STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–19.
Published online in International Academic Press (www.IAPress.org)

Multi-period portfolio optimization under uncertainty using Markowitz-based stochastic equipment replacement

Mochamad Nizar Palefi Ma'ady*, Amiratus Sholiha

*Information Systems Study Program, Telkom University, Surabaya Campus, Indonesia

Abstract The stochastic dynamic programming framework, conventionally applied to industrial equipment replacement decisions involving periodic maintenance choices, is adapted here for pension fund portfolio optimization. Markowitz mean-variance optimization remains the standard for single-period portfolio allocation, balancing expected returns against risk. Its single-period scope motivates the proposed portfolio replacement approach, which combines (i) Markowitz risk-return optimization, (ii) multi-period stochastic dynamic programming, and (iii) probabilistic uncertainty modeling. This integrated framework supports three annual strategic decisions—rebalancing existing allocations, maintaining current weights, or executing complete portfolio replacements—while accounting for stochastic market conditions. The methodology is applied to six years of Indonesian pension fund data from the Financial Services Authority of Indonesia (OJK). It generates optimal multi-period policies at low computational cost, traces the decision path by backward induction, and quantifies risk in each period. The case study shows how the framework produces an explicit, interpretable year-by-year policy for a pension fund portfolio; a quantitative comparison against benchmark strategies and liability-based objectives is identified as the next step of this research.

Keywords Equipment replacement, Markowitz method, multi-period optimization, portfolio rebalancing, stochastic dynamic programming.

AMS 2010 subject classifications 90C39, 90C40, 91G10

DOI: 10.19139/soic-2310-5070-4453

1. Introduction

Pension and insurance fund management faces a critical dilemma that has become increasingly pronounced in today's complex financial landscape. On one hand, these institutional funds have a fundamental obligation to maintain security and stability, as they safeguard the retirement savings and insurance protections of millions of individuals. The fiduciary responsibility requires conservative stewardship to ensure capital preservation and reliable payouts, a view consistent with the treatment of pension liabilities as an integral part of the asset allocation problem [20]. On the other hand, there is intensifying pressure to generate higher returns to maximize benefits for stakeholders [1], particularly in an era of rising life expectancies and increasing benefit obligations. This dual mandate creates inherent tension in investment decision-making. The challenge is exacerbated by market fluctuations that have become more volatile in recent years. A persistent low-interest-rate environment has also depressed yields on traditional fixed-income investments [2].

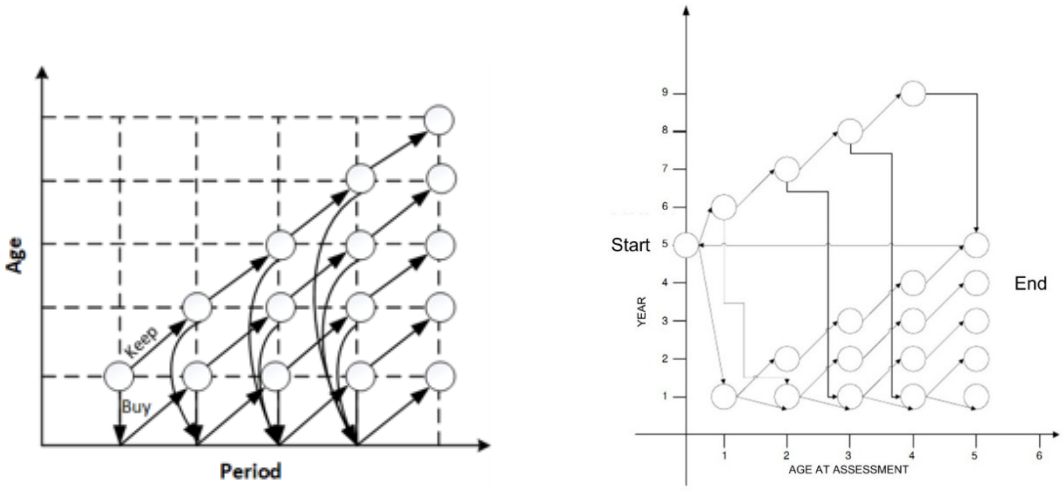
However, the pursuit of higher returns also introduces substantial risks, including market volatility and liquidity constraints [3]. Effective risk management is therefore essential to maximize expected returns without compromising the stability of the fund [4]. This dilemma is not unique to a single country but is observed globally, including in Croatia, Portugal, and Canada [5, 6]. Each country often employs its own mathematical frameworks to

*Correspondence to: Mochamad Nizar Palefi Ma'ady (Email: mnizarpm@telkomuniversity.ac.id). Information Systems Study Program, Telkom University, Surabaya Campus, Indonesia

ISSN 2310-5070 (online) ISSN 2311-004X (print)
Copyright © 202x International Academic Press

identify optimal solutions. The core challenge for fund managers is to maximize expected returns while minimizing risk, typically measured by the standard deviation. Balancing these two objectives ensures substantial benefits for both the company and the fund participants [7].

Currently, the Markowitz method [18], which formalized the risk-return trade-off later extended by the capital asset pricing framework [19], is widely used in portfolio optimization literature. Effective pension fund management is critical for ensuring financial stability, mitigating underfunding risks, and maintaining participant trust [8]. Investment allocations must account for liquidity needs while also supporting economic and social development through instruments like hedge funds, real estate, and private equity. Sustainable pension fund management can even serve as a significant source of funding for national development [9]. Retirement planning further underscores the importance of balanced portfolio management aligned with long-term goals. Investment horizons play a key role in asset selection and strategy. Long-term investors, such as those decades from retirement, can afford higher risk for aggressive growth. Those nearing retirement typically prioritize capital preservation and stable income.



(a) (b)

Figure 1. Two equipment replacement graphs: (a) the standard graph from dynamic programming, which establishes the foundational framework; (b) a modified graph adapted to the problem examined by Naibaho [12].

Stochastic dynamic programming (SDP) is a numerical method traditionally used to optimize equipment replacement in industrial applications, following the original formulation of the equipment replacement policy problem [23] and its later extensions [10]. It offers a robust framework for complex decision-making. Its flexibility and mathematical rigor make it particularly valuable, as it not only identifies optimal end values but also tracks long-term progress toward objectives [11]. This article proposes adopting the equipment replacement problem methodology to optimize insurance and pension fund management. By leveraging stochastic dynamic programming, we aim to provide an effective solution for multi-period portfolio optimization, enhancing decision-making in fund management.

The contribution of this paper is applied and methodological rather than theoretical. We do not derive new theoretical results for stochastic dynamic programming. Combining mean–variance allocation with multi-period optimization is well established in multistage stochastic programming for asset–liability management [24, 32] and in dynamic asset allocation [11]. Our contribution is threefold: (i) an explicit mapping of the equipment replacement model onto a pension fund portfolio, in which the state is the *portfolio age*; (ii) an extension of the classical keep-or-replace decision to three actions—rebalance, maintain, or sell-and-buy—combined with a selling (cut-loss) probability; and (iii) an illustrative application to Indonesian pension fund data, an emerging-market setting that is still rarely studied with operations research methods.

2. Related Work

2.1. Portfolio optimization and pension fund management

Putrihadiningrum and Violita [17] conducted a comprehensive performance optimization study examining asset allocation strategies for investment portfolios. Their research employed a mixed-methods approach, combining descriptive analysis with quantitative techniques, to evaluate pension fund portfolio data spanning 2019-2022. The investigation compared two distinct optimization methodologies: a baseline equal-weight allocation strategy versus a constrained Markowitz mean-variance optimization framework. Findings indicated suboptimal investment performance attributable to inefficient asset selection, resulting in an inadequate risk-return tradeoff within the portfolio composition. In a related Indonesian setting, Krisnawan [16] optimized the investment portfolio of the old-age savings (THT) program of PT Taspen (Persero).

Draženović et al. [6], in their empirical investigation "The Efficiency of Mandatory Pension Funds: Case of Croatia," employed data envelopment analysis (DEA) to assess the technical efficiency of Croatia's pension system. The study analyzed twelve decision-making units (DMUs) representing four pension funds across three classification categories (A, B, and C) during the 2015-2018 period. Only five of the twelve funds achieved scale efficiency. A comparison of the Charnes-Cooper-Rhodes (CCR) and Banker-Charnes-Cooper (BCC) models showed that the BCC model was more sensitive in identifying efficient funds during 2017-2018. The research identified three primary constraints affecting operational homogeneity: (1) regulatory limitations, (2) prevalent herding behavior among fund managers, and (3) insufficient market competition. The study contributed by recommending expanded research directions, including comprehensive efficiency evaluations of all Croatian pension funds and comparative analyses with European counterparts.

2.2. Equipment replacement problem

Fan et al. [10] proposed an innovative stochastic equipment replacement framework under operational uncertainty. Their model specifically addresses the critical challenge of variable vehicle utilization patterns in fleet management systems, as demonstrated through a case study with the Texas Department of Transportation (TxDOT). To keep real-world applications computationally tractable, the authors used three techniques. State space reduction handles high-dimensional problems, adaptive discretization treats continuous decision variables, and parallel computing accelerates solution times. These methodological innovations allow the model to maintain solution quality while scaling complex, large-scale replacement problems characteristic of modern fleet operations. The proposed SDP approach demonstrates effectiveness in environments with significant uncertainty and non-stationary utilization patterns, outperforming traditional deterministic replacement strategies.

Naibaho [12] employed dynamic programming methodology to optimize heavy vehicle replacement strategies for oil palm logistics operations. The study specifically analyzed two different trucks engaged in palm fruit harvesting and crude palm oil (CPO) transportation, where replacement costs ranged significantly from IDR 494 million to IDR 585 million per unit. A multi-stage dynamic programming formulation evaluated maintenance-versus-replacement decisions. It incorporated age-dependent maintenance costs, the capital expenditure for new acquisitions, and the residual values of aging equipment. The optimization framework generated a replacement policy maximizing total profitability across the equipment lifecycle. This case study notably confirmed the methodology's practical adaptability across varying operational constraints and economic conditions characteristic of tropical agricultural logistics. The author further emphasized the approach's scalability to other capital equipment replacement scenarios in developing economies with similar cost structures and utilization patterns.

Figure 1 contrasts (a) the standard equipment replacement graph of dynamic programming with (b) its adaptation to vehicle replacement in palm oil transportation logistics [12]. This adaptability allows the methodology to maintain theoretical rigor while being tailored to different operational scenarios. The graphical modifications reflect necessary adjustments to state transitions and decision nodes to accommodate real-world constraints like maintenance cost patterns, replacement thresholds, and operational requirements specific to industrial equipment management.

2.3. *Multi-period portfolio optimization and asset–liability management*

The single-period mean–variance model [18] was extended to multi-period and dynamic settings by Samuelson [21] and Merton [22], who derived optimal consumption and investment rules by dynamic programming in discrete and continuous time, respectively. These classic results rely on stylized return processes and do not model the discrete, institution-level decision of whether to keep or rebuild a portfolio. For institutional investors, multistage stochastic programming has become the dominant approach to asset–liability management (ALM). The Russell–Yasuda Kasai model [24] and the applications collected by Zenios and Ziemba [32] optimize asset allocations over a scenario tree subject to liability constraints. Infanger [11] solves dynamic asset allocation with stochastic dynamic programming and Monte Carlo sampling. More recent studies have extended risk-based allocation to multi-period settings with transaction costs, liquidity and diversification constraints [39]. Others have combined machine-learning asset selection with dynamic mean–variance weighting [40], or formulated chance-constrained asset–liability management models in which regulatory reliability levels are decision variables [41]. Alternative multi-period approaches include robust portfolio optimization, which protects allocations against uncertainty sets of parameters [33]; regret-minimizing online portfolio selection, such as universal portfolios [34]; and reinforcement learning, which learns trading policies directly from data [35]. A further strand studies rebalancing under transaction costs, showing that it is optimal to trade only when weights leave a no-trade region [36].

2.4. *Research gap and positioning*

Table 1 compares these approaches with the framework proposed here. Existing multi-period methods choose continuous asset weights at each stage; none of them treats the *construction of a new portfolio* as a distinct, discrete decision whose value depends on how long the current portfolio has been held. The equipment replacement analogy makes this decision explicit through the state variable *portfolio age*, and it extends the classical two-action structure (keep or replace) to three actions: rebalance (R), maintain without rebalancing (no-R), and sell-and-buy (S-B). Its potential advantage over purely financial formulations is interpretability and a small state space, which suits pension fund committees that review allocations once a year. Its limitations—a stylized uncertainty model, no explicit liability side, and a heuristic analogy—are discussed in Sections 3 and 5.3. Section 3 builds on this gap by mapping the equipment replacement model onto the portfolio problem.

Table 1. Positioning of the proposed framework relative to multi-period portfolio approaches.

Approach	Uncertainty model	Decision per period	Limitation for this setting
Dynamic programming [21, 22]	Stylized return process	Continuous weights	No discrete replace decision; strong distributional assumptions
Multistage SP for ALM [24, 32]	Scenario tree	Continuous weights with liabilities	Scenario tree grows exponentially with stages
SDP asset allocation [11]	Sampled return distributions	Continuous weights	Requires distribution estimation and sampling
Robust optimization [33]	Parameter uncertainty sets	Continuous weights	Conservative; single- or few-period
Online/regret minimization [34]	None (adversarial)	Continuous weights	No risk or liability objective
Reinforcement learning [35]	Learned from data	Trading policy	Needs long data histories; low interpretability
This paper	Scenarios and selling probability	Discrete: R, no-R, S-B	Heuristic analogy; no liability side yet

3. Conceptual Framework

Modern portfolio theory, developed by Markowitz, provides the fundamental framework for determining optimal portfolio allocations through quantitative risk-return analysis. Its original single-period formulation was subsequently extended to multi-period and dynamic settings [21, 22], and large-scale multistage stochastic programming models have since been adopted by institutional investors such as insurers and pension funds for asset-liability management [24]. When integrated with stochastic dynamic programming principles adapted from equipment replacement models—an approach we term *portfolio replacement optimization*—this combined methodology becomes particularly relevant for pension fund management due to its long-term investment horizon. Pension fund managers face annual decisions among three strategic options, as illustrated in Figure 2.

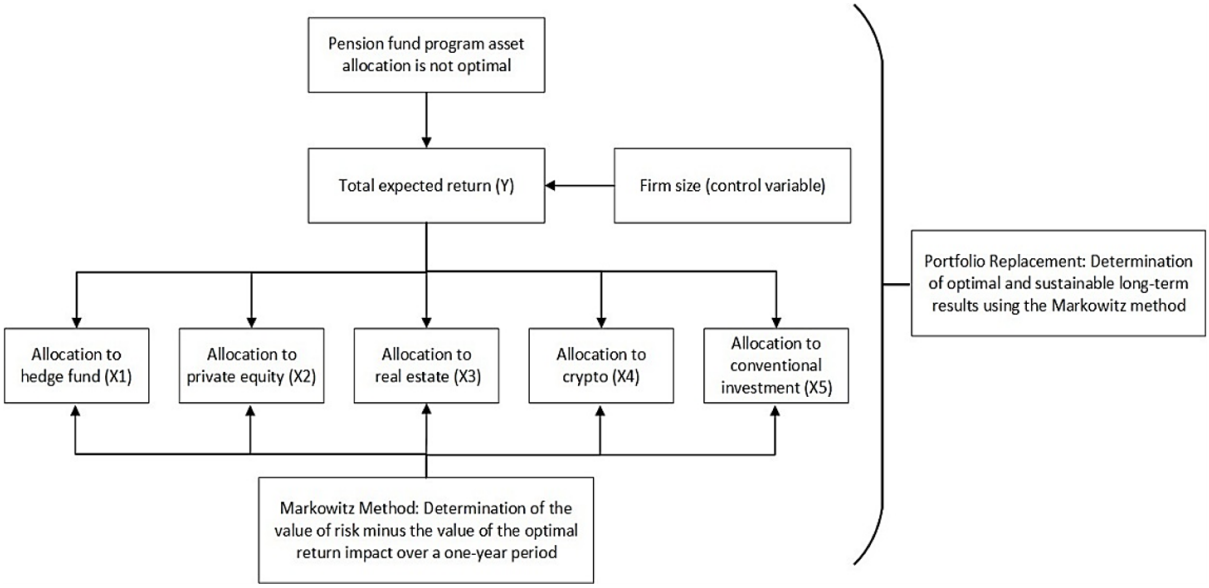


Figure 2. Conceptual framework.

We apply this framework to historical Indonesian pension fund portfolio data that span years i (initial period) to j (terminal year, specifically year 6 in this study). The investment portfolio comprises five alternative asset classes [15] with distinct characteristics. Hedge funds $(X1)$ employ absolute return strategies. Private equity $(X2)$ offers illiquid capital appreciation [27]. Real estate $(X3)$ provides tangible assets that hedge inflation. Cryptocurrency $(X4)$ represents high-volatility digital assets whose diversification benefits within mixed portfolios have been documented [28]. Conventional investments $(X5)$ include traditional stocks and bonds. This methodological approach enables systematic evaluation of replacement decisions while accounting for the unique constraints and objectives of institutional pension management.

The analytical process has three steps. First, returns and risks are calculated (Table 5). Second, the portfolio is optimized annually with the Markowitz method, Eq. (7). Third, three annual decision options are evaluated: rebalancing based on Eq. (7), maintaining the current allocation, or liquidating and acquiring a new portfolio. The core research problem focuses on determining which annual decision strategy yields the most optimal. Using multi-period optimization techniques, we aim to identify whether annual rebalancing, maintaining the status quo, or periodic complete portfolio replacement produces superior results over the time horizon.

The dynamic programming approach from machine replacement theory is typically employed to optimize the replacement of aging equipment in industrial settings. This methodology incorporates key variables including trade-in value (ti), purchase price (P), operational costs (ci), and salvage value (si) over discrete time periods (i). We apply the same conceptual framework to pension fund management. There, annual decisions over a fixed time horizon must combine several related variables (the alternative investments) through cumulative optimization with min or max operators. This adaptation serves as a foundational study for the problem context.

Table 2 defines each equipment replacement parameter in terms of portfolio quantities. The central state variable is the *portfolio age* i: the number of years since the portfolio was last constructed from scratch, i.e., since the last S-B decision. A newly constructed portfolio has age 1; after an R or no-R decision the same portfolio is carried forward and its age increases by one; after an S-B decision the age resets. Economically, age proxies the accumulated drift of the holdings away from their original target, the costs accumulated while holding them, and the staleness of the estimates used to construct them. We stress that this mapping is heuristic rather than derived: unlike physical equipment, a portfolio does not deteriorate mechanically, and its value is driven by market returns rather than by wear. The results should therefore be read as decision support under the stated assumptions, not as a structural model of portfolio value.

Table 2. Mapping of equipment replacement parameters to portfolio quantities.

Equipment replacement	**Portfolio replacement**	**Interpretation**
Equipment age i	Portfolio age i	Years since the last full construction (S-B) of the portfolio
Purchase price P	Initial portfolio value P	Capital committed when a new portfolio is constructed
Trade-in value $t(i)$	Markowitz trade-off value $t(i)$	Risk-adjusted value of the portfolio of age i from Eq. (7)
Operating cost $c(i)$	Operating cost $c(i)$	Cost of holding and managing a portfolio of age i in a year
Salvage value $s(i)$	Salvage value $s(i)$	Total return realized when a portfolio of age i is liquidated at the horizon
— (no analogue)	Selling probability $q(i)$	Probability that the portfolio must be sold (cut loss) in year i

By extending the standard decision graph from two options (keep or replace) to three alternatives (rebalance, maintain, or sell-and-buy), this research generalizes the solution approach for pension fund management. The study demonstrates how stochastic dynamic programming can yield optimal solutions in case studies involving uncertainty, while accommodating diverse input variables. Two hypotheses are tested. H1) The proposed portfolio replacement methodology better accounts for market uncertainties; it improves portfolio performance through more effective dynamic asset allocation over the investment horizon. H2) The optimal multi-period policy produced by the portfolio replacement model differs from a fixed annual-rebalancing policy, i.e., there is at least one year in which maintaining (no-R) or replacing (S-B) the portfolio is optimal. H2 is rejected if the backward induction selects rebalancing (R) in every year of the horizon.

4. Methodology

4.1. Independent and dependent variables

The definitions of dependent and independent variables and their measurements are explained in Table 3 below:

4.2. Data sources

The data were obtained from financial reports published on the Financial Services Authority (OJK) website [14] and from the data sets of Kirana [4]. The regulatory and asset-allocation context of Indonesian pension funds is further documented in [31]. The study covers the six annual periods 2017–2022. Decisions are made for 2017–2021, and 2022 is the terminal period that provides the boundary condition of the dynamic program. All parameters are calibrated on this full sample, so the results in Section 5 are in-sample; an out-of-sample evaluation on years not used for calibration is left to future research (Section 6).

Table 3. Definition of dependent and independent variables.

Dependent variables	Independent variables
• Expected return: Profit divided by number of monthly investment periods. • Standard deviation (investment risk): Volatility level of returns divided by monthly periods.	• Hedge funds: Collective investment vehicles. • Private equity: Direct investment in non-public companies. • Real estate: Physical property investments. • Crypto: Digital assets using cryptography. • Conventional investments: Traditional assets (equities, bonds, deposits).

Table 4 summarizes the OJK pension fund statistics [14]. The data are aggregate figures for all Indonesian pension funds, reported *monthly* in billions of rupiah (IDR) at fair value and broken down by investment instrument; the release shown covers January–September 2015. Each instrument is assigned to one of the five asset classes used in this study, labelled a–e as in Table 5. Instruments outside the asset universe of the study—government and corporate bonds, deposits, sukuk, mutual funds and asset-backed securities—are not used, and instruments with zero holdings throughout the period (Bank Indonesia certificates and stock option contracts) are dropped. The release contains no missing entries; zero values denote the absence of holdings. The five selected instruments represented 22.4% of total pension fund investments in September 2015, of which the less liquid classes (direct placements, property and land) accounted for 9.3%.

Table 4. OJK pension fund investment data by instrument (aggregate of all Indonesian pension funds, monthly, billion IDR, fair value) and mapping to the asset classes of this study. Share = percentage of total investments in September 2015.

OJK instrument	Asset class	Jan 2015	…	Sep 2015	Share (%)
Direct placement in shares	a: Private equity	6,147.61	…	6,800.63	3.67
Buildings; land and buildings	b: Indonesian property	6,971.17	…	7,808.35	4.21
Land	c: Indonesian land	2,718.13	…	2,653.38	1.43
Collective investment contracts (KIK)	d: Real estate funds	40.94	…	144.03	0.08
Listed shares	e: IDX stocks	29,212.84	…	24,092.83	13.00
Other instruments (government and corporate bonds, deposits, sukuk, mutual funds, others)	Not used	138,418.33	…	143,784.18	77.60
Total investments		183,509.02	…	185,283.40	100.00

4.3. Research flow

The research framework of this study follows the comprehensive process illustrated in Figure 3, which outlines the investment portfolio optimization approach incorporating diversification, rebalancing, Markowitz methods, and portfolio replacement strategies. This systematic procedure encompasses:

4.3.1 Initial portfolio construction The process begins with identifying and evaluating various alternative investment options. Through diversification, selected assets are strategically combined to form an optimal portfolio. Diversification mitigates investment risk by allocating capital across non-correlated asset classes, which enables more effective risk management [13]. This is consistent with evidence that common risk factors drive a substantial

Table 5. Investment portfolio scenario analysis.

Scenario	a	b	c	d	e	f	g	h	i
Aggressive	2.4	3.05	2.25	0.02	11.94	19.66	10.56	11.11	0.5
Balanced	3.5	2.67	1.51	0.04	11.94	19.66	10.71	11.53	0.495
Conservative	3.3	2.49	1.65	0.28	11.94	19.66	10.65	11.41	0.495

*a: Private Equity
b: Indonesian Property
c: Indonesian Land
d: Real Estate Funds
e: IDX Stocks
f: Total Investment
g: E(r) (%)
h: Risk (%)
i: F

share of asset return co-movement [30]. The foundation for this optimization is laid by calculating critical financial parameters for the selected assets, including their individual returns, variances, and the correlations between them.

4.3.2 Portfolio parameter calculation Before optimization can occur, the methodology requires the calculation of several critical financial parameters to quantify risk and return (R). These metrics begin with the expected return (Er) of individual assets, which represent their mean projected outcome, and their variance (σ_i^2), which measures their specific risk levels. To understand how assets interact within a portfolio, correlation coefficient (ρ_{ij}) and covariance (σ_{ij}) are computed, providing essential insights into the relationships between different investments. These inputs are then aggregated to calculate the overall portfolio risk (σ_p), capturing the combined volatility of the selected asset mix.

Table 6. Portfolio parameters.

No.	Parameter	Definition
1.	Return (R)	Expected investment outcomes
2.	Expected return (Er)	Mean projected results
3.	Correlation coefficient (ρ) and covariance (σ_{ij})	Inter-asset relationship metrics crucial for diversification analysis
4.	Portfolio risk (σ_p)	Aggregate portfolio volatility
5.	Variance (σ_i^2)	Measurement of individual asset risk

4.3.3 Markowitz optimization With all necessary parameters calculated, the framework applies the principles of Modern Portfolio Theory to construct optimal portfolios. This optimization process, grounded in Markowitz formulas, is designed to identify the asset weightings that maximize expected returns for any given level of risk. To keep the portfolio on the efficient frontier, these weightings are reviewed and adjusted annually through a formal rebalancing procedure. The effect of rebalancing on risk-adjusted performance has been shown to vary across asset classes [29]. This disciplined approach maintains strategic allocation, ensuring the portfolio's composition continues to align with its original risk-return objectives.

4.3.4 Continuous portfolio management The final component of the framework establishes a dynamic management process that extends across both short-term and long-term investment horizons. On an annual basis, the portfolio undergoes periodic rebalancing to return to its optimal asset weights, correcting for any market-driven drifts.

Beyond these tactical adjustments, the framework allows for strategic decision-making, including the potential replacement of the entire portfolio should investment objectives or market conditions warrant such a change. This ongoing cycle of performance monitoring against stated goals ensures the investment strategy remains responsive and effective over time.

4.4. Formula for determining return and risk

This section presents the key formulas for calculating investment returns and risks, following Modern Portfolio Theory principles. The equations progress from basic return calculations to comprehensive portfolio risk assessment.

4.4.1 Single period return The simplest measure of investment performance, calculating the percentage change in price between two periods. This serves as the foundation for all subsequent calculations.

$$R = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

4.4.2 Expected return The arithmetic means of historical returns, used to estimate future performance. This measures the central tendency of an asset's returns over time.

$$E(R_i) = \frac{\sum_{t=1}^{N} R_{it}}{N} \tag{2}$$

where,

$E(R_i)$ = expected return of investment i.

R_{it} = realized return in period t.

N = number of observation periods.

4.4.3 Investment risk (variance) Variance quantifies the dispersion of returns around the mean, serving as the fundamental measure of an asset's standalone risk.

$$\sigma_i^2 = \frac{\sum_{t=1}^{N} \big(R_{it} - E(R_i)\big)^2}{N} \tag{3}$$

where,

σ_i^2 = variance of investment i; N is the number of observation periods, as in Eq. (2).

4.4.4 Portfolio risk This comprehensive formula demonstrates how portfolio risk depends not just on individual asset risks (σ^2), but crucially on their correlations (ρ). The covariance terms highlight diversification benefits.

$$\sigma_p^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} w_i w_j \sigma_{ij} = \sum_{i=1}^{M} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{M}\sum_{j>i} w_i w_j \rho_{ij} \sigma_i \sigma_j \tag{4}$$

where,

σ_p^2 = portfolio variance.

w_i = portfolio weight of asset class i, with $\sum_i w_i = 1$.

M = number of asset classes ($M = 5$ in this study).

4.4.5 Correlation coefficient Measures the linear relationship (-1 to +1) between two assets' returns. This determines the diversification potential when combining assets in a portfolio.

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \tag{5}$$

4.4.6 Covariance The unscaled measure of co-movement between assets, used directly in portfolio risk calculations.

$$\sigma_{ij} = \frac{\sum_{t=1}^{N} \big(R_{it} - E(R_i)\big)\big(R_{jt} - E(R_j)\big)}{N} \tag{6}$$

where,

σ_{ij} = covariance between assets i and j.

4.4.7 Markowitz method Markowitz [18] formulated portfolio selection as a quadratic program that either minimizes portfolio variance for a target expected return or, equivalently, minimizes variance penalized by expected return through a risk-tolerance parameter. We use the latter (penalty) form, optimizing over the portfolio weights w_i in one period:

$$\min_{w} \; F(w) = \sum_{i=1}^{M} \sum_{j=1}^{M} \sigma_{ij}\, w_i w_j \; - \; \varsigma \sum_{i=1}^{M} E(R_i)\, w_i \quad \text{s.t.} \quad \sum_{i=1}^{M} w_i = 1, \;\; w_i \geq 0 \tag{7}$$

where,

F = portfolio variance minus ς times expected return ("risk minus return").

σ_{ij} = covariance between asset classes i and j (Eq. (6)).

w_i = portfolio weight of asset class i.

ς = Risk tolerance, $\varsigma \geq 0$.

$E(R_i)$ = expected return of asset class i (Eq. (2)).

Each value of ς yields one mean–variance efficient portfolio: $\varsigma = 0$ gives the minimum-variance portfolio, and increasing ς moves the solution along the efficient frontier toward portfolios with higher expected return and higher risk. The three scenarios in Table 5 (aggressive, balanced, conservative) correspond to different points on this frontier.

4.5. Portfolio replacement method

The optimization process extends over multiple time periods using stochastic dynamic programming methods, with portfolio replacement decisions being made periodically. At each stage, three options are considered. The Sell-Buy (S-B) decision replaces the whole portfolio through asset transactions. The No Rebalancing (no-R) decision maintains the current allocation. The Rebalancing (R) decision adjusts asset weights while retaining the existing instruments. The optimal solution for each decision path is determined through backward policy evaluation, which traces the most efficient investment strategy for each period based on available data. The methodology then generates year-by-year investment recommendations, together with the expected return and risk of the selected portfolios (Section 5). This integrated approach combines Markowitz portfolio theory for asset allocation, periodic rebalancing for risk management, and stochastic dynamic programming for multi-period optimization.

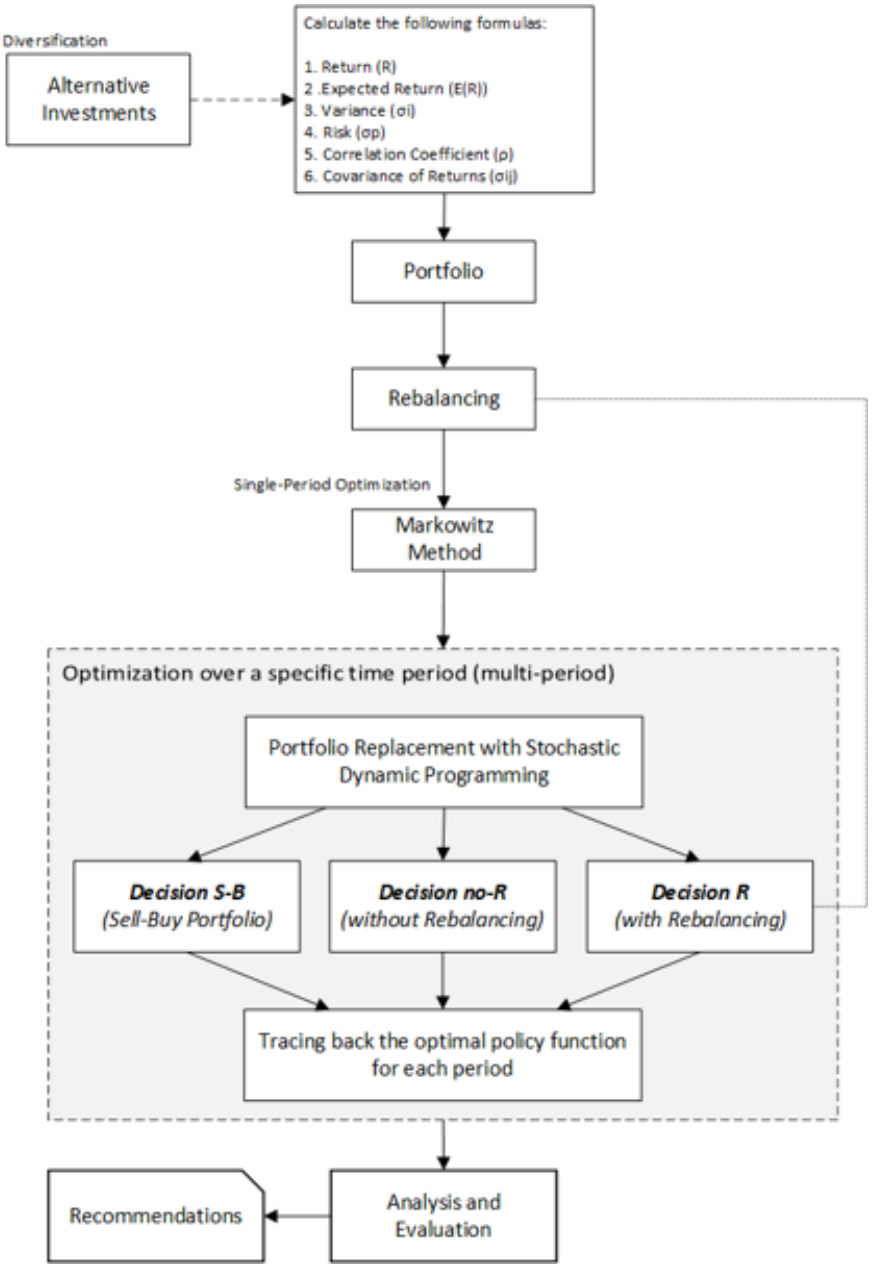


Figure 3. Research flow.

Within this dynamic programming framework, the key terms follow the standard Markov decision process formalism [25]. A stage is a discrete period that contains decision variables. An alternative is a possible choice at a stage, together with its valuation. The state links consecutive stages, so that each stage can be optimized independently while the overall solution remains feasible. This portfolio replacement method operates according to four fundamental application rules. The optimization framework begins by defining the objective functions.

1.) We consider the standard equipment replacement problem's value function:
$V_k(i) \equiv$ minimum cost from year k to N when starting with equipment of age i.

This establishes our baseline for comparison. Building on this foundation, we adapt the formulation to our portfolio replacement by introducing two new value functions:

$W_k(i, A) \equiv$ maximum risk-adjusted return from year k to N for portfolio of age i under scenario $A \in$ {a(agg.), b(bal.), c(cons.)}.

Assumes no cut loss occurs. Its counterpart:

$F_k(i, A) \equiv$ maximum risk-adjusted return including cut loss probability.

The recursive relationships form the computational core of our approach. The modified recurrence relation evaluates three possible actions at each stage: portfolio rebalancing (R), maintaining current allocations (non-R), or complete portfolio replacement (S-B). Key parameters in our model include those in Table 7:

Table 7. Notation of the portfolio replacement model.

Symbol	Description
$c(i)$	Operating cost in year i
$t(i)$	Markowitz trade-off value in year i
$s(i)$	Salvage value (total returns) in year i
$q(i)$	Probability of selling the portfolio in year i; constant across ages, $q(i) = q$ (Eq. (12))
P	Initial portfolio value
N	Total number of periods
y	Age of the portfolio held at the start of the horizon
ρ_k	Reward from rebalancing the portfolio to Markowitz-optimal weights in year k
$v(i)$	Terminal value of a portfolio of age i liquidated under a cut-loss sale

2.) Modify the recursive relationship formula to suit the problem context (recurrence relation). The origin of recursive formula (equipment replacement, [23]):

$$V_k(i) = \min \begin{cases} \text{Keep: } c(i) + V_{k+1}(i+1) \\ \text{Replace: } P - t(i) + c(0) + V_{k+1}(1) \end{cases} \tag{8}$$

Modified portfolio replacement formula:

$$W_k(i) = \max \begin{cases} \text{R}: & t(i) + \rho_k + W_{k+1}(i+1) \\ \text{no-R}: & t(i) + 0 + W_{k+1}(i+1) \\ \text{S-B}: & -c(i) + q(i) \cdot F_{k+1}(1) + \big(1 - q(i)\big) \cdot W_{k+1}(1) \end{cases} \tag{9}$$

The argument of W and F is always the portfolio age. Rebalancing (R) and maintaining (no-R) carry the same portfolio forward, so its age increases from i to $i+1$; the two actions differ only in the rebalancing reward ρ_k, which no-R does not receive. Selling and buying (S-B) replaces the portfolio, so the next stage starts with a new portfolio of age 1. This matches Figure 4, where the green (R) and blue (no-R) arrows move diagonally to age $i+1$ and the orange (S-B) arrow returns the portfolio to age 1.

Derivation and sign convention. Eq. (9) follows from Bellman's principle of optimality [23, 25]: the optimal value from year k onward equals the best one-period reward plus the optimal value of the state reached in year $k+1$. For R and no-R the next state is certain, so the continuation value is $W_{k+1}(i+1)$. For S-B, a cut-loss sale occurs with probability $q(i)$, so the continuation value is the expectation $q(i)F_{k+1}(1) + \big(1 - q(i)\big)W_{k+1}(1)$. Because Eq. (9) is a maximization, benefits enter with a positive sign ($t(i)$, ρ_k, and the terminal values $s(i)$ and $v(i)$

in Eq. (10)) and costs with a negative sign ($-c(i)$). This reverses the signs of the cost-minimization origin, Eq. (8), where costs are positive and salvage values negative. All terms are expressed in the common unit of value of the parameters in Table 8, so that they can be added.
3.) Boundary conditions:

$$\begin{aligned} W_{N+1}(i) &= s(i) \\ F_{N+1}(i) &= v(i) \end{aligned} \tag{10}$$

4.) Answer is given by:

$$W_1(y) \tag{11}$$

where y is the age of the portfolio held at the start of the horizon ($y = 2$ in the case study; see Figure 5).

Formally, the model is a finite-horizon Markov decision process [25]. The stages are the years $k = 1, \ldots, N$; the state is the pair (i, A) of portfolio age and scenario; and the action set is $\{\text{R}, \text{no-R}, \text{S-B}\}$. The transition is deterministic in the age: R and no-R move the state from age i to $i + 1$, and S-B moves it to age 1. The one-period reward is $t(i) + \rho_k$ for R, $t(i)$ for no-R, and $-c(i)$ for S-B (Eq. (9)). Uncertainty enters through the selling probability $q(i)$, which weights the value with and without a cut-loss sale. The objective is the maximum cumulative risk-adjusted return $W_1(y)$. Because the horizon, the state set and the action set are all finite, backward induction from the boundary condition (10) yields an optimal value function that exists and is unique. It also yields an optimal deterministic Markov policy, which may be non-unique when two actions tie [25].

Uncertainty model. Annual asset-class returns are modeled as independent and identically distributed across years. Their means and covariances are estimated by the sample moments in Eqs. (2)–(6) from the OJK [14] and Kirana [4] data. Market conditions are represented by the three scenarios A (aggressive, balanced, conservative) of Table 5. The selling probability is assumed constant across portfolio ages, $q(i) = q$ for all i. It is estimated from the same historical data as the relative frequency of years with a negative portfolio return, i.e., years in which a cut-loss sale would have been triggered:

$$\hat{q} = \frac{1}{N} \sum_{t=1}^{N} \mathbf{1}\{R_{p,t} < 0\}, \tag{12}$$

where $R_{p,t}$ is the portfolio return in year t and $\mathbf{1}\{\cdot\}$ is the indicator function. The complete computational procedure is given in Appendix A.

As explained in Figure 1, the author proposes the modification of the graph based on the problem context as a portfolio replacement, as illustrated in Figure 4. The orange line represents the decision to sell and buy a portfolio within the same year, resulting in the portfolio's age resetting to one in the following period. Meanwhile, the green and blue lines indicate the decision to continue the portfolio from the previous period, with the portfolio's age increasing. However, this decision still presents the choice of whether to perform rebalancing or proceed without rebalancing.

5. Result

Before introducing the Portfolio Replacement formulas (8)–(11), we first apply Markowitz optimization (1)–(7) to obtain the Markowitz trade-off values $t(i)$ for each portfolio age and period. The historical expected returns for each asset category are as follows: direct private equity (10%), property including buildings (8%), land only (7%), collective real estate investment funds (6%), and stocks listed on the Indonesia Stock Exchange (12%).

The numerical computation procedure can be visualized using the graph in Figure 4, where each node represents the objective function $W(i)$ from the computation. The stage-wise calculations correspond to the number of points in this graph. In this case study the horizon has $N = 6$ annual periods (2017–2022), with period 6 serving as the boundary condition, and the maximum portfolio age is $k = 7$. These values allow systematic evaluation of replacement decisions across the entire investment horizon.

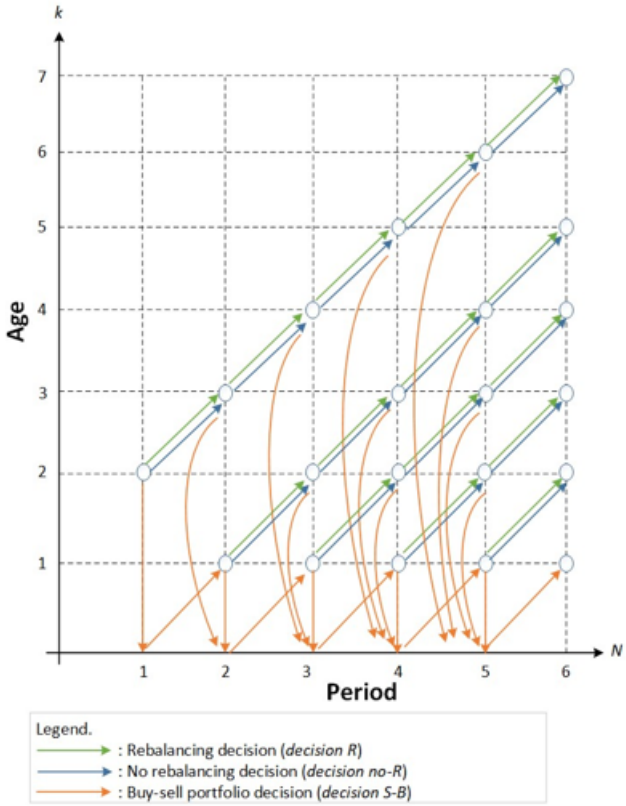


Figure 4. Modified equipment replacement graph with three decisions.

Table 8. Model parameters

Type	Parameters	Values	Range
Constants	N, k	6, 7	–
Costs	$c(i)$	5, 10, 12, 20.5, 24, 34, 20, 25	$i = 0$ to 7
Trade-offs	$t(i)$	5.6, 5.3, 4.8, 4.4, 6.3, 5.0, 5.2	$i = 1$ to 7
Salvage	$s(i)$	125, 680, 711, 156, 290, 450, ...	$i = 1$ to 7

5.1. Discussion

Based on the demonstration from the previous chapter, the backward procedure can be traced back, as concluded at point (1, 2). For this case study, the sequence of decisions is as follows: decision no-R, decision R, decision R, decision R, and decision S-B. Year 1 is held without rebalancing, and Years 2–4 are rebalanced with the Markowitz method. In Year 5 the portfolio is sold and a new one is bought (Figure 5). H1 is addressed through experimentation on a case study sample. The SDP model is solved while adjusting the probability value q in the formulas for the return $W_k(i)$ and the risk $F_k(i)$. Meanwhile, H2 is demonstrated in Figure 5, which shows the result of the calculation starting from period 6 (as the boundary condition) to node (1, 2). The decision-making process is then traced backward from period 6 to each preceding period.

Hypothesis outcomes. H2 is supported. The optimal policy (no-R, R, R, R, S-B for 2017–2021) contains two years, 2017 and 2021, in which annual rebalancing is not optimal. The replacement model therefore yields a different policy from fixed annual rebalancing. H1 cannot yet be accepted: it has been examined only by varying the selling probability q within a single in-sample case study, without benchmark strategies, performance metrics,

out-of-sample evaluation or statistical tests. We therefore report H1 as not confirmed by the present evidence and leave its formal test to future work (Section 5.3).

Benchmark comparison. Table 9 reports two baselines under the same input data. The first is single-period Markowitz optimization with annual rebalancing to the scenario weights, i.e., the allocations used by the model. The second is an equal-weight (1/5) portfolio, a standard reference strategy in the portfolio literature [38]. The inputs are the expected returns per asset class reported above and the scenario allocations of Table 5. Portfolio expected return is $E(R_p) = \sum_i w_i E(R_i)$, and cumulative expected growth over the five decision years 2017–2021 is $(1 + E(R_p))^5 - 1$, assuming annual rebalancing to fixed weights. The Markowitz allocations yield 10.56–10.70% per year, reproducing column g of Table 5, against 8.60% for equal weighting: an expected advantage of about 2.0–2.1 percentage points per year, or 14–15 percentage points over five years. Using the portfolio risk reported in column h of Table 5, the Markowitz allocations achieve a return-to-risk ratio $E(R_p)/\sigma_p$ of 0.93–0.95, i.e., a Sharpe-type ratio without a risk-free rate. This comparison is ex ante. Asset-level variances and correlations are not available for the full sample, so the risk and return-to-risk ratio of the benchmark, the full Sharpe ratio, maximum drawdown and a backtest on realized returns are left to future work (Section 5.3).

Economic significance. To judge whether this difference matters in practice, we scale it to the holdings in Table 4: the five asset classes amounted to IDR 41.5 trillion in September 2015. An expected advantage of 2.0–2.1 percentage points per year then corresponds to roughly IDR 0.8–0.9 trillion of additional expected investment income per year. Over the five decision years this amounts to about IDR 5.9–6.3 trillion, which is economically material for participants' benefits. Statistical significance is a different matter. We use the annual portfolio volatility of about 11% (Table 5) as a rough proxy for the volatility of the return difference. With five annual observations, a gap of 2 percentage points could not be distinguished from zero ($t \approx 0.4$). The advantage also comes with a higher concentration in listed equities (about 61% versus 20%) and would be reduced by the transaction costs of rebalancing (Section 5.3). We therefore interpret it as economically relevant but not statistically established.

Robustness across scenarios. The three Markowitz scenarios correspond to different risk tolerances ς, i.e., different points on the efficient frontier. The advantage over equal weighting is stable across them: 1.96, 2.10 and 2.05 percentage points per year for the aggressive, balanced and conservative allocations. The return-to-risk ratio varies only between 0.93 and 0.95. The comparison therefore does not hinge on the choice of ς or of the scenario. Sensitivity to the selling probability q and to transaction costs requires re-solving the recursion and is left to future research (Section 5.3).

Table 9. Comparison of two baselines: single-period Markowitz with annual rebalancing (three scenarios) and an equal-weight portfolio (weights in %; a–e as in Table 5; annual rebalancing, 2017–2021). σ_p is taken from column h of Table 5; it cannot be computed for the equal-weight portfolio without asset-level covariances (–).

Strategy	**a**	**b**	**c**	**d**	**e**	$E(R_p)$ (%/yr)	σ_p (%)	$E(R_p)/\sigma_p$	**5-yr growth** (%)
Markowitz – aggressive	12.2	15.5	11.4	0.1	60.7	10.56	11.11	0.95	65.2
Markowitz – balanced	17.8	13.6	7.7	0.2	60.7	10.70	11.53	0.93	66.3
Markowitz – conservative	16.8	12.7	8.4	1.4	60.7	10.65	11.41	0.93	65.9
Equal weight (1/5)	20.0	20.0	20.0	20.0	20.0	8.60	–	–	51.1

5.2. Computational complexity

The computational complexity of stochastic dynamic programming algorithms for equipment replacement problems is conventionally expressed using Big O notation, consistent with established complexity results for solving finite-horizon Markov decision problems [26]. This complexity depends on three parameters. The time horizon T is the number of discrete periods in the analysis. The state space size S is the number of possible machine states, such as wear levels or age categories. The action space size A is the number of decisions available at each stage, such as replace, maintain, or repair.

The asymptotic upper bound of the algorithm's complexity is given by $\mathcal{O}(T \times S \times A)$. This bound comes from three nested loops. The algorithm processes all T periods sequentially in the backward induction. In each period

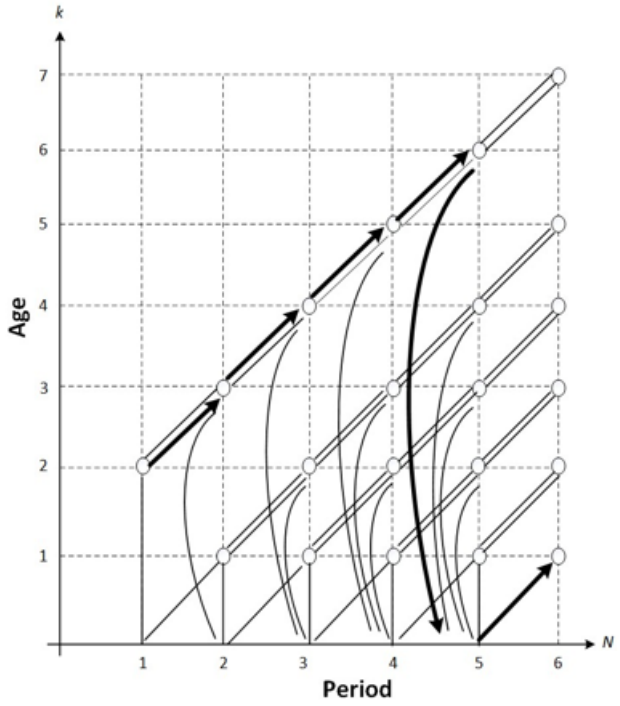


Figure 5. Tracking back optimal values from year to year (tracing-back policy function) for optimal decision making from year to year.

$t \in \{1, \ldots, T\}$, it evaluates all S possible states. For each state $s \in S$, it considers all A actions to optimize the value function. The product of these parameters reflects the exhaustive evaluation in dynamic programming: every combination of period, state, and action must be considered to guarantee optimality. This polynomial complexity is characteristic of solutions to Markov decision processes, where the state-action pairs must be fully enumerated at each decision epoch.

For the case study, $T = N = 6$ stages, at most $k = 7$ portfolio ages and three scenarios give $S \leq 7 \times 3 = 21$ states, and $A = 3$ actions. Backward induction therefore requires at most $6 \times 21 \times 3 = 378$ evaluations of the recurrence (9), plus one Markowitz quadratic program (7) per age–scenario pair to obtain $t(i)$. Adding asset classes does not enlarge the dynamic programming state space, because the weights are re-optimized inside each Markowitz subproblem; it only increases the dimension M of each quadratic program, which is solvable in polynomial time. The state space would instead grow combinatorially—as $\binom{m+M-1}{M-1}$ for weights discretized into m levels—if the current weights were added to the state. By comparison, scenario-tree formulations of multistage stochastic programming grow exponentially with the number of stages [32], and reinforcement learning avoids enumeration but requires long data histories for training [35].

Because wall-clock time depends on hardware and implementation, we report a hardware-independent operation count instead. Each of the at most 378 evaluations of the recurrence (9) compares three actions with a few additions and multiplications, i.e., on the order of 10 elementary operations, or about 4×10^3 in total. Each of the $7 \times 3 = 21$ Markowitz subproblems (7) is a convex quadratic program in $M = 5$ variables. An interior-point solver needs $\mathcal{O}(M^3) \approx 125$ operations per iteration and typically a few tens of iterations. This gives about $21 \times 30 \times 125 \approx 8 \times 10^4$ operations. The whole computation therefore requires on the order of 10^5 elementary operations, which on any current computer (about 10^9 operations per second) amounts to well under one second. Computational effort is thus not a practical constraint at this problem size; it becomes relevant only if the state space is enlarged as discussed above.

5.3. Practical limitations

Several practical considerations qualify the results. *Transaction costs and taxes:* rebalancing and, above all, sell-and-buy decisions incur trading costs and possibly taxes. With proportional costs it is optimal to trade only when weights leave a no-trade region [36]. Explicit costs would therefore make R and S-B less frequent than in the present policy, where costs enter only through $c(i)$. Multi-period models that include transaction costs and liquidity explicitly [39] indicate how such terms could be added. *Liquidity:* private equity and direct real estate cannot be sold quickly or at observed valuations [27], so a full S-B decision may be infeasible within one year for these asset classes. *Estimation error:* mean–variance weights are highly sensitive to errors in expected returns [37]. With short samples, optimized portfolios often fail to beat equal weighting out of sample [38]. Six annual observations are a short sample, so the inputs in Table 5 carry considerable uncertainty. *High-volatility assets and regulation:* cryptocurrency exhibits extreme volatility and regulatory uncertainty [28], which conflict with the capital-preservation mandate of pension funds, and Indonesian pension fund investments are subject to OJK rules on eligible instruments [31]. *Liabilities:* the present model optimizes the asset side only; liability matching and funding-ratio objectives are not modeled, although chance-constrained formulations such as [41] show how regulatory and liability requirements can enter an allocation model. Future work will add these constraints and compare the policy against buy-and-hold, equal-weight, annual-rebalancing and single-period Markowitz benchmarks. The comparison will use return, risk, Sharpe ratio and maximum drawdown, following backtesting designs such as [40]. It will also include sensitivity analyses on q, ς and transaction costs with out-of-sample evaluation.

6. Conclusion

This research applies a stochastic equipment-replacement framework to multi-period pension fund portfolio decisions in Indonesia. The mathematical results demonstrate that the stochastic dynamic programming method can be implemented. This study uses samples from OJK and Kirana [4] during the period 2017-2022, with decisions for 2017–2021 and 2022 as the terminal period. The calculation results show that in 2017 the portfolio was recommended without rebalancing, then from 2018 to 2020 it was recommended to optimize using the Markowitz formula (7). Finally, in the last decision year, 2021, it is recommended to sell and buy a new portfolio. The method could also be applied to the portfolios of other companies. Future studies can extend it by adjusting the dependent and independent variables and by adding more detailed financial aspects. The main limitations are the heuristic nature of the equipment-replacement analogy, the stylized uncertainty model, a benchmark comparison limited to expected return, and the absence of liability objectives. In particular, the optimal policy was computed for a single set of selling probabilities $q(i)$. Future research should test its sensitivity by re-solving the recursion (9) over a range of q values. It should also evaluate the policy out of sample, calibrating the model on an earlier window and testing it on later years. The Markowitz subproblem should also be extended with explicit liquidity, transaction-cost and regulatory constraints, such as the OJK investment limits per asset class. Finally, the policy should be compared with further baselines, including a static (no-rebalancing) allocation and a conventional multistage stochastic programming model, using risk-adjusted metrics on realized returns.

Acknowledgement

This study was supported by a grant from the Research Institute and Community Service (LPPM) at Telkom University, Surabaya Campus, Indonesia.

Disclosure of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

A. Computational procedure

Algorithm 1 gives the complete procedure used to compute the optimal portfolio replacement policy, so that the results can be reproduced from the inputs reported in the paper. The inputs are the scenario allocations and expected returns (Table 5 and Section 5), the model parameters $c(i)$, $t(i)$ and $s(i)$ (Table 8), the rebalancing reward ρ_k, the cut-loss terminal value $v(i)$, the initial portfolio age y, and the horizon N. The cut-loss value function $F_k(i)$ is computed by the same recursion as $W_k(i)$, starting from its own boundary condition $F_{N+1}(i) = v(i)$. Ages are bounded by the largest age reachable within the horizon, $i_{\max} = y + N$.

Algorithm 1 Backward induction for portfolio replacement and forward policy tracing

Require: Scenarios $\mathcal{A} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$; expected returns and covariances (Eqs. (2)–(6)); $c(i)$, $s(i)$, $v(i)$, ρ_k, N, y

Step 1 (single-period optimization). For each scenario $A \in \mathcal{A}$ and age $i = 1, \dots, i_{\max}$, solve the Markowitz problem (7) and record the trade-off value $t(i)$

Step 2 (uncertainty). Estimate the selling probability $\hat{q}$ by Eq. (12) and set $q(i) = \hat{q}$ for all i

Step 3 (boundary). For $i = 1, \dots, i_{\max}$: $W_{N+1}(i) \leftarrow s(i)$, $\quad F_{N+1}(i) \leftarrow v(i)$ ▷ Eq. (10)

Step 4 (backward induction).

for $k = N, N-1, \dots, 1$ **do**

  for each scenario $A \in \mathcal{A}$ **do**

    for $i = 1, \dots, i_{\max}$ **do**

      $V_{\text{R}} \leftarrow t(i) + \rho_k + W_{k+1}(i+1)$

      $V_{\text{no-R}} \leftarrow t(i) + W_{k+1}(i+1)$

      $V_{\text{S-B}} \leftarrow -c(i) + q(i)\, F_{k+1}(1) + \big(1 - q(i)\big)\, W_{k+1}(1)$

      $W_k(i) \leftarrow \max\{V_{\text{R}}, V_{\text{no-R}}, V_{\text{S-B}}\}$ ▷ Eq. (9)

      $\pi_k(i) \leftarrow \arg\max\{V_{\text{R}}, V_{\text{no-R}}, V_{\text{S-B}}\}$ ▷ store the optimal action

      compute $F_k(i)$ by the same three lines with F_{k+1} in place of W_{k+1}

    end for

  end for

end for

Step 5 (forward tracing). $i_1 \leftarrow y$

for $k = 1, \dots, N$ **do**

  $a_k \leftarrow \pi_k(i_k)$

  if $a_k \in \{\text{R}, \text{no-R}\}$ **then**

    $i_{k+1} \leftarrow i_k + 1$ ▷ portfolio carried forward

  else

    $i_{k+1} \leftarrow 1$ ▷ S-B: new portfolio

  end if

end for

Ensure: Optimal value $W_1(y)$ (Eq. (11)) and optimal decision path $(a_1, \dots, a_N)$

For the case study, Step 1 solves 21 small quadratic programs and Step 4 evaluates the recurrence at most 378 times (Section 5.2); the whole procedure can be implemented in a spreadsheet or in a few dozen lines of code. Step 5 produces the decision path of the type shown in Figure 5.

REFERENCES

1. A. Bilqis, N. Aliputri, R. A. Andini, S. Andini, and D. Lestari, *Problematika pengelolaan dana pensiun: Studi kasus tinjauan kendala sistematika dana pensiun badan usaha milik negara (BUMN)*, INNOVATIVE: Journal of Social Science Research, vol. 4, no. 3, pp. 6775–6784, 2024.
2. U. Islam, N. Sultan, and S. K. Riau, *Perkembangan dan tantangan pasar modal Indonesia*, Al Masraf: Jurnal Lembaga Keuangan dan Perbankan, vol. 1, no. 1, 2016.
3. S. Wahyudi, Hasanudin, and I. R. D. Pangestuti, *Asset allocation and strategies on investment portfolio performance: A study on the implementation of employee pension fund in Indonesia*, Accounting, vol. 6, no. 5, pp. 839–850, 2020. https://doi.org/10.5267/j.ac.2020.5.010
4. D. J. Kirana, *Penerapan portofolio investasi aset progam dana pensiun pada optimalisasi expected return*, JIAFE (Jurnal Ilmiah Akuntansi Fakultas Ekonomi), vol. 5, no. 2, pp. 131–144, 2020. https://doi.org/10.34204/jiafe.v5i2.1499
5. C. P. Barros and M. T. M. Garcia, *Analysing the performance of the pension fund industry with a stochastic frontier model: A case study for Portugal*, The Geneva Papers on Risk and Insurance - Issues and Practice, vol. 32, no. 2, pp. 190–210, 2007. https://doi.org/10.1057/palgrave.gpp.2510126
6. B. O. Draženović, S. Hodžić, and D. Maradin, *The efficiency of mandatory pension funds: Case of Croatia*, South East European Journal of Economics and Business, vol. 14, no. 2, pp. 82–94, 2019. https://doi.org/10.2478/jeb-2019-0015
7. R. D. P. Aritonang, *Konsep Investasi dalam Keuangan*, 1st ed., Qianzy Sains, 2024.
8. M. Syafii, *Manajemen Investasi Pensiun: Membangun Dana Pensiun yang Kokoh untuk Masa Depan*, 1st ed., Takaza Innovatix Labs, 2024.
9. E. W. Yuliasari, *Pengelolaan dana pensiun perlu perubahan fundamental guna dukung pembangunan nasional*, Direktorat Jenderal Kekayaan Negara (DJKN), Kemenkeu, 2020. Available: https://www.djkn.kemenkeu.go.id/berita/baca/22131/Pengelolaan-Dana-Pensiun-Perlu-Perubahan-
10. W. D. Fan, R. Machemehl, M. Gemar, and L. Brown, *A stochastic dynamic programming approach for the equipment replacement optimization under uncertainty*, Journal of Transportation Systems Engineering and Information Technology, vol. 14, no. 3, pp. 76–84, 2014. https://doi.org/10.1016/S1570-6672(13)60137-3
11. G. Infanger, *Chapter 5 - Dynamic asset allocation strategies using a stochastic dynamic programming approach*, in Handbook of Asset and Liability Management, S. A. Zenios and W. T. Ziemba, Eds. North-Holland, 2008, pp. 199–251. https://doi.org/10.1016/B978-044453248-0.50011-3
12. F. R. Naibaho, *Optimalisasi keputusan untuk memperoleh keuntungan maksimal dalam penentuan kelayakan operasional kendaraan dengan menggunakan metode dynamic programming pada PT. Tor Ganda*, ResearchGate, 2015. https://www.researchgate.net/publication/286480844
13. G. L. Clark and T. Hebb, *New pension funds and urban revitalization New York case study: Competitive returns and a revitalized New York*, ResearchGate, 2005. https://www.researchgate.net/publication/266095210
14. Otoritas Jasa Keuangan, *Tabel laporan aset bersih, perhitungan hasil usaha dan neraca dana pensiun*, 2015. [Online]. Available: https://ojk.go.id/id/kanal/iknb/data-dan-statistik/dana-pensiun/Pages/Laporan-Aset-Neto-Neraca-Hasil-Usaha-Dapen-JanSep2015.aspx
15. U. July and J. Chen, *What are alternative investments? Definition and examples*, Investopedia, 2024. [Online]. Available: https://www.investopedia.com/terms/a/alternative_investment.asp
16. O. Krisnawan, *Optimalisasi portofolio investasi program tunjangan hari tua (THT) pada PT Taspen (persero)*, Indonesian Treasury Review: Jurnal Perbendaharaan, Keuangan Negara dan Kebijakan Publik, vol. 4, no. 1, pp. 59–69, 2019. https://doi.org/10.33105/itrev.v4i1.117
17. D. Putrihadiningrum and C. E. Violita, *Optimizing portfolio performance at PT. IAI Financial*, GREENOMIKA, vol. 5, no. 2, pp. 206–211, 2023. https://doi.org/10.55732/unu.gnk.2023.05.2.9
18. H. Markowitz, *Portfolio selection*, The Journal of Finance, vol. 7, no. 1, pp. 77–91, 1952. https://doi.org/10.2307/2975974
19. W. F. Sharpe, *Capital asset prices: A theory of market equilibrium under conditions of risk*, The Journal of Finance, vol. 19, no. 3, pp. 425–442, 1964. https://doi.org/10.1111/j.1540-6261.1964.tb02865.x
20. W. F. Sharpe and L. G. Tint, *Liabilities—A new approach*, The Journal of Portfolio Management, vol. 16, no. 2, pp. 5–10, 1990. https://doi.org/10.3905/jpm.1990.409248
21. P. A. Samuelson, *Lifetime portfolio selection by dynamic stochastic programming*, The Review of Economics and Statistics, vol. 51, no. 3, pp. 239–246, 1969. https://doi.org/10.2307/1926559
22. R. C. Merton, *Lifetime portfolio selection under uncertainty: The continuous-time case*, The Review of Economics and Statistics, vol. 51, no. 3, pp. 247–257, 1969. https://doi.org/10.2307/1926560
23. R. Bellman, *Equipment replacement policy*, Journal of the Society for Industrial and Applied Mathematics, vol. 3, no. 3, pp. 133–136, 1955. https://doi.org/10.1137/0103015
24. D. R. Cariño, T. Kent, D. H. Myers, C. Stacy, M. Sylvanus, A. L. Turner, K. Watanabe, and W. T. Ziemba, *The Russell-Yasuda Kasai model: An asset/liability model for a Japanese insurance company using multistage stochastic programming*, Interfaces, vol. 24, no. 1, pp. 29–49, 1994. https://doi.org/10.1287/inte.24.1.29
25. M. L. Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, John Wiley & Sons, 1994.
26. M. L. Littman, T. L. Dean, and L. P. Kaelbling, *On the complexity of solving Markov decision problems*, in Proceedings of the Eleventh Conference on Uncertainty in Artificial Intelligence (UAI-95), pp. 394–402, 1995.
27. R. S. Harris, T. Jenkinson, and S. N. Kaplan, *Private equity performance: What do we know?*, The Journal of Finance, vol. 69, no. 5, pp. 1851–1882, 2014. https://doi.org/10.1111/jofi.12154
28. A. Petukhina, S. Trimborn, W. K. Härdle, and H. Elendner, *Investing with cryptocurrencies – evaluating their potential for portfolio allocation strategies*, Quantitative Finance, vol. 21, no. 11, pp. 1825–1853, 2021. https://doi.org/10.1080/14697688.2021.1880023
29. A. Bányai, T. Tatay, G. Thalmeiner, and L. Pataki, *The impact of rebalancing strategies on ETF portfolio performance*, Journal of Risk and Financial Management, vol. 17, no. 12, article 533, 2024. https://doi.org/10.3390/jrfm17120533

30. E. F. Fama and K. R. French, *Common risk factors in the returns on stocks and bonds*, Journal of Financial Economics, vol. 33, no. 1, pp. 3–56, 1993. https://doi.org/10.1016/0304-405X(93)90023-5
31. OECD, *Retirement Savings Protection Gaps and Challenges to Achieve Adequate Retirement Outcomes: Addressing the Challenges for Asset-backed Pensions in Indonesia*, OECD Publishing, 2025. Available: https://www.oecd.org/en/publications/addressing-the-challenges-for-asset-backed-pensions-in-indonesia_37f4700d-en.html
32. S. A. Zenios and W. T. Ziemba, Eds., *Handbook of Asset and Liability Management, Volume 2: Applications and Case Studies*, North-Holland (Elsevier), Amsterdam, 2007. ISBN 978-0-444-52802-5.
33. D. Goldfarb and G. Iyengar, *Robust portfolio selection problems*, Mathematics of Operations Research, vol. 28, no. 1, pp. 1–38, 2003. https://doi.org/10.1287/moor.28.1.1.14260
34. T. M. Cover, *Universal portfolios*, Mathematical Finance, vol. 1, no. 1, pp. 1–29, 1991. https://doi.org/10.1111/j.1467-9965.1991.tb00002.x
35. J. Moody and M. Saffell, *Learning to trade via direct reinforcement*, IEEE Transactions on Neural Networks, vol. 12, no. 4, pp. 875–889, 2001. https://doi.org/10.1109/72.935097
36. M. H. A. Davis and A. R. Norman, *Portfolio selection with transaction costs*, Mathematics of Operations Research, vol. 15, no. 4, pp. 676–713, 1990. https://doi.org/10.1287/moor.15.4.676
37. V. K. Chopra and W. T. Ziemba, *The effect of errors in means, variances, and covariances on optimal portfolio choice*, The Journal of Portfolio Management, vol. 19, no. 2, pp. 6–11, 1993. https://doi.org/10.3905/jpm.1993.409440
38. V. DeMiguel, L. Garlappi, and R. Uppal, *Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?*, The Review of Financial Studies, vol. 22, no. 5, pp. 1915–1953, 2009. https://doi.org/10.1093/rfs/hhm075
39. K. Belabbes, M. El Hachloufi, and Z. E. A. Guennoun, *Mean-TVaR models for diversified multi-period portfolio optimization with realistic factors based on uncertainty theory*, Statistics, Optimization & Information Computing, vol. 11, no. 4, pp. 963–977, 2023. https://doi.org/10.19139/soic-2310-5070-1657
40. A. Bouhmady, H. Kadiri, K. Belkhoutout, and N. Raissi, *Dynamic portfolio optimisation in Morocco's stock market through machine-learning selection and complex mean-variance allocation*, Statistics, Optimization & Information Computing, vol. 14, no. 2, pp. 663–676, 2025. https://doi.org/10.19139/soic-2310-5070-2529
41. A. Ghania, N. Albehery, and M. Helal, *A lexicographic bi-objective chance-constrained model for asset-liability management with endogenous reliability: Theory and application to Banque Misr*, Statistics, Optimization & Information Computing, vol. 15, no. 6, pp. 4975–4992, 2026. https://doi.org/10.19139/soic-2310-5070-3337