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Generalized Coupled System of ψ-Hilfer Fractional Differential Equations with Self and Cross-Dependence

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Abstract In this paper, we extend the study of coupled systems of ψ-Hilfer fractional differential equations with terminal conditions to a more general case where each equation depends on both unknown functions. This formulation incorporates both self- and cross-coupling, which is essential for many realistic models in fields such as biology, physics, and engineering. Furthermore, we establish Ulam–Hyers stability of our generalized coupled system. Finally, we provide two concrete numerical examples (a linear one with $\psi(t) = t$ and a nonlinear one with $\psi(t) = t^2$) that illustrate the applicability of the theoretical results to both linear and genuinely nonlinear problems . The extension presented here opens the door to modeling more complex interactions in fractional dynamical systems while preserving the rigorous analysis of existence, uniqueness, and stability.

Keywords Fractional Differential Equations, Riemann-Liouville, Terminal Value Problems, Ulam-Hyers Stability, Generalized Ulam-Hyers Stability.

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1. Introduction

Fractional calculus has become an indispensable tool for modeling complex phenomena in various fields such as physics, economics, chemistry, electrochemistry, biology, bioengineering, etc. [7, 12, 15].

Fractional differential equations (FDEs) have the ability to capture memory and hereditary properties, which are often absent in classical integer-order models, making them particularly attractive for real-world applications.

Several recent works have contributed to the numerical and analytical treatment of fractional differential equations. For example, Swain and Jassim [25] developed numerical solutions for multi-dimensional fractional telegraph equations, while Saadeh et al. [20] proposed a hybrid approach for solving fractional Caputo partial differential equations with convergence analysis. Additionally, Malkawi and Rabaiah [14] established fixed-point theorems in Neutrosophic MR-metric spaces with applications to fractional differential equations, further demonstrating the importance of fixed-point methods in this field.

These equations may involve various fractional derivative operators. the most famous are the Riemann-Liouville (RL), Hilfer, Caputo, Katugampola, and Hadamard operators. Researchers concerned with the topic of FDEs have presented several generalizations of fractional derivatives, such as ψ-Hilfer, Hilfer–Hadamard,

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Hilfer–Katugampola, and ψ-Caputo [6, 17, 24]. In addition, Ulam-Hyers stability results for generalized FDEs have been studied [1, 5, 13]. Among these many fractional operators, the Hilfer fractional derivative and more generally its ψ-Hilfer extension has attracted significant interest because it unifies the Riemann–Liouville and Caputo fractional derivatives and allows for a flexible choice of the kernel function ψ, enabling the adaptation of the fractional model to specific experimental data or geometric constraints.

Terminal value problems (TVPs) for fractional differential equations, where the condition is specified at the right endpoint of the interval, arise naturally in areas such as optimization, inverse problems, and certain physical contexts. The combination of terminal conditions with the ψ-Hilfer fractional operator has been the subject of recent research, leading to existence, uniqueness, and stability results for single equations as well as for coupled systems.

For single equations, fundamental existence results for ψ-Hilfer terminal value problems (TVPs) were established by Almalahi, Abdo, and Panchal [4]. Extensions to Riemann-Liouville TVPs in variable-exponent Lebesgue spaces were obtained by Refice, Souid, and Guirao [19], while Bouriah, Benchohra, and Özyurt [8] studied nonlinear neutral TVPs with Hilfer–Katugampola derivatives. Additionally, Guedim et al. [9] proposed and analyzed initial-terminal value problems for fractional differential equations of variable order.

For coupled systems, the pioneering work by Abdo, Shah, Panchal, and Wahash [2] established existence, uniqueness, and Ulam–Hyers stability for ψ-Hilfer coupled TVPs. More recently, Shah and Irshad [21] investigated Ulam–Hyers–Rassias stability for coupled ψ-Hilfer systems with multipoint boundary conditions, and Aldwoah et al. [3] carried out a symmetry analysis of nonlinear terminal fractional systems under weighted ψ-Hilfer derivatives in different weighted Banach spaces. A broad overview of recent developments in Ulam–Hyers stability for fractional-order differential equations is provided by Shankar et al. [22].

Coupled systems of fractional differential equations arise in various applications, such as multi-agent systems [28], coupled oscillators [18], competition models [16], and epidemiological dynamics. Traditionally, the coupling is assumed to be "cross" only: one equation depends on the other variable, but not on its own state. Yet, in many realistic scenarios, each component of the system is influenced both by the other component and by its own dynamics. For example, in a predator–prey model, the prey population may be limited by both predation and its own carrying capacity, while the predator population depends on prey availability and its own intraspecific competition [16]. Such self- and cross-dependence cannot be captured by the simpler cross-coupled structure.

Several recent studies have further extended the theory of ψ-Hilfer coupled systems. For instance, Sombut et al. [23] established existence and uniqueness results for a coupled system of ψ-Hilfer fractional differential equations under uncoupled non-local multi-point conditions. Latha Maheswari et al. [11] investigated coupled sequential ψ-Hilfer fractional impulsive boundary value problems using topological degree theory. More recently, Verma and Sumelka [27] analyzed the qualitative behavior of nonlinear ψ-Hilfer equations with complex fractional order, including Ulam–Hyers stability.

Motivated by these considerations, the present work proposes and analyzes the following more general coupled system of ψ-Hilfer fractional differential equations with terminal conditions, where each equation depends explicitly on both unknown functions.

$$\begin{cases} D_{a^+}^{\Theta_1,\eta_1;\psi} y(t) = f_1(t, x(t), y(t)), \quad a < t \leq T, \quad a > 0, \\ D_{a^+}^{\Theta_2,\eta_2;\psi} x(t) = f_2(t, y(t), x(t)), \quad a < t \leq T, \quad a > 0. \end{cases} \tag{1}$$

with the terminal conditions

$$\begin{cases} y(T) = w_1 \in \mathbb{R}, \\ x(T) = w_2 \in \mathbb{R} \end{cases} \tag{2}$$

Where $0 < \Theta_i < 1, 0 \leq \eta_i \leq 1, D_{a^+}^{\Theta_i,\eta_i;\psi}$ is the Hilfer fractional derivative of order Θ_i and type η_i with respect to ψ and $f_i : (a,T]\times\mathbb{R}^2 \to \mathbb{R}$, (i=1,2).

This generalization is essential for modeling more complex interactions in fractional dynamical systems while preserving the rigorous analysis of existence, uniqueness, and stability.

This formulation encompasses the usual cross-coupled systems (when f_1 does not depend on y and f_2 does not depend on x) as a special case, but allows for richer interactions. The presence of both self- and cross-dependence introduces new mathematical challenges, in particular when applying fixed-point methods, because the resulting integral equations become mutually coupled in a more intricate way.

1.1. Comparison with Existing Results and Degenerate Cases

Throughout this subsection, L_{ij}, M_{ij}, and Λ denote, respectively, the Lipschitz constants, the constants arising from the fractional integral operators, and the contraction matrix, all are formally defined in Section 3. The reader may consult Section 3 for their precise definitions, here we only use them to compare our results with those of [2].

• Comparison with Abdo et al. (2020)

The most closely related work is the pioneering paper by Abdo, Shah, Panchal, and Wahash [2], which studied a coupled system of ψ-Hilfer fractional differential equations with terminal conditions under a *cross-coupled* structure:

$$D_{a^+}^{\Theta_1,\eta_1;\psi} y(t) = f_1(t, x(t)), \qquad D_{a^+}^{\Theta_2,\eta_2;\psi} x(t) = f_2(t, y(t)).$$

In that formulation, each equation depends only on the other unknown function. The present work generalizes this to the *self- and cross-dependent* system

$$D_{a^+}^{\Theta_1,\eta_1;\psi} y(t) = f_1(t, x(t), y(t)), \qquad D_{a^+}^{\Theta_2,\eta_2;\psi} x(t) = f_2(t, y(t), x(t)),$$

where each equation depends on both unknown functions. This is not merely a notational extension: the presence of self-dependence introduces additional coupling terms in the fixed-point operator, which are reflected in the four entries of the Lipschitz matrix L .
When the self-dependence terms vanish, i.e., $L_{12} = 0$ and $L_{22} = 0$, our system reduces to the cross-coupled case. In that limit, the matrix Λ becomes

$$\Lambda_{\text{cross}} = \begin{pmatrix} 0 & L_{11}M_{12} \\ L_{21}M_{21} & 0 \end{pmatrix},$$

and the spectral radius condition $\rho(\Lambda_{\text{cross}}) < 1$ reduces to the scalar inequality

$$\sqrt{L_{11}M_{12}\, L_{21}M_{21}} < 1,$$

which is precisely the type of condition obtained in [2] for cross-coupled systems. Thus, our main theorem not only generalizes the result of [2] but also recovers it as a special case.

• Degenerate cases

1. Classical Hilfer case: If $\psi(t) = t$, then the ψ-Hilfer operator reduces to the standard Hilfer fractional derivative. All our results then remain valid with $\psi(t) = t$, and the constants M_{ij} simplify accordingly.

2. Cross-coupled case: If f_1 does not depend on y and f_2 does not depend on x, then $L_{12} = L_{22} = 0$, and our system reduces to the cross-coupled system of [2].
3. Decoupled case: If f_1 depends only on y and f_2 depends only on x, then $L_{11} = L_{21} = 0$, and the system decouples into two independent terminal value problems. In this case Λ is diagonal,

$$\Lambda_{\text{dec}} = \begin{pmatrix} L_{12}M_{11} & 0 \\ 0 & L_{22}M_{22} \end{pmatrix},$$

so

$$\rho(\Lambda_{\text{dec}}) = \max\{L_{12}M_{11},\ L_{22}M_{22}\}.$$

The condition $\rho(\Lambda_{\text{dec}}) < 1$ is therefore equivalent to the two scalar inequalities $L_{12}M_{11} < 1$ and $L_{22}M_{22} < 1$, which are precisely the contraction conditions for each of the two decoupled scalar terminal value problems.
4. Our results complement those of [11, 23, 27] by considering terminal conditions and self–cross coupling, which are not addressed in those works.

These observations confirm that the proposed framework is a consistent and genuine generalization of existing results, while also allowing for richer interactions that are essential in many applications.

Our main goal is to establish existence and uniqueness of solutions for system 1. Moreover, we investigate the Ulam–Hyers stability of the system, a topic of growing importance in fractional analysis because it quantifies the robustness of solutions under small perturbations.

1.2. Contributions

The main contributions of this paper are as follows:

1. **Generalized coupled system with self- and cross-dependence.** We introduce and analyze a coupled system of ψ-Hilfer fractional differential equations with terminal conditions, in which each equation depends on both unknown functions. This formulation strictly generalizes the usual cross-coupled systems studied in the literature, including the pioneering work of Abdo et al.[2], by allowing self-dependence terms that are essential for many realistic models.
2. **Existence and uniqueness under a spectral radius condition.** We establish existence and uniqueness of solutions in a suitable weighted Banach space using Banach's fixed point theorem. The contraction condition is expressed in terms of the spectral radius of a 2×2 contraction matrix, which provides a sharp and easily verifiable criterion.
3. **Ulam-Hyers stability with explicit stability constants.** We prove both Ulam-Hyers and generalized Ulam–Hyers stability for the generalized system and derive explicit stability constants in terms of the model parameters and the spectral radius condition.
4. **Numerical illustration.** We provide two concrete numerical examples (a linear one with $\psi(t) = t$ and a nonlinear one with $\psi(t) = t^2$) that verify the spectral radius condition and yield explicit stability constants, demonstrating the applicability of the theoretical results.

1.3. Organization

The remainder of the paper is organized as follows. Section 2 recalls the necessary definitions and preliminaries of ψ-fractional calculus. Section 3 contains the main existence and uniqueness theorem, together with the construction of the fixed-point operator and the contraction condition. Section 4 is devoted to Ulam-Hyers stability, including the derivation of the integral inequalities and the stability constants, and also establishes the continuous dependence of the solution on the terminal data (Subsection 4.1). Section 5 presents two numerical examples that verify the theoretical results and illustrate the applicability of the framework. Section 6 concludes the paper and outlines directions for future research.

2. Preliminaries

In order to achieve our goal, we recall some notations and auxiliary results needed throughout this paper.

Let $[a,T] \subset \mathbb{R}^+$ with $0 < a < T < \infty$ and C[a,T] be the Banach space of real-valued continuous functions defined on the interval [a,T] endowed with the norm defined by

$$\| \sigma \|_C = \max_{t\in[a,T]} |\sigma(t)| \ , \ \sigma \in C[a,T].$$

Consider $\zeta = \Theta + \eta(1-\Theta)$, where $0 < \Theta < 1$, $0 \leq \eta \leq 1$. Let $\psi \in C^1[a,T]$ be an increasing function with $\psi'(t) \neq 0$ for each $t \in [a,T]$. the weighted spaces $C_{1-\zeta,\psi}[a,T]$ and $C^n_{1-\zeta,\psi}[a,T]$ of continuous functions are defined by [24]

$C_{1-\zeta,\psi}[a,T] = \{\sigma : (a,T] \to \mathbb{R} \ s.th \ [\psi(t)-\psi(a)]^{1-\zeta}\sigma(t) \in C[a,T]\}$, and

$C^n_{1-\zeta,\psi}[a,T] = \{\sigma \in C^{n-1}[a,T], \sigma^{(n)} \in C_{1-\zeta,\psi}[a,T]\}$.

It is easy to prove that the above two classes are Banach spaces with the norms

$$\| \sigma \|_{C_{1-\zeta,\psi}} = \max_{t\in[a,T]} |[\psi(t)-\psi(a)]^{1-\zeta}\sigma(t)|,$$

$$\| \sigma \|_{C^n_{1-\zeta,\psi}} = \sum_{j=0}^{n-1} \| \sigma^{(j)} \|_C + \| \sigma^{(n)} \|_{C_{1-\zeta,\psi}},$$

respectively.

Definition 1 ([26] ,Definition 3)
The weighted continuous spaces $C^{\zeta}_{1-\zeta,\psi}[a,T]$ and $C^{\Theta,\eta}_{1-\zeta,\psi}[a,T]$ are given by

$C^{\zeta}_{1-\zeta,\psi}[a,T] := \{\sigma \in C_{1-\zeta,\psi}[a,T], \ D^{\zeta;\psi}_{a^+}\sigma \in C_{1-\zeta,\psi}[a,T]\}$, and

$C^{\Theta,\eta}_{1-\zeta,\psi}[a,T] := \{\sigma \in C_{1-\zeta,\psi}[a,T], \ D^{\Theta,\eta;\psi}_{a^+}\sigma \in C_{1-\zeta,\psi}[a,T]\}$.

Obviously,

$C^{\zeta}_{1-\zeta,\psi}[a,T] \subset C^{\Theta,\eta}_{1-\zeta,\psi}[a,T] \subset C_{1-\zeta,\psi}[a,T] \subset C[a,T]$.

Definition 2 ([10], Sec. 2.5, Eq. (2.5.1))
Let $0 < \Theta$, $\sigma \in L_1[a,T]$. Then the left-sided ψ-RL fractional integral of order Θ of a function σ w.r.t. ψ is described by

$$I^{\Theta;\psi}_{a^+}\sigma(t) = \frac{1}{\Gamma(\Theta)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\Theta-1} \ \sigma(s) \ ds$$

Definition 3 ([10] , Eq. (2.5.17)), ([6] , Definition 1)
Let $n-1 < \Theta < n$ and $\sigma \in C^n[a,T]$. Then the left-sided ψ-fractional derivatives in the concept ψ-RL and ψ-Caputo of order Θ of a function σ w.r.t. ψ are described by

$$D^{\Theta;\psi}_{a^+}\sigma(t) = D^{n,\psi} I^{n-\Theta;\psi}_{a^+}\sigma(t)$$

and

$$^{C}D_{a^+}^{\Theta;\psi}\sigma(t) = I_{a^+}^{n-\Theta;\psi}\sigma_{\psi}^{[n]}(t),$$

respectively, where $\sigma_{\psi}^{[n]}(t) = D^{n,\psi}\sigma(t)$, $D^{n,\psi} = [\frac{1}{\psi'(t)}\frac{d}{dt}]^n$, and $n = [\Theta] + 1$

Definition 4 ([24])
let $\zeta = \Theta + \eta(n - \Theta)$, where $n - 1 < \Theta < n \in \mathbb{N}$, $0 \leq \eta \leq 1$ and $\sigma \in C^n[a, T]$. Then the left-sided ψ-fractional derivative in the concept ψ-Hilfer of order Θ and type η of a function σ w.r.t. ψ is described by

$$\begin{aligned} D_{a^+}^{\Theta,\eta;\psi}\sigma(t) &= I_{a^+}^{\eta(n-\Theta);\psi} D^{n,\psi} I_{a^+}^{(1-\eta)(n-\Theta);\psi}\sigma(t) \\ &= I_{a^+}^{\eta(n-\Theta);\psi} D^{\zeta;\psi}\sigma(t) \end{aligned}$$

Where $D^{\zeta;\psi}\sigma(t) = D^{n,\psi} I_{a^+}^{(1-\eta)(n-\Theta);\psi}\sigma(t)$

Remark 1
The above operator $D_{a^+}^{\Theta,\eta;\psi}$ is a generalization that interpolates between the RL and Caputo definitions, that is, η (between 0 and 1) is the type which acts as a "slider".

If $\eta = 0$, the definition simplifies to the ψ-Riemann-Liouville derivative.
If $\eta = 1$, the definition simplifies to the ψ-Caputo derivative.
For $0 < \eta < 1$, one obtains an operator that is a mix of both.

This enables the study of fractional differential equations, allowing the results to be applied to both the RL and Caputo cases and everything in between.

Theorem 1 ([29], Theorem 1.41)
Let Ξ be a Banach space and ϕ be a nonempty closed subset of Ξ. Then any contraction operator $\pi : \phi \to \phi$ has a unique fixed point.

Throughout this paper, we assume $0 < \Theta < 1$, so that $n = 1$.

3. Main Results(Existence and Uniquness)

To simplify the forthcoming equations, we set

$$\kappa_{\psi}^{\zeta}(t, a) := [\psi(t) - \psi(a)]^{\zeta-1} \text{ and } H_{\psi}^{\Theta}(t, s) := \psi'(s)(\psi(t) - \psi(s))^{\Theta-1}$$

Theorem 2 ([2], Theorem 3)
Let $\zeta = \Theta + \eta(1 - \Theta)$ where $0 < \Theta < 1$ and $0 \leq \eta \leq 1$. If $\sigma : (a, T] \to \mathbb{R}$ is a function such that $\sigma(\cdot) \in C_{1-\zeta,\psi}[a, T]$, then $y \in C_{1-\zeta,\psi}^{\zeta}(a, T]$ satisfies the TVP for ψ-Hilfer FDEs

$$\begin{aligned} D_{a^+}^{\Theta,\eta;\psi} y(t) &= \sigma(t), \qquad t \in (a, T], a > 0, \\ y(T) &= w \in \mathbb{R}, \end{aligned}$$

if and only if y satisfies the following fractional integral equation:

$$y(t) = \frac{\kappa_{\psi}^{\zeta}(t, a)}{\kappa_{\psi}^{\zeta}(T, a)}\left[w - \frac{1}{\Gamma(\Theta)}\int_a^T H_{\psi}^{\Theta}(T, s)\,\sigma(s) \quad ds\right] + \frac{1}{\Gamma(\Theta)}\int_a^t H_{\psi}^{\Theta}(t, s)\,\sigma(s) \quad ds. \tag{3}$$

Applying Theorem 2 to each equation in our generalized coupled system with $\sigma_1(t) = f_1(t, x(t), y(t))$ and $\sigma_2(t) = f_2(t, y(t), x(t)$, we obtain the following generalized integral formulation:

Lemma 1 (Generalized integral formulation for coupled system with self- and cross-dependence)
Let i= 1,2, $\zeta_i = \Theta_i + \eta_i(1 - \Theta_i)$ where $0 < \Theta_i < 1$ and $0 \leq \eta_i \leq 1$. Let f_1,f_2: $(a, T]\times\mathbb{R}^2 \to \mathbb{R}$ satisfy

$$f_1(\cdot, x(\cdot), y(\cdot)) \in C_{1-\zeta_1,\psi}[a, T]$$
$$f_2(\cdot, y(\cdot), x(\cdot)) \in C_{1-\zeta_2\psi}[a, T]$$

Then $(y, x) \in C^{\zeta_1}_{1-\zeta_1,\psi}(a, T] \times C^{\zeta_2}_{1-\zeta_2,\psi}(a, T]$ satisfies the generalized ψ-Hilfer coupled terminal value problem (1)-(2) if and only if (y,x) satisfies the following coupled system of fractional integral equations:

$$\begin{cases} y(t) = \dfrac{\kappa^{\zeta_1}_\psi(t, a)}{\kappa^{\zeta_1}_\psi(T, a)}\left[w_1 - \dfrac{1}{\Gamma(\Theta_1)}\int_a^T H^{\Theta_1}_\psi(T, s)\, f_1(s, x(s), y(s)) \quad ds\right] + \dfrac{1}{\Gamma(\Theta_1)}\int_a^t H^{\Theta_1}_\psi(t, s)\, f_1(s, x(s), y(s)) \quad ds, \\ x(t) = \dfrac{\kappa^{\zeta_2}_\psi(t, a)}{\kappa^{\zeta_2}_\psi(T, a)}\left[w_2 - \dfrac{1}{\Gamma(\Theta_2)}\int_a^T H^{\Theta_2}_\psi(T, s)\, f_2(s, y(s), x(s)) \quad ds\right] + \dfrac{1}{\Gamma(\Theta_2)}\int_a^t H^{\Theta_2}_\psi(t, s)\, f_2(s, y(s), x(s)) \quad ds. \end{cases} \tag{4}$$

where $\kappa^{\zeta_i}_\psi(t, a) := [\psi(t) - \psi(a)]^{\zeta_i - 1}$ and $H^{\Theta_i}_\psi(t, s) := \psi'(s)(\psi(t) - \psi(s))^{\Theta_i - 1}$.

Proof.
Since $f_1(\cdot, x(\cdot), y(\cdot)) \in C_{1-\zeta_1,\psi}[a, T]$ and $f_2(\cdot, y(\cdot), x(\cdot)) \in C_{1-\zeta_2,\psi}[a, T]$, we can apply Theorem 2 to each equation separately:

- For y, Theorem 2 with $\sigma(t) = f_1(t, x(t), y(t))$ gives the equivalence between the differential equation for y and its integral form.
- For x, Theorem 2 with $\sigma(t) = f_2(t, y(t), x(t))$ gives the equivalence between the differential equation for x and its integral form.

Thus, (y,x) satisfies (1)-(2) if and only if it satisfies (4).

□

Before presenting the main theorem, we assume that the following assumptions are satisfied:
Let $i = 1, 2$, $\zeta_i = \Theta_i + \eta_i(1 - \Theta_i)$, where $0 < \Theta_i < 1$ and $0 \leq \eta_i \leq 1$.

(H1) f_1,f_2: $(a, T]\times\mathbb{R}^2 \to \mathbb{R}$ are such that:
For all $y \in C_{1-\zeta_1,\psi}[a, T]$ *and* $x \in C_{1-\zeta_2,\psi}[a, T]$
$f_1(\cdot, x(\cdot), y(\cdot)) \in C^{\eta_1(1-\Theta_1)}_{1-\zeta_1,\psi}[a, T]$ and $f_2(\cdot, y(\cdot), x(\cdot)) \in C^{\eta_2(1-\Theta_2)}_{1-\zeta_2,\psi}[a, T]$,

(H2) Lipschitz condition
There exist constants $L_{11}, L_{12}, L_{21}, L_{22} \geq 0$ such that for all $t \in (a, T]$ and all $p_1, p_2, q_1, q_2 \in \mathbb{R}$
$|f_1(t, p_1, q_1) - f_1(t, p_2, q_2)| \leq L_{11}|p_1 - p_2| + L_{12}|q_1 - q_2|$,
$|f_2(t, q_1, p_1) - f_2(t, q_2, p_2)| \leq L_{21}|q_1 - q_2| + L_{22}|p_1 - p_2|$.

Remark 2 (Growth condition is not independent)
In earlier versions, a growth condition (H3) was imposed alongside the Lipschitz condition (H2). However, (H3) is automatically satisfied under (H2) provided that $f_1(\cdot, 0, 0) \in C_{1-\zeta_1,\psi}[a, T]$ and $f_2(\cdot, 0, 0) \in C_{1-\zeta_2,\psi}[a, T]$. Indeed,

for any $p, q \in \mathbb{R}$ and $t \in (a, T]$,

$$|f_1(t,p,q)| \leq |f_1(t,0,0)| + L_{11}|p| + L_{12}|q|,$$

and similarly

$$|f_2(t,q,p)| \leq |f_2(t,0,0)| + L_{21}|q| + L_{22}|p|.$$

Thus (H3) holds with

$$\varphi_1(t) = |f_1(t,0,0)|, \quad c_{11} = L_{11}, \quad c_{12} = L_{12},$$
$$\varphi_2(t) = |f_2(t,0,0)|, \quad c_{21} = L_{21}, \quad c_{22} = L_{22}.$$

Therefore, we do not need to assume (H3) separately, it is a consequence of (H1) and (H2). In the sequel, all estimates that previously used (H3) are understood with the above identifications.

• Matrix formulation for contraction analysis
Define the Lipschitz matrix

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

In the following theorem, we prove the existence and uniqueness of the solution for the generalized ψ-Hilfer coupled terminal value problem (1) - (2).

Theorem 3 (Generalized existence and uniqueness theorem)
Let $\zeta_i = \Theta_i + \eta_i(1 - \Theta_i)$, where $0 < \Theta_i < 1$ and $0 \leq \eta_i \leq 1$ *for* $i = 1, 2$. Assume that hypotheses (H1) and (H2) hold. (The growth condition (H3) is automatically satisfied as noted in Remark 2) . Define the constants

$$M_{11} = 2(\psi(T) - \psi(a))^{\Theta_1} \cdot \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1 + \Theta_1)},$$
$$M_{12} = 2(\psi(T) - \psi(a))^{\Theta_1 + \zeta_2 - \zeta_1} \cdot \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2 + \Theta_1)},$$
$$M_{21} = 2(\psi(T) - \psi(a))^{\Theta_2 + \zeta_1 - \zeta_2} \cdot \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1 + \Theta_2)},$$
$$M_{22} = 2(\psi(T) - \psi(a))^{\Theta_2} \cdot \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2 + \Theta_2)},$$

form the contraction matrix

$$\Lambda = \begin{pmatrix} L_{12}M_{11} & L_{11}M_{12} \\ L_{21}M_{21} & L_{22}M_{22} \end{pmatrix}$$

If the spectral radius $\rho(\Lambda) < 1$, then the generalized ψ-Hilfer coupled system (1) - (2) has a unique solution in the space $E^{\zeta_1} \times E^{\zeta_2}$, where $E^{\zeta_i} := C^{\zeta_i}_{1-\zeta_i,\psi}[a, T]$ for $i = 1, 2$.

Proof.
We present our proof in the following five steps:

• Step 1: Setup and operator formulation.

Let E^{ζ_1} and E^{ζ_2} be equipped with the weighted norms
$\|y\|_{E^{\zeta_1}} = \sup_{t\in(a,T]} |(\psi(t) - \psi(a))^{1-\zeta_1} y(t)|$, $\|x\|_{E^{\zeta_2}} = \sup_{t\in(a,T]} |(\psi(t) - \psi(a))^{1-\zeta_2} x(t)|$.

Define the product space $E := E^{\zeta_1} \times E^{\zeta_2}$ with the norm

$$\|(y,x)\|_E = \max\{\|y\|_{E^{\zeta_1}}, \|x\|_{E^{\zeta_2}}\},$$

define the operator $\tau := (\tau_1, \tau_2) : E \to E$ by

$$\tau(y,x) := (\tau_1(x,y), \tau_2(y,x)),$$

where $\tau_1 : E^{\zeta_2} \times E^{\zeta_1} \to E^{\zeta_1}$ and $\tau_2 : E^{\zeta_1} \times E^{\zeta_2} \to E^{\zeta_2}$ are given by

$$\begin{cases} \tau_1(x,y)(t) = \dfrac{\kappa_\psi^{\zeta_1}(t,a)}{\kappa_\psi^{\zeta_1}(T,a)}\left[w_1 - \dfrac{1}{\Gamma(\Theta_1)}\int_a^T H_\psi^{\theta_1}(T,s)\ f_1(s,x(s),y(s)) \quad ds\right] + \dfrac{1}{\Gamma(\Theta_1)}\int_a^t H_\psi^{\theta_1}(t,s)\ f_1(s,x(s),y(s)) \quad ds, \\ \tau_2(y,x)(t) = \dfrac{\kappa_\psi^{\zeta_2}(t,a)}{\kappa_\psi^{\zeta_2}(T,a)}\left[w_2 - \dfrac{1}{\Gamma(\Theta_2)}\int_a^T H_\psi^{\theta_2}(T,s)\ f_2(s,y(s),x(s)) \quad ds\right] + \dfrac{1}{\Gamma(\Theta_2)}\int_a^t H_\psi^{\theta_2}(t,s)\ f_2(s,y(s),x(s)) \quad ds. \end{cases} \tag{5}$$

For clarity, we present the proof in the product norm $\|\cdot\|_E$. If $\|\Lambda\|_\infty < 1$ does not hold, the same argument applies after replacing $\|\cdot\|_E$ by the weighted norm $\|\cdot\|_\varepsilon$ discussed in the remark following the proof.

The remaining steps show that τ is a contraction, so that Banach's fixed-point theorem yields a unique fixed point (y^*, x^*). This fixed point is the unique solution of the generalized ψ-Hilfer coupled terminal value problem (1) - (2).

• **Step 2: Define a suitable closed ball.**

Define the closed ball

$$\S_R := \{(y,x) \in E : \|(y,x)\|_E \le R\},$$

where $R > 0$ will be chosen later. This set is closed, bounded, convex, and nonempty in the complete normed metric (Banach) space E.

• **Step 3: Self-mapping property (τ maps S_R into itself).**

We need to show that for a suitable R, if $(y,x) \in S_R$, then $\tau(y,x) \in S_R$. Let $(y,x) \in S_R$.
For $\tau_1(x,y)$, consider its two parts:

$$A_1(t) = \frac{\kappa_\psi^{\zeta_1}(t,a)}{\kappa_\psi^{\zeta_1}(T,a)}\left[w_1 - \frac{1}{\Gamma(\Theta_1)}\int_a^T H_\psi^{\Theta_1}(T,s) f_1(s,x(s),y(s))\, ds\right],$$

$$B_1(t) = \frac{1}{\Gamma(\Theta_1)}\int_a^t H_\psi^{\Theta_1}(t,s) f_1(s,x(s),y(s))\, ds.$$

Using the Lipschitz condition (H2) and the fact that $f_1(\cdot,0,0) \in C_{1-\zeta_1,\psi}[a,T]$, we have

$$|f_1(s,x(s),y(s))| \le |f_1(s,0,0)| + L_{11}|x(s)| + L_{12}|y(s)|.$$

Since $(y,x) \in S_R$, we have

$$|x(s)| \le R(\psi(s)-\psi(a))^{\zeta_2-1}, \qquad |y(s)| \le R(\psi(s)-\psi(a))^{\zeta_1-1}.$$

Let $\Delta = \psi(T) - \psi(a)$. We compute

$$\begin{aligned}\|A_1\|_{E^{\zeta_1}} &\leq |w_1| \cdot \frac{(\psi(t)-\psi(a))^{\zeta_1-1}}{\Delta^{\zeta_1-1}} \cdot (\psi(t)-\psi(a))^{1-\zeta_1} \\ &\quad + \frac{1}{\Gamma(\Theta_1)} \frac{(\psi(t)-\psi(a))^{\zeta_1-1}}{\Delta^{\zeta_1-1}} \cdot (\psi(t)-\psi(a))^{1-\zeta_1} \int_a^T |H_\psi^{\Theta_1}(T,s)|\, |f_1(s,x(s),y(s))|\, ds \\ &\leq \frac{|w_1|}{\Delta^{\zeta_1-1}} + \frac{1}{\Gamma(\Theta_1)\Delta^{\zeta_1-1}} \int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta_1-1} \\ &\qquad \times \Big[|f_1(s,0,0)| + L_{11}R(\psi(s)-\psi(a))^{\zeta_2-1} + L_{12}R(\psi(s)-\psi(a))^{\zeta_1-1}\Big]\, ds.\end{aligned}$$

The integrals can be evaluated using Beta functions. Let

$$\Phi_1 = \int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta_1-1} |f_1(s,0,0)|\, ds,$$

which is finite since $f_1(\cdot,0,0) \in C_{1-\zeta_1,\psi}[a,T]$. For the other terms,

$$\int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta_1-1}(\psi(s)-\psi(a))^{\zeta_2-1}\, ds = \Delta^{\Theta_1+\zeta_2-1}\beta(\zeta_2,\Theta_1),$$

$$\int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta_1-1}(\psi(s)-\psi(a))^{\zeta_1-1}\, ds = \Delta^{\Theta_1+\zeta_1-1}\beta(\zeta_1,\Theta_1),$$

where $\beta(\cdot,\cdot)$ is the Beta function, $\beta(\zeta,\Theta) = \frac{\Gamma(\zeta)\Gamma(\Theta)}{\Gamma(\zeta+\Theta)}$. Thus

$$\begin{aligned}\|A_1\|_{E^{\zeta_1}} \leq \frac{|w_1|}{\Delta^{\zeta_1-1}} + \frac{1}{\Gamma(\Theta_1)\Delta^{\zeta_1-1}}\Big[&\Phi_1 + L_{11}R\,\Delta^{\Theta_1+\zeta_2-1}\frac{\Gamma(\zeta_2)\Gamma(\Theta_1)}{\Gamma(\zeta_2+\Theta_1)} \\ &+ L_{12}R\,\Delta^{\Theta_1+\zeta_1-1}\frac{\Gamma(\zeta_1)\Gamma(\Theta_1)}{\Gamma(\zeta_1+\Theta_1)}\Big].\end{aligned}$$

Similarly, for $B_1(t)$,

$$\begin{aligned}\|B_1\|_{E^{\zeta_1}} &\leq \frac{1}{\Gamma(\Theta_1)} \sup_{t\in(a,T]} (\psi(t)-\psi(a))^{1-\zeta_1} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\Theta_1-1} \\ &\qquad \times \Big[|f_1(s,0,0)| + L_{11}R(\psi(s)-\psi(a))^{\zeta_2-1} + L_{12}R(\psi(s)-\psi(a))^{\zeta_1-1}\Big]\, ds.\end{aligned}$$

The worst case is when $t = T$, giving the same bound as for the integral in A_1. Hence

$$\|B_1\|_{E^{\zeta_1}} \leq \frac{\Delta^{1-\zeta_1}}{\Gamma(\Theta_1)}\Big[\Phi_1 + L_{11}R\,\Delta^{\Theta_1+\zeta_2-1}\frac{\Gamma(\zeta_2)\Gamma(\Theta_1)}{\Gamma(\zeta_2+\Theta_1)} + L_{12}R\,\Delta^{\Theta_1+\zeta_1-1}\frac{\Gamma(\zeta_1)\Gamma(\Theta_1)}{\Gamma(\zeta_1+\Theta_1)}\Big].$$

Combining,

$$\|\tau_1(x,y)\|_{E^{\zeta_1}} \leq C_1 + K_{11}R,$$

where

$$C_1 = \frac{|w_1|}{\Delta^{\zeta_1-1}} + \frac{\Phi_1}{\Gamma(\Theta_1)}\left(\frac{1}{\Delta^{\zeta_1-1}} + \Delta^{1-\zeta_1}\right),$$

and

$$K_{11} = L_{12}M_{11} + L_{11}M_{12},$$

with

$$M_{11} = \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_1)} \cdot 2\Delta^{\Theta_1}, \qquad M_{12} = \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_1)} \cdot 2\Delta^{\Theta_1+\zeta_2-\zeta_1}.$$

Similarly, for $\tau_2(y,x)$,

$$\|\tau_2(y,x)\|_{E^{\zeta_2}} \leq C_2 + K_{22}R,$$

where

$$C_2 = \frac{|w_2|}{\Delta^{\zeta_2-1}} + \frac{\Phi_2}{\Gamma(\Theta_2)}\left(\frac{1}{\Delta^{\zeta_2-1}} + \Delta^{1-\zeta_2}\right),$$

$$K_{22} = L_{21}M_{21} + L_{22}M_{22},$$

and

$$M_{21} = \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_2)} \cdot 2\Delta^{\Theta_2+\zeta_1-\zeta_2}, \qquad M_{22} = \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_2)} \cdot 2\Delta^{\Theta_2}.$$

Choose

$$R \geq \max\left(\frac{C_1}{1-K_{11}}, \frac{C_2}{1-K_{22}}\right),$$

provided $K_{11} < 1$ and $K_{22} < 1$. This ensures $\|\tau(y,x)\|_E \leq R$, that is, $\tau(y,x) \in S_R$, hence $\tau(S_R) \subseteq S_R$.

- **Step 4: Contraction property.**

Take two pairs $(y_1,x_1),(y_2,x_2) \in \S_R$,
let

$$Y = \|y_1 - y_2\|_{E^{\zeta_1}},\ X = \|x_1 - x_2\|_{E^{\zeta_2}},$$

for τ_1, using the Lipschitz condition (H2),

$$|f_1(s,x_1(s),y_1(s)) - f_1(s,x_2(s),y_2(s))| \leq L_{11}|x_1(s)-x_2(s)| + L_{12}|y_1(s)-y_2(s)|,$$

since $|x_1(s)-x_2(s)| \leq X(\psi(s)-\psi(a))^{\zeta_2-1}$ and $|y_1(s)-y_2(s)| \leq Y(\psi(s)-\psi(a))^{\zeta_1-1}$,
we have, for the A_1 part

$$\begin{aligned}\|A_1(x_1,y_1) - A_1(x_2,y_2)\|_{E^{\zeta_1}} &\leq \frac{1}{\Gamma(\Theta_1)\,\Delta^{\zeta_1-1}} \int_a^T \psi'(s)\,(\psi(T)-\psi(s))^{\Theta_1-1}\Big[L_{11}X\,(\psi(s)-\psi(a))^{\zeta_2-1} \\ &\qquad + L_{12}Y\,(\psi(s)-\psi(a))^{\zeta_1-1}\Big]\,ds \\ &\leq \frac{1}{\Gamma(\Theta_1)}\Big[L_{11}X\,\Delta^{\Theta_1+\zeta_2-\zeta_1}\,\frac{\Gamma(\zeta_2)\Gamma(\Theta_1)}{\Gamma(\zeta_2+\Theta_1)} \\ &\qquad + L_{12}Y\,\Delta^{\Theta_1}\,\frac{\Gamma(\zeta_1)\Gamma(\Theta_1)}{\Gamma(\zeta_1+\Theta_1)}\Big].\end{aligned}$$

Similarly, for the B_1 part, we get the same bound. Thus

$$\|\tau_1(x_1,y_1) - \tau_1(x_2,y_2)\|_{E^{\zeta_1}} \leq 2\left[L_{11}X\Delta^{\Theta_1+\zeta_2-\zeta_1}\frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_1)} + L_{12}Y\Delta^{\Theta_1}\frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_1)}\right] = L_{11}M_{12}X + L_{12}M_{11}Y.$$

Similarly, for τ_2

$$\|\tau_2(y_1,x_1) - \tau_2(y_2,x_2)\|_{E^{\zeta_2}} \leq L_{21}M_{21}Y + L_{22}M_{22}X.$$

Therefore,

$$\begin{pmatrix}\|\tau_1(x_1,y_1) - \tau_1(x_2,y_2)\|_{E^{\zeta_1}} \\ \|\tau_2(y_1,x_1) - \tau_2(y_2,x_2)\|_{E^{\zeta_2}}\end{pmatrix} \leq \begin{pmatrix}L_{12}M_{11} & L_{11}M_{12} \\ L_{21}M_{21} & L_{22}M_{22}\end{pmatrix}\begin{pmatrix}Y \\ X\end{pmatrix} = \Lambda\begin{pmatrix}Y \\ X\end{pmatrix}.$$

Since the spectral radius $\rho(\Lambda) < 1$, there exists a vector norm on $\mathbb{R}^2$ such that the induced matrix norm of Λ is less than 1. Equivalently, there exists $\lambda \in [0, 1)$ such that

$$\begin{aligned}
\|\tau(y_1, x_1) - \tau(y_2, x_2)\|_E &= \|(\tau_1(x_1, y_1), \tau_2(y_1, x_1)) - (\tau_1(x_2, y_2), \tau_2(y_2, x_2))\|_E \\
&= \|(\tau_1(x_1, y_1) - \tau_1(x_2, y_2), \tau_2(y_1, x_1) - \tau_2(y_2, x_2))\|_E \\
&= \max\{\|\tau_1(x_1, y_1) - \tau_1(x_2, y_2)\|_{E^{\zeta_1}}, \|\tau_2(y_1, x_1) - \tau_2(y_2, x_2)\|_{E^{\zeta_2}}\} \\
&\leq \lambda \max\{Y, X\} \\
&= \lambda \max\{\|y_1 - y_2\|_{E^{\zeta_1}}, \|x_1 - x_2\|_{E^{\zeta_2}}\} \\
&= \lambda \|(y_1 - y_2, x_1 - x_2)\|_E \\
&= \lambda \|(y_1, x_1) - (y_2, x_2)\|_E \\
&< \|(y_1, x_1) - (y_2, x_2)\|_E.
\end{aligned}$$

Hence, τ is a contraction on $\S_R$.

- **Step 5: Application of Banach fixed-point theorem**

Since $\S_R$ is a closed subset of the Banach space E (being a product of two Banach spaces), and $\tau : \S_R \to \S_R$ is a contraction, by theorem 1 it has a unique fixed point in $\S_R$. We show that this fixed point is the unique solution of the generalized ψ-Hilfer coupled system (1) - (2).

Let (y^*, x^*) be the fixed point of τ. Then

$$\tau(y^*, x^*) = (y^*, x^*) = (\tau_1(x^*, y^*), \tau_2(y^*, x^*)).$$

Hence,

$$y^* = \tau_1(x^*, y^*) \; and \; x^* = \tau_2(y^*, x^*).$$

So,

$$\begin{aligned}
\tau_1(x^*, y^*)(t) &= y^*(t) \\
&= \frac{\kappa_\psi^{\zeta_1}(t, a)}{\kappa_\psi^{\zeta_1}(T, a)} \left[w_1 - \frac{1}{\Gamma(\Theta_1)} \int_a^T H_\psi^{\theta_1}(T, s) \; f_1(s, x^*(s), y^*(s)) \quad ds \right] + \frac{1}{\Gamma(\Theta_1)} \int_a^t H_\psi^{\theta_1}(t, s) \; f_1(s, x^*(s), y^*(s)) \quad ds,
\end{aligned}$$

And,

$$\begin{aligned}
\tau_2(y^*, x^*)(t) &= x^*(t) \\
&= \frac{\kappa_\psi^{\zeta_2}(t, a)}{\kappa_\psi^{\zeta_2}(T, a)} \left[w_2 - \frac{1}{\Gamma(\Theta_2)} \int_a^T H_\psi^{\theta_2}(T, s) \; f_2(s, y^*(s), x^*(s)) \quad ds \right] + \frac{1}{\Gamma(\Theta_2)} \int_a^t H_\psi^{\theta_2}(t, s) \; f_2(s, y^*(s), x^*(s)) \quad ds.
\end{aligned}$$

That is,
(y^*, x^*) satisfies (4) and by lemma (1) it satisfies the system (1)-(2).

□

Remark (On the hypothesis $\rho(\Lambda) < 1$). The proof of Theorem 3 is written for the product norm $\|(y, x)\|_E = \max\{\|y\|_{E^{\zeta_1}}, \|x\|_{E^{\zeta_2}}\}$, under which the contraction constant is $\|\Lambda\|_\infty$, which is in general larger than $\rho(\Lambda)$. When $\|\Lambda\|_\infty \geq 1$ but $\rho(\Lambda) < 1$, the same argument applies after replacing the norm by the weighted norm

$$\|(y, x)\|_\varepsilon = \max\{\|y\|_{E^{\zeta_1}}, \; \varepsilon \|x\|_{E^{\zeta_2}}\}, \qquad \varepsilon > 0.$$

Under this norm the contraction constant is

$$\|\Lambda\|_\varepsilon = \max\{\Lambda_{11} + \Lambda_{12}/\varepsilon, \; \varepsilon\Lambda_{21} + \Lambda_{22}\}.$$

For a nonnegative 2×2 matrix with $\rho(\Lambda) < 1$, one has $\Lambda_{11} < 1$ and $\Lambda_{22} < 1$ (since $\rho(\Lambda) \geq \Lambda_{ii}$), and the admissible range of ε is

$$\frac{\Lambda_{12}}{1-\Lambda_{11}} < \varepsilon < \frac{1-\Lambda_{22}}{\Lambda_{21}}.$$

This range is nonempty because

$$\det(I-\Lambda) = (1-\lambda_1)(1-\lambda_2) > 0,$$

where λ_1, λ_2 are the eigenvalues of Λ and both satisfy $|\lambda_i| < 1$. With such a weight, Steps 3–5 of the proof go through verbatim, the constants K_{11}, K_{22} being replaced by $\Lambda_{11} + \Lambda_{12}/\varepsilon$ and $\varepsilon\Lambda_{21} + \Lambda_{22}$, both of which are < 1. Hence the conclusion of Theorem 3 holds under the stated hypothesis $\rho(\Lambda) < 1$.

Remark (Global uniqueness). The uniqueness established in Theorem 3 is in fact global, not merely on the closed ball S_R. Indeed, under the Lipschitz condition (H2), any solution (y, x) of the generalized system (1) - (2) satisfies the *a priori* estimates

$$\|y\|_{E^{\zeta_1}} \leq C_1 + \Lambda_{11}\|y\|_{E^{\zeta_1}} + \Lambda_{12}\|x\|_{E^{\zeta_2}}, \qquad \|x\|_{E^{\zeta_2}} \leq C_2 + \Lambda_{21}\|y\|_{E^{\zeta_1}} + \Lambda_{22}\|x\|_{E^{\zeta_2}},$$

where C_1, C_2 are the constants defined in Step 3 of the proof and Λ_{ij} denotes the (i,j)-entry of the contraction matrix Λ introduced in Theorem 3. Indeed, writing $y = \tau_1(x, y)$ and $x = \tau_2(y, x)$, and using the contraction estimates from Step 4 together with the bound $\|\tau_i(0,0)\|_{E^{\zeta_i}} \leq C_i$, these inequalities follow at once. In matrix form,

$$(I-\Lambda)\begin{pmatrix}\|y\|_{E^{\zeta_1}} \\ \|x\|_{E^{\zeta_2}}\end{pmatrix} \leq \begin{pmatrix}C_1 \\ C_2\end{pmatrix}.$$

Since $\rho(\Lambda) < 1$, the matrix $I - \Lambda$ is invertible with nonnegative inverse, so

$$\begin{pmatrix}\|y\|_{E^{\zeta_1}} \\ \|x\|_{E^{\zeta_2}}\end{pmatrix} \leq (I-\Lambda)^{-1}\begin{pmatrix}C_1 \\ C_2\end{pmatrix},$$

which yields an explicit constant $R > 0$, independent of the solution, such that $\|(y, x)\|_E \leq R$. Consequently, every possible solution lies in the closed ball S_R, and the contraction property established in Step 4 forces any two solutions in S_R to coincide. Hence uniqueness holds in the whole space $E^{\zeta_1} \times E^{\zeta_2}$, not only on the ball S_R.

4. Ulam-Hyers Stability for Generalized System

Definition 5
The generalized ψ-Hilfer coupled system (1) - (2) is said to be Ulam–Hyers (U-H) stable if there exists a vector $\lambda = (\lambda_{f_1}, \lambda_{f_2}) > 0$ (with $\lambda_{f_1}, \lambda_{f_2} > 0$) such that for any $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and for every $(\widetilde{y}, \widetilde{x}) \in E^{\zeta_1} \times E^{\zeta_2}$ satisfying the perturbed inequalities

$$\begin{cases} |D_{a^+}^{\Theta_1,\eta_1;\psi}\widetilde{y}(t) - f_1(t, \widetilde{x}(t), \widetilde{y}(t))| \leq \epsilon_1, \\ |D_{a^+}^{\Theta_2,\eta_2;\psi}\widetilde{x}(t) - f_2(t, \widetilde{y}(t), \widetilde{x}(t))| \leq \epsilon_2, \quad t \in (a, T], \end{cases} \tag{6}$$

there exists a unique solution $(y, x) \in E^{\zeta_1} \times E^{\zeta_2}$ of the generalized system (1) - (2) with the same terminal values:

$$y(T) = \widetilde{y}(T), \quad x(T) = \widetilde{x}(T)$$

such that

$$\|(\widetilde{y}, \widetilde{x}) - (y, x)\|_{E^{\zeta_1} \times E^{\zeta_2}} \leq \lambda_{f_1}\epsilon_1 + \lambda_{f_2}\epsilon_2$$

Equivalently, in vector form,

$$\|(\widetilde{y},\widetilde{x})-(y,x)\|_{E^{\zeta_1}\times E^{\zeta_2}}\leq \lambda\cdot\epsilon,$$

where $\lambda\cdot\epsilon=\lambda_{f_1}\epsilon_1+\lambda_{f_2}\epsilon_2$.

The system (1) - (2) is Generalized Ulam–Hyers (G-U-H) stable if there exist continuous functions $\pi_{f_1},\pi_{f_2}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ with $\pi_{f_1}(0)=\pi_{f_2}(0)=0$ such that for every $\epsilon=(\epsilon_1,\epsilon_2)>0$ and for each $(\widetilde{y},\widetilde{x})\in E^{\zeta_1}\times E^{\zeta_2}$ satisfying the perturbed inequalities above, there exists a solution $(y,x)\in E^{\zeta_1}\times E^{\zeta_2}$ of (1) - (2) with $y(T)=\widetilde{y}(T)$, $x(T)=\widetilde{x}(T)$, and

$$\|(\widetilde{y},\widetilde{x})-(y,x)\|_{E^{\zeta_1}\times E^{\zeta_2}}\leq \pi_{f_1}(\epsilon_1)+\pi_{f_2}(\epsilon_2).$$

Remark 3
A function $(\widetilde{y},\widetilde{x})\in E^{\zeta_1}\times E^{\zeta_2}$ is a solution of the perturbed inequalities (6) if and only if there exists a pair of functions $(h_1,h_2)\in E^{\zeta_1}\times E^{\zeta_2}$ (where h_1 depends on the solution $\widetilde{y}$ whereas h_2 depends on the solution $\widetilde{x}$) such that:

1.
$$|h_1(t)|\leq\epsilon_1\quad and\quad |h_2(t)|\leq\epsilon_2\quad \text{for all } t\in(a,T],$$

2. For $t\in(a,T]$,
$$\begin{cases} D_{a^+}^{\Theta_1,\eta_1;\psi}\widetilde{y}(t)=f_1(t,\widetilde{x}(t),\widetilde{y}(t))+h_1(t),\\ D_{a^+}^{\Theta_2,\eta_2;\psi}\widetilde{x}(t)=f_2(t,\widetilde{y}(t),\widetilde{x}(t))+h_2(t).\end{cases}$$

Lemma 2
Let $(\widetilde{y},\widetilde{x})\in E^{\zeta_1}\times E^{\zeta_2}$ be a solution of the perturbed inequalities (6). Then the following fractional integral inequalities hold

$$\begin{cases} |\widetilde{y}(t)-\tau_1(\widetilde{x},\widetilde{y})(t)|\leq \dfrac{2\epsilon_1(\psi(T)-\psi(a))^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)}\,(\psi(t)-\psi(a))^{\zeta_1-1},\\ |\widetilde{x}(t)-\tau_2(\widetilde{y},\widetilde{x})(t)|\leq \dfrac{2\epsilon_2(\psi(T)-\psi(a))^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)}\,(\psi(t)-\psi(a))^{\zeta_2-1},\end{cases}\tag{7}$$

where τ_1 and τ_2 are the operators defined in (5).

Consequently, in the weighted norms of E^{ζ_1} and E^{ζ_2}

$$\begin{cases} \|\widetilde{y}-\tau_1(\widetilde{x},\widetilde{y})\|_{E^{\zeta_1}}\leq \dfrac{2\epsilon_1(\psi(T)-\psi(a))^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)},\\ \|\widetilde{x}-\tau_2(\widetilde{y},\widetilde{x})\|_{E^{\zeta_2}}\leq \dfrac{2\epsilon_2(\psi(T)-\psi(a))^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)}.\end{cases}\tag{8}$$

Proof.
We prove the inequality for $\widetilde{y}$, the proof for $\widetilde{x}$ is completely analogous by symmetry.

• Step 1: Reformulation via perturbation functions

By Remark (3), the inequalities (6) imply the existence of functions h_1,h_2 with:

$$|h_1(t)|\leq\epsilon_1\quad and\quad |h_2(t)|\leq\epsilon_2\quad \text{for all } t\in(a,T],$$

Such that

$$\begin{cases} D_{a^+}^{\Theta_1,\eta_1;\psi}\widetilde{y}(t) = f_1(t,\widetilde{x}(t),\widetilde{y}(t)) + h_1(t), \\ D_{a^+}^{\Theta_2,\eta_2;\psi}\widetilde{x}(t) = f_2(t,\widetilde{y}(t),\widetilde{x}(t)) + h_2(t). \end{cases} \tag{9}$$

This equivalence is standard in Ulam–Hyers stability theory, it converts the inequality constraints into equalities with bounded perturbation terms.

• **Step 2: Apply Theorem 2 to the perturbed equation**

Apply Theorem (2) to the first equation in (9) with:
· $\Theta = \Theta_1$, $\eta = \eta_1$, $\zeta = \zeta_1$
· $\sigma(t) = f_1(t,\widetilde{x}(t),\widetilde{y}(t)) + h_1(t)$
· $w = \widetilde{y}(T)$ (since the approximate solution satisfies the same terminal condition as the exact solution)
Then $\widetilde{y}$ satisfies

$$\widetilde{y}(t) = \frac{\kappa_\psi^{\zeta_1}(t,a)}{\kappa_\psi^{\zeta_1}(T,a)}\left[\widetilde{y}(T) - \frac{1}{\Gamma(\Theta_1)}\int_a^T H_\psi^{\Theta_1}(T,s)\ (f_1(s,\widetilde{x}(s),\widetilde{y}(s)) + h_1(s)) \quad ds\right] + \frac{1}{\Gamma(\Theta_1)}\int_a^t H_\psi^{\Theta_1}(t,s)\ (f_1(s,\widetilde{x}(s),\widetilde{y}(s)) + h_1(s)) \quad ds$$

• **Step 3: Separate the exact operator and the perturbation term**

Recall the definition of τ_1

$$\tau_1(\widetilde{x},\widetilde{y})(t) = \frac{\kappa_\psi^{\zeta_1}(t,a)}{\kappa_\psi^{\zeta_1}(T,a)}\left[\widetilde{y}(T) - \frac{1}{\Gamma(\Theta_1)}\int_a^T H_\psi^{\Theta_1}(T,s)\ f_1(s,\widetilde{x}(s),\widetilde{y}(s)) \quad ds\right] + \frac{1}{\Gamma(\Theta_1)}\int_a^t H_\psi^{\Theta_1}(t,s)\ f_1(s,\widetilde{x}(s),\widetilde{y}(s)) \quad ds.$$

Substituting this into the expression for $\widetilde{y}(t)$, we obtain

$$\widetilde{y}(t) = \tau_1(\widetilde{x},\widetilde{y})(t) + H_1(t),$$

where the term $H_1(t)$ collects all contributions from h_1 :

$$H_1(t) = -\frac{\kappa_\psi^{\zeta_1}(t,a)}{\kappa_\psi^{\zeta_1}(T,a)}.\frac{1}{\Gamma(\Theta_1)}\int_a^T H_\psi^{\Theta_1}(T,s)h_1(s) \quad ds + \frac{1}{\Gamma(\Theta_1)}\int_a^t H_\psi^{\Theta_1}(t,s)\ h_1(s) \quad ds.$$

Thus,

$$\widetilde{y}(t) - \tau_1(\widetilde{x},\widetilde{y})(t) = H_1(t).$$

• **Step 4: Estimate** $|H_1(t)|$

We now bound $|H_1(t)|$ using:
· $H_\psi^{\Theta}(u,s) = \psi'(s)(\psi(u)-\psi(s))^{\Theta-1}$
· $|h_1(s)| \leq \epsilon_1$

$\cdot\ \kappa_{\psi}^{\zeta_1}(t,a) = (\psi(t)-\psi(a))^{\zeta_1-1}$
$\cdot\ \kappa_{\psi}^{\zeta_1}(T,a) = (\psi(T)-\psi(a))^{\zeta_1-1} := \Delta^{\zeta_1-1}$

4.1 Bound the first term of $H_1(t)$.

$$\left| -\frac{\kappa_{\psi}^{\zeta_1}(t,a)}{\kappa_{\psi}^{\zeta_1}(T,a)} \cdot \frac{1}{\Gamma(\Theta_1)} \int_a^T H_{\psi}^{\Theta_1}(T,s)h_1(s) \quad ds \right| \leq \frac{(\psi(t)-\psi(a))^{\zeta_1-1}}{\Delta^{\zeta_1-1}} \cdot \frac{1}{\Gamma(\Theta_1)} \int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta-1}|h_1(s)| \quad ds$$
$$\leq \frac{(\psi(t)-\psi(a))^{\zeta_1-1}}{\Delta^{\zeta_1-1}} \cdot \frac{\epsilon_1}{\Gamma(\Theta_1)} \int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta-1} \quad ds$$

Computing the integral,

$$\int_a^T \psi'(s)(\psi(T)-\psi(s))^{\Theta-1} \quad ds = \left[-\frac{(\psi(T)-\psi(s))^{\Theta_1}}{\Theta_1} \right]_{s=a}^{s=T} = \frac{(\psi(T)-\psi(s))^{\Theta_1}}{\Theta_1} = \frac{\Delta^{\Theta_1}}{\Theta_1}$$

Therefore, the first term $\leq \dfrac{(\psi(t)-\psi(a))^{\zeta_1-1}}{\Delta^{\zeta_1-1}} \cdot \dfrac{\epsilon_1\Delta^{\Theta_1}}{\Gamma(\Theta_1)\Theta_1} = \epsilon_1(\psi(t)-\psi(a))^{\zeta_1-1} \cdot \dfrac{\Delta^{\Theta_1-\zeta_1+1}}{\Gamma(\Theta_1+1)}$.

4.2 Bound the second term of $H_1(t)$.

$$\left| \frac{1}{\Gamma(\Theta_1)} \int_a^t H_{\psi}^{\Theta_1}(t,s)h_1(s) \quad ds \right| \leq \frac{1}{\Gamma(\Theta_1)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\Theta-1}|h_1(s)| \quad ds$$
$$\leq \frac{\epsilon_1}{\Gamma(\Theta_1)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\Theta-1} \quad ds$$

computing the integral

$$\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\Theta-1} \quad ds = \frac{(\psi(t)-\psi(a))^{\Theta_1}}{\Theta_1}$$

Thus,

second term $\leq \epsilon_1 \dfrac{(\psi(t)-\psi(a))^{\Theta_1}}{\Gamma(\Theta_1)\Theta_1} = \epsilon_1 \dfrac{(\psi(t)-\psi(a))^{\Theta_1}}{\Gamma(\Theta_1+1)}$.

4.3 Combine both terms.

$$|H_1(t)| \leq \epsilon_1 \left[(\psi(t)-\psi(a))^{\zeta_1-1} \cdot \frac{\Delta^{\Theta_1-\zeta_1+1}}{\Gamma(\Theta_1+1)} + \frac{1}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\Theta_1} \right]$$

To obtain a uniform bound in the weighted space E^{ζ_1}, we must express both terms with the same factor $(\psi(t)-\psi(a))^{\zeta_1-1}$.

Use the inequality: for any $t \in (a,T]$,

$$(\psi(t)-\psi(a))^{\Theta_1} \leq (\psi(t)-\psi(a))^{\zeta_1-1}\Delta^{\Theta_1-\zeta_1+1}$$

applying this to the second term:

$$\epsilon_1 \cdot \frac{(\psi(t)-\psi(a))^{\Theta_1}}{\Gamma(\Theta_1+1)} \leq \epsilon_1 \cdot \frac{\Delta^{\Theta_1-\zeta_1+1}}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\zeta_1-1}$$

now both terms share the common factor $(\psi(t)-\psi(a))^{\zeta_1-1}$, summing them:

$$|H_1(t)| \leq \epsilon_1 \cdot \frac{\Delta^{\Theta_1-\zeta_1+1}}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\zeta_1-1} + \epsilon_1 \cdot \frac{\Delta^{\Theta_1-\zeta_1+1}}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\zeta_1-1},$$

hence,

$$|H_1(t)| \leq \frac{2\epsilon_1\Delta^{\Theta_1-\zeta_1+1}}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\zeta_1-1}.$$

Since $\widetilde{y}(t)-\tau_1(\widetilde{x},\widetilde{y})(t) = H_1(t)$, we have proved

$$|\widetilde{y}(t)-\tau_1(\widetilde{x},\widetilde{y})(t)| \leq \frac{2\epsilon_1\Delta^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\zeta_1-1}.$$

• **Step 5: Weighted norm estimate**.

The norm on $E^{\zeta_1} := C^{\zeta_1}_{1-\zeta_1,\psi}[a,T]$ is

$$\|u\|_{E^{\zeta_1}} = \sup_{t\in(a,T]} |(\psi(t)-\psi(a))^{1-\zeta_1}u(t)|,$$

apply this to $u(t) = \widetilde{y}(t)-\tau_1(\widetilde{x},\widetilde{y})(t)$

$$\begin{aligned}
\|\widetilde{y}-\tau_1(\widetilde{x},\widetilde{y})\|_{E^{\zeta_1}} &= \sup_{t\in(a,T]} |(\psi(t)-\psi(a))^{1-\zeta_1}(\widetilde{y}(t)-\tau_1(\widetilde{x},\widetilde{y})(t))| \\
&= \sup_{t\in(a,T]} |(\psi(t)-\psi(a))^{1-\zeta_1}|\,|\widetilde{y}(t)-\tau_1(\widetilde{x},\widetilde{y})(t)| \\
&\leq \sup_{t\in(a,T]} (\psi(t)-\psi(a))^{1-\zeta_1}\frac{2\epsilon_1\Delta^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)}(\psi(t)-\psi(a))^{\zeta_1-1} \\
&= \frac{2\epsilon_1\Delta^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)}
\end{aligned}$$

• **Step 6: Proof for $\widetilde{x}$ (symmetry)** .

The proof for $\widetilde{x}$ is identical by swapping indices: $\Theta_1 \to \Theta_2, \zeta_1 \to \zeta_2, \epsilon_1 \to \epsilon_2$ *and* $\tau_1 \to \tau_2$. The perturbation term $H_2(t)$ is estimated analogously, yielding

$$|\widetilde{x}(t)-\tau_2(\widetilde{y},\widetilde{x})(t)| \leq \frac{2\epsilon_2\Delta^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)}(\psi(t)-\psi(a))^{\zeta_2-1},$$

and consequently,

$$\|\widetilde{x}-\tau_2(\widetilde{y},\widetilde{x})\|_{E^{\zeta_2}} \leq \frac{2\epsilon_2\Delta^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)}.$$

Theorem 4 (Ulam–Hyers stability for generalized system)
Let the assumptions of Theorem 3 hold, and let the spectral radius $\rho(\Lambda) < 1$. Then the generalized ψ-Hilfer coupled system (1) - (2) is Ulam–Hyers (U-H) stable and generalized Ulam–Hyers (G-U-H) stable in the product space $E^{\zeta_1} \times E^{\zeta_2}$.

Proof.
• **Step 1: Setup – Exact and approximate solutions**.

Let $(\widetilde{y},\widetilde{x}) \in E^{\zeta_1} \times E^{\zeta_2}$ be an approximate solution satisfying the perturbed inequalities (6). Let $(y,x) \in E^{\zeta_1} \times E^{\zeta_2}$ be the unique exact solution of (1) - (2) with the same terminal values:

$$y(T) = \widetilde{y}(T) = w_1, \qquad x(T) = \widetilde{x}(T) = w_2.$$

This exact solution exists and is unique by Theorem (3).

• **Step 2: Apply Lemma 2.**

$$\begin{cases} \|\widetilde{y} - \tau_1(\widetilde{x}, \widetilde{y})\|_{E^{\zeta_1}} \leq k_1\epsilon_1, \\ \|\widetilde{x} - \tau_2(\widetilde{y}, \widetilde{x})\|_{E^{\zeta_2}} \leq k_2\epsilon_2, \end{cases} \tag{10}$$

where

$$k_1 := \frac{2\Delta^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)}, \qquad k_2 := \frac{2\Delta^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)}$$

• **Step 3: Subtract the exact operator**.

The exact solution satisfies, by definition of the fixed point,

$$y = \tau_1(x, y), \quad x = \tau_2(y, x)$$

therefore, for the first component

$$\widetilde{y} - y = (\widetilde{y} - \tau_1(\widetilde{x}, \widetilde{y})) + (\tau_1(\widetilde{x}, \widetilde{y}) - \tau_1(x, y))$$

taking the E^{ζ_1}-norm and using the triangle inequality

$$\|\widetilde{y} - y)\|_{E^{\zeta_1}} \leq \|\widetilde{y} - \tau_1(\widetilde{x}, \widetilde{y})\|_{E^{\zeta_1}} + \|\tau_1(\widetilde{x}, \widetilde{y}) - \tau_1(x, y)\|_{E^{\zeta_1}}$$

similarly for the second component

$$\|\widetilde{x} - x)\|_{E^{\zeta_2}} \leq \|\widetilde{x} - \tau_2(\widetilde{y}, \widetilde{x})\|_{E^{\zeta_2}} + \|\tau_2(\widetilde{y}, \widetilde{x}) - \tau_2(y, x)\|_{E^{\zeta_2}}$$

• **Step 4: Apply the contraction estimates**.

From the proof of Theorem (3)(contraction property), we have the key estimates:

$$\begin{cases} \|\tau_1(\widetilde{x}, \widetilde{y}) - \tau_1(x, y)\|_{E^{\zeta_1}} \leq L_{11}M_{12}\|\widetilde{x} - x\|_{E^{\zeta_2}} + L_{12}M_{11}\|\widetilde{y} - y\|_{E^{\zeta_1}}, \\ \|\tau_2(\widetilde{y}, \widetilde{x}) - \tau_2(y, x)\|_{E^{\zeta_2}} \leq L_{21}M_{21}\|\widetilde{y} - y\|_{E^{\zeta_1}} + L_{22}M_{22}\|\widetilde{x} - x\|_{E^{\zeta_2}}. \end{cases} \tag{11}$$

• **Step 5: Form the coupled system of norm inequalities**.

Let

$$U := \|\widetilde{y} - y\|_{E^{\zeta_1}}, \quad V := \|\widetilde{x} - x\|_{E^{\zeta_2}}$$

substituting (10) and (11) into the inequalities from Step 3, we obtain

$$\begin{cases} U \leq k_1\epsilon_1 + L_{11}M_{12}V + L_{12}M_{11}U, \\ V \leq k_2\epsilon_2 + L_{21}M_{21}U + L_{22}M_{22}V. \end{cases} \tag{12}$$

• **Step 6: Rearrange and solve the system**.

Rewrite (12) as

$$\begin{cases} (1 - L_{12}M_{11})U - L_{11}M_{12}V \leq k_1\epsilon_1, \\ -L_{21}M_{21}U + (1 - L_{22}M_{22})V \leq k_2\epsilon_2. \end{cases} \tag{13}$$

this is a linear system in U, V. In matrix form

$$(I - \Lambda)\begin{pmatrix} U \\ V \end{pmatrix} \leq \begin{pmatrix} k_1\epsilon_1 \\ k_2\epsilon_2 \end{pmatrix} \tag{14}$$

where $\Lambda = \begin{pmatrix} L_{12}M_{11} & L_{11}M_{12} \\ L_{21}M_{21} & L_{22}M_{22} \end{pmatrix}$ and I is the 2×2 identity matrix.

• **Step 7: The key condition: invertibility of** $I - \Lambda$.

Since $\rho(\Lambda) < 1$, the matrix $I - \Lambda$ is invertible and its inverse is given by the Neumann series

$$(I - \Lambda)^{-1} = \sum_{k=0}^{\infty} \Lambda^k,$$

and this series converges. Moreover, $(I - \Lambda)^{-1}$is a nonnegative matrix (since all entries of Λ are nonnegative), let

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} := (I - \Lambda)^{-1}.$$

Explicitly, for a 2×2 matrix

$$(I - \Lambda)^{-1} = \frac{1}{det(I - \Lambda)}\begin{pmatrix} 1 - L_{22}M_{22} & L_{11}M_{12} \\ L_{21}M_{21} & 1 - L_{12}M_{11} \end{pmatrix}.$$

• **Step 8: Solve for U and V**.

Multiplying both sides of (14) by $(I - \Lambda)^{-1}$, we obtain

$$\begin{cases} U \leq \alpha_{11}(k_1\epsilon_1) + \alpha_{12}(k_2\epsilon_2), \\ V \leq \alpha_{21}(k_1\epsilon_1) + \alpha_{22}(k_2\epsilon_2). \end{cases} \tag{15}$$

• **Step 9: Define the Ulam–Hyers stability constants**

From (15), we have

$$U \leq \alpha_{11}k_1\epsilon_1 + \alpha_{12}k_2\epsilon_2, \qquad V \leq \alpha_{21}k_1\epsilon_1 + \alpha_{22}k_2\epsilon_2.$$

then

$$U := \|\widetilde{y} - y\|_{E^{\varsigma_1}} \leq \frac{1}{det(I - \Lambda)}\left((1 - L_{22}M_{22})k_1\epsilon_1 + L_{11}M_{12}k_2\epsilon_2\right)$$

$$V := \|\widetilde{x} - x\|_{E^{\varsigma_2}} \leq \frac{1}{det(I - \Lambda)}\left(L_{21}M_{21}k_1\epsilon_1 + (1 - L_{12}M_{11})k_2\epsilon_2\right).$$

We now define the Ulam–Hyers stability constants λ_{f_1} and λ_{f_2} such that

$$\|(\widetilde{y},\widetilde{x})-(y,x)\|_{E^{\zeta_1}\times E^{\zeta_2}} \leq \lambda_{f_1}\epsilon_1+\lambda_{f_2}\epsilon_2$$

from the bounds on U and V, we have
$\max\{U,V\} \leq \dfrac{1}{det(I-\Lambda)}\max\{(1-L_{22}M_{22})k_1\epsilon_1+L_{11}M_{12}k_2\epsilon_2, L_{21}M_{21}k_1\epsilon_1+(1-L_{12}M_{11})k_2\epsilon_2\}$
hence,
$\max\{U,V\} \leq \dfrac{1}{det(I-\Lambda)}\left(\max\{(1-L_{22}M_{22})k_1, L_{21}M_{21}k_1\}\epsilon_1+\max\{L_{11}M_{12}k_2,(1-L_{12}M_{11})k_2\}\epsilon_2\right)$,
thus we define
$\lambda_{f_1} := \dfrac{1}{det(I-\Lambda)}\max\{(1-L_{22}M_{22})k_1, L_{21}M_{21}k_1\}, \lambda_{f_2} := \dfrac{1}{det(I-\Lambda)}\max\{L_{11}M_{12}k_2,(1-L_{12}M_{11})k_2\}$
so,
$\max\{U,V\} \leq \lambda_{f_1}\epsilon_1+\lambda_{f_2}\epsilon_2$, and we have

$$\|(\widetilde{y},\widetilde{x})-(y,x)\|_{E^{\zeta_1}\times E^{\zeta_2}} = \max\{U,V\} \leq \lambda_{f_1}\epsilon_1+\lambda_{f_2}\epsilon_2.$$

proving that the generalized system is Ulam–Hyers stable.

• **Step 10: Generalized Ulam–Hyers (G-U-H) stability**

If we take continuous functions $\pi_{f_1},\pi_{f_2}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ with $\pi_{f_1}(0)=\pi_{f_2}(0)=0$, for example,

$$\pi_{f_1}(\epsilon_1)=\lambda_{f_1}\epsilon_1, \quad \pi_{f_2}(\epsilon_2)=\lambda_{f_2}\epsilon_2.$$

then the same proof yields

$$\|(\widetilde{y},\widetilde{x})-(y,x)\|_{E^{\zeta_1}\times E^{\zeta_2}} \leq \pi_{f_1}(\epsilon_1)+\pi_{f_2}(\epsilon_2)$$

thus, the system is G-U-H stable.

4.1. Continuous dependence on the terminal values

A standard requirement in the qualitative theory of differential equations is that solutions depend continuously on the data of the problem. Since the system studied here is a terminal value problem, the relevant data are the terminal values $w=(w_1,w_2)$ rather than initial values at $t=a$. We establish continuous dependence with respect to these terminal values.

Theorem 5 (Continuous dependence on terminal values)
Let the assumptions of Theorem 3 hold, and let $\rho(\Lambda)<1$. For $w=(w_1,w_2)$ and $\widetilde{w}=(\widetilde{w}_1,\widetilde{w}_2)$ in $\mathbb{R}^2$, let (y,x) and $(\widetilde{y},\widetilde{x})$ denote the unique solutions of the generalized ψ-Hilfer coupled terminal value problem (1)-(2) corresponding to w and $\widetilde{w}$, respectively, with the same nonlinearities f_1, f_2. Then

$$\|(y-\widetilde{y},\, x-\widetilde{x})\|_{E^{\zeta_1}\times E^{\zeta_2}} \;\leq\; C_w\,\|w-\widetilde{w}\|_{\infty},$$

where $\|w-\widetilde{w}\|_{\infty}=\max\{|w_1-\widetilde{w}_1|,\,|w_2-\widetilde{w}_2|\}$ and

$$C_w=\|(I-\Lambda)^{-1}\|_{\infty}\,\max\{\Delta^{1-\zeta_1},\,\Delta^{1-\zeta_2}\}.$$

In particular, if $\widetilde{w}\to w$, then $(\widetilde{y},\widetilde{x})\to(y,x)$ in $E^{\zeta_1}\times E^{\zeta_2}$.

Proof.
Subtracting the integral equations for (y, x) and $(\widetilde{y}, \widetilde{x})$ and using the notation of Theorem 3, we obtain

$$\begin{cases} U \le \Delta^{1-\zeta_1}|w_1 - \widetilde{w}_1| + L_{11}M_{12}V + L_{12}M_{11}U, \\ V \le \Delta^{1-\zeta_2}|w_2 - \widetilde{w}_2| + L_{21}M_{21}U + L_{22}M_{22}V, \end{cases}$$

where $U := \|y - \widetilde{y}\|_{E^{\zeta_1}}$ and $V := \|x - \widetilde{x}\|_{E^{\zeta_2}}$. In matrix form,

$$(I - \Lambda)\begin{pmatrix} U \\ V \end{pmatrix} \le \begin{pmatrix} \Delta^{1-\zeta_1}|w_1 - \widetilde{w}_1| \\ \Delta^{1-\zeta_2}|w_2 - \widetilde{w}_2| \end{pmatrix}.$$

Since $\rho(\Lambda) < 1$, the matrix $I - \Lambda$ is invertible and $(I - \Lambda)^{-1}$ is entrywise nonnegative. Therefore

$$\begin{pmatrix} U \\ V \end{pmatrix} \le (I - \Lambda)^{-1}\begin{pmatrix} \Delta^{1-\zeta_1}|w_1 - \widetilde{w}_1| \\ \Delta^{1-\zeta_2}|w_2 - \widetilde{w}_2| \end{pmatrix},$$

and taking the maximum of the two components gives the asserted estimate.

Remark 4 (Role of the terminal condition)
Since the problem studied here is a terminal value problem, the natural data are $w_1 = y(T)$ and $w_2 = x(T)$, not initial values at $t = a$. Theorem 5 therefore establishes continuous dependence with respect to the terminal data that are actually prescribed in this setting. This addresses the standard requirement in the qualitative theory of differential equations in the context of terminal value problems.

5. Examples

We present two examples for our generalized system

Example1

$$\begin{cases} D_{a^+}^{0.5,0.5;\psi} y(t) = 0.2x + 0.1y, \quad t \in (0.1, 1], \ \ a > 0, \\ D_{a^+}^{0.6,0.4,;\psi} x(t) = 0.15y + 0.25x, \ \ t \in (0.1, 1], \ \ a > 0. \end{cases}$$

with the terminal conditions

$$\begin{cases} y(1) = 1, \\ x(1) = 2 \end{cases}$$

1. Choice of parameters.
We take
• $a = 0.1, T = 1, \psi(t) = t$, so $\Delta = \psi(1) - \psi(0.1) = 1 - 0.1 = 0.9$.
• Fractional orders.
$\Theta_1 = 0.5, \eta_1 = 0.5 \Rightarrow \zeta_1 = \Theta_1 + \eta_1(1 - \Theta_1) = 0.75$.
$\Theta_2 = 0.6, \eta_2 = 0.4 \Rightarrow \zeta_2 = \Theta_2 + \eta_2(1 - \Theta_2) = 0.76$.
• Terminal values: $w_1 = 1, w_2 = 2$.
and

$$f_1(t, x(t), y(t)) = 0.2x + 0.1y, \quad f_2(t, y(t), x(t)) = 0.15y + 0.25x$$

These functions satisfy the Lipschitz conditions with
$L_{11} = 0.2, L_{12} = 0.1, L_{21} = 0.15, L_{22} = 0.25.$

2. Computation of the contraction matrix Λ.

$$M_{11} = 2\Delta^{\Theta_1} \cdot \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_1)}, \qquad M_{12} = 2\Delta^{\Theta_1+\zeta_2-\zeta_1} \cdot \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_1)},$$
$$M_{21} = 2\Delta^{\Theta_2+\zeta_1-\zeta_2} \cdot \frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_2)}, \qquad M_{22} = 2\Delta^{\Theta_2} \cdot \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_2)}.$$

And

$\Delta^{\Theta_1} = 0.9^{0.5} \approx 0.9487$
$\Delta^{\Theta_1+\zeta_2-\zeta_1} = 0.9^{0.5+0.76-0.75} \approx 0.9477$
$\Delta^{\Theta_2+\zeta_1-\zeta_2} = 0.9^{0.6+0.75-0.76} \approx 0.9397$
$\Delta^{\Theta_2} = 0.9^{0.6} \approx 0.9387$

The Gamma ratios are

$$\frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_1)} = \frac{\Gamma(0.75)}{\Gamma(1.25)} \approx 1.352, \qquad \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_1)} = \frac{\Gamma(0.76)}{\Gamma(0.76+0.5)} \approx 1.340$$
$$\frac{\Gamma(\zeta_1)}{\Gamma(\zeta_1+\Theta_2)} = \frac{\Gamma(0.75)}{\Gamma(0.75+0.6)} \approx 1.375, \qquad \frac{\Gamma(\zeta_2)}{\Gamma(\zeta_2+\Theta_2)} = \frac{\Gamma(0.76)}{\Gamma(0.76+0.6)} \approx 1.362.$$

Thus

$$M_{11} = 2 \times 0.9487 \times 1.352 \approx 2.565,$$
$$M_{12} = 2 \times 0.9477 \times 1.340 \approx 2.540,$$
$$M_{21} = 2 \times 0.9397 \times 1.375 \approx 2.584,$$
$$M_{22} = 2 \times 0.9387 \times 1.362 \approx 2.557.$$

The contraction matrix Λ is

$$\Lambda = \begin{pmatrix} L_{12}M_{11} & L_{11}M_{12} \\ L_{21}M_{21} & L_{22}M_{22} \end{pmatrix} = \begin{pmatrix} 0.1 \times 2.565 & 0.2 \times 2.540 \\ 0.15 \times 2.584 & 0.25 \times 2.557 \end{pmatrix} = \begin{pmatrix} 0.2565 & 0.508 \\ 0.3876 & 0.63925 \end{pmatrix}$$

3. Spectral radius check.
$tr\Lambda = 0.2565 + 0.63925 = 0.89575, \quad det\Lambda = 0.2565 \times 0.63925 - 0.508 \times 0.3876 = -0.0329.$
Eigenvalues:

$$\lambda_{1,2} = \frac{0.89575 \pm \sqrt{(0.89575)^2 - 4(-0.0329)}}{2}$$
$$= \frac{0.89575 \pm \sqrt{0.8024 - 4(-0.0329)}}{2}$$
$$= \frac{0.89575 \pm \sqrt{0.8024 + 0.1316}}{2}$$
$$= \frac{0.89575 \pm \sqrt{0.934}}{2}$$

Thus
$\lambda_1 \approx 0.93109, \quad \lambda_2 \approx -0.03534.$

The spectral Radius is $\approx 0.93109 < 1$. Hence, Theorem 3 guarantees existence and uniqueness of a solution in $E^{0.75} \times E^{0.76}$.

4. Ulam–Hyers stability constants.
Computing

$$k_1 = \frac{2\Delta^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)} = \frac{2\times 0.9^{0.5+1-0.75}}{\Gamma(1.5)} = \frac{2\times 0.9^{0.75}}{0.8862269} = \frac{2\times 0.9240}{0.8862269} \approx 2.085.$$

$$k_2 = \frac{2\Delta^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)} = \frac{2\times 0.9^{0.6+1-0.76}}{\Gamma(1.6)} = \frac{2\times 0.9^{0.84}}{0.893515} = \frac{2\times 0.9153}{0.893515} \approx 2.049.$$

Now,

$$(I-\Lambda) = \begin{pmatrix} 1-0.2565 & -0.508 \\ -0.3876 & 1-0.63925 \end{pmatrix} = \begin{pmatrix} 0.7435 & -0.508 \\ -0.3876 & 0.36075 \end{pmatrix}$$

$det(I-\Lambda) = 0.7435\times 0.36075 - 0.508\times 0.3876 = 0.0713$

The stability constants are

$$\lambda_{f_1} = \frac{k_1}{det(I-\Lambda)}\max\{1-\Lambda_{22},\Lambda_{21}\} = \frac{2.085}{0.0713}\max\{1-0.63925, 0.3876\} = 29.24\max\{0.36075, 0.3876\}$$
$$= 29.24\times 0.3876 \approx 11.33,$$

$$\lambda_{f_2} = \frac{k_2}{det(I-\Lambda)}\max\{\Lambda_{12},1-\Lambda_{11}\} = \frac{2.049}{0.0713}\max\{0.508, 1-0.2565\} = \frac{2.049}{0.0713}\max\{0.508, 0.7435\}$$
$$= 28.74\times 0.7435 \approx 21.37.$$

Thus, for any $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and for every $(\widetilde{y}, \widetilde{x})$ satisfying the perturbed inequalities, there exists an exact solution (y, x) of the generalized system with the same terminal values suth that

$$\|(\widetilde{y}, \widetilde{x}) - (y, x)\|_{E^{0.75}\times E^{0.76}} \leq 11.33\ \epsilon_1 + 21.37\ \epsilon_2.,$$

This establishes Ulam–Hyers stability (and, with $\pi_{f_i}(\epsilon_i) = \lambda_{f_i}\epsilon_i, generalized\ Ulam–Hyers\ stability$).

Example2 (Nonlinear Numerical Example with $\psi(t) = t^2$)

To demonstrate that the proposed framework is applicable to genuinely nonlinear problems and to a non-trivial kernel function, we consider the following nonlinear coupled system:

$$\begin{cases} D_{a^+}^{\Theta_1,\eta_1;\psi} y(t) = f_1(t, x(t), y(t)), & a < t \leq T, \\ D_{a^+}^{\Theta_2,\eta_2;\psi} x(t) = f_2(t, y(t), x(t)), & a < t \leq T, \end{cases} \tag{16}$$

with terminal conditions

$$y(T) = w_1, \qquad x(T) = w_2.$$

Choice of parameters. We take

$$a = 0.1, \quad T = 1, \quad \psi(t) = t^2, \quad \Delta = \psi(1) - \psi(0.1) = 1 - 0.01 = 0.99.$$

The fractional orders and types are chosen as

$$\Theta_1 = 0.5,\ \eta_1 = 0.5\ \Rightarrow\ \zeta_1 = 0.75, \qquad \Theta_2 = 0.6,\ \eta_2 = 0.4\ \Rightarrow\ \zeta_2 = 0.76.$$

The terminal values are $w_1 = 1$ and $w_2 = 2$. The nonlinearities are

$$f_1(t, x, y) = 0.1\sin(x) + 0.05\frac{y}{1+y^2} + 0.02t,$$
$$f_2(t, y, x) = 0.08y + 0.06\cos(x) + 0.01t^2.$$

These functions satisfy the Lipschitz condition (H2) with

$$L_{11} = 0.1, \quad L_{12} = 0.05, \quad L_{21} = 0.08, \quad L_{22} = 0.06.$$

Indeed, $|\sin(x_1) - \sin(x_2)| \leq |x_1 - x_2|$, and the derivative of $y/(1+y^2)$ is bounded by 1, so the Lipschitz constants are as stated.

Contraction matrix. Using the formulas from Theorem 3 with $\psi(t) = t^2$, we compute

$$M_{11} \approx 2.690, \quad M_{12} \approx 2.666, \quad M_{21} \approx 2.734, \quad M_{22} \approx 2.714.$$

Thus the contraction matrix is

$$\Lambda = \begin{pmatrix} L_{12}M_{11} & L_{11}M_{12} \\ L_{21}M_{21} & L_{22}M_{22} \end{pmatrix} = \begin{pmatrix} 0.1345 & 0.2666 \\ 0.2187 & 0.1628 \end{pmatrix}.$$

Its eigenvalues are

$$\lambda_1 \approx 0.3905, \qquad \lambda_2 \approx -0.0932,$$

so the spectral radius is

$$\rho(\Lambda) \approx 0.3905 < 1.$$

By Theorem 3, the nonlinear system (16) has a unique solution in $E^{0.75} \times E^{0.76}$.

Ulam–Hyers stability constants. Following the proof of Theorem 4, we compute

$$k_1 = \frac{2\Delta^{\Theta_1+1-\zeta_1}}{\Gamma(\Theta_1+1)} \approx \frac{2(0.99)^{0.75}}{\Gamma(1.5)} \approx 2.240, \qquad k_2 = \frac{2\Delta^{\Theta_2+1-\zeta_2}}{\Gamma(\Theta_2+1)} \approx \frac{2(0.99)^{0.84}}{\Gamma(1.6)} \approx 2.219.$$

Since

$$\det(I - \Lambda) \approx 0.6663,$$

the stability constants are

$$\lambda_{f_1} = \frac{k_1}{\det(I-\Lambda)}\max\{1-\Lambda_{22}, \Lambda_{21}\} \approx 2.814, \qquad \lambda_{f_2} = \frac{k_2}{\det(I-\Lambda)}\max\{\Lambda_{12}, 1-\Lambda_{11}\} \approx 2.882$$

Therefore, for any $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and every $(\widetilde{y}, \widetilde{x})$ satisfying the perturbed inequalities, there exists the exact solution (y, x) with the same terminal values such that

$$\|(\widetilde{y}, \widetilde{x}) - (y, x)\|_{E^{0.75}\times E^{0.76}} \leq 2.814\,\epsilon_1 + 2.882\,\epsilon_2.$$

This confirms both Ulam–Hyers and generalized Ulam–Hyers stability for the nonlinear system. The use of the non-trivial kernel $\psi(t) = t^2$ demonstrates the flexibility and advantage of the ψ-Hilfer framework over classical derivatives.

6. Conclusion

In this paper, we studied a generalized coupled system of ψ-Hilfer fractional differential equations with terminal conditions, in which each equation depends on both unknown functions. We established existence and uniqueness of solutions via Banach's fixed-point theorem under a spectral radius condition on the associated contraction matrix, proved that uniqueness is global, and derived explicit Ulam-Hyers and generalized Ulam-Hyers stability constants. We also established continuous dependence of the solution on the terminal values. Two numerical examples, one linear with $\psi(t) = t$ and one nonlinear with $\psi(t) = t^2$, verified the theoretical results and demonstrated the flexibility of the framework. Future work includes extensions to nonlocal boundary conditions and applications to specific models in biology and physics.

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