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Homotopy analysis method for solving Atangana-Baleanu Caputo derivative with fractional order integro-differential equations

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Abstract In this paper, we presented the homotopy analysis method, namely (HAM) for solving nonlinear fractional-order integro-differential equations with fractional derivatives of the Atangana-Baleanu Caputo sense. In this approach determines solutions as convergent power series, with components that are straightforward to compute. Some illustrative examples have been given to demonstrate the accuracy of the proposed method in solving such fractional-order integro-differential equations (FIDEs) and can be applied to other nonlinear problems.

Keywords Fractional integro-differential, Atangana-Baleanu Caputo, Homotopy analysis method

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1. Introduction

Fractional calculus is a rapidly growing field of mathematics that focuses on derivatives and integrals of real or complex order. Interestingly, this branch of calculus, despite its long history dating back to classical calculus, which has attracted considerable attention from researchers across various disciplines. This increased interest stems from the remarkable results achieved when applying fractional operators to model real-world problems, [1–17]. The most significant difference between fractional analysis and classical analysis lies in the absence of a single definition for the differointegration, as is the case of classical analysis. The existence of multiple definitions of such derivatives in fractional analysis allows for the use of the definition which is most appropriate to the problem, leading to the optimal solution. Some of the most prominent classical definitions of fractional derivatives are those mentioned by Arqub and Maayah in 2019, [18]. Where if we do a simple search, it will be found that there are many well-known fractional order derivatives such as Caputo, Riemann–Liouville, Conformable and Atangana–Baleanu Caputo (ABC) fractional derivatives, [19–21]. In addition to the above, many fundamental phenomena in nature are solved utilizing fractional order integro-differential equations (FIDEs), [1, 22]. For example, applications of FIDEs can be found in electromagnetism, [23] thermal conductivity, [24] and others. Becomes of the difficulties encountered when solving analytically problems modeled using fractional derivative or integrals, many numerical and approximate methods are proposed by several authors. Liao in 1992, [25] used the fundamental concepts of Homotopy in topology serves as the foundation for introducing a versatile analytical approach to addressing nonlinear problems, known as the Homotopy Analysis Method. (HAM). The HAM [26] offers an effective approach for determining both explicit and numerical solutions to a wide and diverse range of differential systems that model real-world physical phenomena, [27,28]. Based on the principle of topological Homotopy, the validity of

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homogeneous approximate solution is independent of the presence or absence of micro coefficients in the equation under consideration. Therefore, this method can overcome the limitations of perturbation techniques, enabling the analysis of complex nonlinear problems, [29]. In this paper, we will study derive which is the approximate solutions to the FIDEs associated to ABC

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) = \mathcal{F}(t) + \int_0^t \mathcal{K}(t,\tau)\mathcal{G}(u(t))d\tau $$

Subject to

$$ u(0) = u_0 $$

$\mathcal{F}$, $\mathcal{K}$, $\mathcal{G}$ are a continuous function, $u_0 \in \mathbb{R}$ represents an unknown function that needs to be determined, and ${}^{ABC}\mathcal{D}_t^{\alpha}u(t)$ it refers to the fractional derivative ABC of order α.

2. PRELIMINARIES

In this section, we will provide a brief overview of the basic principles of fractional calculus, essential for understanding the content of this research paper.

Definition 2.1. [30,31] The fractional derivative of order $\alpha \in (0,1]$ which means the ABC of $u \in$ Q[0,1] The base point $t = 0$ is defined as:

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) = \frac{\mathcal{W}(\alpha)}{1-\alpha}\int_0^t u'(\tau)\, E_{\alpha}\Big(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\Big)\, d\tau. $$

Which is the $\mathcal{W}(\alpha)$ is a normalization function, and Q^1 represents a Sobolev-space of first-order fitted with L^2 norm which is defined over the set $Q \in \mathbb{R}$. The Sobolev-space is:

$$ Q^1(\Omega) = \left\{u \in L^2(\Omega) \ : \ u' \in L^2(\Omega)\right\}. $$

The normalization function $\mathcal{W}(\alpha)$ is then defined as follows:

$$ \mathcal{W}(\alpha) = \frac{\alpha}{\Gamma(\alpha)} + 1 - \alpha. $$

and here, $E_{\alpha}(t)$ is the standard Mittag-Leffler function defined as follows:

$$ E_{\alpha}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i\alpha+1)}, \quad \alpha > 0, \ -\infty < t < \infty. $$

Definition 2.2. [30,31] The fractional integral of order $\alpha \in (0,1]$, which means the ABC of $u \in Q(0,1]$. The base point $t = 0$ is defined as:

$$ {}_0^{ABC}\mathcal{J}_t^{\alpha}u(t) = \frac{1-\alpha}{\mathcal{W}(\alpha)}u(t) + \frac{\alpha}{\mathcal{W}(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)\, d\tau, \quad 0 < \alpha < 1. $$

The results of the next lemmas given in literature's will be used in this work, hence they are only stated without proof.

Lemma 2.3. [32] The fractional derivative and integral, based on the ABC of the function $u \in Q[0,1]$ exhibit the following properties or relationships of the Newton-Leibniz equality:

$$ {}_0^{ABC}\mathcal{J}_t^{\alpha}({}_0^{ABC}\mathcal{D}_t^{\alpha}u(t)) = u(t) - u(0) $$

Lemma 2.4. [33] The fractional derivative and integral, based on the ABC of the function $u \in Q[0,1]$ satisfy the following:

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}\left({}_0^{ABC}\mathcal{J}_t^{\alpha}u(t)\right) = u(t) - u(0)\,\frac{\mathcal{W}(\alpha)}{1-\alpha}E_{\alpha}\Big(-\frac{\alpha}{1-\alpha}t^{\alpha}\Big) $$

The above property mentioned that the fractional integer operator ${}_0^{ABC}\mathcal{J}_t^{\alpha}u(t)$ is not the right-hand inverse of ${}_0^{ABC}\mathcal{D}_t^{\alpha}$ .The fractional operator was derived based on by solving of the fractional differential equation, as referenced in [34].

3. DESCRIPTION OF THE PROBLEM SOLUTION

In this section, the HAM approach will be introduced to solving FIDEs based on ABC fractional derivatives.

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) = \mathcal{F}(t) + \int_0^t \mathcal{K}(t,\tau)\mathcal{G}(u(t))d\tau \tag{3.1} $$

Given the initial condition $u(0) = u_0$
To make the issue easier to understand, let's assume that for Eq.(3.1) for all $\alpha \in (0,1], 0 \le \tau, t \le 1, \mathcal{F}, \mathcal{K}, \mathcal{G}$ are a continuous function within the necessary dimensional domains, $u_0 \in \mathbb{R}$, and u is an unknown function to be evaluated. The operator ${}_0^{ABC}\mathcal{D}_t^{\alpha}$ it refers to the fractional derivative ABC of $u(t)$ of order α.
The nonlinear operator related to the problem under consideration is:

$$ \mathcal{N}[u(t)] = {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) - \mathcal{F}(t) - \int_0^t \mathcal{K}(t,\tau)\mathcal{G}(u(\tau))d\tau \tag{3.2} $$

Hence, based on the HAM, [28,35,36] the corresponding m^{th} order deformation given by Homotopy technic. It is read as follows:

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t)[u_m(t) - \chi_m u_{m-1}(t)] = \hbar\mathcal{H}(t)\mathcal{R}_m(\vec{u}_{m-1}) \tag{3.3} $$

with the initial conditions $u_m(0) = u_0$, where

$$ \mathcal{R}_m\big(\vec{u}_{m-1}(t)\big) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\left({}_0^{ABC}\mathcal{D}_t^{\alpha}\Phi(t,p) - \mathcal{F}(t) - \int_0^t \mathcal{K}(t,\tau)\,\mathcal{G}\big(\Phi(\tau,p)\big)\,d\tau\right)\Bigg|_{p=0} \tag{3.4} $$

Therefore, the Homotopy series is provided by:

$$ u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) \tag{3.5} $$

It is worthwhile to introduce a simple iteration scheme for $u_m(t)$. To achieve this, we take a non-zero auxiliary function to be $\mathcal{H} = 1$ and the non-zero auxiliary parameter $\hbar = -1$. Back substituting of these assumed parameters into Eq. (3.3), implies to:

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u_m(t) = \chi_m\,{}_0^{ABC}\mathcal{D}_t^{\alpha}u_{m-1}(t) - \mathcal{R}_m\big(\vec{u}_{m-1}(t)\big) \tag{3.6} $$

with $u_m(0) = \beta$. By applying the ABC fractional order integral operator ${}_0^{ABC}\mathcal{J}_t^{\alpha}$ on the both sides of Eq. (3.6), obtaining:

$$ u_m(t) = \chi_m\big(u_{m-1}(t) - u_{m-1}(0)\big) - {}_0^{ABC}\mathcal{J}_t^{\alpha}\left[\mathcal{R}_m\big(\vec{u}_{m-1}(t)\big)\right], \quad m = 1,2,3,\ldots \tag{3.7} $$

By evaluating Eq. (3.7) for the successive iterations $m = 1,2,3,\ldots$, we obtain $u_1, u_2, \ldots$. Finally, these functions are substituted with $u_0(t)$ in Eq. (3.5) yields the approximate solution to the problem of the initial value given by Eq. (3.1).

4. CONVERGENCE ANALYSIS

Suppose that the solution $u(t)$ of FIDE can be expressed using the HAM series

$$u(t) = \sum_{m=0}^{\infty} u_m(t)$$

Let the linear operator ${}_0^{ABC}\mathcal{D}_t^{\alpha} u(t)_m$ convergence control auxiliary parameter $\hbar$, and the nonlinear operator $\mathcal{N}[u(t)]$ be clearly defined so that

$$\mathcal{N}[u(t)] = {}_0^{ABC}\mathcal{D}_t^{\alpha} u(t) - \mathcal{F}(t) - \int_0^t \mathcal{K}(t,\tau)\,\mathcal{G}\big(u(\tau)\big)\,d\tau.$$

According to HAM, the m^{th} order deformation formula is

$${}_0^{ABC}\mathcal{D}_t^{\alpha}\big[u_m(t) - \chi_m u_{m-1}(t)\big] = h\,\mathcal{R}_m\big(\vec{u}_{m-1}(t)\big),$$

where

$$\mathcal{R}_m\big(\vec{u}_{m-1}(t)\big) = {}_0^{ABC}\mathcal{D}_t^{\alpha} u_{m-1}(t) - (1-\chi_m)\mathcal{F}(t) - \int_0^t \mathcal{K}(t,\tau)\,\mathcal{G}\big(u_{m-1}(\tau)\big)\,d\tau,$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

In the next theorem, we will be state and prove the convergence of the derived series of the approximate solution to the exact solution, as established by the proposed method which is the HAM.

Theorem 4.1
If the series $u(t) = \sum_{m=0}^{\infty} u_m(t)$ is convergent for $t \in [0,T]$, then the sum of the series is the exact solution of the FIDE.

Proof
Suppose that the HAM series converges. Then:

$$lim_{m\to\infty} u_m(t) = 0.$$

From the m^{th} order deformation formula

$${}_0^{ABC}\mathcal{D}_t^{\alpha}\big[u_m(t) - u_{m-1}(t)\big] = \hbar\,\mathcal{R}_m\big(\vec{u}_{m-1}(t)\big).$$

By adding the values from $m = 1$ to n, we obtain

$$\sum_{m=1}^{n} {}_0^{ABC}\mathcal{D}_t^{\alpha}\big[u_m(t) - u_{m-1}(t)\big] = \hbar \sum_{m=1}^{n} \mathcal{R}_m\big(\vec{u}_{m-1}(t)\big).$$

Now, using the linearity [37] of ${}_0^{ABC}\mathcal{D}_t^{\alpha}$

$${}_0^{ABC}\mathcal{D}_t^{\alpha}\left[\sum_{m=1}^{n}\big(u_m(t) - u_{m-1}(t)\big)\right] = \hbar \sum_{m=1}^{n} \mathcal{R}_m\big(\vec{u}_{m-1}(t)\big).$$

Since the series is converges

$$\sum_{m=1}^{n}\big(u_m(t) - u_{m-1}(t)\big) = u_n(t) - u_0(t).$$

Therefore

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) - \mathcal{F}(t) - \int_0^t \mathcal{K}(t,\tau)\,\mathcal{G}\big(u(\tau)\big)\,d\tau = 0. $$

Therefore,

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) = \mathcal{F}(t) + \int_0^t \mathcal{K}(t,\tau)\,\mathcal{G}\big(u(\tau)\big)\,d\tau. $$

Therefore, solving the HAM series satisfies the original equation, thus proving the convergence. □

5. ILLUSTRATIVE EXAMPLES AND NUMERICAL SIMULATION

In this section, we will present three illustrative examples. The HAM analyzes in the context of an infinite power series; there remains a scientific necessity to assess the solution. The accuracy of the obtained results is referred to in these examples.

Example 1. Consider the linear FIDE problem involving the ABC derivative:

$$ \begin{cases} {}_0^{ABC}\mathcal{D}_t^{\alpha}u(t) = 1 - \int_0^t u(\tau)\,d\tau, \\ u(0) = 0, \end{cases} \tag{5.1} $$

where, $t, \tau \in [0,1]$ and $u(t) = \sin t$, is the exact solution of Eq. (5.1) when $\alpha = 1$. For which the statement of this example is given in, [18, 30].

Initially, we select the starting initial guess approximation u_0 to be: $u_0(t) = 0$. Using the HAM we construct:

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}\big[u_m(t) - \chi_m u_{m-1}(t)\big] = -\mathcal{R}_m(\vec{u}_{m-1}(t)), \tag{5.2} $$

where

$$ \mathcal{R}_m(\vec{u}_{m-1}(t)) = ({}_0^{ABC}\mathcal{D}_t^{\alpha})u_{m-1}(t) + \int_0^t u_{m-1}(\tau)\,d\tau + [1-\chi_m](-1) \tag{5.3} $$

Now, applying the operator of fractional integral ${}_0^{ABC}\mathcal{J}_t^{\alpha}$ on the both sides of Eq. (5.2), we acquire the following iterations:

$$ u_1(t) = \frac{(1-\alpha)\Gamma(\alpha) + t^{\alpha}}{(1-\alpha)\Gamma(\alpha) + \alpha}, $$

$$ u_2(t) = -\frac{(1-\alpha)^2\Gamma^2(\alpha)\,t + \frac{2(1-\alpha)\Gamma(\alpha)}{\alpha+1}t^{\alpha+1} + \frac{\alpha^2\Gamma^2(\alpha)}{\Gamma(2\alpha+2)}t^{2\alpha+1}}{\big[(1-\alpha)\Gamma(\alpha) + \alpha\big]^2}, $$

$$ u_3(t) = \frac{\frac{(1-\alpha)^3\Gamma^3(\alpha)}{2}t^2 + \frac{(1-\alpha)^2\Gamma^2(\alpha)}{\alpha+1}t^{\alpha+2} + \frac{3\alpha^2(1-\alpha)\Gamma^3(\alpha)}{2(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+2} + \frac{\alpha^3\Gamma^3(\alpha)}{\Gamma(3\alpha+3)}t^{3\alpha+2}}{\big[(1-\alpha)\Gamma(\alpha) + \alpha\big]^3}. $$

And so on. Similarly, programs such as Maple or (PTC) Mathcad it can be utilized for plot graphs and for determining the additional components of the repetition formula (5.2).

And now, applying the Homotopy series in formula (3.5) to get the final approximate solution. Figure 1, illustrates that the $u(t)$ of problem (5.1) which is the approximate solution that aligns closely with the exact solution, highlighting the effectiveness and precision of the proposed method. Table 1, provides a comparison of various α values between the HAM and the exact solution for solving problem (5.1).

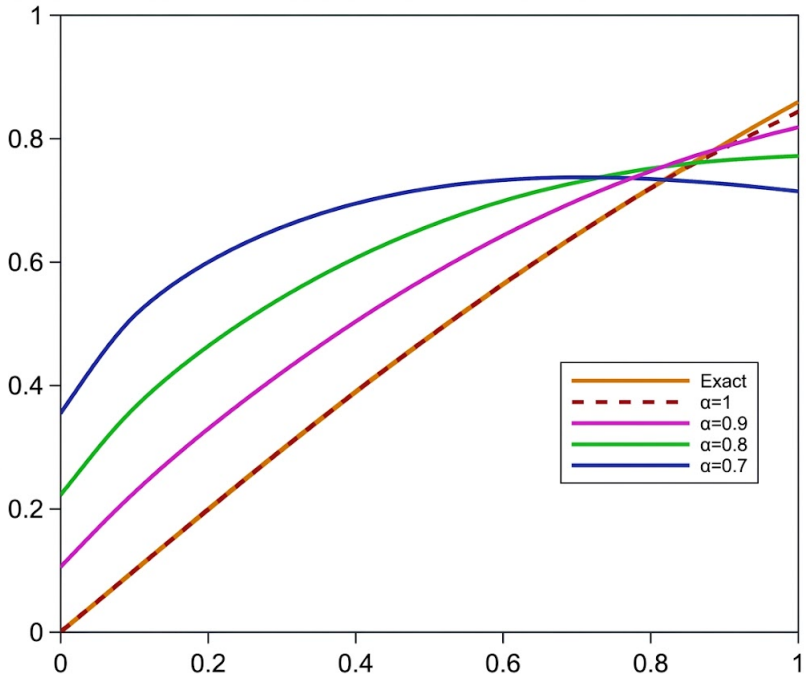


Figure 1. Different values for α of approximate solution and comparison between $\alpha = 1$ and the exact solution.

$t = 0$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	Absolute error
0	0.3574542557	0.2254411167	0.106134483	0	0	0
0.1	0.5195924462	0.3695262001	0.2283537342	0.0998334167	0.0998334166	1.984×10^{-11}
0.2	0.6016452761	0.4664003133	0.3299666788	0.1986693333	0.1986693308	2.538×10^{-9}
0.3	0.6575736242	0.5444789211	0.4213340408	0.29552025	0.2955202067	4.334×10^{-8}
0.4	0.6957135093	0.6083021847	0.503796375	0.3894186667	0.3894183423	3.244×10^{-7}
0.5	0.7201030361	0.6599085224	0.5776658066	0.4794270833	0.4794255386	1.545×10^{-6}
0.6	0.7334031803	0.7005594428	0.6429585929	0.564648	0.5646424734	5.527×10^{-6}
0.7	0.7377046561	0.7312281486	0.6996117297	0.6442339167	0.6442176872	1.623×10^{-5}
0.8	0.7348392182	0.7527914506	0.7475724481	0.7173973333	0.7173550909	4.124×10^{-5}
0.9	0.7265240387	0.7661203983	0.786843573	0.78342075	0.7833269096	9.384×10^{-5}
1	0.7144371623	0.7721287254	0.8175102477	0.8416666667	0.8414709848	1.957×10^{-4}

Table 1. Numerical results of $u(t)$ for various values of α with the exact solution, and the result of absolute error between $\alpha = 1$ and exact solution.

Example 2. Consider the nonlinear FIDE problem involving the ABC derivative:

$$\begin{cases} {}_0^{ABC}\mathcal{D}_t^{\alpha} u(t) = -1 + \int_0^t u^2(\tau)\, d\tau, \\ u(0) = 0, \end{cases} \tag{5.4}$$

where, $t, \tau \in [0, 1]$ and $u(t) = \frac{(\frac{1}{28}t^4 - t)}{(\frac{1}{12}t^3 + 1)}$., is the exact solution of Eq. (5.4) when $\alpha = 1$. For which the statement of this example is given in, [18, 30].

Initially, we select the starting initial guess approximation u_0 to be: $u_0(t) = 0$. Using the HAM we construct:

$${}_0^{ABC}\mathcal{D}_t^{\alpha}\left[u_m(t) - \chi_m u_{m-1}(t)\right] = -\mathcal{R}_m(\vec{u}_{m-1}(t)), \tag{5.5}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}(t)) = {}_0^{ABC}\mathcal{D}_t^{\alpha} u_{m-1}(t) - \int_0^t \sum_{i=0}^{m-1} u_i(\tau)\, u_{m-1-i}(\tau)\, d\tau + [1-\chi_m](1) \tag{5.6}$$

Now, applying the operator of fractional integral ${}_0^{ABC}\mathcal{J}_t^{\alpha}$ on the both sides of Eq. (5.5), we acquire the following iterations:

$$u_1(t) = -\frac{(1-\alpha)\Gamma(\alpha) + t^{\alpha}}{(1-\alpha)\Gamma(\alpha) + \alpha}$$

$$u_2(t) = 0$$

$$u_3(t) = \frac{(1-\alpha)^3 t + \frac{3\alpha(1-\alpha)^2}{\Gamma(\alpha+2)} t^{\alpha+1} + \frac{3\alpha^2(1-\alpha)}{\Gamma(2\alpha+2)} t^{2\alpha+1} + \frac{\alpha^3\Gamma(2\alpha+2)}{\Gamma(\alpha+1)^2(2\alpha+1)\Gamma(3\alpha+2)} t^{3\alpha+1}.}{\left(\frac{\alpha}{\Gamma(\alpha)} + 1 - \alpha\right)^3}$$

It is so complicated to solving the other iterations by hand. Therefore, in this example up to nine iterations, will be evaluated using the computer programs software such as Maple or (PTC) Mathcad it can be utilized for plot graphs and for determining the supplementary components of the repetition Eq. (5.4).

And now, applying the Homotopy series in formula (3.5) to get the final approximate solution. Figure 2, illustrates that the $u(t)$ of problem (5.4) which is the approximate solution that aligns closely with the exact solution, highlighting the effectiveness and precision of the proposed method. Table 2, provides a comparison of various α values between the HAM and the exact solution for solving problem (5.4).

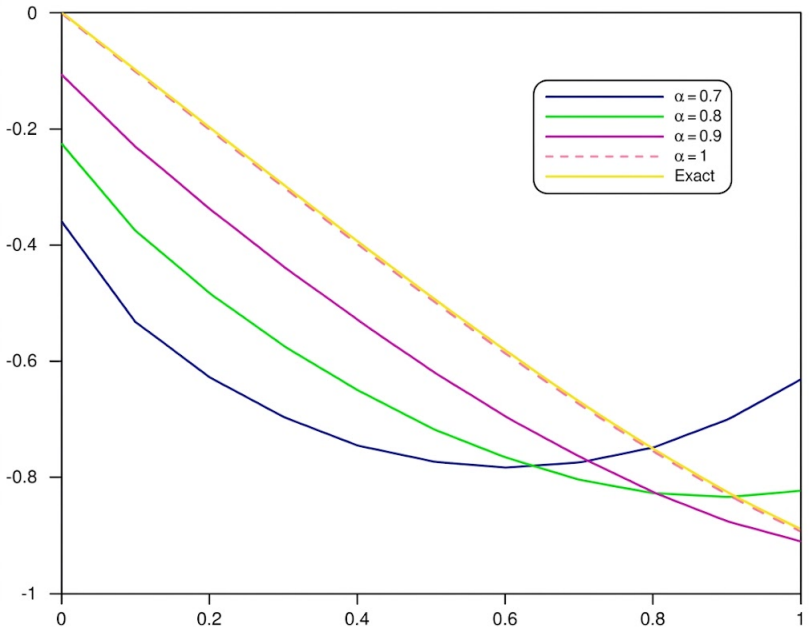


Figure 2. Different values for α of approximate solution and comparison between $\alpha = 1$ and the exact solution.

$t=0$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	Exact solution	Absolute error
0	-0.357454256	-0.225441117	-0.106134483	0	0	0
0.1	-0.530798632	-0.376022049	-0.230707661	-0.099988101	-0.099988096	5.17×10^{-9}
0.2	-0.626211183	-0.483002558	-0.337428802	-0.199809779	-0.199809651	1.284×10^{-7}
0.3	-0.694907832	-0.573551439	-0.43669643	-0.299039156	-0.299037879	1.277×10^{-6}
0.4	-0.742939517	-0.650927401	-0.52960954	-0.396976433	-0.396968549	7.884×10^{-6}
0.5	-0.772103569	-0.715713394	-0.615961257	-0.492670898	-0.49263623	3.467×10^{-5}
0.6	-0.782785464	-0.7675277	-0.694981851	-0.584963464	-0.584844232	1.192×10^{-4}
0.7	-0.774737276	-0.805512432	-0.765561978	-0.672552813	-0.67221097	3.418×10^{-4}
0.8	-0.747376317	-0.828527835	-0.826345926	-0.754086239	-0.753233467	8.528×10^{-4}
0.9	-0.699923189	-0.835245512	-0.875778765	-0.828271578	-0.826366116	1.905×10^{-3}
1	-0.631473881	-0.824199114	-0.912133505	-0.894	-0.89010989	3.89×10^{-3}

Table 2. Numerical result of $u(t)$ for different values of α with the exact solution, and the result of absolute error between $\alpha=1$ and exact solution.

Example 3. Consider the nonlinear FIDE problem involving the ABC derivative:

$$\begin{cases} ({}_0^{ABC}\mathcal{D}_t^{\alpha}u(t)) = e^t(1+e^t) - \dfrac{t^2}{4}\left(1-e^{2t}+2te^{2t}\right) - u^2(t) + \displaystyle\int_0^t t^2\tau\, u^2(\tau)\, d\tau, \\ u(0) = e^0 = 1, \end{cases} \tag{5.7}$$

where, $t,\tau\in[0,1]$ and $u(t)=e^t$., is the exact solution of Eq. (5.7) when $\alpha=1$. For which the statement of this example is given in, [30].

Initially, we select the starting initial guess approximation u_0 to be: $u_0(t)=1$. Using the HAM we construct:

$${}_0^{ABC}\mathcal{D}_t^{\alpha}\left[u_m(t)-\chi_m u_{m-1}(t)\right] = -\mathcal{R}_m(\vec{u}_{m-1}(t)), \tag{5.8}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}(t)) = ({}_0^{ABC}\mathcal{D}_t^{\alpha}u(t)) - e^t(1+e^t) + \frac{t^2}{4}\left(1-e^{2t}+2te^{2t}\right) + \sum_{i=0}^{m-1} u_i(t)\, u_{m-1-i}(t) - \int_0^t t^2\tau \sum_{j=0}^{m-1} u_j(t)\, u_{m-1-j}(t)\, d\tau.$$

Now, applying the operator of fractional integral ${}_0^{ABC}\mathcal{J}_t^{\alpha}$ on the both sides of Eq. (5.8), we acquire the following iterations:

$$u_1(t) = \frac{(1-\alpha)}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)} + \frac{\alpha}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)\Gamma(\alpha+1)}t^{\alpha},$$

$$u_2(t) = \frac{(1-\alpha)^2}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^2} + \frac{2\alpha(1-\alpha)}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^2\Gamma(\alpha+1)}t^{\alpha} + \frac{\alpha^2}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^2\Gamma(2\alpha+1)}t^{2\alpha}$$

$$u_3(t) = \frac{(1-\alpha)^3}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^3} + \frac{3\alpha(1-\alpha)^2}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^3\Gamma(\alpha+1)}t^{\alpha} + \frac{3\alpha^2(1-\alpha)}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^3\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\alpha^3}{\left(\frac{\alpha}{\Gamma(\alpha)}+1-\alpha\right)^3\Gamma(3\alpha+1)}t^{3\alpha}.$$

It is so complicated to solving the other iterations by hand. Therefore, in this example up to five iterations, will be evaluated using the computer programs software such as Maple or (PTC) Mathcad it can be utilized for plot graphs and for determining the supplementary components of the repetition Eq. (5.7).

And now, applying the Homotopy series in formula (3.5) to get the final approximate solution. Figure 3, illustrates that the $u(t)$ of problem (5.7) which is the approximate solution that aligns closely with the exact solution, highlighting the effectiveness and precision of the proposed method.Table 3, provides a comparison of various α values between the HAM and the exact solution for solving problem (5.7).

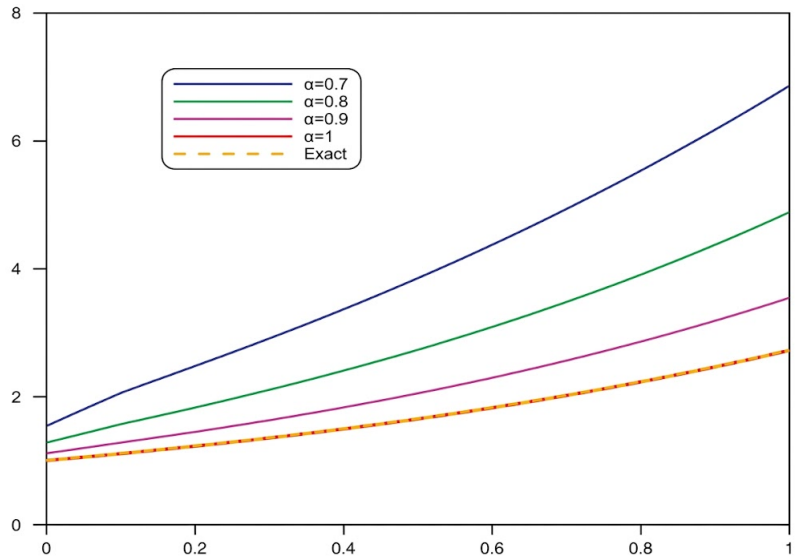


Figure 3. Different values for α of approximate solution and comparison between $\alpha = 1$ and the exact solution.

$t = 0$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	Absolute error
0	1.553062903	1.290887938	1.118734923	1	1	0
0.1	2.067484799	1.578357458	1.287936612	1.105170917	1.105170918	1.409×10^{-9}
0.2	2.486530564	1.839275021	1.45718856	1.221402667	1.221402758	9.149×10^{-8}
0.3	2.917263337	2.115777503	1.640208304	1.34985775	1.349858808	1.058×10^{-6}
0.4	3.373107671	2.415271055	1.840665618	1.491818667	1.491824698	6.031×10^{-6}
0.5	3.860071964	2.742010994	2.061266164	1.648697917	1.648721271	2.335×10^{-5}
0.6	4.381907003	3.099306769	2.304493351	1.822048	1.8221188	7.08×10^{-5}
0.7	4.941422848	3.490103601	2.572814042	2.013571417	2.013752707	1.813×10^{-4}
0.8	5.540965521	3.917197616	2.868758592	2.225130667	2.225540928	4.103×10^{-4}
0.9	6.18262644	4.383331847	3.194958951	2.45875825	2.459603111	8.449×10^{-4}
1	6.868346885	4.891244664	3.554169646	2.716666667	2.718281828	1.615×10^{-3}

Table 3. Numerical result of $u(t)$ for different values of α with the exact solution, and the result of absolute error between $\alpha = 1$ and exact solution.

6. Conclusion

The HAM is shown to be an effective and reliable method for solving FIDEs based on ABC whether is linear or nonlinear, an analytical approximate solution can be obtained. Both of the obtained results can be verified by comparing the approximate outcomes with the exact solution. as well as the almost coincidence between them when $\alpha = 1$. it was concluded that the HAM is a an extremely effective and highly efficient technique, as it can be seen from section four.

The researchers encountered some difficulties in solving nonlinear fractional equations, therefore the primary limitation of the approach discussed in this paper is its challenge in addressing nonlinear problems. These issues could potentially be managed using a linearization technique or by applying the Variational iteration method to handle the nonlinear components, [38], and Homotopy perturbation method [39]. This observation could serve as a suggestion for future research in this area.

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