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On the Analysis of Caputo–Hadamard Fractional Differential Equation with Delay and Pantograph

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Abstract This study investigates Caputo–Hadamard fractional differential equations with pantograph delay arguments. Such equations arise in the modeling of systems whose present state depends on both the current value and scaled past states, reflecting memory and delay effects. The problem is formulated in an appropriate Banach space framework to analyze the behavior of its solutions. By converting the given fractional differential equation into an equivalent integral equation, sufficient conditions for the existence of solutions are established using fixed-point techniques. Furthermore, the uniqueness of the solution is obtained under suitable Lipschitz-type conditions by applying the Banach contraction principle. The stability of the system is investigated in the sense defined by Ulam–Hyers, showing that small perturbations in the system lead to small deviations in the corresponding solutions. In addition, controllability results are derived using a fixed-point approach and a numerical illustration is provided to validate the theoretical conclusions.

Keywords Caputo–Hadamard fractional derivative; boundary value problem; pantograph delay; fixed point theorem

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1. Introduction

Fractional differential equations (FDEs) provide an effective framework designed to describe dynamical systems that exhibit memory effects and hereditary behavior, which often occur in engineering, physics, and related applied sciences. Unlike classical integer-order models, fractional-order systems capture long term dependence and non-local effects, leading to more realistic descriptions of complex processes. The basic mathematical foundations of fractional calculus and its applications have been well established in the literature [14, 18]. Among the various fractional operators, the Hadamard and Caputo–Hadamard derivatives have gained increasing attention due to their logarithmic kernel, which is particularly suitable for modeling ultra-slow diffusion and scale-invariant phenomena. The mathematical foundations of Hadamard-type fractional calculus were developed by Kilbas [12], and subsequent studies have demonstrated the effectiveness of Caputo–Hadamard operators in problems involving boundary and initial conditions are not sufficient. Recent advances have further demonstrated the applicability of the pantograph system and related fractional dynamics [3, 1, 16].

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Delay effects play a crucial role in many real-world systems, where the evolution of the state depends on both present and past values. In recent years, FDE delays have been the subject of considerable research activity that involves proportional delays, constant delays, or a combination of both. In particular, various analytical approaches have been developed to effectively study such equations with mixed proportional and constant delays, highlighting the importance of rigorous qualitative analysis for such models [8, 10]. Moreover, systems that involve multiple delays, including fractional differential equations of two-delays, have been shown to exhibit rich dynamical behavior and additional mathematical challenges [4, 2].

The Fractional pantograph $(0 < \lambda < 1)$ and delay systems have also been investigated with respect to the existence, stability, and robustness properties using various fractional operators and fixed point techniques [15] [16] [1, 3]. Modern developments in functional analysis and FPT provide an essential mathematical framework for establishing such qualitative results in nonlinear fractional systems [11]. In parallel, controllability problems for fractional differential equations have attracted significant attention in recent years. Controllability results for fractional diffusion processes governed by Caputo–Hadamard derivatives have been investigated in the context of ultra-slow diffusion and related applications [5, 6]. Furthermore, Duman [7] and Mairi and Monsif [13] studied controllability aspects of fractional-order delay differential equations. Controllability issues for impulsive and neutral fractional systems with infinite delay have also been addressed by Gunasekar et al. [9]. These studies demonstrate the growing importance of controllability analysis in fractional dynamical systems.Furthermore, Stability concepts in the sense of UH (Ulam–Hyers) and UH-R (Ulam–Hyers–Rassias) futher provide useful tools for assessing the sensitivity of fractional systems under perturbations [17].

Although several studies have investigated fractional differential equations involving delay arguments, pantograph terms, and Caputo–Hadamard derivatives, limited attention has been given to Caputo–Hadamard fractional differential equations with pantograph delay under boundary conditions. Moreover, the combined investigation of well-posedness, Ulam–Hyers stability, and controllability for such problems remains limited in the current literature. Motivated by this gap, the present work focuses on a Caputo–Hadamard fractional differential equation involving pantograph delay under boundary conditions. By employing suitable fixed-point theorems and fractional integral techniques, sufficient conditions are established for the existence, uniqueness, Ulam–Hyers stability, and controllability of solutions. The obtained results extend and complement several existing studies on fractional differential equations with delay effects.

$$\begin{aligned} &{}^{CH}D^{\kappa}_{1+}\mathfrak{u}(\rho) = \psi(\rho, \mathfrak{u}(\rho), \mathfrak{u}(\lambda\rho), {}^{H}D^{\kappa}_{1+}[\mathfrak{u}(\rho - \lambda\rho)]), \quad \rho \in \mathfrak{J} = [1, T] \\ &\mathfrak{u}(1) = \xi, \qquad \mathfrak{u}(T) = \eta \end{aligned} \tag{1.1}$$

where ξ and η are given constants, $\kappa \in (0, 1)$ is the order of the Caputo–Hadamard fractional derivative ${}^{CH}D^{\kappa}_{1+}$, and $\lambda \in (0, 1)$ denotes the pantograph delay parameter.
(H1) The nonlinear function $\psi : \mathfrak{J} \times \Re^3 \to \Re$ is continuous. where $\psi(\rho, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$ corresponds to $\mathfrak{u}_1 = \mathfrak{u}(\rho)$, $\mathfrak{u}_2 = \mathfrak{u}(\lambda\rho)$, and $\mathfrak{u}_3 = {}^{H}D^{\kappa}_{1+}[\mathfrak{u}(\rho - \lambda\rho)]$. The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries. Section 3 establishes the existence and uniqueness results. Section 4 investigates the stability analysis. Section 5 is devoted to controllability results. Section 6 provides numerical examples to illustrate the applicability of the theoretical results. Finally, Section 7 concludes the paper and discusses possible future research directions.

2. Preliminaries

The definitions and preliminary lemmas necessary for use throughout the analysis are collected in this section.

Definition 2.1
[11] Consider $\mathfrak{P}$ to be proper function. Then, the κ-order $\mathrm{H-FIE}$ *is defined by:*

$$\left({}^{H}I_{l+}^{\kappa}\mathfrak{P}\right)(\theta)=\frac{1}{\Gamma(p_1)}\int_{l}^{\theta}\left(\log\left(\frac{\theta}{r}\right)\right)^{\kappa-1}\frac{\mathfrak{P}(r)}{r}dr$$

Definition 2.2
[11] For a function $\mathfrak{P}$, the CH-FDE of order $a-1<\kappa\leq a$ is expressed as:

$$\left({}^{CH}D_{1+}^{\kappa}\mathfrak{P}\right)(\theta)=\theta\frac{d}{d\theta}\left({}^{H}I_{1+}^{a-\kappa}\mathfrak{P}\right)(\theta)=\frac{1}{\Gamma(a-p_1)}\theta\frac{d}{d\theta}\int_{1}^{\theta}\left(\log\left(\frac{\theta}{r}\right)\right)^{a-\kappa-1}\frac{\mathfrak{P}(r)}{r}dr$$

Definition 2.3
The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\qquad \alpha>0,\ \beta>0,\ z\in\mathbb{C}.$$

In particular, for $\beta=1$, it reduces to

$$E_{\alpha}(z)=E_{\alpha,1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}.$$

Lemma 2.1
[12] *Let* $\mathrm{n}-1<\kappa\leq\mathrm{n}$. *Then:*

$${}^{H}I_{1+}^{\kappa}\left({}^{CH}D_{1+}^{\kappa}\mathfrak{P}(\theta)\right)=\mathfrak{P}(\theta)-\sum_{j=0}^{\mathrm{n}-1}\kappa_j(\log\theta)^j,\quad \kappa_j\in\Re,\quad j=0,1,...,\mathrm{n}-1$$

Theorem 2.1
[18] [Banach's FPT] Consider a complete metric space $(\mathfrak{u},\mathrm{d})$ and a contraction mapping $\mathfrak{T}:\mathfrak{u}\to\mathfrak{u}$. Under these conditions $\mathfrak{T}$ has a unique fixed point $\breve{\mathfrak{u}}\in\mathfrak{u}$ satisfying

$$\mathfrak{T}(\breve{\mathfrak{u}})=\breve{\mathfrak{u}}.$$

Theorem 2.2
[18][Arzelà–Ascoli Theorem] Given a family of functions on the compact interval $[\mathrm{a},\mathrm{b}]$ *that are both uniformly bounded and equicontinuous, there exists a subsequence which converges uniformly over* $[\mathrm{a},\mathrm{b}]$.

Theorem 2.3
[18] *[Schaefer's] (FPT) Assume a Banach space denoted by X, and a mapping $\mathfrak{T}:X\to X$ that is completely continuous. consider*

$$\Theta=\{\mathfrak{u}\in X:\mathfrak{v}=\lambda\mathfrak{T}\mathfrak{v},\ \textit{for some}\ \mathfrak{J}\in(0,1)\}$$

Since Θ is bounded. As a consequence, a fixed point of the operator $\mathfrak{T}$ exists.

Theorem 2.4
[18] (Schauder FPT)
Consider a Banach space $\Re$ with a closed and convex subset B *that is nonempty. Assume that the operator* $\mathfrak{T}:\mathrm{B}\to\mathrm{B}$ *is a continuous and that its image* $\mathfrak{T}(\mathrm{B})$ *is relatively compact in $\Re$. Then there exists at least one point in* B *that is fixed by $\mathfrak{T}$.*

Remark 2.1
The analytical framework is based on classical fixed-point and compactness results. The Arzelà–Ascoli theorem is used to establish compactness, Schaefer's fixed-point theorem ensures existence of solutions, the Banach contraction principle guarantees uniqueness, and Schauder's fixed-point theorem is applied in the controllability analysis.

Lemma 2.2
Suppose that $\mathfrak{P} \in C(\mathfrak{J}, \Re)$. Then the solution of the boundary value problem

$$\begin{aligned} &{}^{CH}D^{\kappa}_{1+}\mathfrak{u}(\rho) = \mathfrak{P}(\rho), \quad \rho \in \mathfrak{J} \\ &\mathfrak{u}(1) = \xi, \qquad \mathfrak{u}(T) = \eta \end{aligned} \tag{2.2}$$

Then the solution can be represented as

$$\mathfrak{u}(\rho) = \xi + \frac{\log \rho}{\log T}\left[\eta - \xi - \frac{1}{\Gamma(\kappa)}\int_1^T \left(\log \frac{T}{\mathsf{s}}\right)^{\kappa-1} \mathfrak{P}(\mathsf{s})\frac{d\mathsf{s}}{\mathsf{s}}\right] + \frac{1}{\Gamma(\kappa)}\int_1^{\rho} \left(\log \frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \mathfrak{P}(\mathsf{s})\frac{d\mathsf{s}}{\mathsf{s}}.$$

Proof: Applying the Hadamard fractional integral operator ${}^{H}I^{\kappa}_{1+}$ to both sides of (2.2) and using Lemma (2.1), we obtain

$${}^{H}I^{\kappa}_{1+}\left({}^{CH}D^{\kappa}_{1+}\mathfrak{u}(\rho)\right) = {}^{H}I^{\kappa}_{1+}\mathfrak{P}(\rho), \qquad \rho \in \mathfrak{J}.$$

$$\mathfrak{u}(\rho) - \mathrm{C}_0 - \mathrm{C}_1 \log(\rho) = \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\mathfrak{P}(\mathsf{s})\frac{ds}{\mathsf{s}} \tag{2.3}$$

Setting $\rho = 1$ in (2.3) and using the boundary condition $\mathfrak{u}(1) = \xi$, we obtain

$$\mathrm{C}_0 = \xi.$$

Next, setting $\rho = T$ in (2.3) and using $\mathfrak{u}(T) = \eta$, we get

$$\eta = \xi + \mathrm{C}_1 \log T + \frac{1}{\Gamma(\kappa)}\int_1^T \left(\log\frac{T}{s}\right)^{\kappa-1}\mathfrak{P}(s)\frac{ds}{s}.$$

Therefore,

$$\mathrm{C}_1 = \frac{1}{\log T}\left[\eta - \xi - \frac{1}{\Gamma(\kappa)}\int_1^T \left(\log\frac{T}{s}\right)^{\kappa-1}\mathfrak{P}(s)\frac{ds}{s}\right].$$

Substituting the obtained values of C_0 and C_1 into (2.3), we obtain

$$\mathfrak{u}(\rho) = \xi + \frac{\log \rho}{\log T}\left[\eta - \xi - \frac{1}{\Gamma(\kappa)}\int_1^T \left(\log \frac{T}{\mathsf{s}}\right)^{\kappa-1} \mathfrak{P}(\mathsf{s})\frac{d\mathsf{s}}{\mathsf{s}}\right] + \frac{1}{\Gamma(\kappa)}\int_1^{\rho} \left(\log \frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \mathfrak{P}(\mathsf{s})\frac{d\mathsf{s}}{\mathsf{s}}. \tag{2.4}$$

Hence, the required integral representation of the solution is obtained. This completes the proof.

3. Well-posedness results

Based on Lemma (2.2), we introduce a fixed-point formulation as follows. We define

$$\phi = \{\mathfrak{u}(\rho) : \mathfrak{u}(\rho) \in\subset (\mathfrak{J}, \Re)\}$$

under the norm

$$\|\mathfrak{u}\| = sup_{1\leq\rho\leq T}|\mathfrak{u}(\rho)|$$

Then $(\phi, \|.\|)$ is Banach space. Next, We consider the operator given by:

$$\begin{aligned}\mathfrak{T}\mathfrak{u}(\rho) = \xi + \frac{\log\rho}{\log T}\left[\eta - \xi - \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right]\\ + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\end{aligned} \tag{3.5}$$

Using Theorem (2.3),We demonstrate that solutions exist for system (1.1) under the assumptions stated below.
H2 There exists a constant $\mathfrak{M} > 0$ such that
$|\psi(\rho,\mathfrak{u}(\rho),\mathfrak{u}(\lambda\rho),{}^H D_{1+}^{\kappa}[\mathfrak{u}(\rho-\lambda\rho)])| \leq \mathfrak{M}, \quad \forall \quad \rho\in\mathfrak{J},\forall \quad \mathfrak{u}\in\phi$
H3 There exist non-negative constants $\mathfrak{L}_{\psi 1}+\mathfrak{L}_{\psi 2}+\mathfrak{L}_{\psi 3} > 0$ such that $\forall\rho\in\mathfrak{J}$ and all $\mathfrak{u},\breve{\mathfrak{u}}\in\Re$, the following conditions holds.

$$\begin{aligned}|\psi(\rho,\mathfrak{u}(\rho),\mathfrak{u}(\lambda\rho),{}^H D_{1+}^{\kappa}[\mathfrak{u}(\rho-\lambda\rho)]) - \psi(\rho,\breve{\mathfrak{u}}(\rho),\breve{\mathfrak{u}}(\lambda\rho),{}^H D_{1+}^{\kappa}[\breve{\mathfrak{u}}(\rho-\lambda\rho)])| &\leq \mathfrak{L}_{\psi 1}|\mathfrak{u}(\rho)-\breve{\mathfrak{u}}(\rho)|\\ &+ \mathfrak{L}_{\psi 2}|\mathfrak{u}(\lambda\rho)-\breve{\mathfrak{u}}(\lambda\rho)|\\ &+ \mathfrak{L}_{\psi 3}|H[\mathfrak{u}(\rho-\lambda\rho)]-H[\breve{\mathfrak{u}}(\rho-\lambda\rho)]|\\ &\leq (\mathfrak{L}_{\psi 1}+\mathfrak{L}_{\psi 2}+\mathfrak{L}_{\psi 3}\mathfrak{L}_{\mathrm{H}})\|\mathfrak{u}-\breve{\mathfrak{u}}\|.\end{aligned} \tag{3.6}$$

H4 The functions $H:\mathfrak{J}\times\Re^3\to\Re$ is continuous Furthermore, there exists a constant $\mathfrak{L}_{\mathrm{H}} > 0$, such that

$$|H[\mathfrak{u}(\rho-\lambda\rho)] - H[\breve{\mathfrak{u}}(\rho-\lambda\rho)]| \leq \mathfrak{L}_{\mathrm{H}}|\mathfrak{u}_1-\breve{\mathfrak{u}_1}|,\forall \quad \mathfrak{u},\breve{\mathfrak{u}}\in\Re$$

.

Theorem 3.1
Provided that hypotheses H1–H2 hold, the system described by equation (1.1) *admits a non-negative solution.*

Proof: Consider

$$\mathbb{B}_{\epsilon} = \{\mathfrak{u}\in\phi : \|\mathfrak{u}\|\leq\epsilon\}, \quad \|\mathfrak{u}\| = \max_{\rho\in\mathfrak{J}}|\mathfrak{u}(\rho)|.$$

It follows that, $\|\mathfrak{u}\|\leq\epsilon$ for all $\mathfrak{u}\in\mathbb{B}_{\epsilon}$. Hence, $\mathbb{B}_{\epsilon}$ is a bounded, convex, and a subset that is closed in the Banach space $C(\mathfrak{J})$.

Step 1: We verify that $\mathfrak{T}:\phi\to\phi$ is completely continuous.

Provided that $\{\mathfrak{u}_{\mathrm{n}}\}\subset\mathbb{B}_{\epsilon}$ is a convergent sequence with $\mathfrak{u}_{\mathrm{n}}\to\mathfrak{u}$ as $\mathrm{n}\to\infty$. By H2, $|\psi(\mathfrak{u}_{\mathrm{n}}(s))|\leq\mathfrak{M}$ for all $s\in\mathfrak{J}$, and ψ is continuous. Therefore, by an application of the Lebesgue Dominated Convergence Theorem, this yields the desired result.

$$\lim_{n\to\infty}\mathfrak{T}(\mathfrak{u}_{\mathrm{n}})(\rho) = \mathfrak{T}\mathfrak{u}(\rho), \quad \rho\in\mathfrak{J},$$

Thus, $\mathfrak{T}$ is continuous.

$$\begin{aligned}\lim_{\mathrm{n}\to\infty}\mathfrak{T}(\mathfrak{u}_{\mathrm{n}})(\rho) &= \xi + \frac{\log\rho}{\log T}\left[\eta - \xi - \frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\lim_{\mathrm{n}\to\infty}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right]\\ &+ \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\lim_{\mathrm{n}\to\infty}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\Big)\frac{ds}{\mathsf{s}}\\ &= \mathfrak{T}(\mathfrak{u})(\rho).\end{aligned}$$

$\lim_{\mathrm{n}\to\infty}\mathfrak{T}(\mathfrak{u}_{\mathrm{n}})(\rho) = \mathfrak{T}(\mathfrak{u})(\rho)$.

Therefore, $\mathfrak{T}$ defines a continuous.

Step 2: We establish that $\mathfrak{T}\mathbb{B}_{\epsilon}\subset\mathbb{B}_{\epsilon}$. Let $\mathfrak{u}\in\mathbb{B}_{\epsilon}$, $\rho\in\mathfrak{J}$, we have

$$|\mathfrak{T}(\mathfrak{u})(\rho)| = |\xi| + \left|\frac{\log \rho}{\log T}\right|\left[\left|\eta - \xi\right| + \frac{1}{\Gamma\kappa}\int_1^T \left(\log \frac{T}{\mathsf{s}}\right)^{\kappa-1} |\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])|\frac{ds}{\mathsf{s}}\right]$$
$$+\frac{1}{\Gamma\kappa}\int_1^\rho \left(\log \frac{\rho}{\mathsf{s}}\right)^{\kappa-1} |\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])|\frac{ds}{\mathsf{s}}$$
$$\leq 2|\xi| + |\eta| + \frac{2\mathfrak{M}(\log T)^\kappa}{\Gamma(\kappa+1)} \leq \epsilon.$$

Thus, $\|\mathfrak{T}(\mathfrak{u})\| \leq \epsilon$, which means that $\mathfrak{T}\mathbb{B}_\epsilon \subset \mathbb{B}_\epsilon$.
Step 3: We proceed to show that $\mathfrak{T}:\phi\to\phi$ is equicontinuous on $\mathbb{B}_\epsilon$.
Let $1 \leq \rho_1 \leq \rho_2 \leq T$ and $\forall \mathfrak{u} \in \mathbb{B}_\epsilon$, we examine

$$|\mathfrak{T}(\mathfrak{u})(\rho_2) - \mathfrak{T}(\mathfrak{u})(\rho_1)| = \left|\frac{\log(\rho_2)-\log(\rho_1)}{\log T}\right|\Bigg[\eta - \xi -$$
$$\frac{1}{\Gamma\kappa}\int_1^T \left(\log \frac{T}{\mathsf{s}}\right)^{\kappa-1} \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg]$$
$$+\Bigg|\frac{1}{\Gamma\kappa}\int_1^{\rho_2} \left(\log \frac{\rho_2}{\mathsf{s}}\right)^{\kappa-1} \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}$$
$$-\frac{1}{\Gamma\kappa}\int_1^{\rho_1} \left(\log \frac{\rho_1}{\mathsf{s}}\right)^{\kappa-1} \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg|$$
$$\leq \frac{1}{\log T}\log\left(\frac{\rho_2}{\rho_1}\right)\left[\eta - \xi + \frac{\mathfrak{M}(\log T)^\kappa}{\Gamma(\kappa+1)}\right] + \frac{\mathfrak{M}}{\Gamma(\kappa+1)}\left(\log\left(\frac{\rho_2}{\rho_1}\right)^\kappa - \log(\rho_2) + \log(\rho_1)\right)$$
$$+\frac{\mathfrak{M}}{\Gamma\kappa+1}\log\left(\frac{\rho_2}{\rho_1}\right).$$

Hence, $|\mathfrak{T}(\mathfrak{u})(\rho_2) - \mathfrak{T}(\mathfrak{u})(\rho_1)| \to 0$ as $\rho_1 \to \rho_2$ and independent of $\mathfrak{u} \in \mathbb{B}_\epsilon$

Therefore, $\mathfrak{T}$ satisfies the equicontinuity property. As a result, Theorem (2.2), ensures that $\mathfrak{T}$ is completely continuous.

Step 4: It can be shown that
$$\mathfrak{S} = \{\mathfrak{u} \in \phi : \mathfrak{u} = \mu\mathfrak{T}(\mathfrak{u}), 0 < \mu < 1\},$$
is contained in a bounded subset. For $\mathfrak{u} \in \mathfrak{S}$ and $\rho \in \mathfrak{J}$, we have
$$\|\mathfrak{u}\| = \mu \sup_{1\leq\rho\leq T} |\mathfrak{T}(\mathfrak{u})(\rho)|.$$
By (3.5) and hypothesis H2,By taking the supremum in terms of $\rho \in \mathfrak{J} = [1,T]$, we get
$$\|\mathfrak{T}(\mathfrak{u})\| \leq |\xi| + |\eta - \xi| + \frac{2\mathfrak{M}(\log T)^\kappa}{\Gamma(\kappa+1)} = \epsilon.$$
Since, $\|\mathfrak{u}\| \leq \|\mathfrak{T}(\mathfrak{u})\|$, we deduce that
$$\|\mathfrak{u}\| \leq \epsilon, \quad \forall \mathfrak{u} \in \mathfrak{S}.$$
Therefore, the set $\mathfrak{S}$ is bounded in $C(\mathfrak{J})$. All condition of theorem (2.3) are satisfied.
i.e, $\mathfrak{T}$ has fixed point in ϕ.

Theorem 3.2
Let H3-H4 holds. If
$$\Delta = \left(\frac{2(\mathfrak{L}_{\psi1} + \mathfrak{L}_{\psi2} + \mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})(\log T)^\kappa}{\Gamma(\kappa+1)}\right) < 1 \tag{3.7}$$

then system (1.1) *has a exactly one solution.*

Proof: By using Theorem (3.1), shows that the problem (1.1) has at least a single non-negative solution. Hence, we have shown that $\mathfrak{T}\mathbb{B}_\epsilon \subset \mathbb{B}_\epsilon$. To complete the proof, it is necessary to verify that $\mathfrak{T}$ is a contraction mapping. Take $\mathfrak{u}, \breve{\mathfrak{u}} \in \mathbb{B}_\epsilon$, Then

$$\begin{aligned}|\mathfrak{T}(\mathfrak{u})(\rho) - \mathfrak{T}(\breve{\mathfrak{u}})(\rho)| &= \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1} |\psi(\mathsf{s}, \mathfrak{u}(\mathsf{s}), \mathfrak{u}(\lambda\mathsf{s}), {}^H D_{1+}^\kappa[\mathfrak{u}\mathsf{s} - \lambda\mathsf{s}]) \\ &\quad - \psi(\mathsf{s}, \breve{\mathfrak{u}}(\mathsf{s}), \breve{\mathfrak{u}}(\lambda\mathsf{s}), {}^H D_{1+}^\kappa[\breve{\mathfrak{u}}\mathsf{s} - \lambda\mathsf{s}])|\frac{ds}{\mathsf{s}} \\ &\quad + \frac{1}{\Gamma\kappa}\int_1^\rho \left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} |\psi(\mathsf{s}, \mathfrak{u}(\mathsf{s}), \mathfrak{u}(\lambda\mathsf{s}), {}^H D_{1+}^\kappa[\mathfrak{u}\mathsf{s} - \lambda\mathsf{s}]) \\ &\quad - \psi(\mathsf{s}, \breve{\mathfrak{u}}(\mathsf{s}), \breve{\mathfrak{u}}(\lambda\mathsf{s}), {}^H D_{1+}^\kappa[\breve{\mathfrak{u}}\mathsf{s} - \lambda\mathsf{s}])|\frac{ds}{\mathsf{s}} \\ &\le \left(\frac{2(\mathfrak{L}_{\psi1} + \mathfrak{L}_{\psi2} + \mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})(\log T)^\kappa}{\Gamma(\kappa+1)}\right)(\|\mathfrak{u} - \breve{\mathfrak{u}}\|).\end{aligned}$$

Thus,

$$|\mathfrak{T}(\mathfrak{u})(\rho) - \mathfrak{T}(\breve{\mathfrak{u}})(\rho)| \le \Delta\|\mathfrak{u} - \breve{\mathfrak{u}}\|$$

Hence, the operator $\mathfrak{T}$ is a contraction mapping by inequality (3.7). Therefore, by the theorem (2.1). Thus, the system (1.1) is guaranteed to have one and only one non-negative solution.

4. Stability

In this section, we recall the notions of Ulam–Hyers (UH), generalized Ulam–Hyers (GUH), and generalized Ulam–Hyers–Rassias (GUHR) stability for the Caputo–Hadamard fractional differential equation with pantograph delay (1.1), as defined in [16, 1]. Let $\varepsilon > 0$ be an arbitrary constant and let $\varphi : \mathfrak{J} \to [0, \infty)$ be a continuous function.

$$|{}^{CH}D_{1+}^\kappa \mathfrak{v}(\rho) - \psi\big(\rho, \mathfrak{v}(\rho), \mathfrak{v}(\lambda\rho), {}^H D_{1+}^\kappa[\mathfrak{v}(\rho - \lambda\rho)]\big)| \le \varepsilon, \quad \rho \in \mathfrak{J} = [1, T]. \text{ (4.8)}$$

$$|{}^{CH}D_{1+}^\kappa \mathfrak{v}(\rho) - \psi\big(\rho, \mathfrak{v}(\rho), \mathfrak{v}(\lambda\rho), {}^H D_{1+}^\kappa[\mathfrak{v}(\rho - \lambda\rho)]\big)| \le \varepsilon\varphi(\rho), \quad \rho \in \mathfrak{J} = [1, T]. \tag{4.9}$$

$$|{}^{CH}D_{1+}^\kappa \mathfrak{v}(\rho) - \psi\big(\rho, \mathfrak{v}(\rho), \mathfrak{v}(\lambda\rho), {}^H D_{1+}^\kappa[\mathfrak{v}(\rho - \lambda\rho)]\big)| \le \varphi(\rho), \quad \rho \in \mathfrak{J} = [1, T]. \tag{4.10}$$

For the stability analysis, the following definitions are required.

Definition 4.4
The system (1.1) *possesses UH stability provided that a constant exists*
$\mathfrak{D}_\mathfrak{f} > 0$*, so that, for every* $\varepsilon > 0$*, for every function* $\mathfrak{v} \in \phi$ *satisfying inequality* (4)*, a solution exists* $\mathfrak{u} \in \phi$ *of system* (1.1)*, such that* $\|\mathfrak{v}(\rho) - \mathfrak{u}(\rho)\| \le \mathfrak{D}_\mathfrak{f}\varepsilon, \quad \rho \in [1, \mathsf{T}]$

Definition 4.5
The system (1.1) *exhibits generalized UH stability provided that there exists* $\mathfrak{F} \in \mathrm{C}(\Re^+, \Re^+)$*, with* $\mathfrak{F}(0) = 0$*, so that for any* $\varepsilon > 0$*, for every function* $\mathfrak{v} \in \phi$ *satisfying inequality* (4)*, a solution exists* $\mathfrak{u} \in \phi$ *of system* (1.1)*, with* $\|\mathfrak{v}(\rho) - \mathfrak{u}(\rho)\| \le \mathfrak{F}(\varepsilon)$

Definition 4.6
The system (1.1) possesses UH stability relative to $\varphi \in \phi$ if there exits a constant $\mathfrak{D}_{\mathfrak{f}} \in \Re$ so that, for any $\epsilon > 0$ and every function ϕ satisfying inequality (4.10) a there solution exists $\mathfrak{u} \in \phi$ of system (1.1) with
$\|\mathfrak{v}(\rho) - \mathfrak{u}(\rho)\| \leq \mathfrak{D}_{\mathfrak{f}}\varepsilon\varphi(\rho), \quad \rho \in \mathfrak{J}$

Definition 4.7
The system (1.1) exhibits generalized UH-R stability relative to $\varphi \in \phi$ if there exists a positive real number $\mathfrak{D}_{\mathfrak{f},\phi} > 0$ exists so that, for any $\mathfrak{v} \in \phi$ satisfying inequality (4.10), a corresponding solution $\mathfrak{u} \in \phi$ of system (1.1) can be found with
$\|\mathfrak{v}(\rho) - \mathfrak{u}(\rho)\| \leq \mathfrak{D}_{\mathfrak{f},\varphi}\,\varphi(\rho), \quad \rho \in \mathfrak{J}$

Theorem 4.1
The solution of (1.1) is Ulam-Hyer stable provided that the hypotheses of Theorem (3.1) are valid.

Proof: For a given$\varepsilon > 0$, let $\mathfrak{v} \in \phi$ be any function fulfilling (4), and let $\mathfrak{u} \in \phi$ be the exactly one function of the corresponding system of CHFDEs with pantograph delay.

$$^{CH}D^{\kappa}_{1+}\mathfrak{u}(\rho) = \psi(\rho, \mathfrak{u}(\rho), \mathfrak{u}(\lambda\rho), {}^{H}D^{\kappa}_{1+}[\mathfrak{u}(\rho - \lambda\rho)]), \quad \rho \in \mathfrak{J}$$

With the help of Lemma (2.2), we deduce

$$\mathfrak{u}(\rho) = \mathrm{A}_{\mathfrak{u}} + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s}, \mathfrak{u}(\mathsf{s}), \mathfrak{u}(\lambda\mathsf{s}), {}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s} - \lambda\mathsf{s}])\frac{ds}{\mathsf{s}}$$

where

$$\begin{aligned}\mathrm{A}_{\mathfrak{u}} &= \xi + \frac{\log\rho}{\log T}\left[\eta - \xi - \frac{1}{\Gamma\kappa}\int_1^{T}\left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s}, \mathfrak{u}(\mathsf{s}), \mathfrak{u}(\lambda\mathsf{s}), {}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s} - \lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right] \\ &+ \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s}, \mathfrak{u}(\mathsf{s}), \mathfrak{u}(\lambda\mathsf{s}), {}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s} - \lambda\mathsf{s}])\frac{ds}{\mathsf{s}}.\end{aligned}$$

Alternatively, if $\mathfrak{u}(1) = \mathfrak{v}(1)$ and $\mathfrak{u}(T) = \mathfrak{v}(T)$, then $\mathrm{A}_{\mathfrak{u}} = \mathrm{A}_{\mathfrak{v}}$. Indeed,

$$\begin{aligned}|\mathrm{A}_{\mathfrak{u}} - \mathrm{A}_{\mathfrak{v}}| &\leq \frac{2(\log T)^{\kappa}}{\Gamma(\kappa+1)}(\mathrm{L}_{\alpha} + \mathrm{L}_{\beta} + \mathrm{L}_{\gamma}\mathrm{L}_{\mathrm{H}})|\mathfrak{u}(T) - \mathfrak{v}(T)| \\ &= 0\end{aligned}$$

Thus, $\mathrm{A}_{\mathfrak{u}} = \mathrm{A}_{\mathfrak{v}}$
Then, we have

$$\mathfrak{v}(\rho) = \mathrm{A}_{\mathfrak{v}} + \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s}, \mathfrak{v}(\mathsf{s}), \mathfrak{v}(\lambda\mathsf{s}), {}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s} - \lambda\mathsf{s}])\frac{ds}{\mathsf{s}}$$

Applying integration to inequality (4), we derive

$$\left|\mathfrak{v}(\rho) - \mathrm{A}_{\mathfrak{v}} - \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s}, \mathfrak{v}(\mathsf{s}), \mathfrak{v}(\lambda\mathsf{s}), {}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s} - \lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right| \leq \frac{\varepsilon(\log T)^{\kappa}}{\Gamma(\kappa+1)}. \tag{4.11}$$

We have

$$\begin{aligned}|\mathfrak{v}(\rho)-\mathfrak{u}(\rho)| &\leq \left|\mathfrak{v}(\rho)-\mathrm{A}_{\mathfrak{v}}-\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right| \\ &\quad+\left|\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}])\right. \\ &\quad\left.-\psi(\mathsf{s},\mathfrak{u}\mathsf{s},\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\right| \\ &\leq \frac{\varepsilon(\log T)^{\kappa}}{\Gamma(\kappa+1)}+\frac{2(\mathfrak{L}_{\psi1}+\mathfrak{L}_{\psi2}+\mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})}{\Gamma(\kappa+1)}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \\ &\quad|\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}])-\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])|\frac{ds}{s}.\end{aligned} \tag{4.12}$$

and to apply Mittag-Leffler function (2.3), we obtain

$$\begin{aligned}|\mathfrak{v}(\rho)-\mathfrak{u}(\rho)| &\leq \frac{\varepsilon(\log T)^{\kappa}\mathrm{E}_{\kappa,1}(2(\mathfrak{L}_{\psi1}+\mathfrak{L}_{\psi2}+\mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})(\log T)^{\kappa})}{\Gamma(\kappa+1)} \\ &= \mathfrak{D}_{\mathfrak{f}}\varepsilon.\end{aligned}$$

Therefore, the UH- stability of equation (1.1) is established.
Remark 1. Under the condition $\mathfrak{F}(\varepsilon)=\mathfrak{D}_{\mathfrak{f}}\varepsilon$, problem (1.1) satisfies generalized UH stability.

(H5) Let $\varphi\in\Phi$ be a non-decreasing function. Assume that there exists a constant $\mu_{\varphi,\kappa}>0$ such that, for every $\rho\in[1,\mathsf{T}]$,

$$^{CH}D^{\kappa}_{1+}\varphi(\rho)\leq\mu_{\varphi,\kappa}\varphi(\rho).$$

Theorem 4.2
Suppose that the assumptions of Theorem (3.1) and hypothesis ***(H5)*** *hold. Then system (1.1) is generalized Ulam–Hyers–Rassias stable.*

Proof: Let $\varepsilon>0$ be arbitrary and let $\mathfrak{v}\in\Phi$ be a function satisfying (4.9). Let $\mathfrak{u}\in\Phi$ denote the unique solution of system (1.1).

$$^{CH}D^{\kappa}_{1+}\mathfrak{u}(\rho)=\psi(\rho,\mathfrak{v}(\rho),\mathfrak{v}(\lambda\rho),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}(\rho-\lambda\rho)]),\quad \rho\in\mathfrak{J}=[1,T]$$

Based on Lemma (2.2), we obtain

$$\mathfrak{u}(\rho)=\mathrm{A}_{\mathfrak{u}}+\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}$$

where

$$\begin{aligned}\mathrm{A}_{\mathfrak{u}} &= \xi+\frac{\log\rho}{\log T}\left[\eta-\xi-\frac{1}{\Gamma\kappa}\int_1^{T}\left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right] \\ &\quad+\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}.\end{aligned}$$

Integration inequality (4.9) yields

$$\left|\mathfrak{v}(\rho)-\mathrm{A}_{\mathfrak{v}}-\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right|\leq\varepsilon\mu_{\varphi}\varphi(\rho). \tag{4.13}$$

On the other side, we have

$$
\begin{aligned}
|\mathfrak{v}(\rho)-\mathfrak{u}(\rho)| \leq & \left|\mathfrak{v}(\rho)-\mathrm{A}_{\mathfrak{v}}-\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right| \\
& +\frac{2(\mathfrak{L}_{\psi1}+\mathfrak{L}_{\psi2}+\mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\Big|\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}]) \\
& -\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\Big|\frac{ds}{s} \\
& \leq \varepsilon\mu_\varphi\varphi(\rho)+\frac{2(\mathfrak{L}_{\psi1}+\mathfrak{L}_{\psi2}+\mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})}{\Gamma(\kappa+1)}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \\
& |\psi(\mathsf{s},\mathfrak{v}(\mathsf{s}),\mathfrak{v}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{v}\mathsf{s}-\lambda\mathsf{s}])-\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])|\frac{ds}{s}.
\end{aligned} \tag{4.14}
$$

Applying the generalized Gronwall inequality involving the Mittag–Leffler function, we obtain

$$
|\mathfrak{v}(\rho)-\mathfrak{u}(\rho)| \leq \varepsilon\mu_\varphi\varphi(\rho)\mathrm{E}_{\kappa,1}(2(\mathfrak{L}_{\psi1}+\mathfrak{L}_{\psi2}+\mathfrak{L}_{\psi3}\mathfrak{L}_{\mathrm{H}})(\log T)^{\kappa}), \quad \rho\in[1,\mathsf{T}]
$$

Hence, system (1.1) is generalized Ulam–Hyers–Rassias stable.

5. Analysis of Controllability

Controllability is an important concept in the analysis of dynamical systems, as it characterizes the ability to steer the state of a system from a given initial configuration to a desired final state through an admissible control function. For fractional systems with memory and pantograph delay effects, controllability provides a theoretical guarantee that the influence of past states does not prevent the system from reaching a prescribed target state. Therefore, establishing controllability is essential for applications in engineering, physics, and other fields where delayed fractional-order dynamics arise.

This section investigates the controllability of the CHFDEs with pantograph delay (1.1) based on the definition presented in [9].

$$
\begin{aligned}
& {}^{CH}D^{\kappa}_{1+}\mathfrak{u}(\rho)=\psi(\rho,\mathfrak{u}(\rho),\mathfrak{u}(\lambda\rho),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}(\rho-\lambda\rho)])+\mathrm{Bz}(\rho), \quad \rho\in\mathfrak{J}=[1,T] \\
& \mathrm{x}(1)=\xi, \qquad \mathrm{x}(T)=\eta
\end{aligned} \tag{5.15}
$$

In the above expressions, ${}^{CH}D^{\kappa}_{1+},\mathfrak{u},\xi,\eta,\lambda$ and $\mathrm{Bz}(\rho)$ are as previously defined. Futhermore, the system state $\mathfrak{u}(.)$ lies in the Banach space $\Re$, whereas the control inputs belongs to the Banach space U. The bounded linear operator B maps U in to $\Re$. The mild solution of the system is given in (5.16).

$$
\begin{aligned}
\mathfrak{u}(\rho)= & \xi+\frac{\log\rho}{\log T}\left[\eta-\xi-\frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\right] \\
& +\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^{H}D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}} \\
& +\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\mathrm{Bz}(\mathrm{s})\frac{ds}{\mathsf{s}} \quad \rho\in[1,\mathsf{T}]
\end{aligned} \tag{5.16}
$$

Definition 5.8
The fractional system described by equation (5.15) is controllable within the interval $\mathfrak{J}$ if, for any given initial state ξ and terminal state η in the Banach space $\Re$, there exists a control function $\mathrm{z}(\rho)\in\mathrm{L}^2([1,\mathsf{T}],\mathrm{U})$ so that the corresponding mild solution $\mathfrak{u}(\rho)$ of the system satisfies $\mathfrak{u}(1)=\xi,\mathfrak{u}(T)=\eta$.

Before presenting the theorem, the hypotheses below are considered :
H6 There exits constants $\mathfrak{L}^*$ such that

$$\sup_{1\leq s\leq T} \|\psi(\rho, \mathfrak{u}(\rho), \mathfrak{u}(\lambda\rho), {}^H D^{\kappa}_{1+}[\mathfrak{u}(\rho-\lambda\rho)])\| = \mathfrak{L}^*$$

$$i.e, \sup_{1\leq s\leq T} \|\psi_x(s)\| = \mathfrak{L}^*$$

H7 The bounded linear operator $\mathfrak{Q} : \mathrm{L}^2([1,\mathsf{T}],\mathrm{U}) \to \Re$ is defined by

$$\mathfrak{Q}_{\mathrm{z}} = \frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{s}\right)\mathrm{Bz}(s)\frac{ds}{s}$$

admits an induced inverse $\mathfrak{Q}^{-1}$ on $\frac{\mathrm{L}^2([1,\mathsf{T}],\mathrm{U})}{ker\mathfrak{Q}}$. Moreover, there exist constants $\mathfrak{M}_1, \mathfrak{M}_2 > 0$ such that $\|\mathrm{B}\| \leq \mathfrak{M}_1$ and $\|\mathfrak{Q}^{-1} \leq \mathfrak{M}_2$. Here, $\mathfrak{Q}$ denotes the controllability operator associated with the control function $z(\rho)$, while $\mathfrak{Q}^{-1}$ represents its induced inverse.

Theorem 5.1
Assuming that hypotheses H1-H4, along with H6, H7 are fulfilled, the system associated with equations (5.15) is controllable on the interval $\mathfrak{J}$.

Proof: Under the assumptions of hypotheses H3, H4, control can be constructed using the properties of arbitrary functions $\mathrm{z}(\rho)$.
Define the set $\Omega_{\upsilon_1} = \{\mathfrak{u} \in \mathrm{C}(\mathfrak{J},\Re) : \|\mathfrak{u}_\infty\| \leq \upsilon_1\}, \quad \|\mathfrak{u}_\infty\| = \max_{\rho\in[1,T]} |\mathfrak{u}(\rho)|$.
Therefore, Ω_{υ_1} the set of all continuous functions mapping into $\Re$ on $\mathfrak{J}$ having a uniform bound given by the constant $\upsilon_1 > 0$.
For any $\mathrm{x} \in \Omega_{\upsilon_1}$, the nonlinear term ψ_x is well defined and bounded on $\mathfrak{J}$ by hypothesis H3, H4 and H6. Consequently, the associated control function $\mathrm{z}(\rho) \in \mathrm{L}^2(\mathfrak{J},\mathrm{U})$ can be constructed. Hence, the system possesses a mild solution satisfies the prescribed boundary conditions $\mathfrak{u}(1) = \xi, \quad \mathfrak{u}(T) = \eta$.

$$\begin{aligned}\delta(\rho) = \mathfrak{Q}^{-1}\Bigg\{&\xi + \frac{\log\rho}{\log T}\Bigg[\eta - \xi - \\ &\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}), {}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg] \\ &+\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}), {}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg\}.\end{aligned} \tag{5.17}$$

We intend to prove that Υ maps Ω_{υ_1} onto itself, ensuring that the operator is well established.

$$\begin{aligned}\Upsilon\mathfrak{u}(\rho) = \xi + &\frac{\log\rho}{\log T}\Bigg[\eta - \xi - \\ &\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}), {}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg] \\ &+\frac{1}{\Gamma\kappa}\int_1^{\rho}\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}), {}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}} \\ &+(\mathrm{B}\delta)(s)\end{aligned} \tag{5.18}$$

We proceed to demonstrate the controllability of the system given by equation (5.15) is controllable on $\mathfrak{J}$. The operator Υ has a fixed point, and the control function $\delta(\rho)$ from equation (5.18), yields the mild solution to the control problem. By applying this control function δ, we obtain the following expression.

$$\begin{aligned}\Upsilon\mathfrak{u}(\rho) = \xi + \frac{\log\rho}{\log T}\Bigg[\eta - \xi - \\ \frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg] \\ +\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}} \\ +\mathrm{B}\mathfrak{Q}^{-1}\Bigg[\xi + \frac{\log\rho}{\log T}\Bigg[\eta - \xi - \\ \frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg] \\ +\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg]\end{aligned} \tag{5.19}$$

It follows that the operator Υ is continuous, since all function appearing in its definition are continuous. For every function $\mathrm{X} \in \Omega_{\upsilon_1}, \forall\rho \in \mathfrak{J}$, the following relation is obtained from equation (5.16):

$$\begin{aligned}\delta(\rho) = \mathfrak{Q}^{-1}\Bigg[\xi + \frac{\log\rho}{\log T}\Bigg[\eta - \xi - \frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg] \\ +\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\frac{ds}{\mathsf{s}}\Bigg].\end{aligned}$$

$$\begin{aligned}\|\delta(\rho)\| \le \|\mathfrak{Q}^{-1}\|\Bigg[\|\xi\| + \left\|\frac{\log\rho}{\log T}\right\|\Bigg[\|\eta\| - \|\xi\| - \frac{1}{\Gamma\kappa}\int_1^T\left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1}\|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\Bigg] \\ +\frac{1}{\Gamma\kappa}\int_1^\rho\left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1}\|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D_{1+}^{\kappa}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\Bigg]. \\ \le \mathfrak{M}_2\left[2\|\xi\| + \|\eta\| + \frac{2\mathfrak{L}^*(\log T)^\kappa}{\Gamma(\kappa+1)}\right]\end{aligned} \tag{5.20}$$

By means of equations (5.19) and (5.20), we derive

$$\begin{aligned}
\|\Upsilon\mathfrak{u}(\rho)\| &\leq \|\xi\| + \left\|\frac{\log\rho}{\log T}\right\|\left[\|\eta\| - \|\xi\| - \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1} \|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\right] \\
&\quad + \frac{1}{\Gamma\kappa}\int_1^\rho \left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}} + \mathfrak{M}_1\|\delta(\rho)\| \\
&\leq \|\xi\| + \left\|\frac{\log\rho}{\log T}\right\|\left[\|\eta\| - \|\xi\| - \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1} \|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\right] \\
&\quad + \frac{1}{\Gamma\kappa}\int_1^\rho \left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}} \\
&\quad + \mathfrak{M}_1\mathfrak{M}_2\Bigg\{\|\xi\| + \left\|\frac{\log\rho}{\log T}\right\|\left[\|\eta\| - \|\xi\| - \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1} \|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\right] \\
&\quad + \frac{1}{\Gamma\kappa}\int_1^\rho \left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \|\psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\Bigg\} \\
&\leq (1+\mathfrak{M}_1\mathfrak{M}_2)\left[2\|\xi\| + \|\eta\| + \frac{2\mathfrak{L}^*(\log T)^\kappa}{\Gamma(\kappa+1)}\right] \\
&\leq \mathfrak{R}
\end{aligned} \tag{5.21}$$

Next, we show that the operator $\|\Upsilon : \Omega_{\upsilon_1} \to \Omega_{\upsilon_1}\|$ fulfills all the requirements of Theorem (2.4), and the proof proceeds in multiple steps, demonstrating that the operator Υ takes the set $\Omega_{\upsilon_1} = \Omega_{\upsilon_1} = \{\mathfrak{u} \in \mathrm{C}(\mathfrak{J},\mathfrak{R}) : \|\mathfrak{u}_\infty\| \leq \upsilon_1\}$ into itself.

Step 1: To prove the continuity of Υ, consider a sequences $\mathfrak{u}_\mathfrak{n}$ in Ω_{υ_1} satisfying $\mathfrak{u}_\mathfrak{n} \to \mathfrak{u}$ in Ω_{υ_1}

$$\begin{aligned}
\|(\Upsilon\mathfrak{u}_\mathfrak{n})(\rho) - (\Upsilon\mathfrak{u})(\rho)\| &\leq \|\xi\| + \left\|\frac{\log\rho}{\log T}\right\|\Bigg[\|\eta\| - \|\xi\| - \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1} \\
&\quad \|\psi(\mathsf{s},\mathfrak{u}_\mathfrak{n}(\mathsf{s}),\mathfrak{u}_\mathfrak{n}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}_\mathfrak{n}\mathsf{s}-\lambda\mathsf{s}]) - \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\Bigg] \\
&\quad + \frac{1}{\Gamma\kappa}\int_1^\rho \left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \\
&\quad \|\psi(\mathsf{s},\mathfrak{u}_\mathfrak{n}(\mathsf{s}),\mathfrak{u}_\mathfrak{n}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}_\mathfrak{n}\mathsf{s}-\lambda\mathsf{s}]) - \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}} \\
&\quad + \mathrm{B}\mathfrak{Q}^{-1}\Bigg\{\|\xi\| + \left\|\frac{\log\rho}{\log T}\right\|\Bigg[\|\eta\| - \|\xi\| - \frac{1}{\Gamma\kappa}\int_1^T \left(\log\frac{T}{\mathsf{s}}\right)^{\kappa-1} \\
&\quad \|\psi(\mathsf{s},\mathfrak{u}_\mathfrak{n}(\mathsf{s}),\mathfrak{u}_\mathfrak{n}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}_\mathfrak{n}\mathsf{s}-\lambda\mathsf{s}]) - \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\Bigg] \\
&\quad + \frac{1}{\Gamma\kappa}\int_1^\rho \left(\log\frac{\rho}{\mathsf{s}}\right)^{\kappa-1} \\
&\quad \|\psi(\mathsf{s},\mathfrak{u}_\mathfrak{n}(\mathsf{s}),\mathfrak{u}_\mathfrak{n}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}_\mathfrak{n}\mathsf{s}-\lambda\mathsf{s}]) - \psi(\mathsf{s},\mathfrak{u}(\mathsf{s}),\mathfrak{u}(\lambda\mathsf{s}),{}^H D^{\kappa}_{1+}[\mathfrak{u}\mathsf{s}-\lambda\mathsf{s}])\|\frac{ds}{\mathsf{s}}\Bigg\}
\end{aligned} \tag{5.22}$$

Due to the continuity of $(\Upsilon\mathfrak{u})(\rho)$, it can be inferred that:

$\|(\Upsilon\mathfrak{u}_\mathfrak{n})(\rho) - (\Upsilon\mathfrak{u})(\rho)\| \to 0$ as $n \to \infty$

Thus, Υ exhibits continuity over the set Ω_{υ_1}.

Step 2: Since $\Upsilon(\Omega_{\upsilon_1}) \subset \Omega_{\upsilon_1}$,it can be observed that the set $\Upsilon(\Omega_{\upsilon_1})$ is uniformly bounded, and therefore it is bounded.

Step 3: To verify equicontinuity of $\Upsilon(\Omega_{\upsilon_1})$, take $\rho_1, \rho_2 \in \mathfrak{J}$ and $x \in \Omega_{\upsilon_1}$, assuming $\rho_1 < \rho_2$. In this scenario, it holds that:

$$\begin{aligned}
\|(\Upsilon u)(\rho_2) - (\Upsilon u)(\rho_1)\| \leq & \left\| \frac{\log(\rho_2) - \log(\rho_1)}{\log T} \right\| \Bigg[\|\eta - \xi\| - \\
& \frac{1}{\Gamma\kappa} \int_1^T \left(\log \frac{T}{s} \right)^{\kappa-1} \|\psi(s, u(s), u(\lambda s), {}^H D_{1+}^{\kappa}[us - \lambda s])\| \frac{ds}{s} \Bigg] \\
& + \Bigg| \frac{1}{\Gamma\kappa} \int_1^{\rho_2} \left(\log \frac{\rho_2}{s} \right)^{\kappa-1} \|\psi(s, u(s), u(\lambda s), {}^H D_{1+}^{\kappa}[us - \lambda s])\| \frac{ds}{s} \\
& - \frac{1}{\Gamma\kappa} \int_1^{\rho_1} \left(\log \frac{\rho_1}{s} \right)^{\kappa-1} \|\psi(s, u(s), u(\lambda s), {}^H D_{1+}^{\kappa}[us - \lambda s])\| \frac{ds}{s} \Bigg\| \\
& + \|B\| \|\mathfrak{Q}^{-1}\| \Bigg\{ \left\| \frac{\log(\rho_2) - \log(\rho_1)}{\log T} \right\| \Bigg[\|\eta - \xi\| - \\
& \frac{1}{\Gamma\kappa} \int_1^T \left(\log \frac{T}{s} \right)^{\kappa-1} \|\psi(s, u(s), u(\lambda s), {}^H D_{1+}^{\kappa}[us - \lambda s])\| \frac{ds}{s} \Bigg] \\
& + \Bigg| \frac{1}{\Gamma\kappa} \int_1^{\rho_2} \left(\log \frac{\rho_2}{s} \right)^{\kappa-1} \|\psi(s, u(s), u(\lambda s), {}^H D_{1+}^{\kappa}[us - \lambda s])\| \frac{ds}{s} \\
& - \frac{1}{\Gamma\kappa} \int_1^{\rho_1} \left(\log \frac{\rho_1}{s} \right)^{\kappa-1} \|\psi(s, u(s), u(\lambda s), {}^H D_{1+}^{\kappa}[us - \lambda s])\| \frac{ds}{s} \Bigg\| \Bigg\} \\
& \leq (1 + \|\eta\| \|\mathfrak{Q}^{-1}\|) \frac{1}{\log T} \log \left(\frac{\rho_2}{\rho_1} \right) \Bigg[\|\eta - \xi\| \\
& + \frac{\mathfrak{M}(\log T)^{\kappa}}{\Gamma(\kappa+1)} \Bigg] + \frac{\mathfrak{M}}{\Gamma(\kappa+1)} \left(\log \left(\frac{\rho_2}{\rho_1} \right)^{\kappa} - \log(\rho_2) + \log(\rho_1) \right) \\
& + \frac{\mathfrak{M}}{\Gamma\kappa + 1} \log \left(\frac{\rho_2}{\rho_1} \right).
\end{aligned}$$

By combining the results obtained in the applying Theorem (2.4) along with the preceding steps we deduce that the operator Υ exhibits continuity and compactness. As ρ_2 approaches ρ_1, the term on the right-hand side converges to zero, that is $\|(\Upsilon u)(\rho_2) - (\Upsilon u)(\rho_1)\| \to 0$. Therefore, by Theorem (2.4), the operator Υ is continuous admits a fixed point X, which provides a solution to the problem described by equations (5.15). Hence, the system defined in (5.15) is controllable on the interval $\mathfrak{J}$.

6. Examples

Example 6.1
Consider the CH-FDEs delay with pantograph

$$ {}^{CH}D_{1+}^{0.5} u(\rho) = \frac{1}{10} + \frac{1}{9} u(\rho) + \frac{1}{8} \cos u(0.5\rho) + {}^{H}D_{1+}^{0.5} \left[\frac{1}{7} u(\rho) - \frac{1}{8} \cos u(0.5\rho) \right], \quad \rho \in [1, 2] \tag{6.23}$$

with the boundary conditions

$$u(1) = 0, \qquad u(2) = 1.$$

Define the nonlinear function

$$\psi(\rho, u_1, u_2, u_3) = \frac{1}{10} + \frac{1}{9} u_1 + \frac{1}{8} \cos u_2 + u_3,$$

where

$$u_1 = u(\rho), \quad u_2 = u(0.5\rho), \quad u_3 = H[u(\rho - 0.5\rho)]$$

and

$$H[u(\rho - 0.5\rho)] = {}^H D^{0.5}_{1+}\left[\frac{1}{7}u(\rho) - \frac{1}{8}\cos u(0.5\rho)\right].$$

$$|\psi(\rho, u_1, u_2, u_3)| \le 0.1 + 0.111 + 0.125 + 0.143 = 0.479.$$

Thus, $|\psi(\rho, u_1, u_2, u_3)| \le M$.
Hence hypothesis H_2 *is satisfied.*
For any $u, \breve{u} \in \mathbb{R}$,

$$\begin{aligned}
&|\psi(\rho, \mathfrak{u}(\rho), \mathfrak{u}(\lambda\rho), {}^H D^{\kappa}_{1+}[\mathfrak{u}(\rho - \lambda\rho)]) - \psi(\rho, \breve{\mathfrak{u}}(\rho), \breve{\mathfrak{u}}(\lambda\rho), {}^H D^{\kappa}_{1+}[\breve{\mathfrak{u}}(\rho - \lambda\rho)])| \\
&\le \left| \frac{1}{9}\big(\mathfrak{u}(\rho) - \breve{\mathfrak{u}}(\rho)\big) + \frac{1}{8}\big(\cos(\mathfrak{u}(0.5\rho)) - \cos(\breve{\mathfrak{u}}(0.5\rho))\big) \right. \\
&\quad \left. + {}^H D^{0.5}_{1+}\left[\frac{1}{7}\mathfrak{u}(\rho) - \frac{1}{8}\cos(\mathfrak{u}(0.5\rho))\right] - {}^H D^{0.5}_{1+}\left[\frac{1}{7}\breve{\mathfrak{u}}(\rho) - \frac{1}{8}\cos(\breve{\mathfrak{u}}(0.5\rho))\right] \right| \\
&\le 0.111 + 0.125 + 0.143 = 0.379.
\end{aligned}$$

$$\left(\frac{2(\mathfrak{L}_{\psi 1} + \mathfrak{L}_{\psi 2} + \mathfrak{L}_{\psi 3}\mathfrak{L}_{\mathrm{H}})(\log T)^{\kappa}}{\Gamma(\kappa + 1)}\right) < 1$$

we obtain

$$\frac{0.758 \times 0.833}{0.886} \approx 0.712 < 1.$$

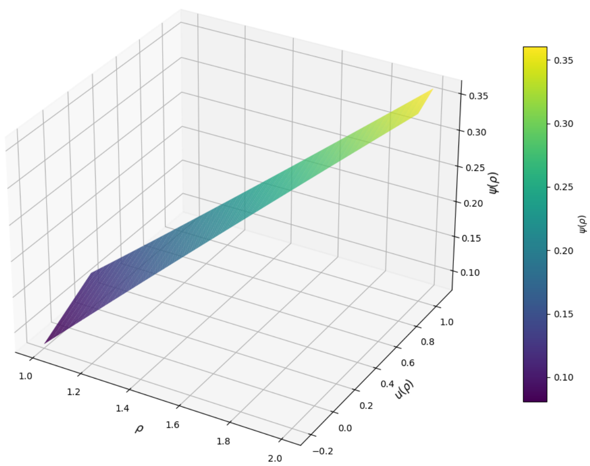

Figure 1. Graphical represented $\psi(\rho, \mathfrak{u}(\rho), \mathfrak{u}(\lambda\rho), {}^H D^{\kappa}_{1+}[\mathfrak{u}(\rho - \lambda\rho)])$ on $\rho = [1, 2]$, illustrating the uniqueness behaviour of the solution

Figure 1 illustrates the smooth variation of the nonlinear operator $\psi(\rho, \mathfrak{u}(\rho), \mathfrak{u}(\lambda\rho), {}^H D^{\kappa}_{1+}[\mathfrak{u}(\rho - \lambda\rho)])$ *over the interval* $\rho = [1, 2]$*. The regular and bounded behaviour of the surface supports the assumptions required for the existence and uniqueness of the Solution is established in the theoretical analysis.*

Next, we compute the constant $\mathfrak{D}_f$ that appears in the UH-stability analysis.

$$\mathfrak{D}_{\mathfrak{f}}\varepsilon. \leq \frac{\varepsilon(\log T)^{\kappa} E_{\kappa,1}\left(2(\mathfrak{L}_{\psi1}+\mathfrak{L}_{\psi2}+\mathfrak{L}_{\psi3}\mathfrak{L}_{H})(\log T)^{\kappa}\right)}{\Gamma(\kappa+1)}.$$

Using the Mittag–Leffler function value

$$E_{0.5,1}(0.631) \approx 1.84,$$

we obtain

$$\mathfrak{D}_f = \frac{(0.833)(1.84)}{0.886} \approx 1.73.$$

$$\mathfrak{D}_f \leq 1.73\,\varepsilon.$$

With $\varepsilon = 0.5$, we obtain: $\mathfrak{D}_f\varepsilon = 1.73(0.5) = 0.865 < 1$.

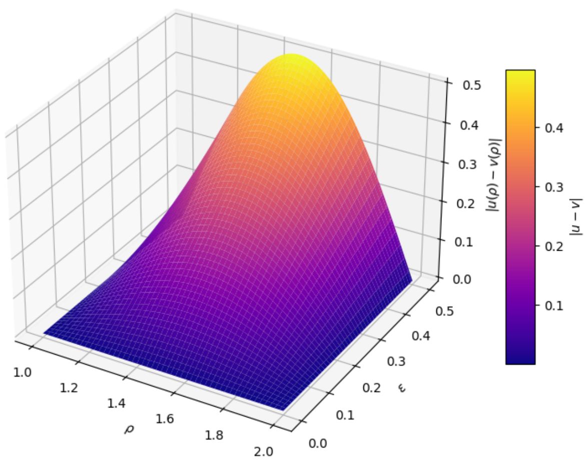

Figure 2. Ulam–Hyers stability behaviour of the solution for Example 6.1.

Figure 2 illustrates the Ulam–Hyers stability behaviour of the considered system. The bounded error surface $|u(\rho) - \tilde{u}(\rho)|$ over $\rho \in [1,2]$ shows that small perturbations produce only small deviations in the solution, confirming the theoretical stability result.
Therefore, all the hypotheses of Theorem (3.1) and Theorem (3.2) are satisfied. Hence the system admits a unique solution on $\rho = [1,2]$. Furthermore, by Theorem (4.1), the system is Ulam–Hyers stable.

Example 6.2
Consider the CH-FDEs delay with pantograph

$${}^{CH}D_{1+}^{0.7}u(\rho) = \frac{1}{50} + \frac{1}{20}u(\rho) + \frac{1}{20}\sin\left(u(0.5\rho)\right) + {}^{H}D_{1+}^{0.7}\left[\frac{1}{25}u(\rho) - \frac{1}{20}\sin\left(u(0.5\rho)\right)\right] + \frac{1}{10}z(\rho), \quad \rho \in [1,2] \tag{6.24}$$

with boundary conditions

$$u(1) = 0, \qquad u(2) = 1. \tag{6.25}$$

The parameters used in the analytical framework established in Theorem (5.1) are chosen as follows:

$$\begin{aligned}
&\mathfrak{M}_1 = 0.2, \qquad \mathfrak{M}_2 = 0.2,\\
&\|\xi\| = 0, \qquad \|\eta\| = 1,\\
&T = 2, \qquad \kappa = 0.7, \qquad \lambda = 0.5,\\
&L^* = L_{\psi 1} + L_{\psi 2} + L_{\psi 3} L_H\\
&\ = 0.05 + 0.05 + 0.04 = 0.14,
\end{aligned}$$

The system of controllability is verified through the evaluation of the relevant parameter.

$$\begin{aligned}
\mathfrak{R} &\leq (1 + \mathfrak{M}_1\mathfrak{M}_2)\left[2\|\xi\| + \|\eta\| + \frac{2\mathfrak{L}^*(\log T)^\kappa}{\Gamma(\kappa+1)}\right]\\
&= (1 + 0.2(0.2))\left(2(0) + 1 + \frac{2(0.774)}{0.909}(0.14)\right)\\
&= 1.04(0.83)\\
&= 0.86 < 1.
\end{aligned}$$

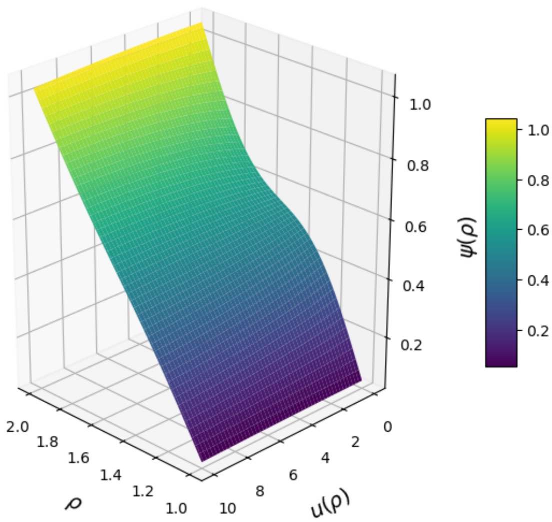

Figure 3. Graphical representation of the nonlinear function $\psi(\rho)$ illustrating the controllability behaviour of the considered Caputo–Hadamard fractional pantograph delay system for Example 6.2.

Figure 3 illustrates the behaviour of the nonlinear function $\psi(\rho)$ over the interval $\rho \in [1, 2]$. The smooth and bounded surface confirms the regular variation of the system dynamics and supports the controllability result established theoretically.

Thus, the conditions required by Theorem (5.1) are fully satisfied.

7. Conclusion

The present work addressed a boundary value problem for a CH-FDEs involving pantograph and delay effects. By reformulating the problem into an appropriate operator equation, standard fixed point arguments were employed to demonstrate the well-posedness of solutions. The stability analysis confirmed that small variation in the problem data lead to only small changes in the solution, indicating that the model is well behaved. Moreover, the controllability results demonstrated that the system can be driven to prescribed states by suitable control function. The obtained results extend existing studies on fractional differential equations with delay effects within the Caputo–Hadamard framework and provide a rigorous analytical foundation for the qualitative analysis of such systems.

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