

Concomitant of Consecutive r -Out-of- $m:W$ System Models through Information-Theoretic Methods with Industrial Modeling Data Application

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Abstract Tsallis entropy is investigated as an uncertainty measure for the concomitants of consecutive r -out-of- $m:W$ systems arising from the Farlie–Gumbel–Morgenstern family, a dependence model frequently encountered in reliability and engineering analyses. A series representation of the proposed entropy measure is established, and its implementation is illustrated through the log-logistic distribution. In addition, stochastic ordering results, theoretical bounds, and characterization properties are derived to reveal the uncertainty behavior of the underlying system. To facilitate practical inference, two distribution-free estimators of the proposed entropy are introduced and examined through extensive Monte Carlo simulation experiments. The usefulness of the proposed methodology is finally demonstrated by analyzing a real dataset.

Keywords Bivariate distributions, Consecutive r -out-of- $m:W$ systems, Concomitants, Stochastic ordering; Tsallis entropy, Nonparametric inferences

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1. Introduction

The analysis of coherent reliability systems has attracted considerable attention because of its importance in modeling the performance of engineering devices, communication infrastructures, manufacturing systems, and other complex networks. Among these models, consecutive systems represent a well-established class in which the operational status depends not only on the condition of individual components but also on their relative positions within the system. According to the underlying success criterion, these systems are categorized into consecutive r -out-of- $m:F$ (fail) and consecutive r -out-of- $m:W$ (good or working) configurations. The former experiences failure whenever a block of at least r adjacent components fails, whereas the latter continues to operate provided that there exists a sequence of at least r consecutive functioning components.

Boundary situations include the well-known series and parallel arrangements: The m -out-of- $m:F$ system (equivalent to 1-out-of- $m:W$) represents a parallel arrangement, whereas the 1-out-of- $m:F$ system (corresponding to m -out-of- $m:W$) corresponds to a classical series structure. Because of their ability to capture local dependence among neighboring components, consecutive system models have been successfully employed in numerous

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practical settings. Representative applications include communication and computer networks, transportation infrastructures, oil and gas distribution systems, and integrated electronic circuits. As an illustration, a pipeline consisting of consecutively connected pumping stations may become inoperative when a sufficiently long sequence of adjacent stations fails, interrupting the transport process. Similar considerations arise in communication systems, where failures occurring in neighboring transmission units may disconnect portions of the network. General accounts of consecutive systems and their theoretical properties are available in [1], while further developments concerning their reliability behavior may be found in [2], [3], and [4].

Suppose that Y denotes the lifetime of a generic component with probability density function $g_Y(y)$, cumulative distribution function $G_Y(y)$, and survival function $\bar{G}_Y(y) = 1 - G_Y(y)$. Let $T_{(r|m:W)}$ denote the lifetime of the corresponding consecutive r -out-of- $m:W$ system. When the condition $2r \geq m$ is satisfied, the survival function of $T_{(r|m:W)}$ admits the explicit representation

$$\bar{G}_{Y,(r|m:W)}(t) = P(T_{(r|m:W)} > t) = (m - r + 1) \bar{G}_Y^r(t) - (m - r) \bar{G}_Y^{r+1}(t), \quad t > 0, \quad (1)$$

which provides a convenient analytical expression for investigating the reliability characteristics of consecutive systems. Consequently, the corresponding probability density function of $T_{(r|m:W)}$ is obtained as

$$g_{Y,(r|m:W)}(t) = r(m - r + 1) \bar{G}_Y^{r-1}(t) g_Y(t) - (m - r)(r + 1) \bar{G}_Y^r(t) g_Y(t), \quad t > 0. \quad (2)$$

In the analysis of bivariate random variables, situations frequently arise in which the marginal distributions are available in advance. Under such circumstances, constructing dependence models based on the known marginals becomes particularly useful. The Farlie–Gumbel–Morgenstern family provides one of the classical approaches for this purpose and has consequently received considerable attention in the statistical literature. Its joint cumulative distribution function and probability density function are given by

$$G_{X,Y}(x, y) = G_X(x) G_Y(y) [1 + \xi(1 - G_X(x))(1 - G_Y(y))], \quad -1 \leq \xi \leq 1, \quad (3)$$

$$g_{X,Y}(x, y) = g_X(x) g_Y(y) [1 + \xi(2G_X(x) - 1)(2G_Y(y) - 1)], \quad -1 \leq \xi \leq 1, \quad (4)$$

where $G_X(x)$ and $G_Y(y)$ denote the marginal cumulative distribution functions of the random variables X and Y , respectively. Although the classical Farlie–Gumbel–Morgenstern model provides a convenient framework for constructing bivariate distributions from specified marginals, its dependence structure is relatively weak. The earliest version of this family was proposed by Morgenstern [5], while Gumbel [6] later investigated the model for exponential marginal distributions. Subsequently, Farlie [7] presented the general formulation given in Eq. (3). To guarantee that $G_{X,Y}(x, y)$ defines a legitimate bivariate cumulative distribution function, the dependence parameter must satisfy $-1 \leq \xi \leq 1$. Under this restriction, the attainable Pearson correlation coefficient between X and Y is bounded in magnitude by $1/3$, which limits the model's capability to describe strong dependence. Motivated by this limitation, a variety of generalized and modified Farlie–Gumbel–Morgenstern families have been introduced to achieve greater flexibility and stronger dependence structures. These extensions have attracted considerable attention in both theoretical investigations and practical applications; representative contributions include those of Johnson and Kotz [8], Huang and Kotz [9, 10], and Bairamov et al. [11].

When addressing problems involving selection, ranking, and prediction, concomitants of order statistics constitute an important statistical concept. The notion of concomitants was originally introduced by David et al. [12] and, independently, by Bhattacharya [13], who referred to them as induced order statistics.

Suppose that $\{(X_k, Y_k), k = 1, 2, \dots, m\}$ is a random sample of independent observations from a bivariate distribution with joint cumulative distribution function $G_{X,Y}(x, y)$. Let $X_{(r:m)}$ denote the r th order statistic obtained from the X -sample. The corresponding observation on the associated variable Y is called the concomitant of the r th order statistic and is denoted by $Y_{[r:m]}$. The probability density function and cumulative distribution

function of $Y_{[r:m]}$ are given, respectively, by

$$g_{Y,[r:m]}(y) = \int_{-\infty}^{\infty} g_{Y|X}(y | x) g_{X,(r:m)}(x) dx, \quad (5)$$

and

$$G_{Y,[r:m]}(y) = \int_{-\infty}^{\infty} G_{Y|X}(y | x) g_{X,(r:m)}(x) dx. \quad (6)$$

Concomitants of order statistics naturally appear in a wide range of practical problems. One important application arises in selection procedures, where observations are selected according to an easily measured variable X , while inference is required for a related characteristic Y that is either difficult, expensive, or impossible to observe immediately. They also play a fundamental role in ranked set sampling, where ranking units can be performed with relatively little effort, whereas obtaining precise measurements of the variable of interest involves considerable cost or time. Consequently, concomitant-based methods provide an efficient framework for statistical inference in such settings.

Interest in concomitant distributions has grown substantially over the past few years, motivated by their usefulness in reliability theory, information theory, and statistical estimation. Numerous studies have investigated their probabilistic and inferential properties under different bivariate distribution families. For example, Almaspoor et al. [14] introduced extropy-based measures for concomitants of generalized order statistics under the Morgenstern family of distributions. Husseiny et al. [15] explored entropy measures together with Fisher information for concomitants arising from the Sarmanov family. Comparative distributional properties for the Huang–Kotz family were established by Barakat and Syam [16]. More recently, Al-Zaydi [17] derived several distributional characteristics and proposed best linear unbiased estimators for concomitants associated with a bivariate generalized Weibull distribution. Subsequently, Al-Zaydi [18] extended these developments by investigating theoretical properties and practical applications of concomitants generated from a bivariate generalized linear exponential distribution. These contributions highlight the growing importance of concomitant distributions as powerful tools for modern statistical inference, reliability modeling, and information-theoretic analysis.

Within information theory, quantifying the uncertainty inherent in probability distributions is one of the principal objectives. In this work, Tsallis entropy is employed as an uncertainty measure for consecutive r -out-of- m : W systems whose component lifetimes are assumed to follow continuous distributions. The introduction of Tsallis entropy by Tsallis [19] renewed significant interest in generalized entropy measures, extending ideas that originated in the work of Havrda and Charvát [20] and were independently explored in ecological diversity studies by Patil and Taillie [21]. Owing to its non-additive nature, Tsallis entropy has become an attractive alternative to Shannon entropy for modeling systems that exhibit non-extensive characteristics. For a random lifetime T , the Tsallis entropy of order γ is defined by

$$Ts_{\gamma}(T) = \frac{1}{1-\gamma} \left[\int_0^{\infty} g_T^{\gamma}(t) dt - 1 \right] = \frac{1}{1-\gamma} \left[\int_0^1 g_T^{\gamma-1}(G_T^{-1}(u)) du - 1 \right], \quad (7)$$

where $\gamma \in \Xi = (0, 1) \cup (1, \infty)$ and $G_T^{-1}(u) = \inf\{t : G_T(t) \geq u\}$, $u \in [0, 1]$, denotes the quantile function associated with the distribution function $G_T(t)$. A fundamental property of Tsallis entropy is that it converges to the classical Shannon differential entropy as the parameter γ approaches unity. Consequently, Shannon entropy [22] is recovered as a limiting case and serves as the standard additive measure of uncertainty. Specifically,

$$H(T) = \lim_{\gamma \rightarrow 1} Ts_{\gamma}(T) = - \int_0^{\infty} g_T(t) \log g_T(t) dt. \quad (8)$$

For applications in reliability theory, it is often convenient to rewrite Tsallis entropy in terms of the hazard rate function. This representation establishes a direct connection between entropy and lifetime characteristics, thereby facilitating reliability analysis. Under this formulation,

$$Ts_\gamma(T) = \frac{1}{1-\gamma} \left(\frac{1}{\gamma} E[\lambda_T^{\gamma-1}(T_\gamma)] - 1 \right), \quad (9)$$

where the hazard rate is defined by $\lambda_T(t) = g_T(t)/\bar{G}_T(t)$, $E(\cdot)$ denotes mathematical expectation, and the random variable T_γ has probability density function

$$g_{T_\gamma}(t) = \gamma g_T(t) \bar{G}_T^{\gamma-1}(t), \quad t > 0, \gamma > 0. \quad (10)$$

The density function in Eq. (10) corresponds to the well-known proportional hazards model. This formulation generates a broad family of lifetime distributions that are commonly referred to as exponentiated survival distributions in the reliability and survival analysis literature.

Alqefari et al. [23] investigated the Tsallis entropy of consecutive r -out-of- $m:W$ good systems together with its statistical inference, theoretical bounds, and characterization results. Motivated by this work, the present article extends the study to the concomitant of consecutive r -out-of- $m:W$ good systems under the Farlie–Gumbel–Morgenstern family and develops several new theoretical and inferential results. The remainder of the paper is structured as follows. Section 2 derives the probability density and cumulative distribution functions of the concomitant of consecutive r -out-of- $m:W$ systems under the Farlie–Gumbel–Morgenstern family. The corresponding Tsallis entropy is then formulated, together with its series expansion and an illustration based on the log-logistic distribution. Stochastic-order comparisons are also established. The proposed development is constructed by exploiting the uniform distribution as a baseline model, leading to tractable analytical expressions. Section 3 establishes several theoretical bounds and characterization results that further clarify the uncertainty properties of the proposed entropy measure. Section 4 presents numerical investigations to support the theoretical findings and introduces two nonparametric estimators for the proposed Tsallis entropy. Their finite-sample performance is examined through Monte Carlo simulations, followed by an application to a real dataset.

2. Concomitant of consecutive r -out-of- $m:W$ systems of Farlie–Gumbel–Morgenstern family

In this section, we will express the probability density function and cumulative distribution function of the concomitant of consecutive r -out-of- $m:W$ systems of the Farlie–Gumbel–Morgenstern family as illustrated in the following theorems.

Theorem 2.1

Consider the absolutely continuous, nonnegative random variable Y that follows the probability density function $g_Y(y)$ and the cumulative distribution function $G_Y(y)$. Based on Farlie–Gumbel–Morgenstern family, the density of the concomitant of consecutive r -out-of- $m:W$ systems, $Y_{[r|m:W]}$, can be expressed as

$$g_{Y_{[r|m:W]}}(y) = g_Y(y) [1 + \xi C_{r,m} (2G_Y(y) - 1)], \quad (11)$$

where $|\xi| \leq 1$, $2r \geq m$, and $C_{r,m} = \frac{-r^2-3r+2m+2}{(r+1)(r+2)}$.

Proof

From (4), the conditional probability density function of the Farlie–Gumbel–Morgenstern family is

$$g_{Y|X}(y|x) = g_Y(y) [1 + \xi (2G_X(x) - 1)(2G_Y(y) - 1)], \quad -1 \leq \xi \leq 1. \quad (12)$$

Following the manner in Eq. (5), and using Eq.'s (12) and (2), we have

$$\begin{aligned} g_{Y,[r|m:W]}(y) &= \int_0^\infty g_{Y|X}(y|x) g_{X,(r|m:W)}(x) dx \\ &= g_Y(y) \int_0^\infty [1 + \xi(2G_X(x) - 1)(2G_Y(y) - 1)] g_{X,(r|m:W)}(x) dx \\ &= g_Y(y) [1 + \xi(2G_Y(y) - 1)I_r], \end{aligned}$$

where

$$I_r = \int_0^\infty (2G_X(x) - 1) g_{X,(r|m:W)}(x) dx.$$

Now,

$$\begin{aligned} I_r &= r(m - r + 1) \int_0^\infty (2G_X(x) - 1) \bar{G}_X^{r-1}(x) g_X(x) dx \\ &\quad - (r + 1)(m - r) \int_0^\infty (2G_X(x) - 1) \bar{G}_X^r(x) g_X(x) dx. \end{aligned}$$

Let $u = \bar{G}_X(x) = 1 - G_X(x)$, $du = -g_X(x) dx$. Since $2G_X(x) - 1 = 1 - 2u$, we have

$$\begin{aligned} \int_0^\infty (2G_X(x) - 1) \bar{G}_X^m(x) g_X(x) dx &= \int_0^1 (1 - 2u) u^m du \\ &= \frac{1}{m+1} - \frac{2}{m+2} = -\frac{m}{(m+1)(m+2)}. \end{aligned}$$

Therefore,

$$\int_0^\infty (2G_X(x) - 1) \bar{G}_X^{r-1}(x) g_X(x) dx = -\frac{r-1}{r(r+1)},$$

and

$$\int_0^\infty (2G_X(x) - 1) \bar{G}_X^r(x) g_X(x) dx = -\frac{r}{(r+1)(r+2)}.$$

Hence,

$$I_r = -\frac{(r-1)(m-r+1)}{r+1} + \frac{r(m-r)}{r+2}.$$

Consequently,

$$g_{Y,[r|m:W]}(y) = g_Y(y) \left[1 + \xi \frac{-r^2 - 3r + 2m + 2}{(r+1)(r+2)} (2G_Y(y) - 1) \right],$$

which completes the proof. $\square$

Theorem 2.2

Consider the absolutely continuous, nonnegative random variable Y that follows the probability density function $g_Y(y)$ and the cumulative distribution function $G_Y(y)$. Based on the Farlie–Gumbel–Morgenstern family, the distribution function of the concomitant of consecutive r -out-of- $m:W$ systems, $Y_{[r|m:W]}$, can be expressed as

$$G_{Y,[r|m:W]}(y) = G_Y(y) [1 + \xi D_{r,m}(1 - G_Y(y))], \quad (13)$$

where $|\xi| \leq 1$, $2r \geq m$, and

$$D_{r,m} = \frac{r^2 + 3r - 2m - 2}{(r+1)(r+2)}.$$

Proof

From (3), the conditional probability density function of the Farlie–Gumbel–Morgenstern family is

$$G_{Y|X}(y|x) = G_Y(y) [1 + \xi(1 - 2G_X(x))(1 - G_Y(y))], \quad -1 \leq \xi \leq 1, \quad (14)$$

Following the manner in Eq. (6), and using Eq.'s (14) and (2), we have

$$\begin{aligned} G_{Y,[r|m:W]}(y) &= \int_0^\infty G_{Y|X}(y|x) g_{X,(r|m:W)}(x) dx \\ &= G_Y(y) \int_0^\infty [1 + \xi(1 - 2G_X(x))(1 - G_Y(y))] g_{X,(r|m:W)}(x) dx \\ &= G_Y(y) [1 + \xi(1 - G_Y(y))I_r], \end{aligned}$$

where

$$I_r = r(m - r + 1) \int_0^\infty (1 - 2G_X(x)) \bar{G}_X^{r-1}(x) g_X(x) dx - (r + 1)(m - r) \int_0^\infty (1 - 2G_X(x)) \bar{G}_X^r(x) g_X(x) dx.$$

Now let $u = \bar{G}_X(x) = 1 - G_X(x)$, $du = -g_X(x) dx$. Then

$$\begin{aligned} I_r &= r(m - r + 1) \int_0^1 (2u - 1)u^{r-1} du - (m - r)(r + 1) \int_0^1 (2u - 1)u^r du \\ &= r(m - r + 1) \left[\frac{2}{r+1} - \frac{1}{r} \right] - (m - r)(r + 1) \left[\frac{2}{r+2} - \frac{1}{r+1} \right]. \end{aligned}$$

Therefore,

$$G_{Y,[r|m:W]}(y) = G_Y(y) \left[1 + \xi \frac{r^2 + 3r - 2m - 2}{(r+1)(r+2)} (1 - G_Y(y)) \right],$$

which completes the proof. $\square$

2.1. Tsallis entropy of the concomitant of consecutive r -out-of- $m:W$ systems

This subsection develops the Tsallis entropy corresponding to the concomitant of consecutive r -out-of- $m:W$ systems, denoted by $Y_{[r|m:W]}$, under the Farlie–Gumbel–Morgenstern family of distributions. To simplify the subsequent derivations, we introduce the transformed random variable

$$V_{[r|m:W]} = G(Y_{[r|m:W]}),$$

where G denotes the continuous cumulative distribution function of the baseline component lifetime. Since G is the parent distribution function rather than the distribution function of $Y_{[r|m:W]}$, the random variable $V_{[r|m:W]}$ is generally not uniformly distributed on $(0, 1)$. Instead, its distribution is induced by that of $Y_{[r|m:W]}$. Assuming that the component lifetimes are independent and identically distributed, and using (11), the probability density function of $V_{[r|m:W]}$ for the case $2r \geq m$ is

$$\Delta_{[r|m:W]}(v) = g_{V,[r|m:W]}(v) = 1 + \xi C_{r,m}(2v - 1), \quad 0 \leq v \leq 1. \quad (15)$$

Consequently, the Tsallis entropy associated with the system lifetime $T_{[r|m:W]}$ can be expressed as

$$\begin{aligned} Ts_\gamma(T_{[r|m:W]}) &= \frac{1}{1-\gamma} \left[\int_0^\infty g_{T,[r|m:W]}^\gamma(t) dt - 1 \right] \\ &= \frac{1}{1-\gamma} \left[\int_0^\infty g_T^\gamma(t) [1 + \xi C_{r,m} (2G_T(t) - 1)]^\gamma dt - 1 \right] \\ &= \frac{1}{1-\gamma} \left[\int_0^1 g_T^{\gamma-1}(G_T^{-1}(v)) [1 + \xi C_{r,m} (2v - 1)]^\gamma dv - 1 \right] \\ &= \frac{1}{1-\gamma} \left[\int_0^1 g_T^{\gamma-1}(G_T^{-1}(v)) \Delta_{[r|m:W]}^\gamma(v) dv - 1 \right], \end{aligned} \quad (16)$$

where the condition $2r \geq m$ is assumed throughout, and $\gamma \in \Xi$.

The next theorem provides an alternative expansion of $Ts_\gamma(T_{[r|m:W]})$ by combining Newton's generalized binomial theorem with the classical binomial expansion and a Beta-distribution identity.

Theorem 2.3

Suppose that $|\xi| \leq 1$, $2r \geq m$, and $|\xi C_{r,m}| < 1$. Then, for every $\gamma \in \Xi$, the Tsallis entropy of $T_{[r|m:W]}$ admits the representation

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\sum_{j=0}^{\infty} \binom{\gamma}{j} (\xi C_{r,m})^j \sum_{k=0}^j \binom{j}{k} 2^k (-1)^{j-k} \frac{E[g_T^{\gamma-1}(G_T^{-1}(U_k))]}{k+1} - 1 \right], \quad (17)$$

where $U_k \sim \text{Beta}(k+1, 1)$.

Proof

From (16), we have

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\int_0^1 g_T^{\gamma-1}(G_T^{-1}(v)) \Delta_{[r|m:W]}^\gamma(v) dv - 1 \right].$$

Since $|\xi C_{r,m}| < 1$ and $|2v - 1| \leq 1$ for $0 \leq v \leq 1$, it follows that $|\xi C_{r,m}(2v - 1)| < 1$, $0 \leq v \leq 1$. Hence, by Newton's generalized binomial theorem,

$$\Delta_{[r|m:W]}^\gamma(v) = \sum_{j=0}^{\infty} \binom{\gamma}{j} (\xi C_{r,m})^j (2v - 1)^j.$$

Moreover, the above series converges uniformly on $[0, 1]$. Consequently,

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\sum_{j=0}^{\infty} \binom{\gamma}{j} (\xi C_{r,m})^j \int_0^1 g_T^{\gamma-1}(G_T^{-1}(v)) (2v - 1)^j dv - 1 \right].$$

Next, applying the classical binomial theorem,

$$(2v - 1)^j = \sum_{k=0}^j \binom{j}{k} (2v)^k (-1)^{j-k} = \sum_{k=0}^j \binom{j}{k} 2^k (-1)^{j-k} v^k.$$

Substituting this expansion into the preceding expression yields

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\sum_{j=0}^{\infty} \binom{\gamma}{j} (\xi C_{r,m})^j \sum_{k=0}^j \binom{j}{k} 2^k (-1)^{j-k} \int_0^1 g_T^{\gamma-1}(G_T^{-1}(v)) v^k dv - 1 \right].$$

Now, let $U_k \sim \text{Beta}(k+1, 1)$. Therefore,

$$\int_0^1 g_T^{\gamma-1}(G_T^{-1}(v)) u^k du = \frac{E[g_T^{\gamma-1}(G_T^{-1}(U_k))]}{k+1}.$$

Substituting this identity into the above expression establishes (17). $\square$

We examine the following exemplary example to show the use of the representation defined in Eq. 16.

Example 2.1

Consider the lifetime $T_{[r|m:W]}$ associated with the proposed model. Suppose that the baseline component lifetimes are independent and identically distributed according to the log-logistic (Fisk) distribution with shape parameter λ and unit scale. Its probability density function is given by

$$g_T(t) = \frac{\lambda t^{\lambda-1}}{(1+t^\lambda)^2}, \quad t > 0, \lambda > 0. \quad (18)$$

The corresponding cumulative distribution function is $G_T(t) = \frac{t^\lambda}{1+t^\lambda}$, and therefore $G_T^{-1}(v) = \left(\frac{v}{1-v}\right)^{1/\lambda}$, $0 < v < 1$. After straightforward algebraic manipulations, it follows that

$$g_T(G_T^{-1}(v)) = \lambda v^{\frac{\lambda-1}{\lambda}} (1-v)^{\frac{\lambda+1}{\lambda}}, \quad 0 < v < 1.$$

Consequently,

$$g_T^{\gamma-1}(G_T^{-1}(v)) = \lambda^{\gamma-1} v^{\frac{(\lambda-1)(\gamma-1)}{\lambda}} (1-v)^{\frac{(\lambda+1)(\gamma-1)}{\lambda}}, \quad 0 < v < 1.$$

Substituting the above identity into (16), we obtain

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\lambda^{\gamma-1} \int_0^1 \Delta_{[r|m:W]}^\gamma(v) v^{\frac{(\lambda-1)(\gamma-1)}{\lambda}} (1-v)^{\frac{(\lambda+1)(\gamma-1)}{\lambda}} dv - 1 \right], \quad (19)$$

where $\Delta_{[r|m:W]}(v) = 1 + \xi C_{r,m}(2v-1)$ is defined in (15). Furthermore, applying Theorem 2.3 yields the series representation

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\lambda^{\gamma-1} \sum_{j=0}^{\infty} \binom{\gamma}{j} (\xi C_{r,m})^j \sum_{k=0}^j \binom{j}{k} 2^k (-1)^{j-k} \frac{E \left[U_k^{\frac{(\lambda-1)(\gamma-1)}{\lambda}} (1-U_k)^{\frac{(\lambda+1)(\gamma-1)}{\lambda}} \right]}{k+1} - 1 \right],$$

where $U_k \sim \text{Beta}(k+1, 1)$.

Since a closed-form expression is generally unavailable, numerical techniques are employed to investigate the behavior of $Ts_\gamma(T_{[r|m:W]})$ with respect to the parameters γ , λ , the dependence parameter ξ , and the system parameters (r, m) . The numerical results presented for concomitants of a consecutive 2-out-of-4:W good system whose lifetime is given by, see Figure 1:

$$T_{(2|4:W)} = \max(\min(T_1, T_2), \min(T_2, T_3), \min(T_3, T_4)).$$

Figure 2 illustrates the behavior of $Ts_\gamma(T_{[2|4:W]})$ as a function of the parameters γ and λ . The figure shows that the Tsallis entropy decreases monotonically as either γ or λ increases. The decline is relatively gradual for small

values of the two parameters but becomes considerably more pronounced when both γ and λ are large. In particular, the effect of increasing γ is amplified for larger values of λ , resulting in a steeper decrease in the entropy surface. These results indicate that both the Tsallis parameter and the shape parameter contribute to reducing the uncertainty associated with the lifetime $T_{[2|4:W]}$, with the influence of γ being more substantial over most of the parameter range.

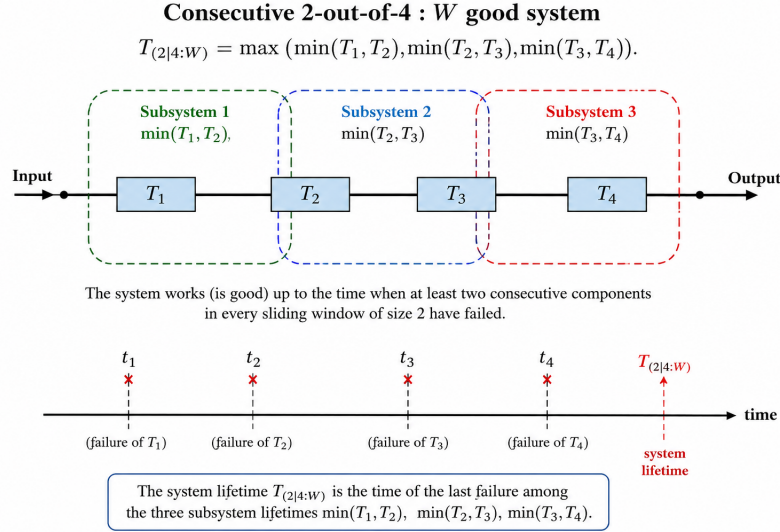


Figure 1. Consecutive 2-out-of-4: W good system.

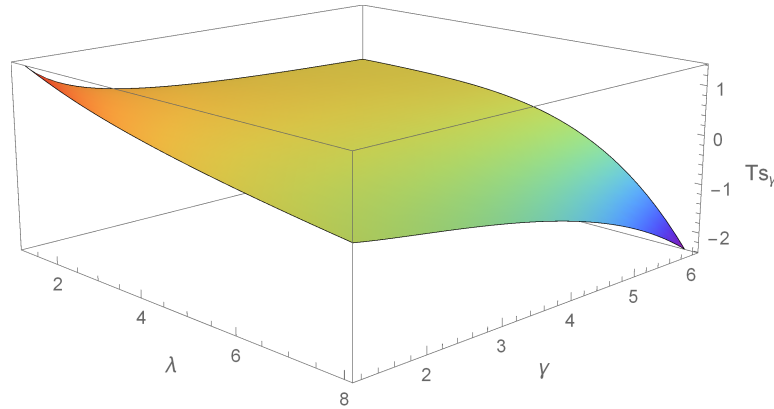


Figure 2. The plot of $Ts_\gamma(T_{[2|4:W]})$ with respect to γ and λ , with $\xi = 0.5$ as demonstrated in Example 2.1.

2.2. Comparisons via stochastic orders

Consider two absolutely continuous nonnegative random variables T_1 and T_2 with probability density functions g_{T_1} and g_{T_2} , cumulative distribution functions G_{T_1} and G_{T_2} , and survival functions $\bar{G}_{T_1}$ and $\bar{G}_{T_2}$, respectively. The following stochastic ordering will be used throughout this subsection. The random variable T_1 is said to be smaller than T_2 in the *dispersive order*, denoted by $T_1 \leq_{\text{DOR}} T_2$, if and only if $g_{T_1}(G_{T_1}^{-1}(w)) \geq g_{T_2}(G_{T_2}^{-1}(w))$, $0 < w < 1$. Further details on this stochastic ordering and its properties may be found in Shaked and Shanthikumar [25]. The following result is an immediate consequence of the representation established in Equation (16).

Theorem 2.4

Let $T_{[r|m:W]}^{(1)}$ and $T_{[r|m:W]}^{(2)}$ be the lifetimes of two consecutive r -out-of- $m:W$ good systems whose component lifetimes are independent and identically distributed with cumulative distribution functions G_{T_1} and G_{T_2} , respectively. If $T_1 \leq_{\text{DOr}} T_2$, then

$$Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) \geq Ts_\gamma\left(T_{[r|m:W]}^{(2)}\right), \quad 0 < \gamma < 1,$$

and

$$Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) \leq Ts_\gamma\left(T_{[r|m:W]}^{(2)}\right), \quad \gamma > 1.$$

Proof

Suppose that $T_1 \leq_{\text{DOr}} T_2$. Then, by the definition of the dispersive order,

$$g_{T_1}(G_{T_1}^{-1}(v)) \geq g_{T_2}(G_{T_2}^{-1}(v)), \quad 0 < v < 1.$$

From Equation (16),

$$(1 - \gamma)Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) = \int_0^1 \Delta_{[r|m:W]}^\gamma(v) g_{T_1}^{\gamma-1}(G_{T_1}^{-1}(v)) dv - 1.$$

Since $g_{T_1}(G_{T_1}^{-1}(v)) \geq g_{T_2}(G_{T_2}^{-1}(v))$, raising both sides to the power $\gamma - 1$ gives

$$g_{T_1}^{\gamma-1}(G_{T_1}^{-1}(v)) \leq g_{T_2}^{\gamma-1}(G_{T_2}^{-1}(v)), \quad 0 < \gamma < 1,$$

whereas

$$g_{T_1}^{\gamma-1}(G_{T_1}^{-1}(v)) \geq g_{T_2}^{\gamma-1}(G_{T_2}^{-1}(v)), \quad \gamma > 1.$$

Consequently,

$$(1 - \gamma)Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) \leq (1 - \gamma)Ts_\gamma\left(T_{[r|m:W]}^{(2)}\right), \quad 0 < \gamma < 1,$$

and

$$(1 - \gamma)Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) \geq (1 - \gamma)Ts_\gamma\left(T_{[r|m:W]}^{(2)}\right), \quad \gamma > 1.$$

Finally, dividing by $1 - \gamma$, which is positive for $0 < \gamma < 1$ and negative for $\gamma > 1$, yields the result. $\square$

An immediate consequence of Equation (16) is that it provides a useful criterion for comparing the Tsallis entropy of two r -out-of- $m:W$ good systems whose independent component lifetimes follow different distributions.

Proposition 2.1

Suppose that the assumptions of Theorem 2.4 hold. If $Ts_\gamma(T_1) \leq Ts_\gamma(T_2)$, and

$$\inf_{v \in A_1} \Delta_{[r|m:W]}(v) \geq \sup_{v \in A_2} \Delta_{[r|m:W]}(v),$$

where

$$A_1 = \{v \in [0, 1] : g_{T_1}(G_{T_1}^{-1}(v)) \leq g_{T_2}(G_{T_2}^{-1}(v))\},$$

and

$$A_2 = \{v \in [0, 1] : g_{T_1}(G_{T_1}^{-1}(v)) > g_{T_2}(G_{T_2}^{-1}(v))\},$$

then $Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) \leq Ts_\gamma\left(T_{[r|m:W]}^{(2)}\right)$, for every $2r \geq m$ and $\gamma \in \Xi$.

Proof

Since $Ts_\gamma(T_1) \leq Ts_\gamma(T_2)$, it follows from the definition of Tsallis entropy that

$$Ts_\gamma(T_2) - Ts_\gamma(T_1) = \frac{1}{1-\gamma} \int_0^1 \zeta_\gamma(v) dv \geq 0, \quad (20)$$

where

$$\zeta_\gamma(v) = g_{T_2}^{\gamma-1}(G_{T_2}^{-1}(v)) - g_{T_1}^{\gamma-1}(G_{T_1}^{-1}(v)), \quad 0 < v < 1.$$

Moreover, from Equation (16),

$$Ts_\gamma(T_{[r|m:W]}^{(2)}) - Ts_\gamma(T_{[r|m:W]}^{(1)}) = \frac{1}{1-\gamma} \int_0^1 \Delta_{[r|m:W]}^\gamma(v) \zeta_\gamma(v) dv.$$

Partitioning the interval $[0, 1]$ into A_1 and A_2 gives

$$Ts_\gamma(T_{[r|m:W]}^{(2)}) - Ts_\gamma(T_{[r|m:W]}^{(1)}) = \frac{1}{1-\gamma} \left[\int_{A_1} \Delta_{[r|m:W]}^\gamma(v) \zeta_\gamma(v) dv + \int_{A_2} \Delta_{[r|m:W]}^\gamma(v) \zeta_\gamma(v) dv \right].$$

We first consider the case $\gamma > 1$. Since the function $x^{\gamma-1}$ is increasing, we have

1. $\zeta_\gamma(v) \geq 0, v \in A_1$,
2. $\zeta_\gamma(v) < 0, v \in A_2$.

Because $1 - \gamma < 0$ and $\inf_{v \in A_1} \Delta_{[r|m:W]}(v) \geq \sup_{v \in A_2} \Delta_{[r|m:W]}(v)$, we obtain

$$\begin{aligned} Ts_\gamma(T_{[r|m:W]}^{(2)}) - Ts_\gamma(T_{[r|m:W]}^{(1)}) &\geq \frac{\left(\inf_{v \in A_1} \Delta_{[r|m:W]}(v)\right)^\gamma}{1-\gamma} \int_{A_1} \zeta_\gamma(v) dv \\ &\quad + \frac{\left(\sup_{v \in A_2} \Delta_{[r|m:W]}(v)\right)^\gamma}{1-\gamma} \int_{A_2} \zeta_\gamma(v) dv \\ &\geq \frac{\left(\sup_{v \in A_2} \Delta_{[r|m:W]}(v)\right)^\gamma}{1-\gamma} \int_0^1 \zeta_\gamma(v) dv \geq 0, \end{aligned}$$

where the last inequality follows from (20). Hence, $Ts_\gamma(T_{[r|m:W]}^{(1)}) \leq Ts_\gamma(T_{[r|m:W]}^{(2)})$. Now suppose that $0 < \gamma < 1$. In this case, $x^{\gamma-1}$ is decreasing. Therefore,

1. $\zeta_\gamma(v) \leq 0, v \in A_1$,
2. $\zeta_\gamma(v) > 0, v \in A_2$.

Since $1 - \gamma > 0$, the preceding argument applies with the signs reversed, yielding

$$Ts_\gamma(T_{[r|m:W]}^{(2)}) - Ts_\gamma(T_{[r|m:W]}^{(1)}) \geq \frac{\left(\sup_{v \in A_2} \Delta_{[r|m:W]}(v)\right)^\gamma}{1-\gamma} \int_0^1 \zeta_\gamma(v) dv \geq 0,$$

again by (20). Consequently, $Ts_\gamma(T_{[r|m:W]}^{(1)}) \leq Ts_\gamma(T_{[r|m:W]}^{(2)})$, for every $\gamma \in \Xi$. This completes the proof. $\square$

3. Some related bounds and characterization

Bounding techniques provide a useful method for approximating the behavior of the entropy over the system's lifetime when closed-form expressions for Tsallis entropy and the proposed model are not available, especially for systems with diverse lifetime distributions or a large number of components. The use of analytical bounds to describe the Tsallis entropy of successive r -out-of- $m:W$ good systems is examined in this subsection. Specifically, we offer the following theorem that provides a lower bound on the Tsallis entropy of the system. This bound promotes a better comprehension of system-level uncertainty and offers insightful information about the entropy structure under practical circumstances.

Lemma 3.1

Let T be a nonnegative absolutely continuous random variable with probability density function g_T and cumulative distribution function G_T . Suppose that $M = \sup_{t \geq 0} g_T(t) < \infty$ and that $\Delta_{[r|m:W]}(v) > 0$, $0 \leq v \leq 1$, which, under the Farlie–Gumbel–Morgenstern model, is guaranteed by $|\xi C_{r,m}| < 1$. Then, for $2r \geq m$,

$$Ts_\gamma(T_{[r|m:W]}) \geq M^{\gamma-1} Ts_\gamma(V_{[r|m:W]}) + \frac{1 - M^{\gamma-1}}{\gamma - 1},$$

where $\gamma \in \Xi$, and $\xi C_{r,m} \neq 0$.

Proof

Since $M = \sup_{t \geq 0} g_T(t)$, it follows that $g_T(t) \leq M$, $t \geq 0$. Hence, $g_T(G_T^{-1}(v)) \leq M$, $0 < v < 1$. First, suppose that $0 < \gamma < 1$. Since $\gamma - 1 < 0$, the function $x \mapsto x^{\gamma-1}$ is decreasing on $(0, \infty)$. Therefore, $g_T^{\gamma-1}(G_T^{-1}(v)) \geq M^{\gamma-1}$, $0 < v < 1$. Using the representation

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left(\int_0^1 \Delta_{[r|m:W]}^\gamma(v) g_T^{\gamma-1}(G_T^{-1}(v)) dv - 1 \right),$$

we obtain

$$Ts_\gamma(T_{[r|m:W]}) \geq \frac{1}{1-\gamma} \left(M^{\gamma-1} \int_0^1 \Delta_{[r|m:W]}^\gamma(v) dv - 1 \right).$$

Next, suppose that $\gamma > 1$. Since $\gamma - 1 > 0$, the function $x \mapsto x^{\gamma-1}$ is increasing on $(0, \infty)$. Hence, $g_T^{\gamma-1}(G_T^{-1}(v)) \leq M^{\gamma-1}$, $0 < v < 1$. Because $\Delta_{[r|m:W]}(v) > 0$, $0 \leq v \leq 1$, we have $\Delta_{[r|m:W]}^\gamma(v) > 0$, and, since $\frac{1}{1-\gamma} < 0$, multiplying the above inequality by $\Delta_{[r|m:W]}^\gamma(v)$ and then by $(1-\gamma)^{-1}$ reverses the inequality. Consequently,

$$Ts_\gamma(T_{[r|m:W]}) \geq \frac{1}{1-\gamma} \left(M^{\gamma-1} \int_0^1 \Delta_{[r|m:W]}^\gamma(v) dv - 1 \right).$$

Since

$$Ts_\gamma(V_{[r|m:W]}) = \frac{1}{1-\gamma} \left(\int_0^1 \Delta_{[r|m:W]}^\gamma(v) dv - 1 \right),$$

it follows that

$$\begin{aligned} Ts_\gamma(T_{[r|m:W]}) &\geq \frac{M^{\gamma-1}}{1-\gamma} \left(\int_0^1 \Delta_{[r|m:W]}^\gamma(v) dv - 1 \right) + \frac{M^{\gamma-1} - 1}{1-\gamma} \\ &= M^{\gamma-1} Ts_\gamma(V_{[r|m:W]}) + \frac{1 - M^{\gamma-1}}{\gamma - 1}. \end{aligned}$$

This completes the proof. $\square$

The following example illustrates the lower bound established in Lemma 3.1. The bound is expressed in terms of the Tsallis entropy of the standardized consecutive r -out-of- $m:W$ good system and depends explicitly on the modal value of the underlying component lifetime density.

Example 3.1

Consider the lifetime $T_{(r|m:W)}$ of a consecutive r -out-of- $m:W$ good system.

$$T_{(r|m:W)} = \max(T_{(1:r)}, T_{(2:r+1)}, \dots, T_{(m-r+1:m)}),$$

where $T_{(p:l)} = \min(T_1, \dots, T_l)$ for $1 \leq p < l \leq m$. Suppose that the component lifetimes are independent and identically distributed according to the two-component Pareto mixture with probability density function

$$g_T(t) = \delta\beta_1 t^{-\beta_1-1} + (1-\delta)\beta_2 t^{-\beta_2-1}, \quad t \geq 1,$$

where $0 \leq \delta \leq 1$, $\beta_1 > \beta_2 > 0$. Since the mode of the mixture distribution occurs at $t = 1$, the modal density is

$$M = g_T(1) = \delta\beta_1 + (1-\delta)\beta_2.$$

Applying Lemma 3.1, we obtain the lower bound

$$Ts_\gamma(T_{[r|m:W]}) \geq (\delta\beta_1 + (1-\delta)\beta_2)^{\gamma-1} Ts_\gamma(V_{[r|m:W]}) + \frac{1 - (\delta\beta_1 + (1-\delta)\beta_2)^{\gamma-1}}{\gamma - 1},$$

which holds for every $\gamma \in \Xi$. Therefore, the lower bound depends explicitly on the mixture parameters δ , β_1 , and β_2 through the modal density $M = \delta\beta_1 + (1-\delta)\beta_2$, while the remaining term is determined by the Tsallis entropy of the corresponding standardized consecutive r -out-of- $m:W$ good system.

The Tsallis entropy of concomitants successive r -out-of- $m:W$ systems is bounded by the following theorem, which relates it to the Tsallis entropy of the individual element lifespans.

Theorem 3.1

Suppose that $|\xi C_{r,m}| < 1$, where $C_{r,m} = \frac{-r^2-3r+2m+2}{(r+1)(r+2)}$. Then, for $\gamma > 1$ ($0 < \gamma < 1$),

$$Ts_\gamma(T_{[r|m:W]}) \geq (\leq) \Delta_{[r|m:W]}^\gamma(v^*) Ts_\gamma(T) + \frac{1}{1-\gamma} (\Delta_{[r|m:W]}^\gamma(v^*) - 1),$$

where

$$v^* = \begin{cases} 1, & \xi C_{r,m} > 0, \\ 0, & \xi C_{r,m} < 0, \\ \text{any } v \in (0, 1), & \xi C_{r,m} = 0. \end{cases}$$

Proof

From (15),

$$\Delta_{[r|m:W]}(v) = 1 + \xi C_{r,m}(2v - 1), \quad 0 \leq v \leq 1.$$

Since $|\xi C_{r,m}| < 1$, it follows that $\Delta_{[r|m:W]}(v) > 0$, $0 \leq v \leq 1$, and therefore $\Delta_{[r|m:W]}^\gamma(v)$ is well defined for every $\gamma > 0$. Moreover,

$$\frac{d}{dv} \Delta_{[r|m:W]}(v) = 2\xi C_{r,m}.$$

Hence, $\Delta_{[r|m:W]}(v)$ is increasing on $[0, 1]$ if $\xi C_{r,m} > 0$, decreasing on $[0, 1]$ if $\xi C_{r,m} < 0$, and constant on $[0, 1]$ if $\xi C_{r,m} = 0$. Consequently,

$$\Delta_{[r|m:W]}(v) \leq \Delta_{[r|m:W]}(v^*), \quad 0 \leq v \leq 1,$$

where

$$v^* = \begin{cases} 1, & \xi C_{r,m} > 0, \\ 0, & \xi C_{r,m} < 0, \\ \text{any } v \in (0, 1), & \xi C_{r,m} = 0. \end{cases}$$

Since the function $x \mapsto x^\gamma$ is increasing on $(0, \infty)$ for every $\gamma > 0$, we also have

$$\Delta_{[r|m:W]}^\gamma(v) \leq \Delta_{[r|m:W]}^\gamma(v^*), \quad 0 \leq v \leq 1.$$

Therefore,

$$\begin{aligned} Ts_\gamma(T_{[r|m:W]}) &= \frac{1}{1-\gamma} \left[\int_0^1 \Delta_{[r|m:W]}^\gamma(v) (g_T(G_T^{-1}(v)))^{\gamma-1} dv - 1 \right] \\ &\geq (\leq) \frac{1}{1-\gamma} \left[\Delta_{[r|m:W]}^\gamma(v^*) \int_0^1 (g_T(G_T^{-1}(v)))^{\gamma-1} dv - 1 \right] \\ &= \frac{\Delta_{[r|m:W]}^\gamma(v^*)}{1-\gamma} \left[\int_0^1 (g_T(G_T^{-1}(v)))^{\gamma-1} dv - 1 \right] + \frac{\Delta_{[r|m:W]}^\gamma(v^*) - 1}{1-\gamma}. \end{aligned}$$

Since

$$Ts_\gamma(T) = \frac{1}{1-\gamma} \left[\int_0^1 (g_T(G_T^{-1}(v)))^{\gamma-1} dv - 1 \right],$$

we obtain

$$Ts_\gamma(T_{[r|m:W]}) \geq (\leq) \Delta_{[r|m:W]}^\gamma(v^*) Ts_\gamma(T) + \frac{1}{1-\gamma} (\Delta_{[r|m:W]}^\gamma(v^*) - 1),$$

where the inequality is interpreted as “ $\geq$ ” for $\gamma > 1$ and “ $\leq$ ” for $0 < \gamma < 1$. This completes the proof. $\square$

Theorem 3.2

Let $T_j, j = 1, 2, \dots, m$, be the lifetimes of components of a consecutive r -out-of- $m:W$ good systems with $T_{[r|m:W]}$ having the common failure rate function $\lambda_T(t)$. Assume that $E(\lambda_T^{2(\gamma-1)}(T)) < \infty$, where $\gamma > 1$ ($0 < \gamma < 1$). Then

$$Ts_\gamma(T_{[r|m:W]}) \geq (\leq) \frac{1}{1-\gamma} \left(\sqrt{\Omega_{r,m;\xi,\gamma} E(\lambda_T^{2(\gamma-1)}(T))} - 1 \right),$$

where

$$\Omega_{r,m;\xi,\gamma} = \int_0^1 (1-u)^{2(\gamma-1)} \Delta_{[r|m:W]}^{2\gamma}(u) du.$$

Proof

Using (9),

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left(\frac{1}{\gamma} E[\lambda_{T,[r|m:W]}^{\gamma-1}(T_{[r|m:W],\gamma})] - 1 \right),$$

where $g_{T,[r|m:W],\gamma}(t) = \gamma g_{T,[r|m:W]}(t) \bar{G}_{T,[r|m:W]}^{\gamma-1}(t)$. Since $g_{T,[r|m:W]}(t) = g_T(t) \Delta_{[r|m:W]}(G_T(t))$, and $\bar{G}_{T,[r|m:W]}(t) = \bar{G}_T(t) (1 - \xi D_{r,m} G_T(t))$, we obtain

$$E[\lambda_{T,[r|m:W]}^{\gamma-1}(T_{[r|m:W],\gamma})] = \gamma \int_0^\infty \lambda_T^{\gamma-1}(t) g_T(t) \bar{G}_T^{\gamma-1}(t) \Delta_{[r|m:W]}^\gamma(G_T(t)) dt.$$

Applying the Cauchy–Schwarz inequality,

$$E\left[\lambda_{T,[r|m:W]}^{\gamma-1}(T_{[r|m:W]},\gamma)\right] \leq \gamma \left(\int_0^\infty \lambda_T^{2(\gamma-1)}(t)g_T(t)dt\right)^{1/2} \left(\int_0^\infty g_T(t)\bar{G}_T^{2(\gamma-1)}(t)\Delta_{[r|m:W]}^{2\gamma}(G_T(t))dt\right)^{1/2}.$$

Finally, using the transformation $u = G_T(t)$, $du = g_T(t)dt$, we have

$$E\left[\lambda_{T,[r|m:W]}^{\gamma-1}(T_{[r|m:W]},\gamma)\right] \leq \gamma \sqrt{E\left(\lambda_T^{2(\gamma-1)}(T)\right)} \sqrt{\Omega_{r,m;\xi,\gamma}},$$

where

$$\Omega_{r,m;\xi,\gamma} = \int_0^1 (1-u)^{2(\gamma-1)} \Delta_{[r|m:W]}^{2\gamma}(u) du,$$

which completes the proof. $\square$

3.1. Characterization Results

In this subsection, we establish several characterization results for the concomitant of consecutive r -out-of- m :W systems using the Tsallis entropy measure. Throughout the discussion, attention is restricted to linear consecutive r -out-of- m :W systems satisfying $2r \geq m$. We first recall a useful lemma based on the classical Müntz–Szász theorem (see Kamps [26]).

Lemma 3.2

Suppose that $\psi(z)$ is an integrable function on the interval (α, β) and satisfies $\int_\alpha^\beta z^{m_k} \psi(z) dz = 0$, $k = 1, 2, \dots$, where $\{m_k\}_{k \geq 1}$ is a strictly increasing sequence of positive integers such that $\sum_{k=1}^\infty \frac{1}{m_k} = \infty$. Then $\psi(z) = 0$ for almost every $z \in (\alpha, \beta)$.

The preceding lemma is a well-known consequence of the Müntz–Szász theorem, which guarantees the completeness of the sequence

$$\{z^{m_1}, z^{m_2}, \dots\}.$$

Moreover, Hwang and Lin [27] extended this theorem to the family $\{\phi^{m_k}(z), m_k \geq 1\}$, where $\phi(z)$ is assumed to be absolutely continuous and monotone on (α, β) .

Throughout this part we consider the constraints $2r \geq m$, $\gamma > 1$, and $\xi C_{r,m} \neq 0$.

Theorem 3.3

Consider two consecutive r -out-of- m :W systems with lifetimes $T_{[r|m:W]}^{(1)}$ and $T_{[r|m:W]}^{(2)}$, constructed from independent and identically distributed components having cumulative distribution functions G_1 and G_2 , and corresponding probability density functions g_1 and g_2 , respectively. Then G_1 and G_2 belong to the same location family, up to a location shift, if and only if

$$Ts_\gamma\left(T_{[r|m:W]}^{(1)}\right) = Ts_\gamma\left(T_{[r|m:W]}^{(2)}\right), \quad 2r \geq m, \gamma > 1, \xi C_{r,m} \neq 0.$$

Proof

Assume first that $G_2(y) = G_1(y - a)$, $a \in \mathbb{R}$. Using the definition of Tsallis entropy together with the change of

variable $x = y - a$, we obtain

$$\begin{aligned} Ts_\gamma(T_{[r|m:W]}^{(2)}) &= \frac{1}{1-\gamma} \left(\int_a^\infty g_{T,[r|m:W],2}^\gamma(y) dy - 1 \right) \\ &= \frac{1}{1-\gamma} \left(\int_0^\infty g_{T,[r|m:W],1}^\gamma(x) dx - 1 \right) \\ &= Ts_\gamma(T_{[r|m:W]}^{(1)}), \end{aligned}$$

which establishes the necessity. To prove the converse, recall that

$$\Delta_{[r|m:W]}(v) = 1 + \xi C_{r,m}(2v - 1), \quad 0 < v < 1.$$

Since $Ts_\gamma(T_{[r|m:W]}^{(1)}) = Ts_\gamma(T_{[r|m:W]}^{(2)})$, it follows that

$$\frac{1}{1-\gamma} \left(\int_0^1 \Delta_{[r|m:W]}^\gamma(v) g_1^{\gamma-1}(G_1^{-1}(v)) dv - 1 \right) = \frac{1}{1-\gamma} \left(\int_0^1 \Delta_{[r|m:W]}^\gamma(v) g_2^{\gamma-1}(G_2^{-1}(v)) dv - 1 \right).$$

Hence,

$$\int_0^1 \Delta_{[r|m:W]}^\gamma(v) [g_1^{\gamma-1}(G_1^{-1}(v)) - g_2^{\gamma-1}(G_2^{-1}(v))] dv = 0,$$

which equivalent to

$$\int_0^1 v^\gamma \tau_{r,m;\xi;\gamma}^\gamma(v) [g_1^{\gamma-1}(G_1^{-1}(v)) - g_2^{\gamma-1}(G_2^{-1}(v))] dv = 0,$$

where

$$\tau_{r,m;\xi}(v) = [(1 - \xi C_{r,m})v^{-1} + 2\xi C_{r,m}].$$

Applying Lemma 3.2 with

$$\psi(v) = \tau_{r,m;\xi;\gamma}^\gamma(v) [g_1^{\gamma-1}(G_1^{-1}(v)) - g_2^{\gamma-1}(G_2^{-1}(v))],$$

yields

$$g_1^{\gamma-1}(G_1^{-1}(v)) = g_2^{\gamma-1}(G_2^{-1}(v)), \quad \text{a.e. } v \in (0, 1),$$

keeping in mind that a.e. stands for almost everywhere. Since $\gamma \neq 1$, the above equality is equivalent to $g_1(G_1^{-1}(v)) = g_2(G_2^{-1}(v))$, $0 < v < 1$. Therefore, G_1 and G_2 belong to the same location family, differing only by a location shift, which completes the proof. $\square$

4. Nonparametric estimation inference

This section introduces two distribution-free estimators for the Tsallis entropy associated with the system lifetime $T_{[r|m:W]}$. Suppose that $Y_1, Y_2, \dots, Y_M$ is a random sample drawn from the baseline lifetime distribution having cumulative distribution function G_T and probability density function g_T . Throughout this section, the sample observations are assumed to be independent and identically distributed. Let $Y_{(1:M)} \leq Y_{(2:M)} \leq \dots \leq Y_{(M:M)}$ denote the corresponding order statistics. These observations are used exclusively for estimating the unknown density function appearing in the Tsallis entropy formula and are assumed to be independent of the lifetime of the system under study. Since the inverse distribution function satisfies

$$\frac{d}{dv} G_T^{-1}(v) = \frac{1}{g_T(G_T^{-1}(v))}, \quad 0 < v < 1,$$

the representation in Equation (16) can equivalently be expressed as

$$Ts_\gamma(T_{[r|m:W]}) = \frac{1}{1-\gamma} \left[\int_0^1 \Delta_{[r|m:W]}^\gamma(v) g_T^{\gamma-1}(G_T^{-1}(v)) dv - 1 \right], \quad (21)$$

where $\Delta_{[r|m:W]}(v) = 1 + \xi C_{r,m}(2v - 1)$, with $2r \geq m$ and $\gamma \in \Xi$. To construct an estimator of $Ts_\gamma(T_{[r|m:W]})$, we adopt the finite-difference approximation introduced by Vasicek [28] for estimating the derivative $\frac{dG_T^{-1}(v)}{dv}$. Substituting the resulting estimate of $g_T(G_T^{-1}(v))$ into Equation (21) yields the first nonparametric estimator,

$$\begin{aligned} \widehat{Ts}_{\gamma,1}(T_{[r|m:W]}) &= \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M \Delta_{[r|m:W]}^\gamma \left(\frac{j}{M+1} \right) \left(\frac{2w}{M(Y_{(j+w:M)} - Y_{(j-w:M)})} \right)^{\gamma-1} - 1 \right] \\ &= \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M \left(1 + \xi C_{r,m} \left(2 \left(\frac{j}{M+1} \right) - 1 \right) \right)^\gamma \left(\frac{2w}{M(Y_{(j+w:M)} - Y_{(j-w:M)})} \right)^{\gamma-1} - 1 \right]. \end{aligned} \quad (22)$$

The integer w , satisfying $w < M/2$, represents the window width employed in the finite-difference approximation. To accommodate observations near the sample boundaries, we define

1. $Y_{(j-w:M)} = Y_{(1:M)}$, whenever $j - w \leq 1$,
2. $Y_{(j+w:M)} = Y_{(M:M)}$, whenever $j + w \geq M$.

A refined version of the above estimator can be obtained by incorporating the boundary adjustment suggested by Ebrahimi et al. [29]. Their modification replaces the constant multiplier 2 with a position-dependent correction factor, thereby improving estimation accuracy near the sample extremes. The resulting estimator is

$$\begin{aligned} \widehat{Ts}_{\gamma,2}(T_{[r|m:W]}) &= \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M \Delta_{[r|m:W]}^\gamma \left(\frac{j}{M+1} \right) \left(\frac{\Lambda_i w}{M(Y_{(j+w:M)} - Y_{(j-w:M)})} \right)^{\gamma-1} - 1 \right] \\ &= \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M \left(1 + \xi C_{r,m} \left(2 \left(\frac{j}{M+1} \right) - 1 \right) \right)^\gamma \left(\frac{\Lambda_i w}{M(Y_{(j+w:M)} - Y_{(j-w:M)})} \right)^{\gamma-1} - 1 \right]. \end{aligned} \quad (23)$$

The correction coefficient Λ_i is defined by

$$\Lambda_i = \begin{cases} 1 + \frac{j-1}{w}, & 1 \leq j \leq w, \\ 2, & w+1 \leq j \leq M-w, \\ 1 + \frac{M-j}{w}, & M-w+1 \leq j \leq M. \end{cases} \quad (24)$$

An essential first step in assessing estimators for parametric functions is establishing estimator consistency. The consistency of the estimator defined in Eq. (22) is confirmed by the following theorem. The method used in the proof is comparable to that of Vasicek's Theorem 1 [28], which established a commonly used method for demonstrating consistency in entropy-based statistics. Park [30], Xiong et al. [31], and Alqefari et al. [23] have also used this technique to confirm the validity of their respective test statistics.

Proposition 4.1

Assume that $T_1, T_2, \dots, T_M$ is a random sample of size M from a population with probability density function g_T and cumulative distribution function G_T . Suppose that $E(g_T^{\gamma-1}(T)) < \infty$, and $\text{Var}(g_T^{\gamma-1}(T)) < \infty$.

Then $\widehat{T}s_{\gamma,1}(T_{[r|m:W]}) \xrightarrow{P} Ts_{\gamma}(T_{[r|m:W]})$, as $M \rightarrow \infty$, $w \rightarrow \infty$, $\frac{w}{M} \rightarrow 0$, for every $\gamma \in \Xi$, and $\xrightarrow{P}$ represents the convergence of probability.

Proof

The proof follows the consistency argument of Noughabi and Arghami [32]. As $M \rightarrow \infty$, $w \rightarrow \infty$, $\frac{w}{M} \rightarrow 0$, the Vasicek estimator satisfies

$$\begin{aligned} \frac{2w}{M(T_{(j+w):M} - T_{(j-w):M})} &= \frac{G_M(T_{(j+w):M}) - G_M(T_{(j-w):M})}{T_{(j+w):M} - T_{(j-w):M}} \approx \frac{G_T(T_{(j+w):M}) - G_T(T_{(j-w):M})}{T_{(j+w):M} - T_{(j-w):M}} \\ &\approx g_T(T_{(j:M)}), \end{aligned}$$

where G_M denotes the empirical distribution function. Moreover, $\frac{j}{M+1} = G_M(T_{(j:M)})$, and $G_M(T_{(j:M)}) \xrightarrow{\text{a.s.}} G_T(T_{(j:M)})$, $M \rightarrow \infty$. Hence,

$$\begin{aligned} \widehat{T}s_{\gamma,1}(T_{[r|m:W]}) &= \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M \Delta_{[r|m:W]}^{\gamma} \left(\frac{j}{M+1} \right) \left(\frac{2w}{M(T_{(j+w):M} - T_{(j-w):M})} \right)^{\gamma-1} - 1 \right] \\ &\approx \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M g_T^{\gamma-1}(T_{(j:M)}) \Delta_{[r|m:W]}^{\gamma}(G_M(T_{(j:M)})) - 1 \right] \\ &= \frac{1}{1-\gamma} \left[\frac{1}{M} \sum_{j=1}^M g_T^{\gamma-1}(T_{(j:M)}) \Delta_{[r|m:W]}^{\gamma}(G_T(T_{(j:M)})) - 1 \right]. \end{aligned} \quad (25)$$

Since the order statistics are simply a permutation of the original sample,

$$\frac{1}{M} \sum_{j=1}^M g_T^{\gamma-1}(T_{(j:M)}) \Delta_{[r|m:W]}^{\gamma}(G_T(T_{(j:M)})) = \frac{1}{M} \sum_{j=1}^M g_T^{\gamma-1}(T_j) \Delta_{[r|m:W]}^{\gamma}(G_T(T_j)).$$

Applying the Strong Law of Large Numbers,

$$\frac{1}{M} \sum_{j=1}^M g_T^{\gamma-1}(T_j) \Delta_{[r|m:W]}^{\gamma}(G_T(T_j)) \xrightarrow{\text{a.s.}} E \left[g_T^{\gamma-1}(T) \Delta_{[r|m:W]}^{\gamma}(G_T(T)) \right].$$

Furthermore,

$$\begin{aligned} E \left[g_T^{\gamma-1}(T) \Delta_{[r|m:W]}^{\gamma}(G_T(T)) \right] &= \int_0^{\infty} g_T^{\gamma-1}(t) \Delta_{[r|m:W]}^{\gamma}(G_T(t)) dt \\ &= \int_0^1 \Delta_{[r|m:W]}^{\gamma}(v) g_T^{\gamma-1}(G_T^{-1}(v)) dv, \end{aligned} \quad (26)$$

where the last equality follows from the transformation $v = G_T(t)$. Therefore,

$$\widehat{T}s_{\gamma,1}(T_{[r|m:W]}) \xrightarrow{P} \frac{1}{1-\gamma} \left[\int_0^1 \Delta_{[r|m:W]}^{\gamma}(v) g_T^{\gamma-1}(G_T^{-1}(v)) dv - 1 \right] = Ts_{\gamma}(T_{[r|m:W]}), \quad (27)$$

which proves the consistency of $\widehat{T}s_{\gamma,1}(T_{[r|m:W]})$. $\square$

Remark 4.1

The preceding discussion offers an intuitive indication that the proposed estimator is consistent. However, a formal

justification would necessitate deriving the asymptotic properties of the spacing-based estimator under appropriate regularity assumptions associated with the concomitant model. Establishing these theoretical conditions is beyond the present scope and is therefore reserved for future research.

In the same manner, we can see that $\widehat{T}s_{\gamma,2}(T_{[r|m:W]}) \xrightarrow{P} Ts_{\gamma}(T_{[r|m:W]})$.

To assess the finite-sample behavior of the proposed estimators $\widehat{T}s_{\gamma,1}(T_{[r|m:W]})$ and $\widehat{T}s_{\gamma,2}(T_{[r|m:W]})$ defined in Eqs. (22) and (23), a Monte Carlo experiment consisting of 1000 replications is conducted under the exponential distribution

$$G_Y(y) = 1 - e^{-y}, \quad y \geq 0.$$

The performance of the estimators is evaluated using three standard accuracy measures, namely the bias, the root mean squared error (R^*), and the standard deviation (S^*). Simulation results are obtained for sample sizes $M = 20, 50, 70$, and 100 , together with several choices of the parameters w, γ , and $\xi = 0.5$. The analysis is carried out for the system lifetimes $T_{[2|4:W]}$ and $T_{[3|4:W]}$, and the corresponding numerical results for $\widehat{T}s_{\gamma,1}(T_{[r|m:W]})$ and $\widehat{T}s_{\gamma,2}(T_{[r|m:W]})$ are presented in Tables 1–4. The simulation results summarized in Table 1 reveal several important features of the proposed estimator:

1. Across all examined combinations of w and γ , enlarging the sample size M consistently lowers the absolute bias, R^* , and S^* . This behavior demonstrates the improved precision and increased stability of the estimator as more observations become available.
2. When the sample size is held constant, the values of the absolute bias, R^* , and S^* tend to decline as the parameter w increases, indicating enhanced estimation performance for larger values of this parameter.
3. For fixed values of M and w , increasing the entropy parameter γ generally results in higher absolute bias, R^* , and S^* . The deterioration is more evident for relatively small values of w , whereas its influence becomes considerably weaker as w increases.
4. A comparison of the two proposed estimators shows that the bias-corrected estimator, $\widehat{T}s_{\gamma,2}(T_{[r|m:W]})$, generally provides smaller absolute bias, lower R^* , and lower S^* than $\widehat{T}s_{\gamma,1}(T_{[r|m:W]})$, indicating superior finite-sample performance.

Overall, the Monte Carlo simulation demonstrates that the performance of the proposed estimators is strongly influenced by the sample size and the model parameters. Both estimators exhibit improved finite-sample performance as the sample size M and the parameter w increase, whereas larger values of the entropy parameter γ generally lead to greater estimation errors, particularly when w is relatively small. Among the two estimators, the bias-corrected estimator $\widehat{T}s_{\gamma,2}(T_{[r|m:W]})$ consistently exhibits better accuracy and stability, making it the preferred estimator for practical applications.

Furthermore, Figure 3 presents the estimated sampling density functions of the proposed estimators $\widehat{T}s_{\gamma,1}(T_{[2|4:W]})$ and $\widehat{T}s_{\gamma,2}(T_{[2|4:W]})$ for sample sizes $M = 20, 50, 70, 100$, and 150 , with $\xi = 0.5$, $r = 2$, $w = 7$, and $\gamma = 2$. For both estimators, the sampling distributions become progressively narrower and more concentrated as the sample size increases, while the corresponding peaks become higher. This behavior indicates a substantial reduction in sampling variability and provides empirical evidence of the consistency of the proposed estimators.

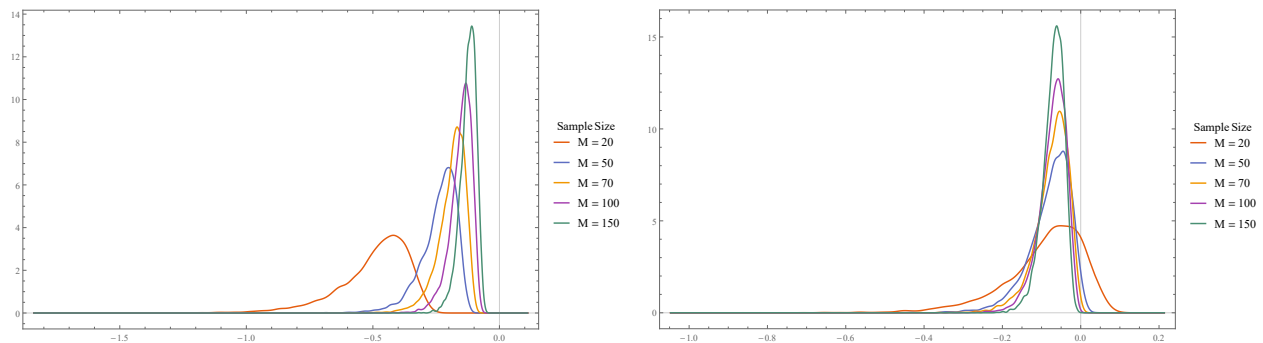
A comparison of the two panels further reveals that $\widehat{T}s_{\gamma,2}(T_{[2|4:W]})$ produces sampling distributions that are slightly more concentrated and exhibit smaller dispersion than those of $\widehat{T}s_{\gamma,1}(T_{[2|4:W]})$, particularly for moderate and large sample sizes. These results suggest that the corrected estimator possesses improved finite-sample precision while maintaining the same limiting behavior as the original estimator.

4.1. Industrial modeling data application

The practical performance of the proposed methodology is illustrated using an industrial dataset consisting of tensile strength measurements (in GPa) of single carbon fibers. Such measurements are of considerable importance

Table 1. Performance of the estimator $\widehat{T}s_{\gamma,1}(T_{[2|4:W]})$ for different values of w , γ , and sample sizes.

w	γ	$M = 20$			$M = 50$			$M = 70$			$M = 100$		
		Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*
2	2	-0.32298	0.47495	0.34840	-0.21943	0.27934	0.17294	-0.21841	0.27706	0.17055	-0.20431	0.24736	0.13951
2	3	-7.58024	193.69072	193.63918	-0.70121	2.16121	2.04531	-0.66455	1.13205	0.91647	-0.45182	0.91924	0.80094
2	4	-76.75853	1603.73930	1602.70288	-9.25893	118.04342	117.73862	-2.98228	16.70770	16.44761	-1.69563	10.56587	10.45951
3	2	-0.21862	0.34003	0.26057	-0.14851	0.19846	0.13171	-0.13996	0.17926	0.11206	-0.12790	0.15562	0.08869
3	3	-0.55973	1.55533	1.45185	-0.35775	0.99954	0.93379	-0.26950	0.56965	0.50212	-0.24713	0.45214	0.37862
3	4	-2.29298	12.51462	12.30273	-2.79822	8.68143	8.21800	-0.82538	3.38121	3.27901	-0.89165	2.99555	2.85975
4	2	-0.19250	0.29118	0.21845	-0.13006	0.18260	0.12815	-0.10813	0.14932	0.10298	-0.10008	0.13155	0.08541
4	3	-0.41836	1.39210	1.32775	-0.20371	0.38837	0.33068	-0.18334	0.29576	0.23208	-0.15581	0.25773	0.20534
4	4	-0.92819	3.74196	3.62497	-0.92934	4.30389	4.20238	-0.59880	3.04911	2.98978	-0.45825	3.01785	2.98287
5	2	-0.16856	0.25345	0.18927	-0.10474	0.15400	0.11288	-0.09382	0.13514	0.09729	-0.08643	0.11818	0.08060
5	3	-0.25832	0.46322	0.38449	-0.18126	0.33135	0.27741	-0.15060	0.24114	0.18835	-0.12996	0.20212	0.15483
5	4	-0.55775	1.65827	1.56168	-0.38072	0.89501	0.81002	-0.36378	0.85780	0.77682	-0.24430	0.51866	0.45753
6	2	-0.15196	0.23508	0.17937	-0.10286	0.15509	0.11608	-0.08667	0.12558	0.09088	-0.07285	0.10698	0.07834
6	3	-0.22371	0.42325	0.35930	-0.16872	0.27662	0.21920	-0.13314	0.20800	0.15985	-0.11029	0.15653	0.11108
6	4	-0.45531	1.61541	1.54991	-0.32638	1.02037	0.96675	-0.25837	0.59526	0.53627	-0.19162	0.35794	0.30233
7	2	-0.13816	0.21870	0.16953	-0.09127	0.13865	0.10437	-0.08287	0.12476	0.09326	-0.06880	0.09903	0.07123
7	3	-0.18172	0.31386	0.25589	-0.14621	0.23654	0.18595	-0.12905	0.19395	0.14481	-0.09841	0.14206	0.10246
7	4	-0.34026	0.72117	0.63584	-0.27325	0.61963	0.55615	-0.23017	0.49860	0.44232	-0.18113	0.31753	0.26081
8	2	-0.12897	0.20615	0.16084	-0.09366	0.14377	0.10911	-0.08080	0.11733	0.08508	-0.07005	0.10149	0.07345
8	3	-0.15898	0.27951	0.22991	-0.14410	0.22004	0.16630	-0.11697	0.17259	0.09506	-0.09506	0.13524	0.09620
8	4	-0.25742	0.62357	0.56794	-0.24586	0.53861	0.47922	-0.19634	0.38155	0.32717	-0.17158	0.32969	0.28156
9	2	-0.11300	0.19436	0.15814	-0.08418	0.13544	0.10609	-0.07929	0.11605	0.08473	-0.06763	0.10044	0.07427
9	3	-0.14094	0.25879	0.21702	-0.13414	0.19828	0.14604	-0.11302	0.16911	0.12580	-0.08957	0.12374	0.08538
9	4	-0.21070	0.44405	0.39091	-0.21897	0.42501	0.36423	-0.19301	0.33009	0.26778	-0.14990	0.26406	0.21741

Figure 3. Estimated sampling density functions of the proposed estimators $\widehat{T}s_{\gamma,1}(T_{[2|4:W]})$ (left) and $\widehat{T}s_{\gamma,2}(T_{[2|4:W]})$ (right) for $M = 20, 50, 70, 100$, and 150 , with $\xi = 0.5$, $r = 2$, $w = 7$, and $\gamma = 2$.

in reliability engineering because the mechanical strength of carbon fibers directly influences the durability and safety of composite structures employed in aerospace, automotive, and other high-performance engineering applications. The dataset contains the tensile strengths of 69 carbon fibers with a gauge length of 20 mm, obtained from impregnated bundles of one thousand fibers. This dataset has been widely used in reliability and lifetime modeling studies and was originally reported by Bader and Priest [33]. The observed tensile strength values are presented in Table 5.

Table 2. Performance of the estimator $\widehat{T}s_{\gamma,1}(T_{[3|4;W]})$ for different values of w , γ , and sample sizes.

w	γ	$M = 20$			$M = 50$			$M = 70$			$M = 100$		
		Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*
2	2	-0.39218	0.59874	0.45264	-0.26433	0.34725	0.22532	-0.26216	0.34731	0.22792	-0.24327	0.30584	0.18545
2	3	-12.39480	318.99162	318.91022	-1.08345	3.61440	3.44992	-0.75495	3.25528	3.17533	-0.67430	1.51552	1.35793
2	4	-148.93417	3119.29545	3117.29695	-18.39106	238.38481	237.79326	-5.71892	33.78196	33.31102	-4.18344	31.28516	31.07863
3	2	-0.26731	0.42944	0.33627	-0.18152	0.24714	0.16780	-0.16961	0.22181	0.14300	-0.15352	0.19009	0.11215
3	3	-0.86426	2.48168	2.32749	-0.55684	1.66169	1.56640	-0.40876	0.93331	0.83946	-0.37125	0.88942	0.02473
3	4	-4.33678	24.19927	23.81941	-5.55151	57.87513	57.63708	-1.57007	6.81358	6.63353	-1.02824	6.30312	6.21937
4	2	-0.23464	0.36354	0.27781	-0.16078	0.22986	0.16436	-0.13382	0.18856	0.13292	-0.12247	0.16469	0.11017
4	3	-0.64397	2.23808	2.14450	-0.31086	0.61820	0.53463	-0.27884	0.47074	0.37945	-0.23573	0.41495	0.34166
4	4	-1.72780	7.15702	6.94881	-1.81808	6.69592	6.59033	-1.15998	4.14836	4.04096	-0.87843	5.46889	5.41218
5	2	-0.20373	0.31109	0.23522	-0.13216	0.19570	0.14440	-0.11697	0.17006	0.12351	-0.10649	0.14734	0.10188
5	3	-0.39028	0.71095	0.59454	-0.27945	0.53131	0.45211	-0.23126	0.38423	0.30700	-0.19779	0.32202	0.25424
5	4	-1.02267	3.10798	2.93638	-0.72215	1.75155	1.59655	-0.69632	1.31697	1.21097	-0.46000	1.03045	0.92254
6	2	-0.17971	0.28598	0.22257	-0.13275	0.19930	0.14872	-0.11047	0.15964	0.11529	-0.09199	0.13478	0.09856
6	3	-0.33885	0.65116	0.55632	-0.26393	0.44240	0.35522	-0.20504	0.32998	0.25867	-0.16836	0.24553	0.17880
6	4	-0.82954	2.99405	2.87828	-0.62287	2.02497	1.92776	-0.49257	1.17791	1.07051	-0.36110	0.70093	0.60106
7	2	-0.16225	0.26070	0.20416	-0.11709	0.17519	0.13037	-0.10819	0.16099	0.11927	-0.08859	0.12644	0.09026
7	3	-0.27205	0.47405	0.38841	-0.22924	0.37546	0.29750	-0.20246	0.30849	0.23287	-0.15273	0.22472	0.16492
7	4	-0.61565	1.32187	1.17034	-0.52081	1.21041	1.09319	-0.43824	0.98396	0.88142	-0.34361	0.62186	0.51857
8	2	-0.14633	0.24087	0.19143	-0.12258	0.18505	0.13869	-0.10625	0.15231	0.10918	-0.09032	0.12979	0.09326
8	3	-0.23644	0.42028	0.34763	-0.22742	0.34729	0.26259	-0.18417	0.27396	0.20292	-0.14863	0.21321	0.15295
8	4	-0.46591	1.15105	1.05307	-0.46579	1.03805	0.92815	-0.37526	0.74767	0.64700	-0.32811	0.64867	0.55985
9	2	-0.12363	0.22125	0.18358	-0.11140	0.17366	0.13329	-0.10471	0.15043	0.10806	-0.08822	0.12859	0.09361
9	3	-0.20634	0.38398	0.32399	-0.21269	0.31148	0.22767	-0.17834	0.26765	0.19968	-0.14017	0.19495	0.13556
9	4	-0.37478	0.79454	0.70094	-0.41509	0.81666	0.70365	-0.36870	0.63910	0.52229	-0.28752	0.51802	0.43112

The tensile strength measurements were analyzed under the Weibull distribution, whose probability density function is defined by

$$g_Y(y) = \frac{\theta_1}{\theta_2} \left(\frac{y}{\theta_2} \right)^{\theta_1-1} \exp \left\{ - \left(\frac{y}{\theta_2} \right)^{\theta_1} \right\}, \quad y > 0, \quad \theta_1, \theta_2 > 0,$$

where θ_1 and θ_2 represent the shape and scale parameters, respectively. The unknown parameters were estimated by the maximum likelihood method. The corresponding estimates were

$$\widehat{\theta}_1 = 3.248879, \quad \widehat{\theta}_2 = 1.617166.$$

To examine the suitability of the Weibull model, several goodness-of-fit procedures were applied. The obtained test statistics and their associated p -values are summarized below.

1. Kolmogorov–Smirnov statistic: $D = 0.043734$, with p -value = 0.9994;
2. Anderson–Darling statistic: $A = 0.20945$, with p -value = 0.9876;
3. Cramér–von Mises statistic: $\Phi^2 = 0.021998$, with p -value = 0.9950.

All three goodness-of-fit tests yield exceptionally large p -values, indicating that the null hypothesis of a Weibull distribution cannot be rejected at conventional significance levels. Consequently, the Weibull model provides an excellent description of the tensile strength data. This conclusion is further supported by the diagnostic plots presented in Figure 4, where the fitted density closely follows the empirical distribution, the probability and quantile plots exhibit an approximately linear pattern, and no substantial departures from the assumed model are observed.

Table 3. Performance of the estimator $\widehat{T}s_{\gamma,2}(T_{[2|4:W]})$ for different values of w , γ , and sample sizes.

w	γ	$M = 20$			$M = 50$			$M = 70$			$M = 100$		
		Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*
2	2	-0.20561	0.33124	0.25983	-0.17103	0.22290	0.14303	-0.18130	0.22456	0.13257	-0.17806	0.21091	0.11309
2	3	-2.03607	48.49831	48.47980	-0.67103	1.73876	1.63914	-0.47274	1.36036	1.33308	-0.30225	0.42759	0.30260
2	4	-10.22135	200.63581	200.47554	-1.71110	14.95828	14.86753	-0.99344	3.42214	3.27641	-0.49191	1.34167	1.30734
3	2	-0.12183	0.23733	0.20377	-0.11029	0.16041	0.11654	-0.11081	0.14823	0.09850	-0.10852	0.13640	0.08267
3	3	-0.23454	0.66180	0.61915	-0.18694	0.39412	0.34713	-0.17198	0.28435	0.22656	-0.16835	0.29992	0.24834
3	4	-0.50423	2.01759	1.95454	-0.55612	1.96440	1.92716	-0.33471	1.89826	1.83399	-0.32507	1.65456	1.62313
4	2	-0.10185	0.20397	0.17682	-0.09348	0.14646	0.11280	-0.08200	0.12355	0.09246	-0.08126	0.11282	0.07829
4	3	-0.17783	0.54012	0.51026	-0.12014	0.22987	0.19608	-0.11848	0.18584	0.14325	-0.10585	0.15933	0.11915
4	4	-0.25328	1.86166	1.82401	-0.25841	1.26782	1.24182	-0.22880	0.79564	0.76241	-0.20851	0.66916	0.64694
5	2	-0.08511	0.18100	0.15981	-0.07143	0.12530	0.10300	-0.06971	0.11393	0.09016	-0.06897	0.10225	0.07552
5	3	-0.11491	0.25070	0.22293	-0.10926	0.19954	0.16705	-0.09683	0.15848	0.12552	-0.08975	0.13812	0.10503
5	4	-0.18088	0.51378	0.48113	-0.16727	0.36197	0.32117	-0.16366	0.40733	0.37319	-0.12679	0.22614	0.18734
6	2	-0.07187	0.16730	0.15116	-0.06894	0.12564	0.10509	-0.06288	0.10502	0.08416	-0.05620	0.09326	0.07446
6	3	-0.09505	0.22541	0.20449	-0.10020	0.17199	0.13986	-0.08593	0.13927	0.10966	-0.07665	0.11341	0.08362
6	4	-0.16360	0.62996	0.60865	-0.14213	0.33614	0.30477	-0.12444	0.24836	0.21504	-0.10596	0.18748	0.15475
7	2	-0.06070	0.15818	0.14614	-0.06020	0.11425	0.09715	-0.05902	0.10495	0.08682	-0.05207	0.08539	0.06771
7	3	-0.07541	0.17687	0.16007	-0.08639	0.15340	0.12682	-0.08348	0.13616	0.10762	-0.06760	0.10513	0.08056
7	4	-0.12007	0.29301	0.26742	-0.12965	0.27341	0.24083	-0.11693	0.21694	0.18282	-0.10253	0.17814	0.14575
8	2	-0.05340	0.14987	0.14010	-0.06163	0.11799	0.10066	-0.05756	0.09816	0.07954	-0.05375	0.08815	0.06991
8	3	-0.06264	0.16045	0.14779	-0.08915	0.15446	0.12621	-0.07702	0.12506	0.09858	-0.06661	0.10288	0.07844
8	4	-0.08441	0.23708	0.22165	-0.12860	0.29087	0.26103	-0.10500	0.19908	0.16922	-0.10062	0.18463	0.15488
9	2	-0.04076	0.14440	0.13860	-0.05306	0.11193	0.09860	-0.05626	0.09703	0.07909	-0.05161	0.08780	0.07107
9	3	-0.05130	0.14966	0.14067	-0.08282	0.14135	0.11461	-0.07467	0.12389	0.09891	-0.06308	0.09504	0.07112
9	4	-0.07462	0.21414	0.20082	-0.11543	0.22917	0.19807	-0.10962	0.19625	0.16286	-0.08910	0.15652	0.12874

To illustrate the proposed methodology, we consider a consecutive 2-out-of-3: W system. In this configuration, the system remains operational whenever at least one pair of consecutive components functions successfully. The corresponding system structure is depicted in Figure 5. Throughout the numerical investigation, the system lifetime is denoted by $T_{(2|3:W)}$ and is used to evaluate the proposed Tsallis entropy estimators. The lifetime of this system is represented by

$$T_{(2|3:W)} = \max\{\min(T_1, T_2), \min(T_2, T_3)\},$$

which serves as the basis for all subsequent theoretical and simulation results. Based on the concomitant of the consecutive 2-out-of-3: W system, Figure 6 depicts the absolute biases of the proposed estimators $\widehat{T}s_{\gamma,1}(T_{[2|3:W]})$ and $\widehat{T}s_{\gamma,2}(T_{[2|3:W]})$ for different values of the window width w , with $\xi = 0.5$ and $\gamma = 2, 3, 4$, and 5. For both estimators, the absolute bias decreases rapidly as the window width increases from $w = 2$ to approximately $w = 5$, after which the curves become nearly stable, indicating that larger window widths lead to only marginal additional improvements in estimation accuracy. Furthermore, for any fixed value of w , the absolute bias increases gradually with the Tsallis parameter γ , implying that smaller values of γ provide more accurate estimates for the considered dataset.

A comparison of the two panels shows that both estimators exhibit almost identical patterns with respect to w and γ . Nevertheless, $\widehat{T}s_{\gamma,2}(T_{[2|3:W]})$ consistently yields slightly smaller absolute biases than $\widehat{T}s_{\gamma,1}(T_{[2|3:W]})$, particularly for moderate and large window widths, demonstrating a modest improvement in finite-sample estimation accuracy.

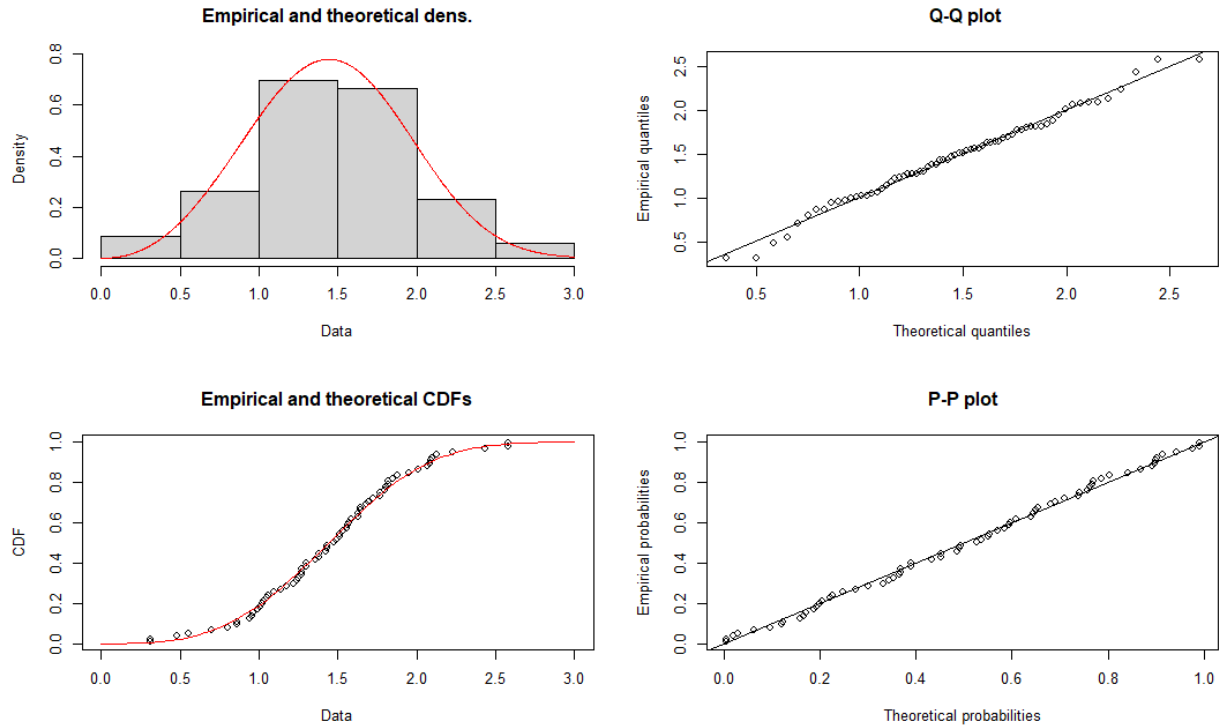


Figure 4. Diagnostic plots for the Weibull model fitted to the tensile strength data. The panels display the fitted density superimposed on the histogram (upper left), the Weibull quantile–quantile (Q–Q) plot (upper right), the empirical and fitted cumulative distribution functions (lower left), and the probability–probability (P–P) plot (lower right).

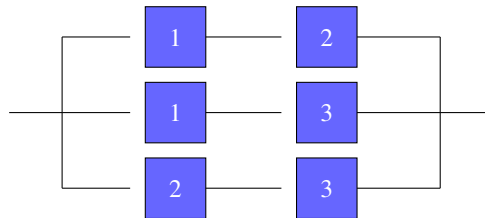


Figure 5. Configuration of the consecutive 2-out-of-3:W system considered throughout the numerical study.

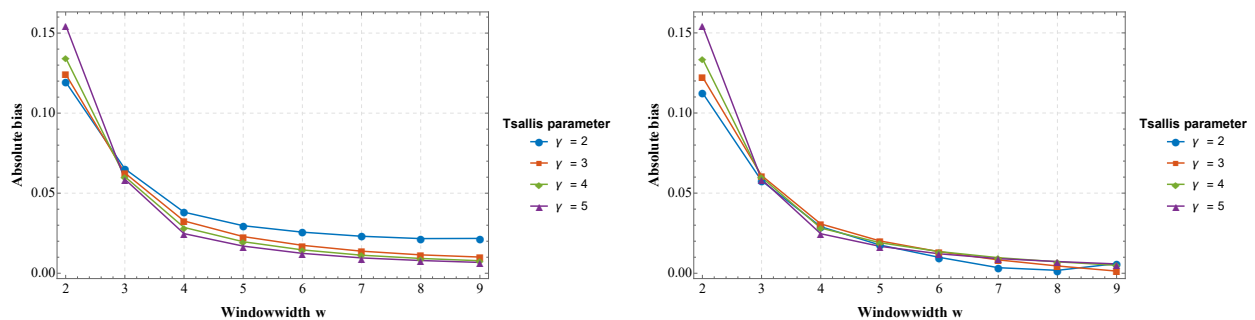


Figure 6. Absolute bias vs w for different γ values of $\widehat{T}s_{\gamma,1}(T_{[2|3:W]})$ (left) and $\widehat{T}s_{\gamma,2}(T_{[2|3:W]})$ (right) with $\xi = 0.5$.

Table 4. Performance of the estimator $\widehat{T}^{s_{\gamma,2}}(T_{[3|4:W]})$ for different values of w , γ , and sample sizes.

w	γ	$M = 20$			$M = 50$			$M = 70$			$M = 100$		
		Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*	Bias	R^*	S^*
2	2	-0.21173	0.37549	0.31026	-0.18656	0.25525	0.17429	-0.20228	0.26166	0.16606	-0.20072	0.24590	0.14212
2	3	-3.22347	79.82036	79.79515	-1.50055	2.12637	2.00955	-1.19054	1.89091	1.85418	-0.40523	0.61693	0.46542
2	4	-19.37728	390.08825	389.80163	-3.02997	30.00588	29.86745	-1.60798	5.82770	5.60428	-0.78938	1.43455	1.37922
3	2	-0.15401	0.26657	0.24107	-0.11230	0.18262	0.14004	-0.11059	0.16931	0.11890	-0.10041	0.15710	0.10096
3	3	-0.28559	0.88695	0.84013	-0.24462	0.57260	0.51798	-0.22427	0.40128	0.33292	-0.22237	0.42973	0.36791
3	4	-0.96099	3.44816	3.36707	-0.95519	2.67333	2.61745	-0.51372	1.47808	1.38663	-0.51131	0.77905	0.73297
4	2	-0.09050	0.22630	0.20752	-0.09869	0.16814	0.13620	-0.08876	0.14392	0.11335	-0.08975	0.13195	0.09678
4	3	-0.20832	0.74531	0.71596	-0.14882	0.30979	0.27184	-0.15110	0.25177	0.20149	-0.13744	0.22050	0.17251
4	4	-0.43565	1.26324	1.21845	-0.40059	1.24321	1.20825	-0.34439	1.22983	1.20245	-0.31931	0.57410	0.54214
5	2	-0.06999	0.19700	0.18424	-0.07478	0.14602	0.12548	-0.07486	0.13280	0.10975	-0.07571	0.11911	0.09200
5	3	-0.12117	0.31011	0.28560	-0.13518	0.26885	0.23251	-0.12258	0.21582	0.17773	-0.11645	0.19111	0.15161
5	4	-0.22363	0.71330	0.67768	-0.23831	0.54883	0.49464	-0.24089	0.43024	0.48268	-0.18908	0.35445	0.29996
6	2	-0.05214	0.18165	0.17410	-0.07417	0.14818	0.12835	-0.06871	0.12324	0.10237	-0.06243	0.11025	0.09091
6	3	-0.09646	0.28030	0.26331	-0.12565	0.23245	0.19567	-0.10872	0.18968	0.15551	-0.09867	0.15512	0.11975
6	4	-0.19428	0.80759	0.78427	-0.20187	0.51597	0.47508	-0.18267	0.38914	0.34377	-0.15864	0.29595	0.24996
7	2	-0.03873	0.16881	0.16439	-0.06311	0.13237	0.11642	-0.06610	0.12563	0.10689	-0.05874	0.10176	0.08314
7	3	-0.06980	0.21428	0.20269	-0.10721	0.20620	0.17623	-0.10804	0.18807	0.15401	-0.08795	0.14583	0.11639
7	4	-0.13663	0.38582	0.36100	-0.18377	0.40273	0.35854	-0.17158	0.33529	0.28821	-0.15463	0.28424	0.23862
8	2	-0.02671	0.15810	0.15590	-0.06686	0.13955	0.12255	-0.06511	0.11789	0.09833	-0.06110	0.10562	0.08619
8	3	-0.05295	0.19398	0.18670	-0.11315	0.20953	0.17644	-0.10022	0.17290	0.14096	-0.08813	0.14371	0.11357
8	4	-0.08756	0.30734	0.29475	-0.18413	0.44000	0.39982	-0.15483	0.30980	0.26847	-0.15418	0.29662	0.25353
9	2	-0.00956	0.15086	0.15063	-0.05726	0.13202	0.11901	-0.06390	0.11620	0.09710	-0.05944	0.10547	0.08716
9	3	-0.03646	0.17944	0.17578	-0.10572	0.19131	0.15953	-0.09736	0.17139	0.14112	-0.08350	0.13296	0.10352
9	4	-0.07334	0.27762	0.26790	-0.16541	0.34861	0.30702	-0.16443	0.30864	0.26133	-0.13705	0.25372	0.21363

Table 5. Tensile strength measurements (GPa) of 69 single carbon fibers with a 20 mm gauge length.

0.312	0.314	0.479	0.552	0.700	0.803
0.861	0.865	0.944	0.958	0.966	0.997
1.006	1.021	1.027	1.055	1.063	1.098
1.140	1.179	1.224	1.240	1.253	1.270
1.272	1.274	1.301	1.301	1.359	1.382
1.382	1.426	1.434	1.435	1.478	1.490
1.511	1.514	1.535	1.554	1.566	1.570
1.586	1.629	1.633	1.642	1.648	1.684
1.697	1.726	1.770	1.773	1.800	1.809
1.818	1.821	1.848	1.880	1.954	2.012
2.067	2.084	2.090	2.096	2.128	2.233
2.433	2.585	2.585			

Conclusion

This consideration examined the Tsallis entropy of the concomitant of consecutive r -out-of- $m:W$ good systems under the Farlie–Gumbel–Morgenstern family. The proposed framework extends the existing entropy formulation for consecutive systems by incorporating dependence through the Farlie–Gumbel–Morgenstern model. Establishing a significant connection between the Tsallis entropy of such systems under broad continuous

lifetime distributions and those governed by the uniform distribution is a key contribution. A series expansion of the proposed entropy was established and illustrated through the log-logistic distribution. In addition, several theoretical properties, including stochastic comparisons, bounds, and characterization results, were derived, providing further insight into the uncertainty behavior of the proposed system.

To support practical implementation, two nonparametric estimators of the proposed Tsallis entropy were developed, and their finite-sample performance was investigated through Monte Carlo simulation. The numerical results showed that both estimators become more accurate and stable as the sample size and the window width increase. In contrast, increasing the Tsallis parameter generally leads to larger estimation errors, particularly for relatively small window widths. The simulation study also demonstrated that the bias-corrected estimator $\widehat{Ts}_{\gamma,2}(T_{[r|m:W]})$ consistently outperforms the original estimator in terms of absolute bias and the associated performance measures, making it the preferred estimator for practical applications.

The proposed methodology was further validated using a real engineering dataset based on the concomitant of the consecutive 2-out-of-3:W system. The empirical analysis confirmed the trends observed in the simulation study. Specifically, the estimation accuracy improved substantially as the window width increased before reaching a nearly stable level, whereas larger values of the Tsallis parameter produced moderately larger absolute biases. Moreover, the bias-corrected estimator consistently yielded slightly smaller absolute biases than the original estimator, demonstrating its improved finite-sample performance in practice.

Overall, the results demonstrate that the proposed entropy framework provides an effective tool for quantifying uncertainty in dependent consecutive systems while preserving analytical tractability and computational feasibility. The proposed methodology may be extended in future work to other dependence models, generalized entropy measures, and broader classes of coherent reliability systems.

In future work, the proposed framework can be extended by investigating its relationships with other classes of ordered variables and probability distributions; see, for example, [34], [35], and [36]. It would also be of interest to consider alternative dependence structures, including asymmetric copula families and copulas capable of modeling stronger dependence, as well as a wider range of lifetime distributions. Moreover, other nonparametric and parametric estimation techniques can be developed for the obtained model. Further research can also address the asymptotic properties of the proposed estimators, including consistency, confidence intervals, and coverage probabilities, together with theoretically justified procedures for selecting the window width and analyzing the convergence and truncation error of the series representation. Such extensions would provide a broader assessment of the proposed information-theoretic framework and its applicability to different reliability and industrial modeling settings.

Author Contributions

Conceptualization, Formal Analysis, Investigation, Methodology, Software, Visualization, Resources, Writing—Original Draft, Validation, and Writing—Review & Editing: H.H.S. and M.S.M.

Competing Interests

The author declares that there are no financial or non-financial competing interests related to this work.

Data Availability

The datasets generated during and/or analyzed during the current study are available within the manuscript.

AI Usage Declaration

The author affirms that no artificial intelligence-based tools were used in the development or writing of this manuscript.

Conflict of Interest

The author reports no conflicts of interest in connection with this research.

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