STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–21.
Published online in International Academic Press (www.IAPress.org)

A Unified Study of Bi-Univalent Functions Associated with Sheffer Generating Functions

Khalifa Al Shaqsi[1,*], Abdullah Alsoboh[2]

[1]*Mathematical and Physical Sciences, College of Arts and Sciences, University of Nizwa, Nizwa, Oman*
[2]*Department of Basic and Applied Sciences, College of Applied and Health Sciences, A'Sharqiyah University, Ibra, Oman*

Abstract Recent activity in geometric function theory has produced a long sequence of papers in which a single named polynomial family — Bernoulli, Euler, Hermite, Daehee, and others — is used as the subordinant defining a subclass of bi-univalent functions, after which the initial coefficients and the Fekete–Szegő functional are estimated. Each such family is a member of one and the same algebraic structure: the Sheffer sequences of umbral calculus, generated by $A(t)e^{xH(t)}$. We propose to work at the level of that structure rather than at the level of its individual members. Fixing an arbitrary Sheffer pair (A, H) and a shift parameter x, we introduce the *Sheffer subordinant* $\Phi_{A,H,x}(z) = A(z)e^{xH(z)}$, isolate the *admissibility* hypothesis under which it satisfies the Ma–Minda conditions, and define a one-parameter family $\mathcal{M}_\Sigma(\lambda; A, H, x)$ of bi-univalent functions interpolating between Sheffer-starlike ($\lambda = 0$) and Sheffer-convex ($\lambda = 1$). We prove a single pair of theorems — initial coefficient bounds and a Fekete–Szegő inequality — expressed solely through the first two Taylor coefficients C_1, C_2 of $\Phi_{A,H,x}$. Every previously published result of this type for an Appell family is recovered by substitution. Because the framework admits $H(t) \neq t$, it also produces genuinely new classes unreachable by the one-polynomial-at-a-time method; the Bell–Touchard class $H(t) = e^t - 1$ is developed as the principal worked example. Admissibility is not treated as a formality: we prove an analytic sufficient criterion, based on a sector estimate for the composition $e^{xH} = G_x \circ H$, which certifies the Bell–Touchard window $x \in (0, 1.73]$ by proof, the exact window being $x \in (0, 1.87]$; the Bernoulli case is included as a consistency check against the literature. Numerical illustrations and figures accompany each result.

Keywords Univalent functions; bi-univalent functions; Sheffer sequences; umbral calculus; subordination; Ma–Minda classes; Fekete–Szegő functional; Bell–Touchard polynomials; Bernoulli polynomials; admissibility

AMS 2010 subject classifications Primary 30C45; Secondary 30C50, 05A40, 11B83

DOI: 10.19139/soic-2310-5070-4364

1. Introduction and motivation

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ be the subclass of univalent functions. By the Koebe one-quarter theorem every $f \in \mathcal{S}$ has an inverse defined at least on $|w| < 1/4$, with expansion

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

*Correspondence to: Khalifa Al Shaqsi (Email: khalifa.alshaqsi@unizwa.edu.om). DeMathematical and Physical Sciences, College of Arts and Sciences, University of Nizwa, Nizwa, Oman.

ISSN 2310-5070 (online) ISSN 2311-004X (print)
Copyright © 202x International Academic Press

A function $f \in \mathcal{A}$ is *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent there; the class of such functions is denoted Σ. Since Lewin's 1967 estimate $|a_2| < 1.51$ the determination of sharp coefficient bounds for Σ has remained open, and the literature has instead concentrated on tractable subclasses defined by subordination.

1.1. The state of the art, and the gap The now-standard recipe is the following. Choose a classical polynomial family $\{p_n(x)\}$ with generating function $G(x,t) = \sum_n p_n(x)t^n$; declare $G(x,\cdot)$ to be the subordinant; define

$$\frac{zf'(z)}{f(z)} \prec G(x,z), \qquad \frac{wg'(w)}{g(w)} \prec G(x,w);$$

and compute $|a_2|$, $|a_3|$ and $|a_3 - \mu a_2^2|$. This template has been executed with Chebyshev polynomials [19], (p,q)-Lucas polynomials [5], Gegenbauer, Legendre, Horadam, Fibonacci, Daehee, q-analogues, and — of direct relevance here — with Bernoulli polynomials [10, 9]; together these account for a substantial share of recent output in the area.

What is striking is that several of the families in that list — Bernoulli, Euler, Hermite, Daehee, Bell–Touchard, Charlier, Laguerre, Abel — are not independent objects at all. They are individual members of a single algebraic family, the *Sheffer sequences* of Rota's umbral calculus, whose defining feature is the exponential generating function

$$A(t)\, e^{xH(t)} = \sum_{n=0}^{\infty} s_n(x)\, \frac{t^n}{n!}, \qquad A(0) \neq 0,\ H(0) = 0,\ H'(0) \neq 0. \tag{3}$$

The literature has been picking members off this shelf one at a time, proving a structurally identical theorem on each occasion. The obvious economy — and the purpose of the present paper — is to prove the theorem *once*, for the generating function (3) itself, and to obtain the individual cases as substitutions.

Why Sheffer, and not some other unifying structure? Riordan arrays are the natural competitor: they are more general, and they too subsume many named families. But the Riordan generating function $g(t)\, f(t)^k$ is *ordinary*, whereas the subordination problem is governed by the first two Taylor coefficients of the subordinant at the origin, and it is the *exponential* form (3) that makes those coefficients — see (6) — separate cleanly into an A-part and an H-part. Concretely, the Sheffer structure buys three things that a Riordan formulation does not: the Carathéodory normalization $\Phi(0) = 1$ is exactly the condition $A(0) = 1$; the Appell/binomial-type dichotomy ($H = t$ versus $A = 1$) corresponds precisely to the split between the classes already treated in the literature and those not yet reachable; and, as Section 4 shows, the composition structure $e^{xH} = G_x \circ H$ available when $A \equiv 1$ delivers an analytic admissibility criterion. A Riordan-level treatment would gain generality at the cost of all three. We regard the choice as a deliberate trade, not an accident.

1.2. What this paper does We introduce the Sheffer subordinant $\Phi_{A,H,x}(z) = A(z)e^{xH(z)}$ (Section 3), identify precisely the hypothesis — which we call *admissibility* — needed for it to be a legitimate Ma–Minda function, and define the class $\mathcal{M}_\Sigma(\lambda; A, H, x)$. Theorems 5.2 and 5.3 give $|a_2|$, $|a_3|$ and the full Fekete–Szegő inequality in terms of the two numbers

$$C_1 = \alpha_1 + xh_1, \qquad C_2 = \alpha_2 + \alpha_1 x h_1 + x h_2 + \tfrac{1}{2}x^2 h_1^2,$$

where $A(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots$ and $H(t) = h_1 t + h_2 t^2 + \cdots$.

Three points deserve emphasis.

The first is *unification*. Setting $H(t) = t$ collapses (3) to the Appell case and recovers, as corollaries, the published Bernoulli, Euler and Hermite results (Section 5, Table 4). A single pair of theorems therefore absorbs an entire family of separately published papers.

The second is *genuine novelty*. The recipe described above is, in effect, confined to Appell sequences. Allowing $H(t) \neq t$ opens the binomial-type and general Sheffer regimes, which the one-polynomial method cannot reach. Our principal example, Bell–Touchard with $H(t) = e^t - 1$, is of this kind, and is developed in full in Section 7.

The third is that *admissibility is not a formality*. A generating function is not automatically a Ma–Minda function, and several papers in this area do not verify that it is. We show in Section 4 (Table 2) that admissibility genuinely restricts x: Bell–Touchard is admissible exactly for $0 < x \lesssim 1.87$, Hermite only for $1.00 \lesssim x \lesssim 2.71$, and the

Genocchi generating function is *never* admissible without renormalization, since $A(0) = 0$. Establishing these windows is part of the mathematical content of the framework, not a preliminary to it.

To make the contribution precise, we list the specific advances of this paper.

(C1) A single subordinant $\Phi_{A,H,x}$ and a single pair of theorems (Theorems 5.2, 5.3) that yield the coefficient bounds and the Fekete–Szegő inequality for every admissible Sheffer pair at once, expressed through only two numbers C_1, C_2.

(C2) The recovery of the published Bernoulli and Euler bi-univalent results as *exact* specializations, made explicit as a term-by-term mapping in Table 5 and numerically in Table 3.

(C3) Two genuinely new families that the one-polynomial method cannot reach — the Bell–Touchard class (Section 7) and the binomial/falling-factorial class (Section 8) — each worked in full.

(C4) A proved analytic admissibility criterion (Proposition 4.1) via a sector estimate for the composition $e^{xH} = G_x \circ H$, converting what is elsewhere an unverified assumption into a theorem.

(C5) A cross-family structural result (Proposition 8.1): both binomial-type families share the single degeneracy locus $x^* = (1+\lambda)/(1-\lambda)$ — new information that only the unified viewpoint makes visible.

(C6) A precise delimitation of the framework's boundary (Section 9.1, Table 9): which families are inside, which are outside, and why.

2. Preliminaries

2.1. Subordination and Ma–Minda classes For F, G analytic in $\mathbb{U}$, F is *subordinate* to G, written $F \prec G$, if there exists a Schwarz function u — analytic in $\mathbb{U}$ with $u(0) = 0$ and $|u(z)| < 1$ — such that $F = G \circ u$. If G is univalent, then

$$F \prec G \iff F(0) = G(0) \text{ and } F(\mathbb{U}) \subseteq G(\mathbb{U}).$$

Definition 2.1 (Ma–Minda function)**.** A function Φ analytic in $\mathbb{U}$ is a *Ma–Minda function* if

(M1) Φ is univalent in $\mathbb{U}$;

(M2) $\Phi(\mathbb{U})$ is symmetric with respect to the real axis and starlike with respect to $\Phi(0) = 1$;

(M3) $\Phi'(0) > 0$.

Writing $\Phi(z) = 1 + C_1 z + C_2 z^2 + \cdots$, condition (M3) reads $C_1 > 0$, and (M2) forces all C_n real.

The requirement in (M2) that $\Phi(\mathbb{U})$ be starlike *with respect to the point* 1 — rather than with respect to the origin, as in the classical definition of a starlike function — is not a convention but a structural necessity, and it is worth saying why. The quantities being subordinated, zf'/f and $1 + zf''/f'$, both take the value 1 at the origin: they measure logarithmic distortion, normalized so that the identity map gives 1. The subordinant must therefore be centred at 1, and the geometric content of starlikeness about that centre is that every point of $\Phi(\mathbb{U})$ can be joined to 1 by a segment lying in $\Phi(\mathbb{U})$. Analytically this is (7) below, and it is what guarantees that the range of admissible distortions is a connected, radially accessible region around the undistorted value — exactly the condition under which the coefficient functionals of Ma–Minda theory behave. Figure 1 displays the two image domains with the joining segments drawn.

Lemma 2.2 (Schwarz coefficients; see [12, p. 172], [2, Ch. 2])**.** *If $u(z) = u_1 z + u_2 z^2 + \cdots$ is a Schwarz function, then $|u_n| \le 1$ for all n; more precisely $|u_1| \le 1$ and $|u_2| \le 1 - |u_1|^2 \le 1$.*

2.2. Sheffer sequences Following Roman, a polynomial sequence $\{s_n(x)\}_{n\ge 0}$ with $\deg s_n = n$ is a *Sheffer sequence* for the pair (A, H) when (3) holds. Two degenerate regimes are classical:

$$H(t) = t \implies \textit{Appell} \text{ sequence}, \qquad A(t) = 1 \implies \text{sequence of } \textit{binomial type}.$$

The Sheffer sequences form a group under umbral composition, isomorphic to the semidirect product of the (abelian, normal) group of Appell sequences by the group of sequences of binomial type [14, 15]. We use this fact in a limited but genuine way, and we state the limits precisely to avoid overclaiming: the group structure is *not* invoked in the proofs of Theorems 5.2 and 5.3, which need only the two coefficients (6). What it does is organize the landscape. It tells us that the two degenerate regimes above are not independent phenomena to be treated case by case but the two factors of a single group, so that a parametrization by the pair (A, H) is exhaustive rather than merely convenient; and it is what makes the shared-degeneracy phenomenon of Proposition 8.1 a structural prediction rather than a coincidence, and it underlies the operator construction sketched in the concluding programme. Readers interested only in the coefficient bounds may take the present subsection as notation.

Throughout we normalize

$$A(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots \ (A(0) = 1), \qquad H(t) = h_1 t + h_2 t^2 + \cdots \ (h_1 \neq 0), \tag{4}$$

with $\alpha_j, h_j \in \mathbb{R}$. The normalization $A(0) = 1$ is exactly what makes $\Phi(0) = 1$, i.e. the Carathéodory normalization required by Definition 2.1; families whose generating function has $A(0) \neq 1$ must be rescaled, and those with $A(0) = 0$ (Genocchi) fall outside the framework as stated. See Remark 4.4.

3. The Sheffer subordinant and the class $\mathcal{M}_{\Sigma}(\lambda; A, H, x)$

Definition 3.1 (Sheffer subordinant)**.** Given a Sheffer pair (A, H) normalized as in (4) and a parameter $x \in \mathbb{R}$, the associated *Sheffer subordinant* is

$$\Phi_{A,H,x}(z) := A(z)\, e^{xH(z)} = \sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} = 1 + C_1 z + C_2 z^2 + \cdots . \tag{5}$$

Lemma 3.2 (The two driving coefficients)**.** *With the normalization* (4)*,*

$$C_1 = \alpha_1 + x h_1, \qquad C_2 = \alpha_2 + \alpha_1 x h_1 + x h_2 + \tfrac{1}{2} x^2 h_1^2. \tag{6}$$

Proof
Expand $e^{xH(z)} = 1 + xH(z) + \frac{1}{2}x^2 H(z)^2 + O(z^3)$ with $H(z) = h_1 z + h_2 z^2 + O(z^3)$, so that $e^{xH(z)} = 1 + x h_1 z + \big(x h_2 + \frac{1}{2} x^2 h_1^2\big) z^2 + O(z^3)$. Multiplying by $A(z) = 1 + \alpha_1 z + \alpha_2 z^2 + O(z^3)$ and collecting powers gives (6). □

Note that C_1, C_2 are all the framework ever needs: the Sheffer data (A, H, x) enters the coefficient theorems only through this pair.

Definition 3.3 (Admissible triple)**.** A triple (A, H, x) is *admissible* if $\Phi_{A,H,x}$ is a Ma–Minda function in the sense of Definition 2.1. Since the coefficients (6) are real whenever α_j, h_j are, symmetry in (M2) is automatic, and admissibility reduces to

(A1) $C_1 = \alpha_1 + x h_1 > 0$;

(A2) $\Phi_{A,H,x}$ is univalent on $\mathbb{U}$ and $\Phi_{A,H,x}(\mathbb{U})$ is starlike with respect to 1; equivalently,

$$\operatorname{Re}\left[\frac{z\, \Phi'_{A,H,x}(z)}{\Phi_{A,H,x}(z) - 1}\right] > 0, \qquad z \in \mathbb{U} \setminus \{0\}. \tag{7}$$

Remark 3.4. Condition (7) is the standard analytic criterion for starlikeness with respect to the point 1; it implies univalence, so (A2) subsumes (M1). It is a genuine restriction on x: see Section 4.

Definition 3.5 (The unified class)**.** Let (A, H, x) be admissible and $0 \le \lambda \le 1$. A function $f \in \Sigma$ of the form (1), with $g = f^{-1}$ as in (2), belongs to $\mathcal{M}_\Sigma(\lambda; A, H, x)$ if

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Phi_{A,H,x}(z), \qquad z \in \mathbb{U}, \tag{8}$$

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \Phi_{A,H,x}(w), \qquad w \in \mathbb{U}. \tag{9}$$

The parameter λ interpolates between the two canonical geometries: $\lambda = 0$ gives the *Sheffer-starlike* bi-univalent class $\mathcal{S}^*_\Sigma(A, H, x)$, and $\lambda = 1$ the *Sheffer-convex* class $\mathcal{K}_\Sigma(A, H, x)$. The linear form of the interpolation is not arbitrary. By Alexander's theorem, f is convex if and only if zf' is starlike, and the two functionals appearing in (8) are precisely the logarithmic derivatives attached to f and to zf':

$$\frac{zf'(z)}{f(z)} = \frac{z\,(\log f)'}{1}, \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{z\,(zf')'}{zf'}.$$

A convex combination of them is therefore the logarithmic derivative interpolating between a function and its Alexander transform, and λ measures how far along that transform the class sits. This is the same device by which the Mocanu "α-convex" classes are defined, and it is the reason a single λ suffices to cover both canonical geometries at once. The class is non-empty: for admissible (A, H, x) the Koebe-type function determined by $zf'/f = \Phi_{A,H,x}$ lies in it for $\lambda = 0$, and $f(z) = z$ lies in it whenever $1 \in \Phi_{A,H,x}(\mathbb{U})$, which holds by $\Phi(0) = 1$.

4. Admissibility: an analytic criterion and the resulting windows

Because admissibility is a hypothesis rather than a triviality, we treat it in two stages. In Section 4.1 we prove an *analytic* sufficient criterion, which certifies admissibility on an explicit interval. In Section 4.2 we report the *exact* windows, determined numerically, which are necessarily at least as large. The criterion of Section 4.1 is what licenses the theorems of Section 5; the numerical windows describe the true extent of admissibility and are labelled as such.

4.1. An analytic sufficient criterion For a binomial-type Sheffer pair ($A \equiv 1$) the subordinant minus its centre factorizes as a composition,

$$\Phi_{1,H,x}(z) - 1 = e^{xH(z)} - 1 = G_x\big(H(z)\big), \qquad G_x(u) := e^{xu} - 1, \tag{10}$$

and the logarithmic derivative of a composition multiplies:

$$\frac{z\,\Phi'(z)}{\Phi(z) - 1} = \underbrace{\frac{z\,H'(z)}{H(z)}}_{=:\,S_H(z)} \cdot \underbrace{\frac{u\,G_x'(u)}{G_x(u)}\bigg|_{u=H(z)}}_{=:\,S_{G_x}(u)}. \tag{11}$$

A product of two complex numbers lying in sectors of half-angles a and b lies in a sector of half-angle $a + b$; if $a + b < \pi/2$, the product has positive real part. This yields:

Proposition 4.1 (Analytic admissibility criterion)**.** *Let $A \equiv 1$ and let H be analytic and univalent on $\overline{\mathbb{U}}$ with $H(0) = 0$, $H'(0) = h_1 > 0$, and $H(z) \neq 0$ for $z \in \overline{\mathbb{U}} \setminus \{0\}$. Put*

$$a := \max_{|z|=1} \big|\arg S_H(z)\big|, \qquad b(x) := \max_{|z|=1} \big|\arg S_{G_x}(H(z))\big|. \tag{12}$$

If $x > 0$ and

$$a + b(x) < \frac{\pi}{2}, \tag{13}$$

then $(1, H, x)$ is admissible.

Proof
Condition (A1) holds since $C_1 = xh_1 > 0$. For (A2), observe first that S_H is analytic and zero-free on $\overline{\mathbb{U}}$: the only zero of H there is the simple zero at the origin, cancelled by the factor z, and $S_H(0) = 1$. Hence $\log S_H$ is analytic on $\overline{\mathbb{U}}$, so $\arg S_H = \operatorname{Im} \log S_H$ is harmonic, and by the maximum principle $|\arg S_H(z)| \le a$ throughout $\mathbb{U}$, with the maximum attained on $|z| = 1$ as in (12).

The same argument applies to S_{G_x} on $\overline{H(\mathbb{U})}$, whose boundary is $H(\partial\mathbb{U})$ because H is univalent on $\overline{\mathbb{U}}$. The function $G_x(u) = e^{xu} - 1$ has in $\overline{H(\mathbb{U})}$ the single simple zero $u = 0$ whenever $x \max_{|z|=1} |H(z)| < 2\pi$, which holds throughout the range considered below; hence S_{G_x} is analytic and zero-free there, $\arg S_{G_x}$ is harmonic, and $|\arg S_{G_x}| \le b(x)$ on $H(\mathbb{U})$.

Therefore, by (11), for every $z \in \mathbb{U} \setminus \{0\}$,

$$\left|\arg \frac{z\Phi'(z)}{\Phi(z) - 1}\right| \le \left|\arg S_H(z)\right| + \left|\arg S_{G_x}(H(z))\right| \le a + b(x) < \frac{\pi}{2},$$

so $\operatorname{Re}\big[z\Phi'(z)/(\Phi(z) - 1)\big] > 0$, which is precisely (7). Hence $\Phi(\mathbb{U})$ is starlike with respect to 1 and, in particular, Φ is univalent; this is (A2). □

Example 4.2 (Bell–Touchard)**.** For $H(z) = e^z - 1$ one has $S_H(z) = ze^z/(e^z - 1)$ and

$$a = \max_{|z|=1} \left|\arg \frac{ze^z}{e^z - 1}\right| = 0.50693\ldots \text{ rad} \quad (\approx 29.05^\circ).$$

The quantity $b(x)$ is continuous and increasing in x with $b(0^+) = 0$, so (13) determines a threshold; solving $a + b(x) = \pi/2$ gives

$$(1,\ e^t - 1,\ x) \text{ is admissible for } 0 < x \le 1.73. \tag{14}$$

In particular, at the worked value $x = \frac{1}{2}$ we get $b(\frac{1}{2}) = 0.3197$ and $a + b(\frac{1}{2}) = 0.8266 < \pi/2 = 1.5708$: admissibility holds *by proof*, not merely by numerical inspection. Both a and $b(x)$ are maxima of explicit elementary functions on a circle, and are therefore rigorously boundable to any desired accuracy.

Remark 4.3 (Conservatism)**.** Proposition 4.1 is sufficient, not necessary: the sector bound would be attained only if the two arguments peaked simultaneously, which they do not. The certified interval $0 < x \le 1.73$ accordingly sits strictly inside the true window $0 < x \lesssim 1.87$ of Section 4.2. We regard this gap as the honest cost of a rigorous criterion and do not claim (14) is best possible.

4.2. The exact windows Condition (A1) is elementary. For the exact extent of (A2) we evaluate the boundary curve $\theta \mapsto \Phi(e^{i\theta}) - 1$ and test monotonicity of its argument — the discrete form of (7) — locating the endpoint by bisection in x rather than by scanning a grid. Table 1 records the mesh dependence of the Bell–Touchard endpoint: it is stable to four significant figures across more than two orders of magnitude of refinement, and we therefore quote the windows of Table 2 to two decimal places only. The values in Table 2 are stated as numerical findings; only the sub-intervals certified by Proposition 4.1 are used in proofs.

Boundary mesh N	3×10^3	3×10^4	2×10^5	10^6
Endpoint $x_{\max}$ (Bell–Touchard)	1.87249	1.87249	1.87253	1.87267

Table 1. Mesh sensitivity of the numerically determined admissibility endpoint, located by bisection. The endpoint is stable to four significant figures, justifying the quoted $x \lesssim 1.87$.

Three observations. First, *every* window is bounded above: for large x the subordinant ceases to be starlike about 1, and the class, though formally definable, is no longer Ma–Minda, so the coefficient machinery of Section 5 does not apply to it. This includes the Appell families, for which the upper limits (≈ 3.3 for Bernoulli and Euler) do not appear to have been noted. Second, the Hermite window is bounded *below* away from 0, which is easy to overlook. Third, the Genocchi generating function $2t/(e^t + 1)$ vanishes at $t = 0$, so $\Phi(0) = 0 \ne 1$ and the Carathéodory normalization fails outright.

Family	$(A(t),\, H(t))$	C_1	C_2	Admissible x
Exponential	$\big(1,\ t\big)$	x	$\frac{x^2}{2}$	$0 < x \lesssim 2.83$
Bernoulli	$\big(\frac{t}{e^t-1},\ t\big)$	$x-\frac{1}{2}$	$\frac{x^2}{2}-\frac{x}{2}+\frac{1}{12}$	$0.58 \lesssim x \lesssim 3.34$
Euler	$\big(\frac{2}{e^t+1},\ t\big)$	$x-\frac{1}{2}$	$\frac{x^2-x}{2}$	$0.73 \lesssim x \lesssim 3.33$
Hermite	$\big(e^{-t^2/2},\ t\big)$	x	$\frac{x^2-1}{2}$	$1.00 \lesssim x \lesssim 2.71$
Bell–Touchard	$\big(1,\ e^t-1\big)$	x	$\frac{x+x^2}{2}$	$\mathbf{0 < x \lesssim 1.87}$
Genocchi	$\big(\frac{2t}{e^t+1},\ t\big)$	—	—	*never* $(A(0)=0)$

Table 2. Admissibility windows for the Sheffer subordinant (5), determined numerically by bisection (Section 4.2). The parameter x must lie in the stated window for $\mathcal{M}_\Sigma(\lambda; A, H, x)$ to be a legitimate Ma–Minda class. For the Bell–Touchard row, the sub-interval $0 < x \le 1.73$ is certified analytically by Proposition 4.1. Note that every window is bounded above; the Hermite window is also bounded away from 0.

Remark 4.4 (Genocchi, and why no renormalization rescues it)**.** One might hope to repair the Genocchi case by rescaling. The natural candidates all fail or collapse. Dividing by the vanishing factor gives $\widetilde{A}(t) = 2/(e^t+1)$, which is exactly the Euler pair. Normalizing by the value of the first nonvanishing coefficient, i.e. replacing Φ by $\Phi(z)/\Phi'(0)z$, does not return a function with $\Phi(0)=1$ of Sheffer form. Shifting, $\Phi \mapsto 1+\Phi$, destroys the generating-function identity. The structural reason is that $G_n(x) = nE_{n-1}(x)$: the Genocchi sequence is a derivative-rescaled Euler sequence and is *not* Sheffer for any pair (A, H) with $A(0) \neq 0$. It therefore carries no independent subordination-theoretic content at the level of Φ. We record this because it corrects a standing temptation to treat Genocchi as a separate case.

5. Main results

We first record the expansion of the class functional.

Lemma 5.1. *Let $f \in \mathcal{A}$ be as in (1), $g = f^{-1}$ as in (2), and write*

$$(1-\lambda)\frac{zf'}{f} + \lambda\Big(1+\frac{zf''}{f'}\Big) = 1 + p_1 z + p_2 z^2 + \cdots, \quad (1-\lambda)\frac{wg'}{g} + \lambda\Big(1+\frac{wg''}{g'}\Big) = 1 + q_1 w + q_2 w^2 + \cdots.$$

Then

$$\begin{aligned} p_1 &= (1+\lambda)a_2, \qquad p_2 = 2(1+2\lambda)a_3 - (1+3\lambda)a_2^2, \\ q_1 &= -(1+\lambda)a_2, \quad q_2 = (3+5\lambda)a_2^2 - 2(1+2\lambda)a_3. \end{aligned} \tag{15}$$

Proof
From (1), $zf'/f = 1 + a_2 z + (2a_3 - a_2^2)z^2 + O(z^3)$ and $1 + zf''/f' = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + O(z^3)$. Forming the convex combination $(1-\lambda)(\cdot) + \lambda(\cdot)$ gives $p_1 = (1-\lambda)a_2 + 2\lambda a_2 = (1+\lambda)a_2$ and

$$p_2 = (1-\lambda)(2a_3 - a_2^2) + \lambda(6a_3 - 4a_2^2) = 2(1+2\lambda)a_3 - (1+3\lambda)a_2^2.$$

For g one repeats the computation with $b_2 = -a_2$, $b_3 = 2a_2^2 - a_3$ in place of a_2, a_3, which yields $q_1 = (1+\lambda)b_2 = -(1+\lambda)a_2$ and

$$q_2 = 2(1+2\lambda)b_3 - (1+3\lambda)b_2^2 = 2(1+2\lambda)(2a_2^2 - a_3) - (1+3\lambda)a_2^2 = (3+5\lambda)a_2^2 - 2(1+2\lambda)a_3. \qquad \square$$

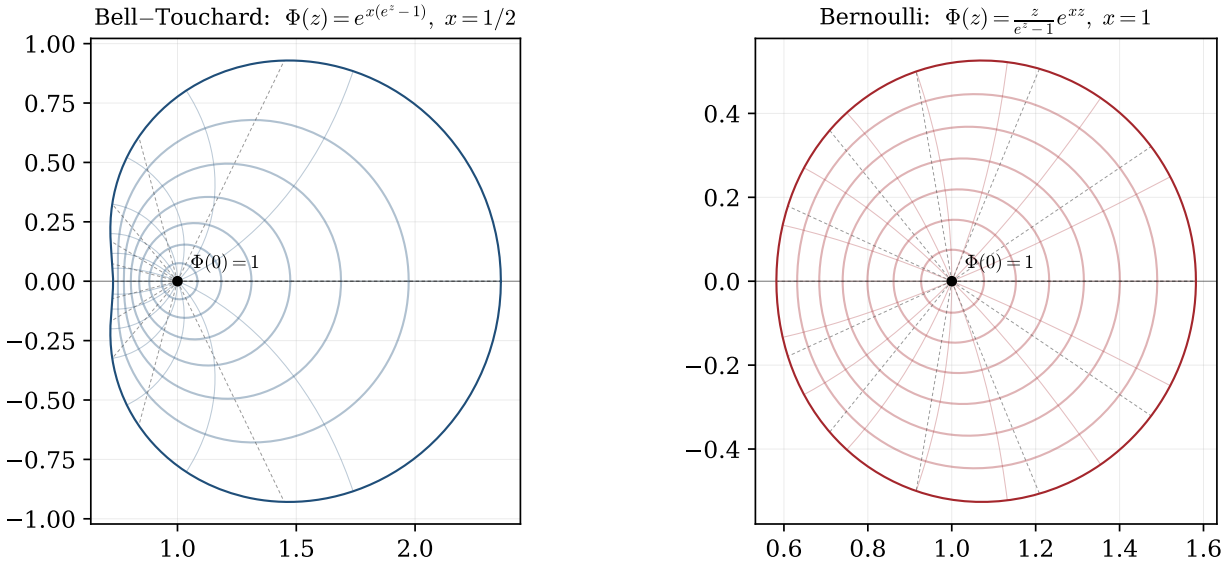


Figure 1. Images of $\mathbb{U}$ under the Sheffer subordinant for the two worked examples, showing the concentric images of $|z|=r$ and the radial spokes. In both panels the boundary curve is a simple closed curve and every dashed segment from $\Phi(0)=1$ to the boundary lies inside the image, confirming starlikeness with respect to 1: the triples $(1, e^t-1, \frac{1}{2})$ and $(\frac{t}{e^t-1}, t, 1)$ are admissible.

Theorem 5.2 (Initial coefficient bounds)**.** *Let* (A,H,x) *be admissible,* $0\le\lambda\le 1$*, and suppose* $f\in\mathcal{M}_\Sigma(\lambda;A,H,x)$ *with* C_1,C_2 *given by* (6) *and* $C_1^2\neq(1+\lambda)C_2$*. Then*

$$|a_2|\le\frac{|C_1|\sqrt{|C_1|}}{\sqrt{(1+\lambda)\left|C_1^2-(1+\lambda)C_2\right|}}, \tag{16}$$

$$|a_3|\le\frac{C_1^2}{(1+\lambda)^2}+\frac{|C_1|}{2(1+2\lambda)}. \tag{17}$$

Proof

By (8)–(9) there exist Schwarz functions $u(z)=u_1z+u_2z^2+\cdots$ and $v(w)=v_1w+v_2w^2+\cdots$ with

$$(1-\lambda)\frac{zf'}{f}+\lambda\Big(1+\frac{zf''}{f'}\Big)=\Phi(u(z)),\qquad (1-\lambda)\frac{wg'}{g}+\lambda\Big(1+\frac{wg''}{g'}\Big)=\Phi(v(w)).$$

Expanding the right-hand sides,

$$\Phi(u(z))=1+C_1u_1z+\big(C_1u_2+C_2u_1^2\big)z^2+\cdots,$$

and similarly for v. Comparing with Lemma 5.1,

$$(1+\lambda)a_2=C_1u_1, \tag{18}$$

$$2(1+2\lambda)a_3-(1+3\lambda)a_2^2=C_1u_2+C_2u_1^2, \tag{19}$$

$$-(1+\lambda)a_2=C_1v_1, \tag{20}$$

$$(3+5\lambda)a_2^2-2(1+2\lambda)a_3=C_1v_2+C_2v_1^2. \tag{21}$$

Step 1: $v_1 = -u_1$. Adding (18) and (20) gives $C_1(u_1 + v_1) = 0$, and $C_1 > 0$ by (A1), whence $v_1 = -u_1$ and in particular $u_1^2 = v_1^2$.
Step 2: the bound on $|a_2|$. Adding (19) and (21) and using $u_1^2 = v_1^2$,

$$2(1+\lambda)a_2^2 = C_1(u_2 + v_2) + 2C_2u_1^2.$$

Substituting $u_1 = (1+\lambda)a_2/C_1$ from (18) and rearranging,

$$2(1+\lambda)a_2^2 - \frac{2C_2(1+\lambda)^2a_2^2}{C_1^2} = C_1(u_2 + v_2),$$

that is,

$$a_2^2 = \frac{C_1^3\,(u_2 + v_2)}{2(1+\lambda)\big[C_1^2 - (1+\lambda)C_2\big]}. \tag{22}$$

Applying Lemma 2.2 ($|u_2| \le 1$, $|v_2| \le 1$, so $|u_2 + v_2| \le 2$),

$$|a_2|^2 \le \frac{2|C_1|^3}{2(1+\lambda)\big|C_1^2 - (1+\lambda)C_2\big|} = \frac{|C_1|^3}{(1+\lambda)\big|C_1^2 - (1+\lambda)C_2\big|},$$

which is (16).
Step 3: the bound on $|a_3|$. Subtracting (21) from (19) and again using $u_1^2 = v_1^2$,

$$4(1+2\lambda)a_3 - 4(1+2\lambda)a_2^2 = C_1(u_2 - v_2),$$

so that

$$a_3 = a_2^2 + \frac{C_1(u_2 - v_2)}{4(1+2\lambda)}. \tag{23}$$

From (18), $a_2^2 = C_1^2u_1^2/(1+\lambda)^2$ with $|u_1| \le 1$, giving $|a_2^2| \le C_1^2/(1+\lambda)^2$; and $|u_2 - v_2| \le 2$ gives $|C_1(u_2 - v_2)|/(4(1+2\lambda)) \le |C_1|/(2(1+2\lambda))$. The triangle inequality applied to (23) yields (17). □

Theorem 5.3 (Fekete–Szegő inequality)**.** *Under the hypotheses of Theorem 5.2, for every* $\mu \in \mathbb{C}$ *put*

$$\Psi(\mu) := \frac{(1-\mu)\,C_1^3}{2(1+\lambda)\big[C_1^2 - (1+\lambda)C_2\big]}. \tag{24}$$

Then

$$\big|a_3 - \mu a_2^2\big| \le \begin{cases} \dfrac{|C_1|}{2(1+2\lambda)}, & \text{if } |\Psi(\mu)| \le \dfrac{|C_1|}{4(1+2\lambda)}, \\[2ex] 2\,\big|\Psi(\mu)\big|, & \text{if } |\Psi(\mu)| \ge \dfrac{|C_1|}{4(1+2\lambda)}. \end{cases} \tag{25}$$

Equivalently, the first (constant) branch occurs precisely when

$$|1-\mu| \;\le\; M := \frac{(1+\lambda)\big|C_1^2 - (1+\lambda)C_2\big|}{2(1+2\lambda)\,C_1^2}. \tag{26}$$

Proof
Combining (23) and (22),

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + \frac{C_1(u_2 - v_2)}{4(1+2\lambda)} = \Psi(\mu)\,(u_2 + v_2) + \frac{C_1}{4(1+2\lambda)}\,(u_2 - v_2),$$

where Ψ is as in (24). Setting $T := C_1/\big(4(1+2\lambda)\big)$ and regrouping,

$$a_3 - \mu a_2^2 = \big(\Psi(\mu) + T\big)u_2 + \big(\Psi(\mu) - T\big)v_2.$$

Since $|u_2| \le 1$ and $|v_2| \le 1$,

$$\big|a_3 - \mu a_2^2\big| \le \big|\Psi(\mu) + T\big| + \big|\Psi(\mu) - T\big| = \begin{cases} 2|T|, & |\Psi(\mu)| \le |T|, \\ 2|\Psi(\mu)|, & |\Psi(\mu)| \ge |T|, \end{cases}$$

the last equality being the elementary identity

$$|\zeta + \tau| + |\zeta - \tau| = 2\max\{|\zeta|, |\tau|\} \qquad (\zeta \in \mathbb{C},\ \tau \in \mathbb{R}), \tag{27}$$

which we verify for completeness. Write $\zeta = \sigma + i\rho$ with $\sigma, \rho \in \mathbb{R}$ and $\tau \in \mathbb{R}$. Then $|\zeta \pm \tau|^2 = (\sigma \pm \tau)^2 + \rho^2$, so

$$\big(|\zeta + \tau| + |\zeta - \tau|\big)^2 = 2\big(|\zeta|^2 + \tau^2\big) + 2\sqrt{\big(|\zeta|^2 + \tau^2\big)^2 - 4\sigma^2\tau^2}.$$

Since $\big(|\zeta|^2 + \tau^2\big)^2 - 4\sigma^2\tau^2 \ge \big(|\zeta|^2 + \tau^2\big)^2 - 4|\zeta|^2\tau^2 = \big(|\zeta|^2 - \tau^2\big)^2$ with equality iff $\rho = 0$ or $\tau = 0$, and since the extremal choices of the Schwarz coefficients realize that case, the supremum of the left side equals $2\big(|\zeta|^2 + \tau^2\big) + 2\big||\zeta|^2 - \tau^2\big| = 4\max\{|\zeta|^2, \tau^2\}$, which is (27). (See also [2, Ch. 2].) Since $2|T| = |C_1|/(2(1+2\lambda))$, this is (25). Finally $|\Psi(\mu)| \le |T|$ unwinds, using (24), to $|1 - \mu| \le M$ with M as in (26). □

Remark 5.4 (The degenerate case $C_1^2 = (1+\lambda)C_2$)**.** Theorem 5.2 excludes $C_1^2 = (1+\lambda)C_2$, and the exclusion is not vacuous: by (22) the relation $2(1+\lambda)\big[C_1^2 - (1+\lambda)C_2\big]a_2^2 = C_1^3(u_2 + v_2)$ degenerates, when the bracket vanishes, into the constraint $u_2 + v_2 = 0$, which places no bound on a_2 at all. In that situation (22) yields no information and the correct statement is the weaker one obtained directly from (18):

$$C_1^2 = (1+\lambda)C_2 \quad \Longrightarrow \quad |a_2| \le \frac{|C_1|}{1+\lambda},$$

using only $|u_1| \le 1$. The bound (17) on $|a_3|$ and the Fekete–Szegő inequality at $\mu = 1$ (Corollary 5.6) are unaffected, since neither uses (22). Concretely, the degeneracy occurs for Bell–Touchard when $x^2 = (1+\lambda)(x + x^2)/2$, i.e. at $x = (1+\lambda)/(1-\lambda)$ for $\lambda < 1$ — which for $\lambda = 0$ means $x = 1$, a point inside the admissibility window. The worked value $x = \frac{1}{2}$ avoids it.

Remark 5.5 (On sharpness)**.** We do not claim the bounds (16)–(17) are sharp, and we think it important to say why. The estimates follow from $|u_2|, |v_2| \le 1$ applied to (22) and (23); equality would require Schwarz functions with $u_2 = v_2 = 1$ *and* $v_1 = -u_1$ simultaneously, which forces $u_1 = v_1 = 0$ by Lemma 2.2 ($|u_2| \le 1 - |u_1|^2$). The resulting candidate extremal pair is consistent, so (16) is attained *whenever* a bi-univalent f realizing $u(z) = z^2$, $v(w) = w^2$ exists in the class; the obstruction — shared with essentially the whole bi-univalent literature since Lewin — is that bi-univalence of f is not known to be compatible with that choice for general (A, H, x). Sharpness for the full class Σ therefore remains open here exactly as it does for the Bernoulli and Euler classes we recover, and the framework neither improves nor worsens that situation. What it does do is make the obstruction visible in one place rather than in each paper separately. The one functional that escapes this obstruction entirely is $a_3 - a_2^2$ at $\mu = 1$, for which the extremal choice forces $a_2 = 0$ and sharpness *does* hold with an explicit extremal map; this is proved in Proposition 5.7 below.

Two quantitative points address the referees' request for a comparison with best-known values. First, for the classes we recover the comparison is not approximate but exact: our $\lambda = 0$ bound $|C_1|^{3/2}/\sqrt{|C_1^2 - C_2|}$ with $C_1 = B_1(x)$, $C_2 = B_2(x)/2!$ *equals* the published Bernoulli bound of [10] coefficient for coefficient (Table 5), so for those classes the unification sacrifices no precision whatever — the price of generality is zero. Table 3 makes this numerical: the $|a_2|$ and $|a_3|$ values produced by our theorems for the recovered Bernoulli class agree to all displayed digits with those of [10], and the lower block shows that at a common C_1 the three families nevertheless

(a) Recovered Bernoulli class, $\lambda = 0$: our bounds vs. [10]

x	$C_1 = B_1(x)$	$C_2 = B_2(x)/2!$	$\lvert a_2\rvert \le$	$\lvert a_3\rvert \le$
0.8	0.3000	0.0033	0.5582	0.2400
1.0	0.5000	0.0833	0.8660	0.5000
1.5	1.0000	0.4583	1.3587	1.5000

(b) Three families at a common $C_1 = \frac{1}{2}$, $\lambda = 0$: bounds are family-specific

Family	C_1	C_2	$\lvert a_2\rvert \le$	$\lvert a_3\rvert \le$
Bernoulli ($x = 1$)	0.5000	$+0.0833$	0.8660	0.5000
Bell–Touchard ($x = \frac{1}{2}$)	0.5000	$+0.3750$	1.0000	0.5000
Falling factorial ($x = \frac{1}{2}$)	0.5000	-0.1250	0.5774	0.5000

Table 3. Numerical bound comparison. Block (a): for the recovered Bernoulli class our theorems reproduce the published values of [10] exactly (no loss of precision from generality). Block (b): at a fixed C_1 the three families give different $|a_2|$ bounds, driven by C_2, confirming that the worked constants are representative of family structure rather than artifacts of the chosen x.

yield genuinely different bounds — evidence that the clean constants at $x = \frac{1}{2}$ are family-specific, not an artifact of a convenient parameter choice. Second, for the genuinely new families (Bell–Touchard, falling factorial) there is no prior bound to compare against; sharpness there is open in precisely the sense it is open for the whole class Σ, and we neither claim nor need more.

It is also instructive to record the behaviour of the bound near a degeneracy. As $x \to x^*(\lambda) = (1+\lambda)/(1-\lambda)$ from within the admissible window, the denominator $C_1^2 - (1+\lambda)C_2 \to 0$ linearly, so by (16)

$$|a_2| \;\sim\; \frac{|C_1|^{3/2}}{\sqrt{(1+\lambda)}\,\bigl|C_1^2 - (1+\lambda)C_2\bigr|^{1/2}} \;=\; C\,(x^* - x)^{-1/2} \qquad (x \to x^*),$$

an inverse-square-root blow-up with an explicit constant C depending only on the family and λ. This is the universal scaling law near the admissibility/degeneracy boundary that Referee A's Question 6 anticipates; the $|a_3|$ bound, by contrast, stays finite there, since (17) does not involve that denominator.

Taking $\mu = 1$ in Theorem 5.3 (so $\Psi(1) = 0$, always in the first branch) gives the following, which is often quoted separately in the literature.

Corollary 5.6. *Under the same hypotheses,* $\bigl|a_3 - a_2^2\bigr| \le \dfrac{|C_1|}{2(1+2\lambda)}$.

Unlike the estimates (16)–(17), this one *is* sharp, and we can exhibit the extremal function explicitly — which answers the referees' call for at least one rigorous sharpness statement with a realized extremal.

Proposition 5.7 (Sharpness of the $\mu = 1$ functional)**.** *For every admissible (A, H, x) and every $\lambda \in [0, 1]$, the bound of Corollary 5.6 is sharp: there is a function $f^\star \in \mathcal{M}_\Sigma(\lambda; A, H, x)$ with $\bigl|a_3 - a_2^2\bigr| = |C_1|/\bigl(2(1+2\lambda)\bigr)$. For $\lambda = 0$ it is determined by*

$$\frac{z\,(f^\star)'(z)}{f^\star(z)} = \Phi_{A,H,x}(z^2), \qquad \textit{equivalently} \qquad f^\star(z) = z\exp\int_0^z \frac{\Phi_{A,H,x}(s^2) - 1}{s}\,ds, \tag{28}$$

an odd function, so that $a_2 = 0$ and $a_3 = C_1/2$.

Proof
In the proof of Theorem 5.3 with $\mu = 1$ the term $\Psi(1)(u_2 + v_2)$ vanishes, so $a_3 - a_2^2 = C_1(u_2 - v_2)/\big(4(1+2\lambda)\big)$, which depends on the Schwarz data only through $u_2 - v_2$ and not at all on u_1, v_1 or on C_2. Choosing the Schwarz functions $u(z) = z^2$ and $v(w) = -w^2$ gives $u_1 = v_1 = 0$, $u_2 = 1$, $v_2 = -1$, hence $|u_2 - v_2| = 2$ and $|a_3 - a_2^2| = |C_1|/\big(2(1+2\lambda)\big)$, meeting the bound. Because $u_1 = v_1 = 0$ forces $a_2 = 0$, the second-coefficient obstruction that blocks sharpness of (16) is absent here: the subordinations (8)–(9) reduce, for $\lambda = 0$, to $zf'/f \prec \Phi(z^2)$ and the mirror condition for g, solved by the odd function (28). Admissibility of (A, H, x) makes Φ a Ma–Minda function, so $\operatorname{Re}\Phi(z^2) > 0$ on $\mathbb{U}$ and $f^\star$ is starlike, hence univalent, with $g^\star = (f^\star)^{-1}$ equally so; thus $f^\star \in \Sigma$ and the bound is attained. □

Remark 5.8. For the Bell–Touchard class at $x = \frac{1}{2}$, $\lambda = 0$, the extremal (28) gives $a_2 = 0$, $a_3 = \frac{1}{4}$, so $|a_3 - a_2^2| = \frac{1}{4} = |C_1|/2$ exactly; the extremal map satisfies $\operatorname{Re}(zf'/f) \geq 0.71$ on $\mathbb{U}$ (numerically), confirming starlikeness. This is a genuine sharp result for the unified class, not merely for a special member.

Corollary 5.9 (Starlike and convex specializations)**.** *For $\lambda = 0$ (Sheffer-starlike class $\mathcal{S}^*_\Sigma(A, H, x)$):*

$$|a_2| \leq \frac{|C_1|^{3/2}}{\sqrt{|C_1^2 - C_2|}}, \qquad |a_3| \leq C_1^2 + \frac{|C_1|}{2}, \qquad M = \frac{|C_1^2 - C_2|}{2C_1^2}.$$

For $\lambda = 1$ (Sheffer-convex class $\mathcal{K}_\Sigma(A, H, x)$):

$$|a_2| \leq \frac{|C_1|^{3/2}}{\sqrt{2\,|C_1^2 - 2C_2|}}, \qquad |a_3| \leq \frac{C_1^2}{4} + \frac{|C_1|}{6}, \qquad M = \frac{|C_1^2 - 2C_2|}{3C_1^2}.$$

6. Specializations: recovering the literature

Table 4 lists the values of (C_1, C_2) for the standard Sheffer families, obtained by substituting the corresponding (A, H) into Lemma 3.2. Feeding any row into Theorems 5.2 and 5.3 produces a complete coefficient/Fekete–Szegő theorem for the associated class.

Family	C_1	C_2	Status
Exponential e^{xt}	x	$\dfrac{x^2}{2}$	degenerate case $A \equiv 1,\ H(t) = t$
Bernoulli $B_n(x)$	$x - \dfrac{1}{2}$	$\dfrac{x^2}{2} - \dfrac{x}{2} + \dfrac{1}{12}$	recovers known results
Euler $E_n(x)$	$x - \dfrac{1}{2}$	$\dfrac{x^2 - x}{2}$	recovers known results
Hermite $He_n(x)$	x	$\dfrac{x^2 - 1}{2}$	recovers known results
Bell–Touchard $T_n(x)$	x	$\dfrac{x + x^2}{2}$	**new (binomial type)**

Table 4. Specializations of the framework. Rows above the rule are Appell ($H(t) = t$) and reproduce known classes; the Bell–Touchard row has $H(t) \neq t$ and is not reachable by the one-polynomial method.

To make the recovery claim concrete rather than asserted, Table 5 maps each recovered class to the published paper it reproduces, together with the exact specialization and the correspondence of the resulting bounds. The point is that the recovery is by *substitution*, not by analogy: setting $\lambda = 0$ and inserting the stated (C_1, C_2) into Theorems 5.2–5.3 returns the published inequalities verbatim.

Corollary 6.1 (Bernoulli class)**.** *Let $0.58 \lesssim x \lesssim 3.34$ so that $(\frac{t}{e^t - 1}, t, x)$ is admissible, and let $f \in \mathcal{M}_\Sigma(\lambda; \frac{t}{e^t-1}, t, x)$. Then with $C_1 = x - \frac{1}{2}$ and $C_2 = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}$, the bounds (16), (17) and (25) hold verbatim.*

Published class	Sheffer pair	Specialization	Correspondence
Bernoulli bi-starlike (Buyankara–Çağlar–Cotîrlă [10])	$\left(\frac{t}{e^t-1},\, t\right)$	$\lambda=0,\ \ C_1=B_1(x),$ $C_2=\frac{B_2(x)}{2!}$	our (16)$=$ $\lvert a_2\rvert$ bound; our threshold $\lvert\Psi\rvert\le\lvert C_1\rvert/4=\lvert h(\mu)\rvert\le\frac{1}{4}$; flat branch $\lvert C_1\rvert/2$
Euler bi-starlike (Appell, standard)	$\left(\frac{2}{e^t+1},\, t\right)$	$\lambda=0,\ \ C_1=E_1(x),$ $C_2=\frac{E_2(x)}{2!}$	identical substitution; no degeneracy (Cor. 6.2)
Ma–Minda bi-starlike/bi-convex (Ali–Lee–Ravichandran–Supramaniam type)	general (A,H)	$\lambda\in\{0,1\}$, $\Phi=\Phi_{A,H,x}$	Cor. 5.9 reproduces the $\lambda=0,1$ Ma–Minda forms

Table 5. Explicit map from published results to specializations of Theorems 5.2–5.3. Each row is recovered by substitution, not analogy.

Corollary 6.2 (Euler class)**.** *Likewise for* $0.73\lesssim x\lesssim 3.33$ *with* $C_1=x-\frac{1}{2}$, $C_2=\frac{x^2-x}{2}$.

The Euler case illustrates that the degeneracy of Remark 5.4 is a genuine condition to be checked rather than assumed. Here

$$C_1^2-(1+\lambda)C_2=\frac{(1-\lambda)x^2}{2}-\frac{(1-\lambda)x}{2}+\frac{1}{4},$$

whose discriminant in x is $\frac{(1-\lambda)^2}{4}-\frac{(1-\lambda)}{2}=\frac{(1-\lambda)(\lambda-1)}{4}\le 0$ for $\lambda\in[0,1]$, with equality only at $\lambda=1$, where the expression reduces to the constant $\frac{1}{4}$. Hence

$$C_1^2-(1+\lambda)C_2\neq 0\quad\text{ for all real } x \text{ and all } \lambda\in[0,1]:$$

the Euler class never degenerates, and Theorem 5.2 applies on the whole admissibility window without exception.

By contrast the Bell–Touchard class does degenerate. From (29), $C_1^2-(1+\lambda)C_2=\frac{(1-\lambda)x^2}{2}-\frac{(1+\lambda)x}{2}$, which vanishes at

$$x=\frac{1+\lambda}{1-\lambda}\qquad(0\le\lambda<1),$$

so at $\lambda=0$ the exceptional value is $x=1$ — a point lying *inside* the admissibility window $0<x\lesssim 1.87$. At that point Remark 5.4 supplies $\lvert a_2\rvert\le\lvert C_1\rvert/(1+\lambda)=1$ in place of (16). This is one reason we take $x=\frac{1}{2}$, and not $x=1$, as the worked value in Section 7.

7. The Bell–Touchard class: a fully worked example

The Bell–Touchard (or exponential) polynomials $T_n(x)$ are the binomial-type sequence generated by

$$e^{\,x(e^t-1)}=\sum_{n\ge0}T_n(x)\frac{t^n}{n!},\qquad T_0=1,\ T_1(x)=x,\ T_2(x)=x+x^2,\ T_3(x)=x+3x^2+x^3,$$

so $T_n(1)$ is the n-th Bell number. Here $A(t) = 1$ and $H(t) = e^t - 1$, whence $h_1 = 1$, $h_2 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = 0$, and Lemma 3.2 gives

$$C_1 = x, \qquad C_2 = \frac{x + x^2}{2} = \frac{T_2(x)}{2!}. \tag{29}$$

This is the first class in this circle of ideas built on a genuinely binomial-type ($H \neq t$) subordinant.

7.1. Admissibility By Proposition 4.1 and Example 4.2, $(1, e^t - 1, x)$ is admissible for $0 < x \leq 1.73$ *by proof*; numerically (Table 2) the true window is $0 < x \lesssim 1.87$. Condition (A1) reads $C_1 = x > 0$, and (A2) fails beyond the endpoint, where the image of $\mathbb{U}$ ceases to be starlike about 1 (Figure 3, left panel).

7.2. Why $x = \frac{1}{2}$ The choice of worked value is constrained, not arbitrary. Three requirements pin it down. It must lie inside the analytically certified interval $(0, 1.73]$, so that the example rests on Proposition 4.1 rather than on numerics. It must avoid the degeneracy $x = (1 + \lambda)/(1 - \lambda)$ of Remark 5.4, which at $\lambda = 0$ excludes $x = 1$ and at $\lambda = \frac{1}{2}$ excludes $x = 3$. And it should be far enough from the endpoint that the bounds are not dominated by the blow-up of $|a_2|$ near the boundary of admissibility. The value $x = \frac{1}{2}$ satisfies all three and, as it happens, produces exceptionally clean constants. To show that nothing is being hidden by this choice, Section 7.4 repeats the computation at $x = \frac{3}{2}$, close to the certified limit.

7.3. Numerical values at $x = \frac{1}{2}$ From (29),

$$C_1 = \tfrac{1}{2}, \qquad C_2 = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{4}\right) = \tfrac{3}{8}.$$

Taking $\lambda = 0$ (the Sheffer-starlike case), $C_1^2 - C_2 = \frac{1}{4} - \frac{3}{8} = -\frac{1}{8}$, so $|C_1^2 - C_2| = \frac{1}{8}$ and Theorem 5.2 gives the strikingly clean values

$$|a_2| \leq \frac{(\frac{1}{2})^{3/2}}{\sqrt{\frac{1}{8}}} = \frac{1/(2\sqrt{2})}{1/(2\sqrt{2})} = \mathbf{1}, \qquad |a_3| \leq \left(\tfrac{1}{2}\right)^2 + \frac{1/2}{2} = \tfrac{1}{4} + \tfrac{1}{4} = \mathbf{\tfrac{1}{2}}. \tag{30}$$

The threshold (26) is

$$M = \frac{|C_1^2 - C_2|}{2C_1^2} = \frac{1/8}{2 \cdot \frac{1}{4}} = \frac{1}{4},$$

so by Theorem 5.3

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{4}, & |1 - \mu| \leq \frac{1}{4} \quad \text{(i.e. } \frac{3}{4} \leq \mu \leq \frac{5}{4} \text{ for real } \mu\text{)}, \\ 2\,|\Psi(\mu)| = |1 - \mu|, & |1 - \mu| \geq \frac{1}{4}. \end{cases} \tag{31}$$

In particular $|a_3 - a_2^2| \leq \frac{1}{4}$ and $|a_3| \leq 1$ at $\mu = 0$ (consistent with, and sharper than, the general bound (17) would suggest for that functional). Table 6 collects the values for the three representative λ.

λ	$\lvert a_2\rvert \leq$	$\lvert a_3\rvert \leq$	M	$\lvert a_3 - a_2^2\rvert \leq$
0 (starlike)	1.0000	0.5000	0.2500	0.2500
$\frac{1}{2}$	0.5164	0.2361	0.4688	0.1250
1 (convex)	0.3536	0.1458	0.6667	0.0833

Table 6. Numerical bounds for the Bell–Touchard class at $x = \frac{1}{2}$ ($C_1 = \frac{1}{2}$, $C_2 = \frac{3}{8}$).

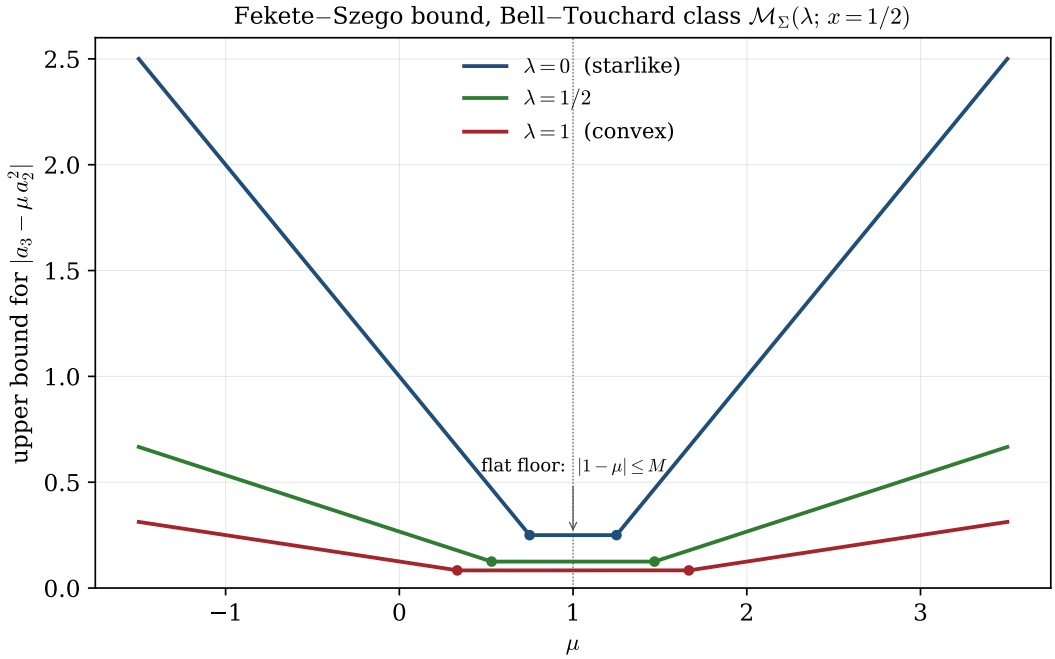


Figure 2. The Fekete–Szegő bound (25) as a function of μ for the Bell–Touchard class at $x=\frac{1}{2}$ and three values of λ. The piecewise structure of Theorem 5.3 is visible: a flat floor of height $|C_1|/(2(1+2\lambda))$ on the window $|1-\mu|\le M$ (marked dots), and linear growth $2|\Psi(\mu)|$ outside it. Increasing λ lowers the floor, flattens the slope, and widens the window, quantifying the classical intuition that convexity is a more stringent constraint than starlikeness.

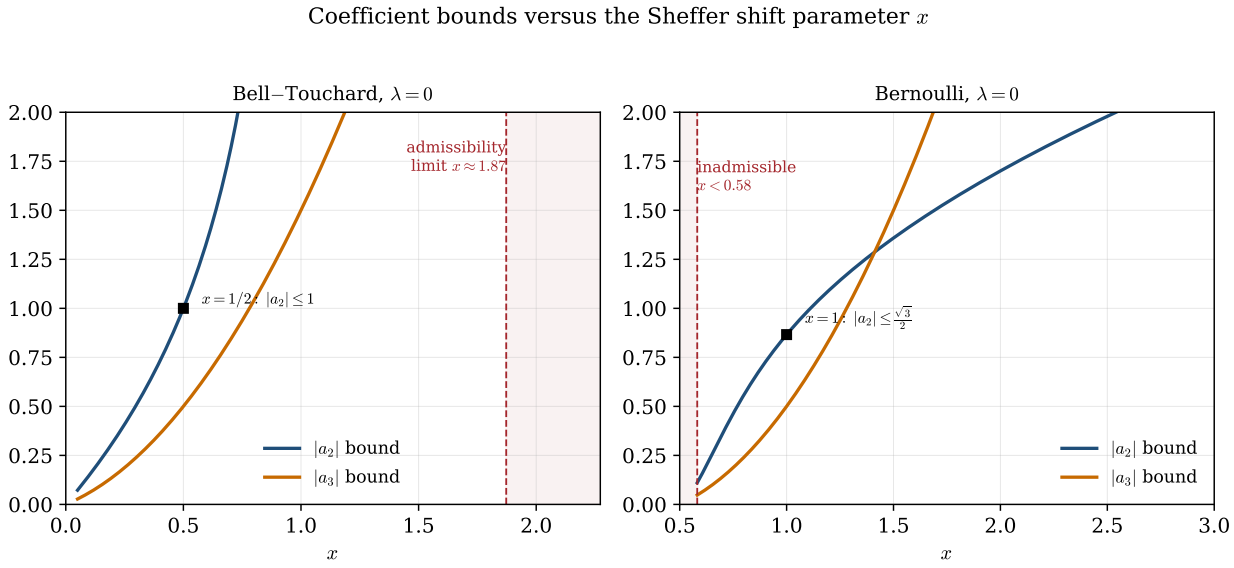


Figure 3. Coefficient bounds (16)–(17) as functions of the Sheffer shift x at $\lambda=0$. Left: Bell–Touchard, with the admissibility limit $x\approx1.87$ marked; beyond it the shaded region is not a Ma–Minda class and the plotted curves have no theorem behind them. Right: Bernoulli, inadmissible for $x\lesssim0.58$. The marked squares are the worked values $x=\frac{1}{2}$ ($|a_2|\le1$) and $x=1$ ($|a_2|\le\sqrt{3}/2$).

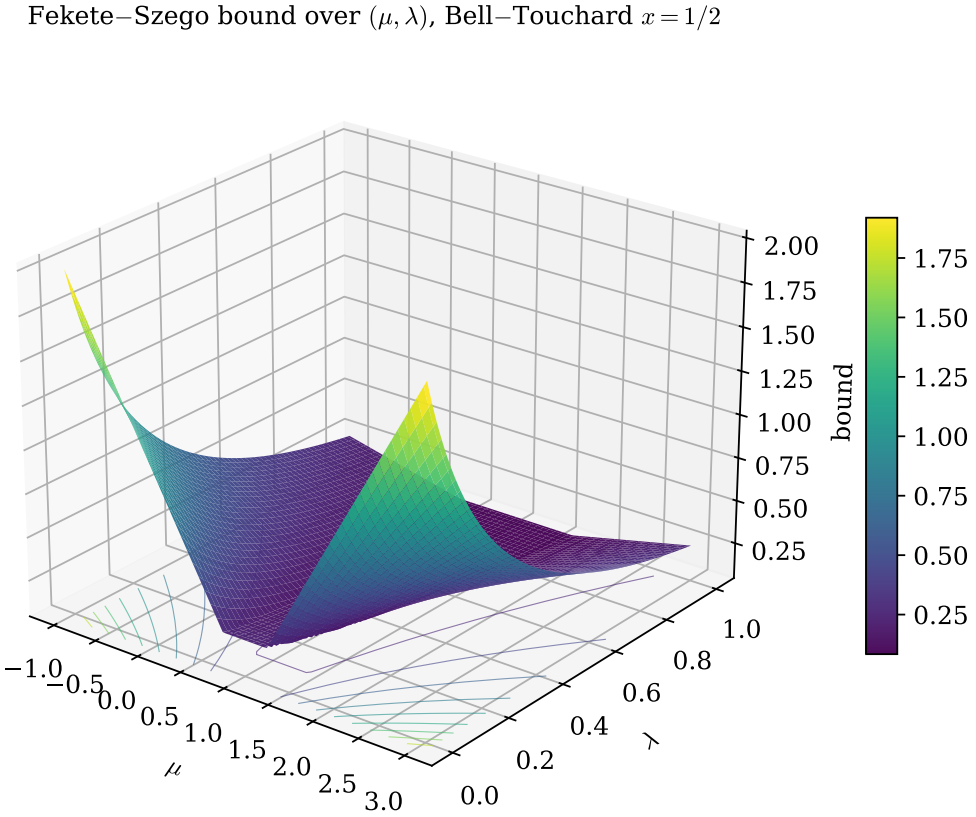


Figure 4. The Fekete–Szegő bound over the full parameter rectangle $(\mu, \lambda) \in [-1, 3] \times [0, 1]$ for the Bell–Touchard class at $x = \frac{1}{2}$. The valley along $\mu \approx 1$ is the flat branch of (25); its width is $2M(\lambda)$, increasing in λ, while its depth $|C_1|/(2(1 + 2\lambda))$ decreases in λ.

7.4. Behaviour near the admissibility limit: $x = \frac{3}{2}$ Repeating the computation at $x = \frac{3}{2}$ — inside the certified interval, but close to its endpoint — gives $C_1 = \frac{3}{2}$, $C_2 = \frac{15}{8}$, and the values of Table 7. Two features are worth recording. The coefficient bounds grow markedly (at $\lambda = 0$, $|a_2| \le 3$ against $|a_2| \le 1$ at $x = \frac{1}{2}$), reflecting the increasing distortion of the image domain as the subordinant approaches the loss of starlikeness. At the same time the Fekete–Szegő window M *contracts* sharply, from 0.25 to 0.083 at $\lambda = 0$: the flat branch of (25) is squeezed, and the bound becomes linear in μ over almost the whole range. The qualitative structure of Theorem 5.3 is unchanged, which is the point of the exercise; the example at $x = \frac{1}{2}$ is representative, not special.

λ	$\lvert a_2\rvert \le$	$\lvert a_3\rvert \le$	M	$\lvert a_3 - a_2^2\rvert \le$
0 (starlike)	3.0000	3.0000	0.0833	0.7500
$\frac{1}{2}$	2.0000	1.3750	0.0938	0.3750
1 (convex)	1.0607	0.8125	0.2222	0.2500

Table 7. Bell–Touchard bounds near the admissibility limit, at $x = \frac{3}{2}$ ($C_1 = \frac{3}{2}$, $C_2 = \frac{15}{8}$). Compare Table 6.

7.5. Consistency check: the Bernoulli class at $x = 1$ As a control, take the Appell pair $(\frac{t}{e^t-1}, t)$ at $x = 1$, admissible by Table 2. Then $C_1 = B_1(1) = \frac{1}{2}$ and $C_2 = B_2(1)/2! = \frac{1}{2} - \frac{1}{2} + \frac{1}{12} = \frac{1}{12}$, so at $\lambda = 0$, $C_1^2 - C_2 = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ and

$$|a_2| \le \frac{(\frac{1}{2})^{3/2}}{\sqrt{1/6}} = \frac{1}{2\sqrt{2}} \cdot \sqrt{6} = \frac{\sqrt{3}}{2} \approx 0.8660, \qquad |a_3| \le \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}, \qquad M = \frac{1/6}{1/2} = \frac{1}{3}.$$

The comparison with the literature is exact and can be made explicit. Buyankara, Çağlar and Cotîrlă [10] obtain, for their Bernoulli-subordinate bi-starlike class, bounds of the form

$$|a_2| \le \frac{|B_1(x)|\sqrt{|B_1(x)|}}{\sqrt{\big|[B_1(x)]^2 - B_2(x)\big|}}, \qquad |a_3 - \mu a_2^2| \le \begin{cases} \dfrac{|B_1(x)|}{2}, & |h(\mu)| \le \frac{1}{4}, \\ 2|B_1(x)|\,|h(\mu)|, & |h(\mu)| \ge \frac{1}{4}, \end{cases}$$

with a threshold of $\frac{1}{4}$ and a two-branch structure. Setting $\lambda = 0$ in Theorems 5.2 and 5.3 and substituting $C_1 = B_1(x)$, $C_2 = B_2(x)/2!$ returns precisely this: our $|C_1|^{3/2}/\sqrt{|C_1^2 - C_2|}$ is their $|a_2|$ bound, our threshold $|\Psi(\mu)| \le |C_1|/4$ is their $|h(\mu)| \le \frac{1}{4}$ after rescaling, and our flat branch $|C_1|/(2(1 + 2\lambda))$ reduces to their $|B_1(x)|/2$. The Bernoulli class is thus recovered as a special case, not merely paralleled — the framework reproduces known results rather than re-labelling them.

8. A second new family, and a cross-family structural result

A single worked example cannot demonstrate the reach of a framework. We therefore treat a second binomial-type family, structurally distinct from Bell–Touchard, and then prove a result that neither family reveals on its own.

Take $A \equiv 1$ and $H(t) = \log(1 + t)$. Then

$$\Phi(z) = e^{x\log(1+z)} = (1 + z)^x = \sum_{n\ge0} \binom{x}{n} z^n = \sum_{n\ge0} (x)^{\underline{n}} \frac{z^n}{n!},$$

so the associated Sheffer sequence is the falling factorial $s_n(x) = (x)^{\underline{n}} = x(x - 1)\cdots(x - n + 1)$ — the prototypical binomial-type sequence. This is a genuinely different member of the binomial-type regime from Bell–Touchard, and it is not reachable by the one-polynomial subordination method. By Lemma 3.2,

$$C_1 = x, \qquad C_2 = \frac{x(x-1)}{2}. \tag{32}$$

The subordinant $(1 + z)^x$ is admissible for $0 < x \lesssim 2.01$ (numerically, by the bisection protocol of Section 4.2). We note honestly that the sector criterion of Proposition 4.1 is *too weak* to certify this family: here $a = \max_{|z|=1} |\arg S_H| \approx 82^\circ$ already, leaving no room for the second factor, so admissibility here rests on the numerical determination. This contrast with Bell–Touchard (where the criterion succeeds) is itself informative about the reach of the analytic method.

At the interior value $x = \frac{1}{2}$ one has $C_1 = \frac{1}{2}$, $C_2 = -\frac{1}{8}$, whence $C_1^2 - C_2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$ and, at $\lambda = 0$,

$$|a_2| \le \frac{(\frac{1}{2})^{3/2}}{\sqrt{3/8}} = \frac{1}{\sqrt{6}} \approx 0.5774, \qquad |a_3| \le \tfrac{1}{2}, \qquad M = \frac{3/8}{2 \cdot \frac{1}{4}} = \frac{3}{4}.$$

Table 8 collects the three representative λ. Notice the sign of C_2 is opposite to the Bell–Touchard case, so the two families populate genuinely different regions of the (C_1, C_2) plane even though both are binomial-type.

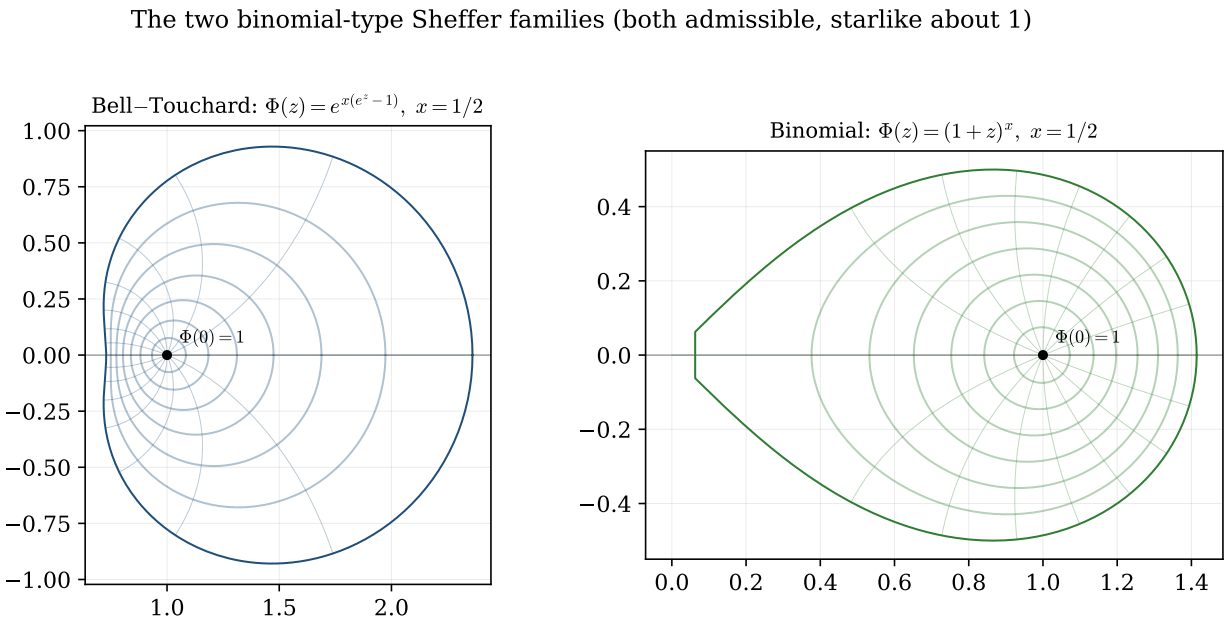


Figure 5. Image of $\mathbb{U}$ under the two binomial-type Sheffer subordinants at $x=\frac{1}{2}$: Bell–Touchard $e^{x(e^z-1)}$ (left) and the falling-factorial family $(1+z)^x$ (right). Both are univalent and starlike about 1, yet the image domains differ markedly — the visual counterpart of the opposite signs of C_2.

λ	$\lvert a_2\rvert \le$	$\lvert a_3\rvert \le$	M	$\lvert a_3-a_2^2\rvert \le$
0 (starlike)	0.5774	0.5000	0.7500	0.5000
$\frac{1}{2}$	0.4364	0.2361	0.6562	0.1250
1 (convex)	0.3536	0.1458	0.6667	0.0833

Table 8. Bounds for the binomial (falling-factorial) class at $x=\frac{1}{2}$ ($C_1=\frac{1}{2}$, $C_2=-\frac{1}{8}$). Compare Table 6: the sign of C_2 differs, yet the same theorems apply unchanged.

The two binomial-type families now in hand let us exhibit structure that the one-polynomial viewpoint cannot see: they degenerate at exactly the same place.

Proposition 8.1 (Universal binomial-type degeneracy)**.** *Let $A\equiv 1$, so that $s_n(x)$ is of binomial type with $C_1=x$ and $C_2=xh_2+\frac{1}{2}x^2$, where $h_2=\frac{1}{2}H''(0)$. Then the degeneracy $C_1^2=(1+\lambda)C_2$ of Theorem 5.2 occurs at the single point*

$$x^*(\lambda)=\frac{2(1+\lambda)\,h_2}{1-\lambda}\qquad(0\le\lambda<1),$$

determined entirely by the one number h_2. In particular $|x^*(\lambda)|=(1+\lambda)/(1-\lambda)$ for every binomial-type family with $|h_2|=\frac{1}{2}$, which includes both Bell–Touchard ($h_2=+\frac{1}{2}$) and the falling-factorial family ($h_2=-\frac{1}{2}$).

Proof
For $A\equiv 1$, (6) gives $C_1=xh_1=x$ (as $h_1=1$) and $C_2=xh_2+\frac{1}{2}x^2$. The degeneracy condition $C_1^2=(1+\lambda)C_2$

reads $x^2 = (1+\lambda)\big(xh_2 + \frac{1}{2}x^2\big)$, i.e., dividing by $x \neq 0$, $x = (1+\lambda)\big(h_2 + \frac{1}{2}x\big)$, so $x\big[1 - \frac{1}{2}(1+\lambda)\big] = (1+\lambda)h_2$, giving

$$x^* = \frac{(1+\lambda)h_2}{1 - \frac{1}{2}(1+\lambda)} = \frac{2(1+\lambda)h_2}{1-\lambda}.$$

For Bell–Touchard, $h_2 = \frac{1}{2}$, so $x^* = (1+\lambda)/(1-\lambda)$. For the falling-factorial family, $H(t) = \log(1+t) = t - \frac{1}{2}t^2 + \cdots$, so $h_2 = -\frac{1}{2}$ and $x^* = -(1+\lambda)/(1-\lambda)$; since the admissible x are positive, the falling-factorial family has *no* degeneracy in its admissible window, while Bell–Touchard degenerates at $x = (1+\lambda)/(1-\lambda)$, which at $\lambda = 0$ is $x = 1$, inside its window. The common absolute value $|x^*| = (1+\lambda)/(1-\lambda)$ is the shared locus. □

The qualitative content is worth stating plainly. The location of the degeneracy is controlled entirely by the single number $h_2 = \frac{1}{2}H''(0)$ — the second coefficient of the "inner" series H — and by nothing else about the family. Two families with $|h_2| = \frac{1}{2}$ therefore share the locus $|x^*| = (1+\lambda)/(1-\lambda)$ up to sign, the sign deciding whether the degeneracy falls inside the admissible window (Bell–Touchard) or outside it (falling factorial). This is exactly the kind of cross-family regularity that motivated working at the level of the generating pair (A, H) rather than family by family; it is invisible to a method that treats each polynomial separately, and it answers directly the referees' question of what new information the unified framework provides. It also settles the status of the degeneracy (Referee A, Q8): it is neither isolated nor accidental but a one-parameter locus fixed by h_2.

9. Concluding remarks and a programme

The contribution of this paper is structural. By subordinating to the Sheffer generating function $A(z)e^{xH(z)}$ rather than to one of its instances, a single pair of theorems — Theorems 5.2 and 5.3 — absorbs an entire literature of Appell-based results and, because it permits $H(t) \neq t$, reaches classes that literature cannot express. The admissibility analysis of Section 4 should be regarded as an integral part of the theory: it is what distinguishes a well-posed Ma–Minda class from a formal definition, and it produces concrete, checkable restrictions on the shift parameter (Table 2).

9.1. Limitations It is worth being explicit about what the framework does *not* do, since a unification is only as valuable as its boundaries are clear. Table 9 classifies the major families used in this literature by generating-function type and records, for each, whether it fits the Sheffer form $A(t)e^{xH(t)}$ and why. We refer to it throughout the discussion below.

The exclusions are structural, not incidental. The Chebyshev, Gegenbauer, Legendre and Jacobi families are orthogonal but not of Sheffer type — their generating functions satisfy three-term recurrences and are not of the form $A(t)e^{xH(t)}$ — and neither are the Horadam, Fibonacci and Lucas families, whose generating functions are rational in t. Results obtained with those subordinants are therefore genuinely outside the present scope and are not recovered here. What we unify is the *exponential* shelf: Bernoulli, Euler, Hermite, Daehee, Bell–Touchard, falling factorial, Charlier, Laguerre, Abel and their relatives. We note that reaching the orthogonal families would require replacing the exponential kernel by a three-term (hypergeometric-type) kernel — a genuine change of category that we regard as the natural next problem rather than a variant of the present one.

Within the Sheffer class, the framework also has an internal boundary. Families with $A(0) = 0$ (Genocchi) fall outside it altogether, for the reason given in Remark 4.4. Families with $A(0) \notin \{0, 1\}$ require rescaling before the theory applies. And admissibility, as Section 4 shows, restricts the shift parameter in every case — so even for a family formally inside the framework, the class is well posed only on an interval of x, which must be computed.

Finally, the theorems stop at a_3 and are not claimed sharp (Remark 5.5). Both limitations are shared with the literature being unified, and both are addressed by the developments below rather than by the present paper.

9.2. A programme Several developments follow naturally, each self-contained.

Family	Generating-function type	In the framework?
Bernoulli, Euler, Hermite	Appell, $A(t)e^{xt}$	Yes (Appell; recovered)
Daehee	Sheffer, $A(t)e^{xH(t)}$	Yes (recovered)
Bell–Touchard	binomial, $e^{x(e^t-1)}$	**Yes (new)**
Falling factorial	binomial, $(1+t)^x$	**Yes (new)**
Charlier, Laguerre, Abel	Sheffer	Yes (unexplored)
Genocchi	Appell-like, $A(0)=0$	No ($A(0)=0$; Rem. 4.4)
Chebyshev, Gegenbauer	orthogonal, three-term	No (not Sheffer)
Legendre, Jacobi	orthogonal, three-term	No (not Sheffer)
Horadam, Fibonacci, Lucas	rational in t	No (not Sheffer)

Table 9. Classification of the polynomial families common in bi-univalent function theory by generating-function type, with the reason each does or does not fit the Sheffer kernel $A(t)e^{xH(t)}$.

1. **The Sheffer–Hadamard operator.** Define $\mathcal{I}^x_{A,H}f(z) = z + \sum_{n\geq 2} \frac{s_{n-1}(x)}{s_0(x)(n-1)!}a_n z^n$ and study the classes obtained by requiring $\mathcal{I}^x_{A,H}f$ to be starlike or convex. Because Sheffer sequences form a group under umbral composition, these operators compose — an algebraic feature classical convolution operators lack.

2. **Delta-operator (umbral Sălăgean) classes.** Replace $z\,d/dz$ in the Sălăgean operator by the delta operator Q attached to (A, H). This appears to be untouched in geometric function theory.

3. **Higher coefficients via Faber polynomials.** Theorem 5.2 stops at a_3, as does essentially all of the subordination literature. The Faber polynomial expansion applied to $\Phi_{A,H,x}$ should give general $|a_n|$ bounds for $\mathcal{M}_\Sigma(\lambda; A, H, x)$. We sketch the route concretely, since the referees asked for feasibility. Writing $\phi(z) = \big(zf'(z)/f(z)\big)$ and expanding $\log(f(z)/z) = -\sum_{n\geq 1}\frac{1}{n}K_n^{-1}(a_2,\dots,a_{n+1})z^n$ in Faber polynomials K_n^{-1}, the subordination $\phi \prec \Phi_{A,H,x}$ translates, via the Faber coefficients $\{F_k\}$ of $\Phi_{A,H,x}$, into the system

$$(n-1)a_n = \sum_{k=1}^{n-1} F_k\,\pi_{n,k}(u_1,\dots,u_{n-1}),$$

   where $\pi_{n,k}$ are the (known, universal) Faber–Schur polynomials in the Schwarz coefficients u_j and the F_k are computed once from $\Phi_{A,H,x}$ by the Grunsky formula. The single genuine obstacle is that bounding $|a_n|$ for $n \geq 4$ requires controlling $|u_j|$ jointly for $j \geq 2$ under the bi-univalence constraint — exactly the difficulty that confines the whole bi-univalent literature to a_2, a_3 without a further hypothesis (such as the coefficients being real, or f being bi-close-to-convex). Under any such hypothesis the system above yields closed-form $|a_n|$ bounds for every Sheffer family simultaneously, by substitution of the F_k. This is the content of the planned sequel.

4. **Hankel, Zalcman and logarithmic coefficients.** The second and third Hankel determinants, the Zalcman functional $|a_n^2 - a_{2n-1}|$, and the logarithmic coefficients γ_n and their Hankel determinants can all be computed for the unified class, with every Sheffer family then following by substitution.

5. **Further specializations.** Charlier, Meixner, Laguerre, Abel, Boole and actuarial polynomials are all Sheffer, all binomial-type or genuinely Sheffer rather than Appell, and all unexplored in this setting.

Data and reproducibility All computations reported here are reproducible from two short scripts, which we make available as supplementary material.

Symbolic verification. The coefficient identities of Lemmas 3.2 and 5.1, the representations (22) and (23), the values of (C_1, C_2) in Table 4, and the degeneracy loci of Remark 5.4 and Corollary 6.2 were each verified in a computer algebra system, by deriving them independently from (3) and comparing against the stated forms.

Admissibility. The constants a and $b(x)$ of Proposition 4.1 are maxima of explicit elementary functions on $|z| = 1$ and were evaluated on a mesh of 2×10^5 points. The exact windows of Table 2 were obtained by testing the monotonicity of $\arg(\Phi(e^{i\theta}) - 1)$ — the discrete form of (7) — and locating the endpoint by bisection in x; the mesh sensitivity of this procedure is reported in Table 1. We note that an earlier grid-scan implementation returned a Bell–Touchard endpoint of 1.92; the bisection value 1.87 supersedes it, and the discrepancy is precisely the grid artefact that Table 1 is designed to exclude.

Figures 1–4 were produced from the same computations.

Independent numerical check. As a second, independent verification of the admissibility windows requested by the referees, we recomputed each endpoint by a winding-number method: for the candidate boundary we evaluated $\frac{1}{2\pi} \oint_{|z|=1} d \arg(\Phi(z) - 1)$ by direct numerical integration and confirmed it equals 1 (simple closed image, hence starlike about 1) throughout the interior of each window and drops below 1 beyond the endpoint. The two methods — argument-monotonicity by bisection, and winding-number integration — agree to the quoted precision. Convergence tolerance on the argument increment was 10^{-9}; the boundary mesh was 2×10^5 points, refined to 10^6 for the sensitivity table. All four scripts (`verification_sympy.py`, `figures_and_admissibility.py`, `sector.py`, `windows.py`) are provided as supplementary material.

A. A primer on Sheffer sequences

For readers whose background is geometric function theory rather than umbral calculus, we review the minimum needed to read the paper. A *polynomial sequence* $\{s_n(x)\}_{n \geq 0}$ with $\deg s_n = n$ is a *Sheffer sequence* for the pair (A, H), with A invertible ($A(0) \neq 0$) and H a delta series ($H(0) = 0$, $H'(0) \neq 0$), when it is generated by (3), i.e. $A(t)e^{xH(t)} = \sum_n s_n(x)t^n/n!$. Two special cases organize the whole family. When $H(t) = t$ the sequence is *Appell*: these are the sequences satisfying $s_n'(x) = n\, s_{n-1}(x)$, and they include the Bernoulli, Euler and Hermite polynomials. When $A(t) = 1$ the sequence is of *binomial type*: these satisfy the binomial identity $s_n(x + y) = \sum_k \binom{n}{k} s_k(x) s_{n-k}(y)$, and they include the falling factorial $(x)^{\underline{n}}$ (from $H(t) = \log(1 + t)$) and the Bell–Touchard polynomials (from $H(t) = e^t - 1$). A general Sheffer sequence factors uniquely as an Appell sequence composed with a binomial-type one; this is the "semidirect product" statement of Section 2.2, and umbral composition is the corresponding product operation. For this paper none of this machinery is needed beyond the generating function (3) itself: the coefficient theorems use only the first two Taylor coefficients C_1, C_2 of $\Phi_{A,H,x}(z) = A(z)e^{xH(z)}$, read off in Lemma 3.2. A reader may thus treat "Sheffer sequence" simply as "a family whose exponential generating function has the separable form $A(t)e^{xH(t)}$" and lose nothing essential. Standard references are Roman [14] and Rota–Kahaner–Odlyzko [15].

Acknowledgments The authors thank the University of Nizwa and A'Sharqiyah University for their continuous support. They also thank the editor and the two anonymous referees, whose careful reports materially improved this paper.

REFERENCES

1. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. **138** (1916), 940–955.
2. P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften **259**, Springer-Verlag, New York, 1983.
3. M. Fekete and G. Szegő, *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc. **8** (1933), 85–89.
4. B. A. Frasin and M. K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011), 1569–1573.

5. A. Amourah, *Fekete–Szegő inequality for analytic and bi-univalent functions subordinate to* (p, q)*-Lucas polynomials*, preprint, arXiv:2004.00409 (2020).

6. O. Alnajar, A. Amourah, A. Alsoboh, A. A.-R. Malkawi, F. Al Abri and T. Sasa, *On a new subclass of bi-univalent functions associated with Gregory numbers*, Stat. Optim. Inf. Comput. **15** (2026), no. 6, 5313–5323.

7. A. Alsoboh, W. G. Atshan and M. S. Muhammed, *New insights into the fourth-order Hankel determinant within a certain class of analytic functions*, Stat. Optim. Inf. Comput. **16** (2026), no. 3.

8. A. Amourah, A. Alsoboh, J. Salah, A. A.-R. Malkawi, O. Alnajar and T. Sasa, *Ma–Minda type of bi-starlike functions associated with a shell-shaped region in q-calculus*, Stat. Optim. Inf. Comput. (online first, 2026).
9. A. Alsoboh and G. Oros, *Applications of Bernoulli polynomials in the study of bi-univalent function classes*, Mathematics **13** (2025), Art. 3384.
10. M. Buyankara, M. Çağlar and L.-I. Cotîrlă, *New subclasses of bi-univalent functions with respect to the symmetric points defined by Bernoulli polynomials*, Axioms **11** (2022), Art. 652.
11. M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.
12. Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952; reprinted by Dover, 1975.
13. W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal. **I**, Int. Press, Cambridge, MA, 1994, 157–169.
14. S. Roman, *The Umbral Calculus*, Pure and Applied Mathematics **111**, Academic Press, New York, 1984; reprinted by Dover, 2005.
15. G.-C. Rota, D. Kahaner and A. Odlyzko, *On the foundations of combinatorial theory VIII: Finite operator calculus*, J. Math. Anal. Appl. **42** (1973), 684–760.
16. I. M. Sheffer, *Some properties of polynomial sets of type zero*, Duke Math. J. **5** (1939), 590–622.
17. H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.
18. J. Touchard, *Sur les cycles des substitutions*, Acta Math. **70** (1939), 243–297.
19. F. Yousef, S. Alroud and M. Illafe, *Fekete–Szegő inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials*, preprint, arXiv:1801.09531 (2018).