiency, indicating that the proposed estimator is more efficient than the existing estimators.

From mean squared error superiority conditions, using three real datasets to examine six conditions, the results show that all conditions are satisfied (Also available from the corresponding author).

6. Percentage Relative Efficiency on Simulated Data

We simulate the data for one study variable and two auxiliary variables as a multivariate normal distribution following Sher et al. [30]:

1. Denote the mean vector and the variance-covariance matrix as

$$\boldsymbol{\mu} = \begin{bmatrix} \bar{Y} & \bar{X} & \bar{Z} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \text{var}(Y) & \text{cov}(Y,X) & \text{cov}(Y,Z) \\ \text{cov}(Y,X) & \text{var}(X) & \text{cov}(X,Z) \\ \text{cov}(Y,Z) & \text{cov}(X,Z) & \text{var}(Z) \end{bmatrix}.$$

Therefore, the mean vector and the variance-covariance matrix are

$$\boldsymbol{\mu} = \begin{bmatrix} 10 & 9 & 8 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 0.5 \\ 1 & 0.5 & 1 \end{bmatrix}.$$

2. Generate a population of size N = 100,000 from a multivariate normal distribution for the one study variable and two auxiliary variables.

3. Calculate information from the population, consisting of the population means of the two auxiliary variables, the coefficient of variation of y, the coefficient of variation of x, the coefficient of variation of z, the correlation

coefficient between x and z, the correlation coefficient between y and x, and the correlation coefficient between y and z.

4. Determine the sample size from

$$n = \frac{N z_{\alpha/2}^2 \sigma^2}{e^2(N-1) + z_{\alpha/2}^2 \sigma^2},$$

where $\alpha = 0.05$, $\sigma^2 = 4$, and $e = 1\%$ of the population mean, we obtain the sample size $n = 1514$.

5. We use simple random sampling, the sample size n = 100, 500, 1514, 2500, and 5000 from a population and calculate the information from this sample, consisting of the sample mean of the study variable, and the sample mean of the two auxiliary variables.

6. Compute the estimate of the proposed estimators and the existing estimators.

7. Repeat 5. and 6., 1,000 times.

8. Calculate the mean squared error and the percentage relative efficiency of all estimators.

9. Compare the performance of all estimators using percentage relative efficiency.

The results are shown in Table 5.

Table 5. Percentage relative efficiency of the estimators for simulated data under different sample sizes

	n				
Estimator	100	500	1514	2500	5000
$\bar{y}$	100	100	100	100	100
$\bar{y}_{pmr}$	226.961	216.519	224.538	224.235	221.476
t_6	209.881	204.311	211.955	211.798	218.865
$\bar{y}_{pr}$	235.962	225.276	237.793	237.316	232.415
$\bar{y}_{SR}$	181.509	179.448	191.481	191.373	195.298
$ESTp2$	202.606	218.952	215.038	198.655	206.552
$\hat{\bar{Y}}_{KC1}$	269.309	267.538	286.620	294.003	250.901
$\hat{\bar{Y}}_{KC2}$	279.719	279.264	297.628	307.455	259.976
$\hat{\bar{Y}}_{KC3}$	270.784	269.181	288.181	295.884	252.191
$\hat{\bar{Y}}_{KC4}$	274.894	273.789	292.528	301.167	255.779
$\hat{\bar{Y}}_{KC5}$	269.309	267.538	286.620	294.003	250.901
$\hat{\bar{Y}}_{KC6}$	**309.101**	**315.056**	**328.517**	**348.926**	**284.994**
$\hat{\bar{Y}}_{KC7}$	298.204	301.092	317.106	332.667	275.854
$\hat{\bar{Y}}_{KC8}$	279.719	279.264	297.628	307.455	259.976
$\hat{\bar{Y}}_{KC9}$	269.309	267.538	286.620	294.003	250.901
$\hat{\bar{Y}}_{KC10}$	287.554	288.324	305.897	317.892	266.749
$\hat{\bar{Y}}_{KC11}$	272.205	270.769	289.684	297.703	253.433
$\hat{\bar{Y}}_{KC12}$	279.719	279.264	297.628	307.455	259.976

Table 5 shows the percentage relative efficiency of the proposed estimators and the existing estimators for simulated data under different sample sizes. The results indicate that all 12 proposed estimators have higher percentage relative efficiency values than the existing estimators, with $\hat{\bar{Y}}_{KC6}$ having the highest percentage relative efficiency value. This finding suggests that the proposed estimators are more efficient than the existing ones.

7. Conclusion and Discussion

In this research, we propose a ratio-cum-log-ratio-type exponential estimator comprising twelve estimators. In addition, we establish conditions under which the mean squared error of the proposed estimator is lower than that of existing estimators. We also compare the performance of the proposed estimator with that of existing estimators based on the percentage of relative efficiency, using real and simulated data. The proposed estimator, developed using the natural logarithm, demonstrates greater efficiency compared to the estimator proposed by Sher et al. [30]. Additionally, it outperforms the estimators proposed by Lu and Yan [21], Noor-ul-Amin et al. [25], Lu [20], and Riyaz et al. [27], suggesting that the proposed estimator is more efficient in the cases considered. This result aligns with the study by Yadav et al. [47], which also employed the natural logarithm to improve the ratio estimator, leading to increased efficiency. The proposed estimator is suitable for estimating the population mean when the data include two auxiliary variables, and the study variable is positively correlated with the auxiliary variables. It is essential to know the sample means of x and z, as well as population information, including the coefficient of variation of population y, x and z, the correlation coefficients between y and x, the correlation coefficients between y and z, and the correlation coefficients between x and z. The reported gains in percentage relative efficiency are based on the simulated and real datasets considered, and the practical significance depends on the availability of the population information of the auxiliary variables.

Conflict of Interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest

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Appendix

To determine the bias of the proposed estimator, the second-order Taylor series expansion is utilized, as illustrated in equation (13),

$$
\begin{aligned}
\hat{\bar{Y}}_{KC} =&(k_1\bar{y}+k_2)\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&+(k_1\bar{y}+k_2)\left[-\frac{\bar{X}}{\bar{x}^2}-\frac{1}{\bar{x}}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})\\
&+k_1\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{y}-\bar{Y})\\
&-(k_1\bar{y}+k_2)\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{2u^2\bar{Z}+2uv}{(u(\bar{z}+\bar{Z})+2v)^2}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{z}-\bar{Z})\\
&+\frac{1}{2}(k_1\bar{y}+k_2)\left[\frac{2\bar{X}}{\bar{x}^3}+\frac{1}{\bar{x}^2}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})^2\\
&+\frac{1}{2}(k_1\bar{y}+k_2)\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{(2u^2\bar{Z}+2uv)^2+(2u^2\bar{Z}+2uv)(2u^2(\bar{z}+\bar{Z}))+4uv}{(u(\bar{z}+\bar{Z})+2v)^4}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{z}-\bar{Z})^2\\
&+k_1\left[-\frac{\bar{X}}{\bar{x}^2}-\frac{1}{\bar{x}}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})(\bar{y}-\bar{Y})\\
&-(k_1\bar{y}+k_2)\left[-\frac{\bar{X}}{\bar{x}^2}-\frac{1}{\bar{x}}\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{2u^2\bar{Z}+2uv}{(u(\bar{z}+\bar{Z})+2v)^2}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{x}-\bar{X})(\bar{z}-\bar{Z})\\
&-k_1\left[\frac{\bar{X}}{\bar{x}}+\left(1+\log\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left[\frac{u(\bar{Z}-\bar{z})}{u(\bar{z}+\bar{Z})+2v}\right]\\
&\left(\frac{2u^2\bar{Z}+2uv}{(u(\bar{z}+\bar{Z})+2v)^2}\right)\Bigg|_{\bar{x}=\bar{X},\bar{y}=\bar{Y},\bar{z}=\bar{Z}}(\bar{y}-\bar{Y})(\bar{z}-\bar{Z}). \qquad (13)
\end{aligned}
$$

From $e_0=\frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $e_1=\frac{\bar{x}-\bar{X}}{\bar{X}}$ and $e_2=\frac{\bar{z}-\bar{Z}}{\bar{Z}}$, we obtain $\bar{y}-\bar{Y}=\bar{Y}e_0$, $\bar{x}-\bar{X}=\bar{X}e_1$ and $\bar{z}-\bar{Z}=\bar{Z}e_2$, which are then substituted in Equation (13), and the result is shown in equation (14),

$$
\begin{aligned}
\hat{\bar{Y}}_{KC} =& 2k_1\bar{Y}+2k_2-2e_1(k_1\bar{Y}+k_2)+2e_0k_1\bar{Y}-e_2(k_1\bar{Y}+k_2)\left(\frac{u\bar{Z}}{u\bar{Z}+v}\right)+\frac{3}{2}e_1^2(k_1\bar{Y}+k_2)\\
&+e_2^2(k_1\bar{Y}+k_2)\left(\frac{3u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}\right)-2e_0e_1k_1\bar{Y}+e_1e_2(k_1\bar{Y}+k_2)\left(\frac{u\bar{Z}}{u\bar{Z}+v}\right)\\
&-e_0e_2k_1\bar{Y}\left(\frac{u\bar{Z}}{u\bar{Z}+v}\right). \qquad (14)
\end{aligned}
$$

From equation (14), we subtract $\bar{Y}$ from both sides of equation (14), resulting in equation (15),

$$\begin{aligned}\hat{\bar{Y}}_{KC} - \bar{Y} &= 2k_1\bar{Y} + 2k_2 - 2e_1(k_1\bar{Y} + k_2) + 2e_0k_1\bar{Y} - e_2(k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right) + \frac{3}{2}e_1^2(k_1\bar{Y} + k_2) \\ &\quad + e_2^2(k_1\bar{Y} + k_2)\left(\frac{3u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\right) - 2e_0e_1k_1\bar{Y} + e_1e_2(k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right) \\ &\quad - e_0e_2k_1\bar{Y}\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right) - \bar{Y}. \end{aligned} \tag{15}$$

The expectation is taken in equation (15), resulting in the bias of $\hat{\bar{Y}}_{KC}$ as shown below,

$$\begin{aligned}Bias\left(\hat{\bar{Y}}_{KC}\right) &= 2k_1\bar{Y} + 2k_2 - 2(k_1\bar{Y} + k_2)E[e_1] + 2k_1\bar{Y}E[e_0] - (k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right)E[e_2] \\ &\quad + \frac{3}{2}(k_1\bar{Y} + k_2)E[e_1^2] + (k_1\bar{Y} + k_2)\left(\frac{3u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\right)E[e_2^2] - 2k_1\bar{Y}E[e_0e_1] \\ &\quad + (k_1\bar{Y} + k_2)\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right)E[e_1e_2] - k_1\bar{Y}\left(\frac{u\bar{Z}}{u\bar{Z} + v}\right)E[e_0e_2] - \bar{Y}. \end{aligned}$$

From $e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}$, $e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$ and $e_2 = \frac{\bar{z} - \bar{Z}}{\bar{Z}}$, we obtain $E[e_0] = 0$, $E[e_1] = 0$, $E[e_2] = 0$, $E[e_0^2] = \lambda C_y^2$, $E[e_1^2] = \lambda C_x^2$, $E[e_2^2] = \lambda C_z^2$, $E[e_0e_1] = \lambda C_{yx}$, $E[e_0e_2] = \lambda C_{yz}$, $E[e_1e_2] = \lambda C_{xz}$, $\lambda = \frac{1-f}{n}$, $f = \frac{n}{N}$, $C_{yx} = \rho_{yx}C_yC_x$, $C_{yz} = \rho_{yz}C_yC_z$ and $C_{xz} = \rho_{xz}C_xC_z$.

Therefore, the bias of the proposed estimator is given as equation (16),

$$\begin{aligned}Bias(\hat{\bar{Y}}_{KC}) &= (k_1\bar{Y} + k_2)\left(2 + \frac{3}{2}\lambda C_x^2 + \frac{3u^2\bar{Z}^2}{(2u\bar{Z} + 2v)^2}\lambda C_z^2 + \frac{u\bar{Z}}{u\bar{Z} + v}\lambda C_{xz}\right) \\ &\quad - k_1\bar{Y}\left(2\lambda C_{yx} + \frac{u\bar{Z}}{u\bar{Z} + v}\lambda C_{yz}\right) - \bar{Y}. \end{aligned} \tag{16}$$

To find the mean squared error of the proposed estimator, we square equation (15), as shown in equation (17),

$$\begin{aligned}
(\hat{\bar{Y}}_{KC}-\bar{Y})^2 &= \left(2k_1\bar{Y}+2k_2-2e_1(k_1\bar{Y}+k_2)+2e_0k_1\bar{Y}-e_2(k_1\bar{Y}+k_2)\frac{u\bar{Z}}{u\bar{Z}+v}-\bar{Y}\right)^2\\
&= (4k_1^2\bar{Y}^2+8k_1k_2\bar{Y}+4k_2^2)+(4e_1^2k_1^2\bar{Y}^2+8e_1^2k_1k_2\bar{Y}+4e_1^2k_2^2)\\
&\quad+4e_0^2k_1^2\bar{Y}^2+(k_1^2\bar{Y}^2+2k_1k_2\bar{Y}+k_2^2)\left(\frac{u^2\bar{Z}^2}{(u\bar{Z}+v)^2}\right)e_2^2+\bar{Y}^2\\
&\quad-2e_1(4k_1^2\bar{Y}^2+8k_1k_2\bar{Y}+4k_2^2)+2e_0(4k_1^2\bar{Y}^2+4k_1k_2\bar{Y})\\
&\quad-2e_2(2k_1^2\bar{Y}^2+4k_1k_2\bar{Y}+2k_2^2)\frac{u\bar{Z}}{u\bar{Z}+v}-2(2k_1\bar{Y}^2+2k_2\bar{Y})\\
&\quad-2e_0e_1(4k_1^2\bar{Y}^2+4k_1k_2\bar{Y})+2e_1e_2(2k_1^2\bar{Y}^2+4k_1k_2\bar{Y}+2k_2^2)\frac{u\bar{Z}}{u\bar{Z}+v}\\
&\quad+2e_1(2k_1^2\bar{Y}^2+2k_2\bar{Y})-2e_0e_2(2k_1^2\bar{Y}^2+2k_1k_2\bar{Y})\frac{u\bar{Z}}{u\bar{Z}+v}\\
&\quad-4e_0k_1\bar{Y}^2+2e_2(k_1\bar{Y}^2+k_2\bar{Y})\frac{u\bar{Z}}{u\bar{Z}+v}\\
&= (4k_1^2\bar{Y}^2+8k_1k_2\bar{Y}+4k_2^2)\left(1+e_1^2+\frac{u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}e_2^2-2e_1-\frac{u\bar{Z}}{u\bar{Z}+v}e_2\right.\\
&\quad\left.+\frac{u\bar{Z}}{u\bar{Z}+v}e_1e_2\right)+\bar{Y}^2+(k_1\bar{Y}^2+k_2\bar{Y})\left(-4+4e_1+\frac{2u\bar{Z}}{u\bar{Z}+v}e_2\right)\\
&\quad+(k_1^2\bar{Y}^2+k_1k_2\bar{Y})\left(8e_0-8e_0e_1-\frac{4u\bar{Z}}{u\bar{Z}+v}e_0e_2\right)+4e_0^2k_1^2\bar{Y}^2\\
&\quad-4e_0k_1\bar{Y}^2. \qquad (17)
\end{aligned}$$

Taking the expectation on both sides of equation (17), we obtain the mean squared error of $\hat{\bar{Y}}_{KC}$ as

$$\begin{aligned}
MSE(\hat{\bar{Y}}_{KC}) &= E\left[(\hat{\bar{Y}}_{KC}-\bar{Y})^2\right]\\
&= (4k_1^2\bar{Y}^2+8k_1k_2\bar{Y}+4k_2^2)\left(1+E[e_1^2]+\frac{u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}E[e_2^2]-\frac{u\bar{Z}}{u\bar{Z}+v}E[e_2]\right.\\
&\quad\left.-2E[e_1]+\frac{u\bar{Z}}{u\bar{Z}+v}E[e_1e_2]\right)+(k_1\bar{Y}^2+k_2\bar{Y})\left(-4+4E[e_1]+\frac{2u\bar{Z}}{u\bar{Z}+v}E[e_2]\right)\\
&\quad+(k_1^2\bar{Y}^2+k_1k_2\bar{Y})\left(8E[e_0]-8E[e_0e_1]-\frac{4u\bar{Z}}{u\bar{Z}+v}E[e_0e_2]\right)\\
&\quad+4k_1^2\bar{Y}^2E[e_0^2]-4e_0k_1\bar{Y}^2E[e_0]+\bar{Y}^2\\
&= \bar{Y}^2+(4k_1^2\bar{Y}^2+8k_1k_2\bar{Y}+4k_2^2)\left(1+\lambda C_x^2+\frac{u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}\lambda C_z^2+\frac{u\bar{Z}}{u\bar{Z}+v}\lambda C_{xz}\right)\\
&\quad-4k_1\bar{Y}^2-4k_2\bar{Y}+4k_1^2\bar{Y}^2\lambda C_y^2-(k_1^2\bar{Y}^2+k_1k_2\bar{Y})\left(8\lambda C_{yx}+\frac{4u\bar{Z}}{u\bar{Z}+v}\lambda C_{yz}\right).
\end{aligned}$$

Therefore, the mean squared error of the proposed estimator is given as

$$MSE(\hat{\bar{Y}}_{KC})=\bar{Y}^2+(4k_1^2\bar{Y}^2+8k_1k_2\bar{Y}+4k_2^2)A-4k_1\bar{Y}^2-4k_2\bar{Y}+4k_1^2\bar{Y}^2\lambda C_y^2-(k_1^2\bar{Y}^2+k_1k_2\bar{Y})B \qquad (18)$$

where $A=1+\lambda C_x^2+\frac{u^2\bar{Z}^2}{(2u\bar{Z}+2v)^2}\lambda C_z^2+\frac{u\bar{Z}}{u\bar{Z}+v}\lambda C_{xz}$ and $B=8\lambda C_{yx}+\frac{4u\bar{Z}}{u\bar{Z}+v}\lambda C_{yz}$.

To find the constants k_1 and k_2 that make the minimum mean squared error of the proposed estimator, we find the first-order partial derivatives of $MSE(\hat{\bar{Y}}_{KC})$ with respect to k_1 and k_2, respectively, as shown in equations (19) and (20),

$$\frac{\partial MSE(\hat{\bar{Y}}_{KC})}{\partial k_1} = (8k_1\bar{Y}^2 + 8k_2\bar{Y})A - 4\bar{Y}^2 + 8k_1\bar{Y}^2\lambda C_y^2 - (2k_1\bar{Y}^2 + k_2\bar{Y})B, \tag{19}$$

and

$$\frac{\partial MSE(\hat{\bar{Y}}_{KC})}{\partial k_2} = (8k_1\bar{Y} + 8k_2)A - 4\bar{Y} - k_1\bar{Y}B. \tag{20}$$

Then, we set equations (19) and (20) equal to zero and simplify them, as shown in equations (21) and (22),

$$\begin{aligned}
(8k_1\bar{Y}^2 + 8k_2\bar{Y})A - 4\bar{Y}^2 + 8k_1\bar{Y}^2\lambda C_y^2 - (2k_1\bar{Y}^2 + k_2\bar{Y})B &= 0 \\
(8k_1\bar{Y}^2 + 8k_2\bar{Y})A &= 4\bar{Y}^2 - 8k_1\bar{Y}^2\lambda C_y^2 + (2k_1\bar{Y}^2 + k_2\bar{Y})B \\
(8k_1\bar{Y} + 8k_2)\bar{Y}A &= 4\bar{Y}^2 - 8k_1\bar{Y}^2\lambda C_y^2 + (2k_1\bar{Y}^2 + k_2\bar{Y})B \\
(8k_1\bar{Y} + 8k_2)A &= \frac{4\bar{Y}^2 - 8k_1\bar{Y}^2\lambda C_y^2 + (2k_1\bar{Y}^2 + k_2\bar{Y})B}{\bar{Y}} \\
(8k_1\bar{Y} + 8k_2)A &= 4\bar{Y} - 8k_1\bar{Y}\lambda C_y^2 + 2k_1\bar{Y}B + k_2B,
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
(8k_1\bar{Y} + 8k_2)A - 4\bar{Y} - k_1\bar{Y}B &= 0 \\
(8k_1\bar{Y} + 8k_2)A &= 4\bar{Y} + k_1\bar{Y}B.
\end{aligned} \tag{22}$$

Now, we equate equation (21) to equation (22) to obtain

$$\begin{aligned}
4\bar{Y} - 8k_1\bar{Y}\lambda C_y^2 + 2k_1\bar{Y}B + k_2B &= 4\bar{Y} + k_1\bar{Y}B \\
- 8k_1\bar{Y}\lambda C_y^2 + k_1\bar{Y}B + k_2B &= 0 \\
k_2 &= \frac{8k_1\bar{Y}\lambda C_y^2 - k_1\bar{Y}B}{B}
\end{aligned} \tag{23}$$

Next, we substitute k_2 from equation (23) into equation (22) to obtain

$$\begin{aligned}
\left[8k_1\bar{Y} + 8\left(\frac{8k_1\bar{Y}\lambda C_y^2 - k_1\bar{Y}B}{B}\right)\right]A - 4\bar{Y} - k_1\bar{Y}B &= 0 \\
8k_1\bar{Y}A + \frac{64k_1\bar{Y}\lambda C_y^2 A - 8k_1\bar{Y}AB}{B} - 4\bar{Y} - k_1\bar{Y}B &= 0 \\
\frac{8k_1\bar{Y}AB + 64k_1\bar{Y}\lambda C_y^2 A - 8k_1\bar{Y}AB - 4\bar{Y}B - k_1\bar{Y}B^2}{B} &= 0 \\
64k_1\bar{Y}\lambda C_y^2 A - 4\bar{Y}B - k_1\bar{Y}B^2 &= 0 \\
k_1(64\lambda C_y^2 A - B^2) - 4B &= 0 \\
k_1 &= \frac{4B}{64\lambda C_y^2 A - B^2}
\end{aligned} \tag{24}$$

Finally, we substitute k_1 from equation (24) into equation (23) to obtain

$$k_2 = \frac{8\left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)\bar{Y}\lambda C_y^2 - \left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)\bar{Y}B}{B} = \frac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}.$$

Therefore, we get k_1 and k_2 as

$$k_1 = \tfrac{4B}{64\lambda C_y^2 A - B^2} \text{ and } k_2 = \tfrac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}.$$

Substituting k_1 and k_2 into equation (18), we get the minimum of $MSE(\hat{\bar{Y}}_{KC})$,

$$\begin{aligned}
MSE_{\min}(\hat{\bar{Y}}_{KC}) &= \bar{Y}^2 + 4\Bigg[\left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)^2 \bar{Y}^2 + 8\left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)\left(\frac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}\right)\bar{Y} \\
&\quad + 4\left(\frac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}\right)^2\Bigg]A - 4\left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)\bar{Y}^2 \\
&\quad - 4\left(\frac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}\right)\bar{Y} + 4\left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)^2 \bar{Y}^2\lambda C_y^2 \\
&\quad - \left[\left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)^2 \bar{Y}^2 + \left(\frac{4B}{64\lambda C_y^2 A - B^2}\right)\left(\frac{32\bar{Y}\lambda C_y^2 - 4\bar{Y}B}{64\lambda C_y^2 A - B^2}\right)\bar{Y}\right]B, \\
&= \bar{Y}^2 + \frac{4A}{(64\lambda C_y^2 A - B^2)^2}\Big[16\bar{Y}^2B^2 + 256\bar{Y}^2\lambda C_y^2 B - 32\bar{Y}^2B^2 \\
&\quad + 1024\bar{Y}^2(\lambda C_y^2)^2 - 265\bar{Y}^2\lambda C_y^2 B + 16\bar{Y}^2B^2\Big] - \frac{16\bar{Y}^2B}{64\lambda C_y^2 A - B^2} \\
&\quad - \frac{128\bar{Y}^2\lambda C_y^2 - 16\bar{Y}^2B}{64\lambda C_y^2 A - B^2} + \frac{64\bar{Y}^2\lambda C_y^2 B^2}{(64\lambda C_y^2 A - B^2)^2} - \frac{B}{(64\lambda C_y^2 A - B^2)^2}\Big[16\bar{Y}^2B^2 \\
&\quad + 128\bar{Y}^2\lambda C_y^2 B - 16\bar{Y}^2B^2\Big], \\
&= \bar{Y}^2 + \frac{4096\bar{Y}^2(\lambda C_y^2)^2 A - 128\bar{Y}^2\lambda C_y^2(64\lambda C_y^2 A - B^2) - 64\bar{Y}^2\lambda C_y^2 B^2}{(64\lambda C_y^2 A - B^2)^2}, \\
&= \bar{Y}^2 + \frac{4096\bar{Y}^2(\lambda C_y^2)^2 A - 8192\bar{Y}^2(\lambda C_y^2)^2 A + 128\bar{Y}^2\lambda C_y^2 B^2 - 64\bar{Y}^2\lambda C_y^2 B^2}{(64\lambda C_y^2 A - B^2)^2}, \\
&= \bar{Y}^2 - \frac{4096\bar{Y}^2(\lambda C_y^2)^2 A - 64\bar{Y}^2\lambda C_y^2 B^2}{(64\lambda C_y^2 A - B^2)^2}, \\
&= \bar{Y}^2 - \frac{64\bar{Y}^2\lambda C_y^2(64\lambda C_y^2 A - B^2)}{(64\lambda C_y^2 A - B^2)^2}, \\
&= \bar{Y}^2 - \frac{64\bar{Y}^2\lambda C_y^2}{64\lambda C_y^2 A - B^2}.
\end{aligned}$$

Therefore, $MSE_{\min}(\hat{\bar{Y}}_{KC}) = \bar{Y}^2\left(1 - \frac{64\lambda C_y^2}{64\lambda C_y^2 A - B^2}\right)$.