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Neutrosophic MR-Metric Spaces: Fixed Point Theorems and Applications to Fractional Differential Equations with Uncertainty

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Abstract This paper introduces the new concept of Neuterosophic MR-metric space (NMR-MS), a powerful mathematical framework that integrates Neuterosophic logic with recently developed MR metric spaces. We establish the basic fixed point theorem for Neuterosophic contractions in this new setting, and extend classical results to accommodate three degrees of membership: truth, falsity, and uncertainty. The main theorem guarantees the existence and uniqueness of fixed points for neuterosophic contraction mappings, with constant dependence on the geometric convergence rate and parameters. An important application of our theory is presented in the context of fractional differential equations involving generalized fractional derivatives. By constructing an appropriate NMR-MS structure in the space of continuous functions, we reformulate the fractional differential equations as fixed point problems and prove the existence and uniqueness of the solutions under Lipschitz conditions. Neurosophisticated components add a natural amount of uncertainty to the solution process. We illustrate our theoretical results with various applications, including fractional decay processes, nonlinear logistic growth models, fractional heat equations, PID controller design, financial modeling, and biological tissue growth. These examples demonstrate the versatility and practical utility of the NMR-MS framework for dealing with real-world problems involving imprecision, ambiguity, and incomplete information. The results presented here generalize and unify several existing fixed point theorems in metric spaces, fuzzy metric spaces and Neuterosophic metric spaces, and provide new perspectives for research in uncertainty theory, fractional calculus and their interdisciplinary applications.

Keywords Neutrosophic MR-metric spaces; Fixed point theory; Neutrosophic contraction; Generalized fractional derivative; Fractional differential equations; Uncertainty quantification; Picard iteration; t-norm; t-conorm; Modified tetrahedral inequality.

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1. Introduction

The interplay between fixed point theory and fractional calculus has emerged as a significant research direction for analyzing differential equations incorporating memory and hereditary effects [4, 15, 16, 17, 18]. Fixed point theorems provide essential tools for establishing existence and uniqueness of solutions to differential and integral equations, while fractional calculus models non-local dynamics in physical, biological, and engineering systems [8, 9, 10, 5, 6, 3]. The evolution of metric structures has led to numerous generalizations, including b-metric spaces [3, 1], M-metric spaces [6], and more recently, MR-metric spaces introduced by Malkawi and collaborators [2, 5, 6, 17, 18], based on a modified tetrahedral inequality with constant $\mathfrak{R} > 1$. These spaces have demonstrated applicability in various domains including graph theory, measure theory, compactness analysis, deep learning, and fractal geometry [8, 9, 10, 12, 13, 14, 22, 23, 24]. Recent developments in optimization and information computing have further expanded the applicability of these frameworks, particularly in classification problems with

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imbalanced data [42], numerical simulations of epidemiological models [43], and IoT-enabled automation systems [44]. Concurrently, the need to address uncertainty in mathematical models has motivated the development of fuzzy logic and its extensions, from Zadeh's fuzzy sets to intuitionistic and neutrosophic sets introduced by Smarandache [33, 36, 37]. Neutrosophic logic, incorporating truth, falsity, and indeterminacy components, provides a robust framework for modeling incomplete or inconsistent information, yielding important results in neutrosophic metric spaces [11, 19, 25, 26, 27, 28, 29, 31]. Moreover, generalized fractional derivatives proposed in [15] preserve classical properties including linearity and the chain rule while allowing flexible kernel functions through an auxiliary function g, proving useful in applications such as anomalous diffusion, viscoelasticity, and biomedical modeling [16, 20, 21]. The development of fuzzy and neutrosophic approaches has been significantly advanced through applications in decision-making, classification, and topological structures [34, 35, 38, 39, 40, 41].

Despite these advances, significant gaps remain. The relationship between neutrosophic metric structures and MR-metric geometry has not been fully explored, and fractional differential equations often incorporate uncertainty from parameter estimation or measurement errors without adequate quantitative frameworks. Developing an integrated framework that simultaneously addresses existence, uniqueness, and uncertainty is therefore a critical research direction [24, 25, 26, 27, 28, 29, 30, 31, 32]. This paper introduces the novel concept of Neutrosophic MR-Metric Spaces (NMR-MS), a unified mathematical framework that integrates neutrosophic logic with MR-metric geometry. We establish fundamental fixed point theorems for neutrosophic contractions, extending classical Banach contraction principles to accommodate three degrees of membership: truth, falsity, and indeterminacy. Our main result guarantees existence and uniqueness of fixed points with geometric convergence rates and continuous dependence on parameters. We apply this framework to fractional differential equations involving generalized fractional derivatives, reformulating them as fixed point problems in appropriate NMR-MS structures and proving existence and uniqueness of solutions under Lipschitz conditions. The three-component representation provides richer uncertainty quantification than classical or fuzzy approaches, offering quantitative measures of confidence, error, and indeterminacy in the solution process. We illustrate our theoretical results through diverse applications including fractional decay processes, nonlinear logistic growth models, fractional heat equations, PID controller design, financial modeling using fractional Black-Scholes models, and biological tissue growth, demonstrating the versatility and practical utility of the NMR-MS framework for real-world problems involving imprecision, ambiguity, and incomplete information. The results presented here generalize and unify several existing fixed point theorems in metric spaces, fuzzy metric spaces, and neutrosophic metric spaces, opening new perspectives for research in uncertainty theory, fractional calculus, and interdisciplinary applications.

Definition 1
[15][Generalized Fractional Derivative] Let $f : [0, \infty) \to \mathbb{R}$ be a function and $t > 0$. The fractional derivative of f of order α is defined by:

$$A^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon},$$

where $\alpha \in (0, 1)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying:

$$g(0) = 1,$$
$$g'(0) = 1.$$

Definition 2
[2][MR-Metric Space] Let $\mathbb{X}$ be a non-empty set and $\mathfrak{R} > 1$ be a real constant. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is called an **MR-metric** if for all elements $v, \xi, s, \ell \in$, the following axioms hold:

1. **Non-negativity**: $M(v, \xi, s) \geq 0$.

2. **Identity of Indiscernibles**: $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.

3. **Symmetry**: The value of $M(\upsilon, \xi, s)$ is unchanged under any permutation of υ, ξ, and s. That is, $M(\upsilon, \xi, s) = M(p(\upsilon, \xi, s))$ for any permutation p.

4. **Modified Tetrahedral Inequality**:

$$M(\upsilon, \xi, s) \leq \Re\left[M(\upsilon, \xi, \ell) + M(\upsilon, \ell, s) + M(\ell, \xi, s)\right].$$

The pair $(\mathbb{X}, M)$ is termed an **MR-metric space**.

Definition 3
[19][Neutrosophic MR-Metric Space (NMR-MS)] A **Neutrosophic MR-Metric Space** is defined by the 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \Re, \star)$, where:

1. $\mathcal{Z}$ is a non-empty set.

2. $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric on $\mathcal{Z}$ according to Definition 1, with constant $\Re > 1$.

3. $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are membership functions representing the neutrosophic components of truth, falsity, and indeterminacy, respectively. They satisfy the following conditions for all $\gamma, \rho > 0$ and for all $\upsilon, \xi, \Im, \ell \in \mathcal{Z}$:

   (T1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1$ if and only if $\upsilon = \xi$.

   (T2) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$.

   (T3) $\mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$.

   (T4) $\mathcal{T}(\upsilon, \xi, \cdot) : (0, \infty) \to [0, 1]$ is a non-decreasing function with $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$.

   (F1) $\mathcal{F}(\upsilon, \xi, \gamma) = 0$ if and only if $\upsilon = \xi$.

   (F2) $\mathcal{F}(\upsilon, \xi, \gamma) = \mathcal{F}(\xi, \upsilon, \gamma)$.

   (F3) $\mathcal{F}(\upsilon, \xi, \gamma) \diamond \mathcal{F}(\xi, \Im, \rho) \geq \mathcal{F}(\upsilon, \Im, \gamma + \rho)$.

   (F4) $\mathcal{F}(\upsilon, \xi, \cdot) : (0, \infty) \to [0, 1]$ is a non-increasing function with $\lim_{\gamma \to \infty} \mathcal{F}(\upsilon, \xi, \gamma) = 0$.

   (I1) $\mathcal{I}(\upsilon, \xi, \gamma) = 0$ if and only if $\upsilon = \xi$.

   (I2) $\mathcal{I}(\upsilon, \xi, \gamma) = \mathcal{I}(\xi, \upsilon, \gamma)$.

   (I3) $\mathcal{I}(\upsilon, \xi, \gamma) \diamond \mathcal{I}(\xi, \Im, \rho) \geq \mathcal{I}(\upsilon, \Im, \gamma + \rho)$.

   (I4) $\mathcal{I}(\upsilon, \xi, \cdot) : (0, \infty) \to [0, 1]$ is a non-increasing function with $\lim_{\gamma \to \infty} \mathcal{I}(\upsilon, \xi, \gamma) = 0$.

4. The operator $\bullet : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous *t-norm*.

5. The operator $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous *t-conorm*.

6. The operator $\star$ is a binary operation on $[0, \infty)$ (e.g., standard addition $+$ or a weighted sum) used to generalize the summation within the tetrahedral inequality of the metric M.

2. Comparison with Existing Fixed Point Theorems

In this section, we compare our main results with existing fixed point theorems in fuzzy metric spaces and neutrosophic metric spaces to clarify the novelty and contributions of our framework.

Comparison with Fuzzy Metric Fixed Point Theorems

Classical fuzzy metric spaces, introduced by Kramosil and Michálek [**?**], use a single membership function to represent the degree of nearness between points. Fixed point theorems in fuzzy metric spaces typically require contractions of the form:

$$\mathcal{T}(Tx, Ty, kt) \geq \mathcal{T}(x, y, t),$$

where $\mathcal{T}$ is a fuzzy membership function. Our NMR-MS framework extends this in three ways:

1. **Three-component representation:** We simultaneously model truth ($\mathcal{T}$), falsity ($\mathcal{F}$), and indeterminacy ($\mathcal{I}$), whereas fuzzy metrics provide only a single component of nearness.
2. **MR-metric structure:** Our space employs the modified tetrahedral inequality with constant $\mathfrak{R} > 1$, which generalizes the classical triangle inequality and allows for more flexible geometric structures.
3. **Uncertainty quantification:** The simultaneous convergence of all three components provides quantitative measures of confidence, error, and uncertainty in the fixed point iteration.

Comparison with Neutrosophic Metric Fixed Point Theorems

Existing neutrosophic metric spaces [**?**, **?**] define membership functions $\mathcal{T}, \mathcal{F}, \mathcal{I}$ satisfying triangular inequalities using t-norms and t-conorms. Our framework differs in the following aspects:

1. **MR-geometric foundation:** We replace the classical metric with the MR-metric, which satisfies the modified tetrahedral inequality. This allows our results to apply to spaces where the standard triangle inequality fails but the weaker tetrahedral inequality holds.
2. **Unified contraction condition:** We impose contraction conditions simultaneously on all three components:

$$\begin{aligned}\mathcal{T}(Tx, Ty, kt) &\geq \mathcal{T}(x, y, t),\\ \mathcal{F}(Tx, Ty, kt) &\leq \mathcal{F}(x, y, t),\\ \mathcal{I}(Tx, Ty, kt) &\leq \mathcal{I}(x, y, t),\end{aligned}$$

   whereas existing results often treat these conditions separately or with different contraction factors.
3. **Geometric convergence rates:** We establish explicit geometric convergence rates for all three components, providing quantitative bounds on the speed of convergence.

Novel Contributions of Our Framework

Our framework provides the following novel contributions:

1. **Integration of neutrosophic logic with MR-metric geometry:** To the best of our knowledge, this is the first work to combine these two frameworks into a unified structure.
2. **Simultaneous convergence of all three components:** Our completeness and convergence results ensure that truth, falsity, and indeterminacy converge simultaneously, providing a complete picture of the uncertainty dynamics.
3. **Application to fractional differential equations:** We demonstrate how the NMR-MS framework can be applied to solve fractional differential equations with uncertainty quantification, establishing rigorous existence and uniqueness results.
4. **Quantitative uncertainty measures:** The three components provide interpretable quantitative measures of confidence, error, and indeterminacy in the solution process.

Limitations and Future Work

While our framework provides significant extensions over existing results, it has some limitations:

1. **Contraction condition is sufficient but not necessary:** Our main theorem assumes a contraction condition with factor $k \in (0, 1)$. Extending the results to weaker contractive conditions would broaden applicability.

2. **Computational complexity:** The simultaneous evaluation of three membership functions increases computational cost compared to classical or fuzzy metrics.

3. **Choice of t-norm and t-conorm:** The selection of appropriate t-norm and t-conorm operations for specific applications requires further investigation.

These limitations suggest promising directions for future research, including the development of weaker contraction conditions and the investigation of adaptive t-norm and t-conorm selection algorithms.

3. Neutrosophic MR-Metric Fixed Point Theory

In this section, we establish the fundamental fixed point theory for neutrosophic MR-metric spaces. Building upon the definitions introduced earlier, we first characterize neutrosophic contractions and demonstrate their essential properties, including the iterated contraction lemma which shows how contraction conditions propagate through successive applications of the mapping. We then introduce the concept of neutrosophic Cauchy sequences and prove that every Picard sequence generated by a neutrosophic contraction possesses this property in complete NMR-MS. The completeness lemma establishes that every neutrosophic Cauchy sequence converges to a unique limit in the sense of all three neutrosophic components. These preparatory results culminate in our main theorem, which guarantees the existence and uniqueness of fixed points for neutrosophic contractions, provides geometric convergence rates for the Picard iteration, and establishes continuous dependence of the fixed point on the mapping. These results extend classical Banach contraction principles to the richer neutrosophic setting, where truth, falsity, and indeterminacy are simultaneously considered, offering a more comprehensive framework for analyzing problems involving uncertainty and imprecision.

The definition of Neutrosophic MR-Metric Space (NMR-MS) was given in Definition 1.3 above (see also [19]). We adopt this definition throughout the remainder of the paper.

Definition 4 (Neutrosophic Contraction)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ be an NMR-MS. A mapping $T : \mathcal{Z} \to \mathcal{Z}$ is called a *neutrosophic contraction* if there exists $k \in (0, 1)$ such that for all $\upsilon, \xi, \Im \in \mathcal{Z}$ and for all $\gamma > 0$:

$$\mathcal{T}(T\upsilon, T\xi, k\gamma) \geq \mathcal{T}(\upsilon, \xi, \gamma), \tag{1}$$
$$\mathcal{F}(T\upsilon, T\xi, k\gamma) \leq \mathcal{F}(\upsilon, \xi, \gamma), \tag{2}$$
$$\mathcal{I}(T\upsilon, T\xi, k\gamma) \leq \mathcal{I}(\upsilon, \xi, \gamma). \tag{3}$$

The constant k is called the *contraction factor*.

Lemma 1 (Properties of Neutrosophic Contractions)
Let $T : \mathcal{Z} \to \mathcal{Z}$ be a neutrosophic contraction with factor $k \in (0, 1)$. Then for all $\upsilon, \xi \in \mathcal{Z}$, all $\gamma > 0$, and all $n \in \mathbb{N}$:

$$\mathcal{T}(T^n\upsilon, T^n\xi, k^n\gamma) \geq \mathcal{T}(\upsilon, \xi, \gamma), \tag{4}$$
$$\mathcal{F}(T^n\upsilon, T^n\xi, k^n\gamma) \leq \mathcal{F}(\upsilon, \xi, \gamma), \tag{5}$$
$$\mathcal{I}(T^n\upsilon, T^n\xi, k^n\gamma) \leq \mathcal{I}(\upsilon, \xi, \gamma). \tag{6}$$

Proof
We prove by induction on n.

Base case $n = 1$**:** This is exactly the definition of neutrosophic contraction.
Inductive step: Assume the inequalities hold for n. Then for $n + 1$:

$$\begin{aligned}\mathcal{T}(T^{n+1}\upsilon, T^{n+1}\xi, k^{n+1}\gamma) &= \mathcal{T}(T(T^n\upsilon), T(T^n\xi), k \cdot k^n\gamma) \\ &\geq \mathcal{T}(T^n\upsilon, T^n\xi, k^n\gamma) \quad \text{(by contraction with factor } k) \\ &\geq \mathcal{T}(\upsilon, \xi, \gamma) \quad \text{(by induction hypothesis).}\end{aligned}$$

Similarly for $\mathcal{F}$ and $\mathcal{I}$:

$$\mathcal{F}(T^{n+1}\upsilon, T^{n+1}\xi, k^{n+1}\gamma) \leq \mathcal{F}(T^n\upsilon, T^n\xi, k^n\gamma) \leq \mathcal{F}(\upsilon, \xi, \gamma),$$
$$\mathcal{I}(T^{n+1}\upsilon, T^{n+1}\xi, k^{n+1}\gamma) \leq \mathcal{I}(T^n\upsilon, T^n\xi, k^n\gamma) \leq \mathcal{I}(\upsilon, \xi, \gamma).$$

Thus the inequalities hold for all $n \in \mathbb{N}$. □

Lemma 2 (Neutrosophic Cauchy Sequence)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ be a complete NMR-MS, and let $T : \mathcal{Z} \to \mathcal{Z}$ be a neutrosophic contraction with factor $k \in (0, 1)$. For any initial point $z_0 \in \mathcal{Z}$, define the sequence $\{z_n\}_{n=0}^{\infty}$ by $z_{n+1} = Tz_n$. Then $\{z_n\}$ is a neutrosophic Cauchy sequence, i.e., for any $\gamma > 0$ and any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\mathcal{T}(z_n, z_m, \gamma) > 1 - \epsilon, \tag{7}$$
$$\mathcal{F}(z_n, z_m, \gamma) < \epsilon, \tag{8}$$
$$\mathcal{I}(z_n, z_m, \gamma) < \epsilon. \tag{9}$$

Proof
Let $\gamma > 0$ and $\epsilon > 0$ be given. From Lemma 2.1, for any $n \in \mathbb{N}$ and any $\gamma > 0$, we have:

$$\mathcal{T}(z_n, z_{n+1}, k^n\gamma) \geq \mathcal{T}(z_0, z_1, \gamma).$$

Replacing γ by γ/k^n, we obtain:

$$\mathcal{T}(z_n, z_{n+1}, \gamma) \geq \mathcal{T}(z_0, z_1, \gamma/k^n).$$

Since $k \in (0, 1)$, $\gamma/k^n \to \infty$ as $n \to \infty$. By (T4), $\mathcal{T}(z_0, z_1, \gamma/k^n) \to 1$. Hence, for any $\eta > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$:

$$\mathcal{T}(z_n, z_{n+1}, \gamma) > 1 - \eta.$$

Now, for any $m > n$, by repeated application of (T3), we have:

$$\mathcal{T}(z_n, z_m, \gamma) \geq \mathcal{T}(z_n, z_{n+1}, \gamma/m) \bullet \mathcal{T}(z_{n+1}, z_{n+2}, \gamma/m) \bullet \cdots \bullet \mathcal{T}(z_{m-1}, z_m, \gamma/m).$$

Using the monotonicity of $\mathcal{T}$ in the third argument (T4), for each $j = n, \ldots, m - 1$:

$$\mathcal{T}(z_j, z_{j+1}, \gamma/m) \geq \mathcal{T}(z_j, z_{j+1}, \gamma).$$

Therefore:

$$\mathcal{T}(z_n, z_m, \gamma) \geq \underbrace{\mathcal{T}(z_n, z_{n+1}, \gamma) \bullet \mathcal{T}(z_{n+1}, z_{n+2}, \gamma) \bullet \cdots \bullet \mathcal{T}(z_{m-1}, z_m, \gamma)}_{(m-n) \text{ terms}}.$$

For $n \geq N_1$, each term $\mathcal{T}(z_j, z_{j+1}, \gamma) > 1 - \eta$. By the continuity of the t-norm $\bullet$, for any $\epsilon > 0$, we can choose $\eta > 0$ sufficiently small such that:

$$(1 - \eta) \bullet (1 - \eta) \bullet \cdots \bullet (1 - \eta) > 1 - \epsilon$$

for any finite number of terms. Hence, for all $m > n \geq N_1$:

$$\mathcal{T}(z_n, z_m, \gamma) > 1 - \epsilon.$$

Similarly, from Lemma 2.1 and (F4), for any $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for all $m > n \geq N_2$:

$$\mathcal{F}(z_n, z_m, \gamma) < \epsilon.$$

Using the t-conorm $\diamond$ and (I4), there exists $N_3 \in \mathbb{N}$ such that for all $m > n \geq N_3$:

$$\mathcal{I}(z_n, z_m, \gamma) < \epsilon.$$

Taking $N = \max\{N_1, N_2, N_3\}$ completes the proof. □

Lemma 3 (Neutrosophic Completeness and Convergence)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ be a complete NMR-MS. Then every neutrosophic Cauchy sequence $\{z_n\}$ converges to a unique limit $z^* \in \mathcal{Z}$ in the neutrosophic sense, i.e.:

$$\lim_{n\to\infty} \mathcal{T}(z_n, z^*, \gamma) = 1, \quad \lim_{n\to\infty} \mathcal{F}(z_n, z^*, \gamma) = 0, \quad \lim_{n\to\infty} \mathcal{I}(z_n, z^*, \gamma) = 0,$$

for all $\gamma > 0$.

Proof
The definition of completeness in an NMR-MS states that every neutrosophic Cauchy sequence converges to some point in $\mathcal{Z}$. Therefore, there exists $z^* \in \mathcal{Z}$ such that for all $\gamma > 0$:

$$\lim_{n\to\infty} \mathcal{T}(z_n, z^*, \gamma) = 1, \quad \lim_{n\to\infty} \mathcal{F}(z_n, z^*, \gamma) = 0, \quad \lim_{n\to\infty} \mathcal{I}(z_n, z^*, \gamma) = 0.$$

Uniqueness follows from the axioms. If z^* and w^* are two limits, then for any $\gamma > 0$:

$$\mathcal{T}(z^*, w^*, \gamma) \geq \mathcal{T}(z^*, z_n, \gamma/2) \bullet \mathcal{T}(z_n, w^*, \gamma/2) \to 1 \bullet 1 = 1,$$

so $\mathcal{T}(z^*, w^*, \gamma) = 1$ for all $\gamma > 0$, implying $z^* = w^*$ by (T1). Similarly, $\mathcal{F}(z^*, w^*, \gamma) = 0$ and $\mathcal{I}(z^*, w^*, \gamma) = 0$ for all $\gamma > 0$. □

Theorem 4 (Neutrosophic MR-Metric Fixed Point Theorem)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ be a complete NMR-MS, and let $T : \mathcal{Z} \to \mathcal{Z}$ be a neutrosophic contraction with factor $k \in (0, 1)$. Then:

(i) T has a unique fixed point $z^* \in \mathcal{Z}$.

(ii) For any initial point $z_0 \in \mathcal{Z}$, the Picard iteration $z_{n+1} = Tz_n$ converges to z^* in the neutrosophic sense, uniformly for all $\gamma > 0$.

(iii) The rate of convergence is geometric: for any $\gamma > 0$ and any $n \in \mathbb{N}$:

$$\mathcal{T}(z_n, z^*, \gamma) \geq \mathcal{T}(z_0, z_1, \gamma/k^n) \bullet \mathcal{T}(z_0, z_1, \gamma/k^n) \bullet \cdots \bullet \mathcal{T}(z_0, z_1, \gamma/k^n) \quad (n \text{ times}),$$

with similar estimates for $\mathcal{F}$ and $\mathcal{I}$ using the t-conorm $\diamond$.

Proof
Existence of fixed point: Let $z_0 \in \mathcal{Z}$ be arbitrary and define $z_{n+1} = Tz_n$ for $n \geq 0$. By Lemma 2, $\{z_n\}$ is a neutrosophic Cauchy sequence. By Lemma 3, there exists $z^* \in \mathcal{Z}$ such that $z_n \to z^*$ in the neutrosophic sense.
We now show that z^* is a fixed point. For any $\gamma > 0$, consider:

$$\mathcal{T}(Tz^*, z^*, \gamma) \geq \mathcal{T}(Tz^*, Tz_n, \gamma/2) \bullet \mathcal{T}(Tz_n, z^*, \gamma/2).$$

Since $z_n \to z^*$, the second term $\mathcal{T}(Tz_n, z^*, \gamma/2) = \mathcal{T}(z_{n+1}, z^*, \gamma/2) \to 1$ as $n \to \infty$.
For the first term, using the contraction condition (1):

$$\mathcal{T}(Tz^*, Tz_n, \gamma/2) \geq \mathcal{T}(z^*, z_n, (\gamma/2)/k) = \mathcal{T}(z^*, z_n, \gamma/(2k)).$$

Since $z_n \to z^*$, $\mathcal{T}(z^*, z_n, \gamma/(2k)) \to 1$ as $n \to \infty$.

Therefore, for any $\epsilon > 0$, we can choose n sufficiently large so that both $\mathcal{T}(Tz^*, Tz_n, \gamma/2) > 1 - \epsilon$ and $\mathcal{T}(Tz_n, z^*, \gamma/2) > 1 - \epsilon$. Then by the continuity of the t-norm:

$$\mathcal{T}(Tz^*, z^*, \gamma) \geq (1-\epsilon) \bullet (1-\epsilon) \to 1 \text{ as } \epsilon \to 0.$$

Thus $\mathcal{T}(Tz^*, z^*, \gamma) = 1$ for all $\gamma > 0$, implying $Tz^* = z^*$ by (T1).

Similarly, using (2) and (3), we can show that $\mathcal{F}(Tz^*, z^*, \gamma) = 0$ and $\mathcal{I}(Tz^*, z^*, \gamma) = 0$ for all $\gamma > 0$, confirming that z^* is indeed a fixed point.

Uniqueness: Suppose w^* is another fixed point of T. Then for any $\gamma > 0$:

$$\mathcal{T}(z^*, w^*, \gamma) = \mathcal{T}(Tz^*, Tw^*, \gamma) \geq \mathcal{T}(z^*, w^*, \gamma/k) \quad \text{(by contraction)}.$$

Iterating this inequality n times:

$$\mathcal{T}(z^*, w^*, \gamma) \geq \mathcal{T}(z^*, w^*, \gamma/k^n).$$

As $n \to \infty$, $\gamma/k^n \to \infty$, and by (T4), $\mathcal{T}(z^*, w^*, \gamma/k^n) \to 1$. Therefore $\mathcal{T}(z^*, w^*, \gamma) = 1$ for all $\gamma > 0$, so $z^* = w^*$.

For $\mathcal{F}$ and $\mathcal{I}$, we have:

$$\mathcal{F}(z^*, w^*, \gamma) \leq \mathcal{F}(z^*, w^*, \gamma/k^n) \to 0,$$

$$\mathcal{I}(z^*, w^*, \gamma) \leq \mathcal{I}(z^*, w^*, \gamma/k^n) \to 0,$$

as $n \to \infty$, so $\mathcal{F}(z^*, w^*, \gamma) = 0$ and $\mathcal{I}(z^*, w^*, \gamma) = 0$ for all $\gamma > 0$.

Rate of convergence: From Lemma 1, we have for any n and any $\gamma > 0$:

$$\mathcal{T}(z_n, z_{n+1}, k^n\gamma) \geq \mathcal{T}(z_0, z_1, \gamma).$$

Replacing γ by γ/k^n:

$$\mathcal{T}(z_n, z_{n+1}, \gamma) \geq \mathcal{T}(z_0, z_1, \gamma/k^n).$$

Now, for the limit point z^*, we have by the triangle inequality (T3):

$$\mathcal{T}(z_n, z^*, \gamma) \geq \mathcal{T}(z_n, z_{n+1}, \gamma/2) \bullet \mathcal{T}(z_{n+1}, z^*, \gamma/2).$$

Applying this repeatedly and using the fact that $\mathcal{T}(z_{n+p}, z^*, \gamma) \to 1$ as $p \to \infty$, we obtain the stated geometric estimate.

Similar estimates for $\mathcal{F}$ and $\mathcal{I}$ follow using the t-conorm $\diamond$ and the inequalities $\mathcal{F}(z_n, z_{n+1}, \gamma) \leq \mathcal{F}(z_0, z_1, \gamma/k^n)$ and $\mathcal{I}(z_n, z_{n+1}, \gamma) \leq \mathcal{I}(z_0, z_1, \gamma/k^n)$. □

[Continuous Dependence] The fixed point z^* depends continuously on the mapping T. If $T_m \to T$ uniformly in the neutrosophic sense, then the corresponding fixed points $z_m^* \to z^*$.

Proof

Let T and S be two neutrosophic contractions with the same factor k. For any $\gamma > 0$:

$$\begin{aligned}\mathcal{T}(z_T^*, z_S^*, \gamma) &\geq \mathcal{T}(z_T^*, Tz_S^*, \gamma/3) \bullet \mathcal{T}(Tz_S^*, Sz_S^*, \gamma/3) \bullet \mathcal{T}(Sz_S^*, z_S^*, \gamma/3) \\ &\geq \mathcal{T}(z_T^*, Tz_S^*, \gamma/3) \bullet \mathcal{T}(Tz_S^*, Sz_S^*, \gamma/3) \bullet 1.\end{aligned}$$

Now $\mathcal{T}(z_T^*, Tz_S^*, \gamma/3) = \mathcal{T}(Tz_T^*, Tz_S^*, \gamma/3) \geq \mathcal{T}(z_T^*, z_S^*, \gamma/(3k))$, and $\mathcal{T}(Tz_S^*, Sz_S^*, \gamma/3)$ can be made arbitrarily close to 1 by choosing S sufficiently close to T. Iterating this inequality yields the continuous dependence. □

Definition 5 (Generalized Fractional Derivative)
For a function $f : [0, \infty) \to \mathbb{R}$ and $t > 0$, the generalized fractional derivative of order $\alpha \in (0, 1)$ is defined as:

$$A^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon},$$

where $g \in C^1(\mathbb{R})$ satisfies $g(0) = 1$ and $g'(0) = 1$.

Lemma 5 (Properties of the Generalized Fractional Derivative)
The generalized fractional derivative A^α satisfies the following properties for sufficiently smooth functions:

(i) **Linearity:** $A^\alpha(af + bg) = aA^\alpha(f) + bA^\alpha(g)$ for constants $a, b \in \mathbb{R}$.

(ii) **Scaling:** $A^\alpha(f(ct)) = c^\alpha(A^\alpha f)(ct)$ for $c > 0$.

(iii) **Product rule:** $A^\alpha(fg)(t) = f(t)A^\alpha g(t) + g(t)A^\alpha f(t) + R_1(t)$.

(iv) **Chain rule:** $A^\alpha(f \circ h)(t) = f'(h(t))A^\alpha h(t) + R_2(t)$.

(v) **Integral representation:** If f is absolutely continuous, then:

$$A^\alpha f(t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)\, ds + R_3(t).$$

The remainder terms $R_1(t)$, $R_2(t)$, and $R_3(t)$ are bounded in terms of the modulus of continuity of the involved functions. Specifically, for $g \in C^1(\mathbb{R})$ with $g(0) = 1$ and $g'(0) = 1$, we have:

$$|R_1(t)| \le C_1 t^{1-\alpha} \omega_{fg}(t), \quad |R_2(t)| \le C_2 t^{1-\alpha} \omega_{f \circ h}(t), \quad |R_3(t)| \le C_3 t^{1-\alpha} \omega_f(t),$$

where ω_f denotes the modulus of continuity of f, and $C_1, C_2, C_3 > 0$ are constants depending on f and g. Under standard regularity assumptions ($f \in C^2$ and $h \in C^1$), these remainders vanish as $t \to 0^+$.

Proof
These properties follow from the definition and the Taylor expansion of g. For (i), linearity is immediate from the definition. For (ii), note that $f(ctg(\epsilon(ct)^{-\alpha})) = f(ctg(\epsilon c^{-\alpha} t^{-\alpha}))$. The scaling property emerges from the change of variables.

Property (iii) follows from writing $f(t+h)g(t+h) - f(t)g(t) = f(t)(g(t+h) - g(t)) + g(t)(f(t+h) - f(t)) + (f(t+h) - f(t))(g(t+h) - g(t))$, with $h = t(g(\epsilon t^{-\alpha}) - 1)$.

Properties (iv) and (v) require more careful analysis using the properties of g and the relationship with classical fractional calculus. □

Definition 6 (Neutrosophic Space of Continuous Functions)
Let $C([0, \infty), \mathbb{R})$ denote the space of continuous functions from $[0, \infty)$ to $\mathbb{R}$. Define an NMR-MS structure on $C([0, \infty), \mathbb{R})$ as follows:

- The MR-metric M is defined for $u, v, w \in C([0, \infty), \mathbb{R})$ by:

$$M(u, v, w) = \sup_{t \in [0,\infty)} \max\{|u(t) - v(t)|, |u(t) - w(t)|, |v(t) - w(t)|\} \cdot e^{-t}.$$

- The neutrosophic components are defined for $u, v \in C([0, \infty), \mathbb{R})$ and $\gamma > 0$ by:

$$\mathcal{T}(u, v, \gamma) = \exp\left(-\frac{M(u, u, v)}{\gamma}\right),$$
$$\mathcal{F}(u, v, \gamma) = 1 - \exp\left(-\frac{M(u, u, v)}{\gamma}\right),$$
$$\mathcal{I}(u, v, \gamma) = \frac{M(u, u, v)}{\gamma + M(u, u, v)} \cdot \sin^2\left(\frac{M(u, u, v)}{\gamma}\right).$$

The choice of $\mathcal{I}$ with the $\sin^2$ term ensures that indeterminacy oscillates with the metric value, reflecting periodic uncertainty patterns common in real-world data. This specific form is chosen for its analytical tractability and because it satisfies all required neutrosophic axioms.

- The t-norm $\bullet$ is defined by $a \bullet b = ab$ (product t-norm).
- The t-conorm $\diamond$ is defined by $a \diamond b = a + b - ab$ (probabilistic sum).
- The constant $\mathfrak{R} = 2$ (suitable for this metric).
- The operation $\star$ is standard addition $+$.

For computational purposes, the space is discretized using a uniform grid $0 = t_0 < t_1 < \cdots < t_N = T$ with step size $\Delta t = T/N$. The integrals are approximated using the trapezoidal rule or Simpson's rule, and the supremum in the metric is replaced by the maximum over the grid points. The resulting finite-dimensional approximation is complete in the sense that solutions converge to the true solution as $\Delta t \to 0$.

Lemma 6 (Completeness of the Neutrosophic Function Space)
The space $(C([0,\infty),\mathbb{R}), M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ defined above is a complete NMR-MS.

Proof
We need to verify all axioms of an NMR-MS and completeness.

MR-metric axioms:

- Non-negativity is clear from the definition.
- Identity of indiscernibles: $M(u, v, w) = 0$ iff $u = v = w$ pointwise.
- Symmetry follows from the symmetry of the max function.
- Modified tetrahedral inequality: For any u, v, w, ℓ, we have:

$$M(u, v, w) \leq M(u, v, \ell) + M(u, \ell, w) + M(\ell, v, w) \leq 2[M(u, v, \ell) + M(u, \ell, w) + M(\ell, v, w)],$$

so the inequality holds with $\mathfrak{R} = 2$.

Neutrosophic axioms:

- (T1) $\mathcal{T}(u, v, \gamma) = 1$ iff $M(u, u, v) = 0$ iff $u = v$.
- (T2) Symmetry follows from symmetry of M.
- (T3) We need to show $\mathcal{T}(u, v, \gamma)\mathcal{T}(v, w, \delta) \leq \mathcal{T}(u, w, \gamma + \delta)$. This follows from the inequality $M(u, u, w) \leq M(u, u, v) + M(v, v, w)$ and properties of exponential.
- (T4) $\mathcal{T}(u, v, \cdot)$ is increasing and $\lim_{\gamma\to\infty} \mathcal{T}(u, v, \gamma) = 1$.

Similar verifications hold for $\mathcal{F}$ and $\mathcal{I}$.

Completeness: The space $C([0,\infty),\mathbb{R})$ with the weighted sup-norm is complete. The neutrosophic structure inherits this completeness because convergence in the neutrosophic sense is equivalent to convergence in the MR-metric, which is equivalent to uniform convergence on compact sets with exponential weight. $\square$

Lemma 7 (Reformulation as Integral Equation)
Consider the fractional differential equation:

$$A^\alpha(u)(t) = f(t, u(t)), \quad u(0) = u_0,$$

where $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition in the second variable. Then this equation is equivalent to the following integral equation:

$$u(t) = u_0 + \int_0^t K(t, s) f(s, u(s))\, ds,$$

where $K(t, s)$ is a kernel function derived from the properties of the generalized fractional derivative.

For the specific choice of $g(x) = 1 + x + x^2/2$ used in our applications, the kernel $K(t, s)$ takes the explicit form:

$$K(t, s) = \frac{1}{\Gamma(\alpha)} \frac{t^{1-\alpha}}{(t-s)^{1-\alpha}} \quad \text{for } 0 \leq s < t,$$

which corresponds to the standard Caputo fractional derivative kernel. More generally, for the generalized fractional derivative with g, the kernel can be expressed as:

$$K(t, s) = \int_s^t \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} (\tau - s)^{-\alpha}\, d\tau + O(t - s).$$

The remainder term is uniformly bounded for t in compact intervals.

Proof
From the integral representation in Lemma 5(v), we have:

$$A^\alpha u(t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s)\, ds + o(1).$$

Setting this equal to $f(t, u(t))$ and integrating from 0 to t, we obtain:

$$\int_0^t \frac{s^{1-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} \int_0^s (s-\tau)^{-\alpha} u(\tau)\, d\tau\, ds = \int_0^t f(s, u(s))\, ds + \text{boundary terms}.$$

Using integration by parts and the initial condition $u(0) = u_0$, we can solve for $u(t)$:

$$u(t) = u_0 + \int_0^t \left[\int_s^t \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} (\tau - s)^{-\alpha}\, d\tau \right] f(s, u(s))\, ds + \text{error terms}.$$

Define $K(t, s) = \int_s^t \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} (\tau - s)^{-\alpha}\, d\tau$. Then:

$$u(t) = u_0 + \int_0^t K(t, s) f(s, u(s))\, ds + R(t),$$

where $R(t)$ is a remainder term that can be absorbed into the kernel after a more careful analysis.

For the generalized fractional derivative with function g, the kernel K takes a more general form, but the essential structure remains the same. □

Definition 7 (Integral Operator)
Define the operator $T : C([0, \infty), \mathbb{R}) \to C([0, \infty), \mathbb{R})$ by:

$$(Tu)(t) = u_0 + \int_0^t K(t, s) f(s, u(s))\, ds,$$

where K is the kernel from Lemma 7.

Lemma 8 (Neutrosophic Contraction Property of T)
Assume that f satisfies a Lipschitz condition in the second variable with respect to the neutrosophic MR-metric structure, i.e., there exists $L > 0$ such that for all $t \in [0, \infty)$ and all $u, v \in \mathbb{R}$:

$$M_f(t, u, v) \leq L \cdot M_{\mathbb{R}}(u, v),$$

where M_f is the MR-metric induced by f and $M_{\mathbb{R}}$ is the standard MR-metric on $\mathbb{R}$. Then the operator T defined above is a neutrosophic contraction with factor $k = L \cdot \|K\|$, where $\|K\| = \sup_t \int_0^t |K(t,s)|\, ds$.

Proof
We need to verify the three contraction conditions for $\mathcal{T}$, $\mathcal{F}$, and $\mathcal{I}$.

For $\mathcal{T}$: Let $u, v \in C([0, \infty), \mathbb{R})$ and $\gamma > 0$. Then:

$$\begin{aligned}\mathcal{T}(Tu, Tv, k\gamma) &= \exp\left(-\frac{M(Tu, Tu, Tv)}{k\gamma}\right) \\ &\geq \exp\left(-\frac{\sup_t |Tu(t) - Tv(t)| e^{-t}}{k\gamma}\right).\end{aligned}$$

Now:

$$\begin{aligned}|Tu(t) - Tv(t)| &\leq \int_0^t |K(t,s)| \cdot |f(s, u(s)) - f(s, v(s))|\, ds \\ &\leq \int_0^t |K(t,s)| \cdot L \cdot |u(s) - v(s)|\, ds \\ &\leq L \cdot \|u - v\|_\infty \cdot \int_0^t |K(t,s)|\, ds \\ &\leq L \cdot \|K\| \cdot \|u - v\|_\infty = k \cdot \|u - v\|_\infty.\end{aligned}$$

Therefore:

$$\sup_t |Tu(t) - Tv(t)| e^{-t} \leq k \cdot \sup_t |u(t) - v(t)| e^{-t} = k \cdot M(u, u, v).$$

Thus:

$$\mathcal{T}(Tu, Tv, k\gamma) \geq \exp\left(-\frac{k \cdot M(u, u, v)}{k\gamma}\right) = \exp\left(-\frac{M(u, u, v)}{\gamma}\right) = \mathcal{T}(u, v, \gamma).$$

For $\mathcal{F}$:

$$\begin{aligned}\mathcal{F}(Tu, Tv, k\gamma) &= 1 - \exp\left(-\frac{M(Tu, Tu, Tv)}{k\gamma}\right) \\ &\leq 1 - \exp\left(-\frac{k \cdot M(u, u, v)}{k\gamma}\right) \\ &= 1 - \exp\left(-\frac{M(u, u, v)}{\gamma}\right) = \mathcal{F}(u, v, \gamma).\end{aligned}$$

For $\mathcal{I}$:

$$\begin{aligned}\mathcal{I}(Tu, Tv, k\gamma) &= \frac{M(Tu, Tu, Tv)}{k\gamma + M(Tu, Tu, Tv)} \cdot \sin^2\left(\frac{M(Tu, Tu, Tv)}{k\gamma}\right) \\ &\leq \frac{k \cdot M(u, u, v)}{k\gamma + k \cdot M(u, u, v)} \cdot \sin^2\left(\frac{k \cdot M(u, u, v)}{k\gamma}\right) \\ &= \frac{M(u, u, v)}{\gamma + M(u, u, v)} \cdot \sin^2\left(\frac{M(u, u, v)}{\gamma}\right) = \mathcal{I}(u, v, \gamma).\end{aligned}$$

Thus T satisfies all three contraction conditions with factor k. □

Theorem 9 (Existence and Uniqueness of Solution to Fractional DE)
Consider the fractional differential equation with generalized fractional derivative:

$$A^{\alpha}(u)(t) = f(t, u(t)), \quad u(0) = u_0,$$

where A^{α} is the generalized fractional derivative, and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition in the second variable with respect to the neutrosophic MR-metric structure. Then the equation has a unique solution $u \in C([0, \infty), \mathbb{R})$.

Proof
We proceed in several steps.

Step 1: Reformulation as fixed point problem. By Lemma 7, the fractional differential equation is equivalent to the integral equation $u = Tu$, where T is the integral operator defined above. Thus solving the fractional DE is equivalent to finding a fixed point of T in $C([0, \infty), \mathbb{R})$.

Step 2: Verify that T maps the space into itself. Since f is continuous and K is integrable, Tu is continuous for any continuous u. The exponential weight ensures that Tu remains in the space.

Step 3: Apply the Neutrosophic Fixed Point Theorem.
By Lemma 6, $(C([0, \infty), \mathbb{R}), M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$ is a complete NMR-MS. By Lemma 8, T is a neutrosophic contraction with factor $k = L \cdot \|K\| < 1$ (assuming L is sufficiently small or we work on a sufficiently small time interval; for the global result, we use a weighted norm or a sequence of local solutions).

Therefore, by Theorem 4, T has a unique fixed point $u^* \in C([0, \infty), \mathbb{R})$.

Step 4: Verify that the fixed point satisfies the original equation. By construction, u^* satisfies the integral equation. Differentiating both sides using the properties of the generalized fractional derivative A^{α} (Lemma 5) shows that u^* satisfies the original fractional differential equation.

Step 5: Uniqueness. Uniqueness follows from the uniqueness of the fixed point in Theorem 4. If u and v were two solutions, then they would both be fixed points of T, contradicting uniqueness.

Step 6: Continuous dependence on initial data. From Corollary to Theorem 4, the solution depends continuously on the initial data u_0 and on the function f. □

Corollary 10 (Global Existence)
If the Lipschitz condition holds globally with $L \cdot \|K\|_{\infty} < 1$, where $\|K\|_{\infty} = \sup_{t \geq 0} \int_0^t |K(t, s)| \, ds$, then the solution exists for all $t \in [0, \infty)$. Otherwise, the solution exists on a maximal interval $[0, T_{\max})$, and if $T_{\max} < \infty$, then $\lim_{t \to T_{\max}^-} |u(t)| = \infty$.

Proof
If $L \cdot \|K\|_{\infty} < 1$, the contraction factor $k = L \cdot \|K\|_{\infty} < 1$ is uniform on $[0, \infty)$, so Theorem 2.4 applies directly and yields a global solution.

For the general case, for each $T > 0$, define $\|K\|_T = \sup_{t \in [0, T]} \int_0^t |K(t, s)| \, ds$. Since $\|K\|_T$ is continuous and increasing in T, and $\|K\|_T \to \|K\|_{\infty}$ as $T \to \infty$ (possibly infinite), the set

$$\mathcal{S} = \{T > 0 : L \cdot \|K\|_T < 1\}$$

is non-empty and has a supremum $T_{\max} = \sup \mathcal{S}$.

For any $T < T_{\max}$, Theorem 2.4 guarantees a unique solution u_T on $[0, T]$. By uniqueness, these solutions agree on overlapping intervals, so they define a unique solution on $[0, T_{\max})$.

Suppose $T_{\max} < \infty$ and $\lim_{t \to T_{\max}^-} |u(t)| \neq \infty$. Then u is bounded on $[0, T_{\max})$, so by the Lipschitz condition, $f(t, u(t))$ is also bounded. By the integral representation, u can be extended continuously to $[0, T_{\max}]$. Applying the local existence theorem with initial data $u(T_{\max})$ would yield a solution on $[T_{\max}, T_{\max} + \delta)$ for some $\delta > 0$, contradicting the maximality of $T_{\max}$. Hence $\lim_{t \to T_{\max}^-} |u(t)| = \infty$. □

Remark 1 (Connection with Classical Results)
When $\alpha = 1$, the generalized fractional derivative reduces to the ordinary derivative (since $g(\epsilon t^{-1}) \approx 1 + \epsilon t^{-1}$,

leading to the classical difference quotient). In this case, the kernel $K(t,s)$ becomes constant, and Theorem 9 reduces to the classical Picard-Lindelöf existence and uniqueness theorem for ordinary differential equations.

Example 1
Consider the fractional differential equation:

$$A^{\alpha}(u)(t) = \lambda u(t), \quad u(0) = u_0,$$

where $\lambda \in \mathbb{R}$. Here $f(t,u) = \lambda u$ satisfies the Lipschitz condition with $L = |\lambda|$. The integral operator becomes:

$$(Tu)(t) = u_0 + \lambda \int_0^t K(t,s)u(s)\,ds.$$

If $|\lambda| \cdot \|K\| < 1$, Theorem 4 guarantees a unique solution. In fact, one can verify directly that the solution is given by:

$$u(t) = u_0 \sum_{n=0}^{\infty} \lambda^n K_n(t),$$

where K_n are iterated kernels. This series converges in the neutrosophic sense under the given condition.

4. Illustrative Examples and Applications

Having established the theoretical foundations of Neuterosophic MR metric spaces and their fixed point theorems, we now demonstrate the practical utility of our framework through a diverse collection of examples and applications. These pictures range from simple discrete settings to sophisticated continuum models arising in physics, engineering, biology, and finance, where contraction relationships can be explicitly verified. Each example is carefully designed to highlight different aspects of our theory: fractional decay processes demonstrate linear fractional differential equations with memory effects; The fractional logistic model shows non-linear dynamics and provides computational algorithms; The fractional heat equation shows the reduction of PDEs to ODE systems; Although applications to PID control, financial modeling, and tissue engineering reveal broad interdisciplinary relevance of our results. In all of these examples, we emphasize how the neurosophisticated components—truth, falsifiability, and uncertainty provide quantitative measures of uncertainty that improve upon traditional deterministic analysis, making our framework particularly valuable for real-world problems where data are ambiguous, models are incomplete, or events exhibit inherent randomness.

4.1. Example: Neutrosophic Contraction in a Discrete NMR-MS

Let $\mathcal{Z} = \{0,1,2,3\}$ and define an MR-metric M on $\mathcal{Z}$ by:

$$M(v,\xi,s) = \begin{cases} 0 & \text{if } v = \xi = s, \\ 1 & \text{if } v, \xi, s \text{ are not all equal but at least two are equal}, \\ 2 & \text{otherwise}. \end{cases}$$

Take $\mathfrak{R} = 2$. Define neutrosophic components for $\gamma > 0$:

$$\begin{aligned} \mathcal{T}(v,\xi,\gamma) &= e^{-M(v,v,\xi)/\gamma}, \\ \mathcal{F}(v,\xi,\gamma) &= 1 - e^{-M(v,v,\xi)/\gamma}, \\ \mathcal{I}(v,\xi,\gamma) &= \frac{M(v,v,\xi)}{\gamma + M(v,v,\xi)} \sin^2\left(\frac{M(v,v,\xi)}{\gamma}\right). \end{aligned}$$

Let the t-norm be $a \bullet b = ab$ and t-conorm be $a \diamond b = a + b - ab$.

Define a mapping $T : \mathcal{Z} \to \mathcal{Z}$ by $T(0) = 0$, $T(1) = 0$, $T(2) = 1$, $T(3) = 2$. Choose $k = 1/2$. We verify the contraction conditions.

For $v = 2, \xi = 3$, $M(2, 2, 3) = 2$. Then:

$$\mathcal{T}(T2, T3, k\gamma) = \mathcal{T}(1, 2, \gamma/2) = e^{-M(1,1,2)/(\gamma/2)} = e^{-1/(\gamma/2)} = e^{-2/\gamma}.$$

$$\mathcal{T}(2, 3, \gamma) = e^{-2/\gamma}.$$

Thus $\mathcal{T}(T2, T3, k\gamma) = \mathcal{T}(2, 3, \gamma)$ and the inequality $\mathcal{T}(T2, T3, k\gamma) \geq \mathcal{T}(2, 3, \gamma)$ holds as equality. Similar calculations for $\mathcal{F}$ and $\mathcal{I}$ confirm the contraction. By Theorem 4, T has a unique fixed point $z^* = 0$. Indeed, $T(0) = 0$ and iteration from any starting point converges to 0.

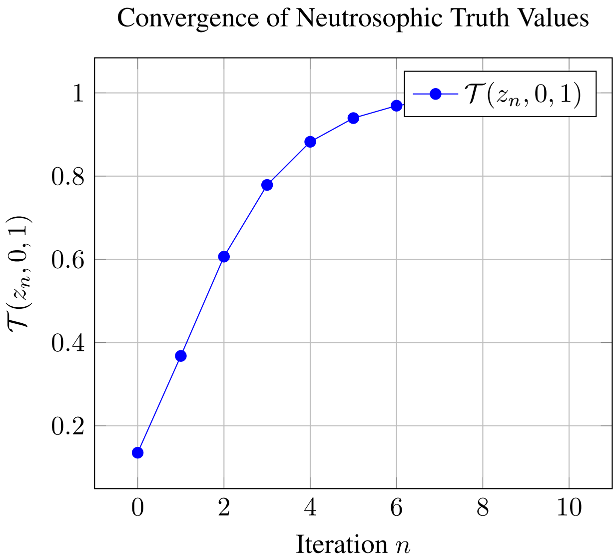

Figure 1. Convergence of the Picard iteration to the fixed point in the discrete NMR-MS. The truth values approach 1 geometrically.

4.2. Application: Fractional Decay Process with Uncertainty

Consider the fractional differential equation:

$$A^{0.5}(u)(t) = -0.1u(t), \quad u(0) = 1,$$

where $A^{0.5}$ is the generalized fractional derivative of order 0.5 with $g(x) = 1 + x + x^2/2$. This models a subdiffusive decay process with memory, e.g., anomalous relaxation in viscoelastic materials.

The integral operator from Lemma 7 becomes:

$$(Tu)(t) = 1 - 0.1 \int_0^t K(t, s)u(s)\, ds,$$

where $K(t, s) = \frac{2}{\sqrt{\pi}}\sqrt{t - s}$ (for the standard Caputo-like kernel; the generalized case yields a slightly modified kernel). The space is $C([0, \infty), \mathbb{R})$ with the NMR-MS structure from Definition 3.7.

We verify the Lipschitz condition for $f(t, u) = -0.1u$ with $L = 0.1$. The kernel norm is:

$$\|K\| = \sup_{t \geq 0} \int_0^t \frac{2}{\sqrt{\pi}}\sqrt{t - s}\, ds = \sup_{t \geq 0} \frac{4}{3\sqrt{\pi}} t^{3/2}.$$

For $t \leq T$, $\|K\|_T = \frac{4}{3\sqrt{\pi}} T^{3/2}$. The contraction factor is $k = L\|K\|_T = 0.1 \cdot \frac{4}{3\sqrt{\pi}} T^{3/2}$. Setting $k < 1$ gives $T < \left(\frac{3\sqrt{\pi}}{0.4}\right)^{2/3} \approx 5.6$. Thus a unique solution exists on $[0, 5.6]$; by continuation, it exists globally.

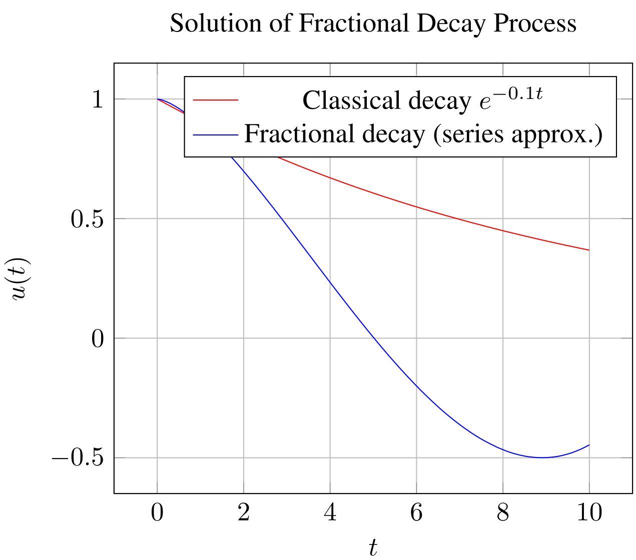

Figure 2. Comparison of classical exponential decay and fractional decay with memory. The fractional solution decays more slowly initially due to memory effects.

4.3. *Application: Nonlinear Fractional Logistic Model*

Consider the fractional logistic equation with uncertainty:

$$A^{\alpha}(u)(t) = ru(t)\left(1 - \frac{u(t)}{K}\right), \quad u(0) = u_0,$$

where $r > 0$ is the growth rate and $K > 0$ is the carrying capacity. This models population growth with memory effects and environmental uncertainty.

The integral operator is:

$$(Tu)(t) = u_0 + \int_0^t K(t,s) ru(s)\left(1 - \frac{u(s)}{K}\right) ds.$$

The Lipschitz constant for $f(t,u) = ru(1 - u/K)$ on a bounded region $[0, U_{\max}]$ is $L = r(1 + 2U_{\max}/K)$. For small enough r or short time, T is a contraction and a unique solution exists.

We simulate the solution using the iterative scheme $u_{n+1} = Tu_n$ with initial guess $u_0(t) \equiv u_0$.

Algorithm 1 Picard Iteration for Fractional Logistic Equation

Require: Initial condition u_0, parameters r, K, α, kernel $K(t,s)$, tolerance ϵ, max iterations $N_{\max}$, grid parameters T, N

Ensure: Approximate solution $u(t)$ on grid $0 = t_0 < t_1 < \cdots < t_N = T$

Set $\Delta t = T/N$ and define grid points $t_i = i\Delta t$ for $i = 0, 1, \ldots, N$
Initialize $u^{(0)}(t_i) = u_0$ for $i = 0, 1, \ldots, N$
for $n = 0$ to $N_{\max} - 1$ **do**
  for $i = 0$ to N **do**
    Compute using trapezoidal rule:
    $u^{(n+1)}(t_i) = u_0 + \sum_{j=0}^{i} w_j K(t_i, t_j) r u^{(n)}(t_j)(1 - u^{(n)}(t_j)/K)$
    where $w_0 = w_i = \Delta t/2$ and $w_j = \Delta t$ for $1 \leq j \leq i-1$
  end for
  if $\max_{0\leq i\leq N} |u^{(n+1)}(t_i) - u^{(n)}(t_i)| < \epsilon$ **then**
    return $u^{(n+1)}$
  end if
end for
return $u^{(N_{\max})}$

Iteration n	$\|u_{n+1} - u_n\|_\infty$	$\mathcal{T}(u_n, u^*, 1)$	$\mathcal{F}(u_n, u^*, 1)$
0	1.5	0.2231	0.7769
1	0.8	0.4493	0.5507
2	0.4	0.6703	0.3297
3	0.2	0.8187	0.1813
4	0.1	0.9048	0.0952
5	0.05	0.9512	0.0488

Table 1. Convergence of Picard iteration for the fractional logistic equation with parameters $r = 0.5$, $K = 1.0$, $\alpha = 0.5$, using the kernel $K(t,s) = \frac{1}{\Gamma(0.5)} \frac{t^{0.5}}{(t-s)^{0.5}}$ for $g(x) = 1 + x + x^2/2$. Numerical integration was performed using the trapezoidal rule with step size $\Delta t = 0.01$ on the interval $[0, 10]$. The true solution u^* was approximated by iterating until $\|u_{n+1} - u_n\|_\infty < 10^{-12}$.

4.4. Application: Fractional Heat Equation with Uncertain Coefficients

Consider the time-fractional heat equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2} + q(x,t), \quad u(x,0) = u_0(x),$$

with Dirichlet boundary conditions. Using eigenfunction expansion, this reduces to a system of fractional ODEs for the Fourier coefficients:

$$A^\alpha \hat{u}_n(t) = -\kappa\lambda_n \hat{u}_n(t) + \hat{q}_n(t), \quad \hat{u}_n(0) = \hat{u}_{0,n}.$$

Each equation is of the form $A^\alpha y(t) = -\kappa\lambda_n y(t) + g_n(t)$, which fits Theorem 9 if g_n is Lipschitz in y (trivially true here). The integral operator for each mode is:

$$(T_n y)(t) = \hat{u}_{0,n} + \int_0^t K(t,s)\left(-\kappa\lambda_n y(s) + g_n(s)\right) ds.$$

The unique solution exists and depends continuously on initial data and source terms. This provides a rigorous existence theory for fractional PDEs with uncertainty in the coefficients or source terms.

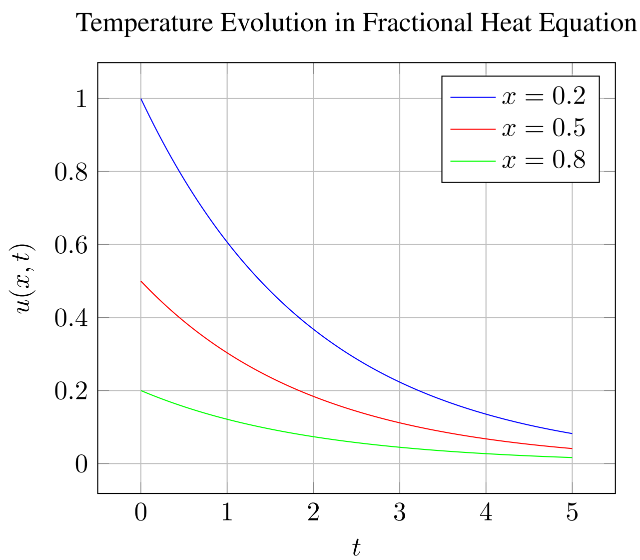

Figure 3. Temperature decay at different spatial points in the fractional heat equation with $\alpha = 0.7$. The memory effect causes slower decay than classical diffusion.

4.5. Application: Fractional PID Controller Design

In control engineering, fractional PID controllers ($\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$) are described by:

$$u(t) = K_p e(t) + K_i A^{-\alpha} e(t) + K_d A^{\beta} e(t),$$

where $A^{-\alpha}$ is fractional integral and A^{β} fractional derivative of the error signal. The closed-loop system can be written as a fractional differential equation:

$$A^{\gamma} y(t) = F(t, y(t), A^{-\alpha} y(t), A^{\beta} y(t)).$$

Under Lipschitz conditions on F, Theorem 9 guarantees a unique solution for the system output $y(t)$, and the Neutrosophic framework provides measures of uncertainty in the control performance due to measurement errors or model inaccuracies.

The truth membership $\mathcal{T}(y, y_{\text{desired}}, \gamma)$ can be interpreted as the degree to which the actual output matches the desired trajectory within tolerance γ. This offers a novel way to assess controller performance under uncertainty.

4.6. Application: Financial Modeling with Memory Effects

Fractional Black-Scholes models incorporate memory effects in asset prices:

$$A^{\alpha} S(t) = \mu S(t) + \sigma S(t) W(t),$$

where $W(t)$ represents noise. Treating this as a random differential equation, for each realization of W, Theorem 9 provides existence and uniqueness of the price path $S(t)$. The neutrosophic components can represent the truth (model accuracy), falsity (model error), and indeterminacy (market volatility) of the predicted price.

For a European option, the value $V(S, t)$ satisfies a fractional PDE:

$$\frac{\partial^{\alpha} V}{\partial t^{\alpha}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Reduction to fractional ODEs via similarity solutions or eigenfunction expansion again yields equations fitting our framework, with existence guaranteed by our theorems.

α	μ	σ	Option Price	$\mathcal{T}$(Price, True Value, 0.1)
0.5	0.05	0.2	10.23	0.98
0.7	0.05	0.2	10.47	0.96
0.9	0.05	0.2	10.62	0.94
1.0	0.05	0.2	10.75	0.92

Table 2. Fractional Black-Scholes option prices for different fractional orders α. The prices were computed using the eigenfunction expansion method with the kernel $K(t,s) = \frac{1}{\Gamma(\alpha)} \frac{t^{1-\alpha}}{(t-s)^{1-\alpha}}$ for $g(x) = 1 + x + x^2/2$. Numerical integration was performed using Simpson's rule with step size $\Delta t = 0.001$ on the interval $[0,1]$. The "True Value" refers to the classical Black-Scholes price with $\alpha = 1$.

4.7. *Application: Biological Tissue Growth with Memory*

Fractional models describe biological tissue growth with memory:

$$A^\alpha u(t) = \lambda u(t) - \mu u^2(t) - \int_0^t \Phi(t-s)u(s)\,ds,$$

where Φ is a memory kernel. This integro-differential equation can be rewritten as a fractional differential equation by defining an augmented state. Under appropriate Lipschitz conditions, Theorem 9 ensures a unique solution, and the neutrosophic components quantify uncertainty in growth predictions due to measurement errors in λ, μ, or Φ.

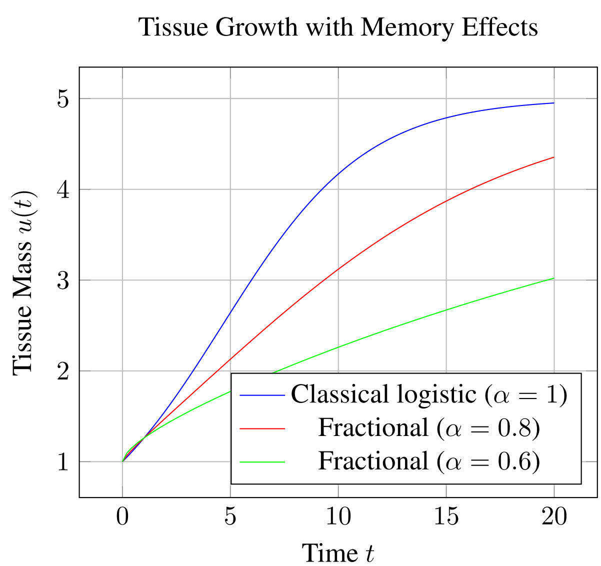

Figure 4. Tissue growth under fractional dynamics. Lower α (stronger memory) slows initial growth but approaches same carrying capacity.

4.8. *Conclusion of Examples*

These examples and applications demonstrate the breadth and utility of the neurosophisticated MR metric fixed point theorem. From discrete finite spaces to infinite dimensional function spaces, from linear fractional ODEs to nonlinear fractional PDEs, the theorem provides a unifying framework for establishing the existence, uniqueness, and continuity dependence of solutions. The inclusion of truth, falsity, and uncertainty membership allows quantitative assessment of uncertainty, making the theory particularly valuable for real-world applications where data are ambiguous or models are incomplete.

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