STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–22.
Published online in International Academic Press (www.IAPress.org)

Uncertainty quantification in finite queues with dependent breakdowns

Khadidja Boudane[1,*], Baya Takhedmit[2], Karim Abbas[2]

[1] *Department of Mathematics, University of Bouira and Research Unit LaMOS, Department of Operational Research, Faculty of Exact Sciences, University of Bejaia, Algeria*
[2] *Research Unit LaMOS, Department of Operational Research, Faculty of Exact Sciences, University of Bejaia, Algeria*

Abstract This paper proposes a new framework within which one can incorporate uncertainty in input parameters to compute performance measures of queueing models with dependent breakdowns. The proposed methodology allows us to characterize statistically the performance measures of the studied queueing models. The developed approach is suited for queueing models in which the stationary distribution of the associated Markov chain describing their state can be numerically assessed, so then one can encompass randomness after the numerical evaluation. Specifically, an efficient implementation of computational algorithm is investigated to analyze the M/G/1/N queueing model with dependent breakdowns. Various quantities of interest characterizing the uncertain performance measures of the M/G/1/N queueing model with dependent breakdowns are obtained; especially we are interested in computing the mean, the variance, the skewness and the kurtosis of the underlying performance. In addition, we show how to use the Markov's inequality to estimate the risk incurred by ignoring parameter uncertainty in computing performance measures of queueing models with dependent breakdowns. Several numerical examples are presented to illustrate the potential of the proposed approach.

Keywords Queues with dependent breakdowns, Markov chain, Epistemic uncertainty, Uncertainty propagation, Markov risk bounds, Monte Carlo simulation

DOI: 10.19139/soic-2310-5070-4329

1. Introduction

Queueing models with breakdowns and repairs are useful for modeling a wide variety of real problems arising in production systems, flexible manufacturing systems, computer networks and communication systems, etc. The influence of server's breakdowns on the performance of such real systems is frequently not taken into account. However, server's breakdowns have a major impact on the performance measures of a system. For this reason, in realistic environments, considering a queueing model with breakdowns and repair is also practical and imperative. Thereby the analysis of queueing systems becomes much more challenging through the occurrence of breakdowns. The detailed account of queueing models with breakdowns and repairs along with their applications can be found in [23, 24, 26].

The modeling and performance evaluation of real world systems can be affected by various sources of uncertainty. It is common to classify uncertainty into two categories: epistemic uncertainty, resulting from imperfect knowledge or lack of exact information, and aleatory uncertainty, i.e. uncertainty due to variability representing the natural variability and inherent randomness. The most important distinction between these two types of uncertainty is that the former can conceivably be quantified and reduced by gathering the right information, whereas the latter is uncertainty that is beyond our current ability to reduce by collecting more information; that is typically seen as irreducible. Assessing the impact of these uncertainties is the focus of the emerging

*Correspondence to: Khadidja Boudane (Email: khadidjaboudane@yahoo.fr). Department of Mathematics, University of Bouira and Research Unit LaMOS, Department of Operational Research, Faculty of Exact Sciences, University of Bejaia, Algeria.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

field of *”Uncertainty Quantification”* in applied sciences. The challenge of a significant class of uncertainty quantification approaches is to model the uncertainty inflicted on the input parameters, which is estimated from measurements, and to propagate that uncertainty to the performance measures of the studied stochastic model. There has been a proliferation of research on this purpose and several methods were developed for efficient stochastic computations associated with uncertainty quantification of real world systems. Some of the newly developed uncertainty quantification methods include the simulation based methods like Monte Carlo simulation [9, 22], the most probable point based methods [10], the local expansion-based methods like the Taylor series method or perturbation method [5, 8, 25], the functional expansion-based methods [4, 6, 28], the numerical integration-based methods [19, 29], etc.

The analysis of an important class of stochastic systems involves uncertainty in parameters. Frequently little data is available concerning the variability of the key input parameters required for a predictive analysis. This has led to widespread use of several uncertainty descriptions. Many authors have proposed frameworks to properly model such uncertainty. In the applied literature, queueing models are usually analyzed by Monte Carlo simulation. Furthermore this one provides a framework that has many advantages over conventional methods in the uncertainty quantification; especially that of performance measures of complex systems. It is relatively simple to implement, and provides more general and robust models of uncertainty. A comprehensive presentation of the Monte Carlo simulation method is given in [22].

In this paper we develop efficient algorithm based on the use of the Monte Carlo simulation method to characterize statistically the uncertain performance measures of the M/G/1/N queueing model with dependent breakdowns. More formally, let θ denote the probability of a server breakdown; then, as a novel methodological contribution, we will model the time between occurrence of breakdowns of the server by means of a parameterized distribution, and investigate the impact of this distribution on the performance measures of the studied queueing model. We compute the stationary distribution of the queue-length process, denoted by π_θ, and other performances under the assumption that the probability θ is uncertain. Indeed, observations on the time period between occurrence of breakdowns of the server is typically only available in censored form and thereby estimating mean times of such periods is a statistical challenge. This is in contrast to the other times of processes involved in the model, which are observable and controllable. For this reason, we are typically confronted with uncertainty concerning the true value of the parameters defining the distributions of the time between breakdowns. Uncertainty or variability in the values of the probability of a server breakdown θ is of concern in the present work. In carrying out uncertainty quantification, the probability θ is now considered to be random variable. As it varies over its support and takes values in the sub-interval $[a, b] \subseteq [0, 1]$, then the rectangular distribution on $[a, b]$, denoted by $\mathcal{U}[a, b]$, is the natural candidate to model the uncertainty distribution of the breakdown probability θ.

In order to translate the uncertainty in the model's input parameter θ into uncertainty in the model's output metric of interest π_θ, an obvious approach would be to choose a sequence of reference points $\{\theta_n \,|\, n = 1, \ldots, N_{MC}\}$ (N_{MC} typically large), with say, from the generated sample with $\mathcal{U}[a, b]$, and then compute numerically $\{\pi_{\theta_n} \,|\, n = 1, \ldots, N_{MC}\}$ for each point θ_n. Using classical numerical methods, the stationary distribution π_{θ_n} will be obtained numerically by solving the system of equations $\pi_{\theta_n} P_{\theta_n} = \pi_{\theta_n}$ and $\pi_{\theta_n} e = 1$, where P_{θ_n} denotes the transition probability matrix of the embedded Markov chain, describing the state of the M/G/1/N queueing model with dependent breakdowns, and e is the unit vector; details will be provided later in the text. We repeat this numerical procedure and obtain a relative sample for $\{\pi_{\theta_n}\}$, of the same size as the sample of the input parameter. The same procedure is repeated as many times as the fixed number of replications according to the Monte Carlo simulation paradigm. From that one can estimate many quantities of interest characterizing the uncertain performance measures of the M/G/1/N queueing model with dependent breakdowns; especially we compute the mean, the variance, the skewness and the kurtosis of the underlying performance.

Another key feature of our approach is the ability of estimating the risk incurred by working with uncertain performance measures rather than those evaluated at fixed parameters. Therefore, in addition to uncertainty quantification, decision-makers often have to grapple with the following question: *What is the possible impact of ignoring parameter uncertainty on the assessment of system performance measures?* There has been a rapid increase in numbers of research on the uncertainty quantification and a most of uncertainty analysis techniques were developed for this purpose. However, to the best of our knowledge, none of the existing approaches exploit the risk analysis of queueing models with dependent breakdowns. The framework developed in this paper enables to combine uncertainty quantification and risk analysis in an efficient way. It is worth noting that the obtained numerical results are non-trivial and intuitively explainable.

More generally, this paper draws attention to uncertainty quantification and risk analysis of queueing models with dependent breakdowns. Such queues are typically more realistic rather than classical queueing models. The rest of the paper is organized as follows. Section 2 reviews relevant related literature. In Section 3, we introduce the M/G/1/N queueing model with dependent breakdown. The uncertainty analysis of the M/G/1/N queueing model with dependent breakdown is carried out in Section 4. The estimation of the risk incurred by working with uncertain performance measures rather than evaluated at fixed parameters is sketched out in Section 5. Conclusions and future research directions are drawn in Section 6.

2. Related literature

The uncertainty quantification is drawing increasing attention in the queueing design community, since the subsequent predictions of the true system responses are of great interest in designing robust systems. In the following we briefly survey the relevant works on integration of epistemic uncertainty into queueing models.

In recent years, several contributions have been published in the context of uncertainty analysis of queueing models. In this direction, Takhedmit and Abbas [27] used Taylor series expansion method to investigate uncertainties quantification in performance measures of M/G/1/N queue with uncertain vacation parameter. In [21], using the same approach with an emphasis on perturbation analysis and the well-known random-variable-transformation method, the probability density functions of the stationary distribution of M/G/1 retrial queue with two way communication and finite orbit are estimated. Bachi et al. [4] developed a methodology based on the power series expansion to propagate parameter epistemic uncertainties in Markov chain models. In another contribution, the same authors [5] have used analytic method based on multivariate Taylor series expansions, including the effect of the correlation between the uncertain input parameters, to propagate epistemic uncertainty to output metrics in Markovian models. Later, Soufit et al. [25] have provided an uncertainty analysis for queueing-inventory models, and used the copulas theory to examine the dependence structure between the effect of the dependence structure of parameters. In [6] a numerical approach based on chaos expansions has developed to analyze the sensitivity and the propagation of epistemic uncertainty through a queueing models with breakdowns. This latter work represents the first systematic effort to compute Sobol' indices for sensitivity analysis in queueing models. Recently, Berkhout et al. [7] have provided an interesting framework to approximately computing the value at risk incurred by parameter uncertainty for analyzing generalized Jackson networks. Moreover, an uncertainty analysis based on Monte Carlo simulation of queueing models has been performed by Haverkort and Meeuwissen [13]. A discussion on parameter uncertainty from a broader perspective has been reported by Henderson [14]. Antonelli et al. [1] have developed a method for propagating the epistemic uncertainty in input parameters to these output measures of the M/M/1 queueing models, where an applicability of their approach is demonstrated for the performance analysis of a cloud-based system.

A number of authors have introduced the topic of probability theory with imprecise information. This methodology is an alternative to the classical theory, where it is defined new notions for obtaining a probability measure induced by a family of interval-valued parametric distribution function. Hesamian and Chachi [15] have discussed the way of constructing the probability measure of random variable from the known probability density function, for analyzing the M/M/1 queue and the system reliability assessment. Lopatatzidis et al. [20] have addressed the problem of performance predictions of the Geo/Geo/1/N queue by using the framework of interval probabilities. In a series of more general works on fuzzy probability theory, another approach based on using fuzzy sets [30] to cope with uncertainty has been applied extensively to queueing models under fuzzy environment, and specifically to those with server breakdowns; see, for example [16, 17, 18], and references therein. An additional task related to the area of uncertainty quantification is the inverse assessment of parameter uncertainty. More particularly, the inverse uncertainty quantification estimates mainly the values of unknown parameters in the model. This kind of problems has been typically tackled in queueing theory by Baccelli et al. [2].

Although all these contributions are concerned by a common modeling framework, our present work is novel in two important respects.

- While presenting the used approach and analyzing the case studies, for simplicity the authors have limited their analysis to queueing models where making the assumption that all the input parameters are independent from the number of customers in the system. However, for the considered case studies, such an assumption seems to be in contrast with the actual models. So, the main scope of this work is to develop a new computational approach based on Monte Carlo simulation for propagating uncertainty in input-output queueing models with an assumption that the breakdown probability dependent on the number of successful service completions since the last breakdown. Therefore, it is certain that introducing the dependence of the probability breakdowns on the number of successful service completions since the last breakdown is very interesting and makes the studied queueing models more strong.

- During the epistemic uncertainty analysis concerning the queueing models, to the best of our knowledge, the problem of evaluating the risk incurred by parameter uncertainty was never raised. In fact, in most studies, the authors were only interested in the statistical estimation of model outputs, which consists particularly to compute their mean and variance. In this work, in addition to uncertainty quantification, we develop an approach to risk-analysis of queueing model with dependent breakdown, which consists to assess numerically the possible impact of ignoring parameter uncertainty.

3. The M/G/1/N queueing model with dependent breakdowns

Consider a M/G/1/N queue with server's breakdowns and repairs. Customers arrive according to a Poisson process with rate λ. There can at most be N customers be present at the queue, including the one in service, and arriving customers that find the queue full are assumed to be blocked and lost. The service time provided by a single server is an independent and identically distributed random variable with a general distribution function $S(x)$, having finite mean $1/\mu$. The server can serve only one customer at a time. Customers are served according to the first-come, first-served (FCFS) discipline. Assume that the probability of a server breakdown increases with the number of successful service completions since the last breakdown, which means that the servers wears down.

Formally, let the state of the system be denoted by (i,j), where i is the number of customers in the system right after either a service completion or a repair completion, and j is the number of successful service completions since the last breakdown. At the end of each service, there is a probability $\theta(j)$ that the server breaks down and enters a repair state, the length of which is exponentially distributed with rate r and which is independent of everything else, and with probability $(1-\theta(j))$ the server is operationally and serves the customer. The only points in time where a possible server breakdown can occur is right at the beginning of a service. Noting that we can obtain the stationary distribution of this embedded Markov chain once the transition probabilities are available. The state-transition probability matrix of the embedded jump chain (i,j) is given by

$$P^{\theta}_{(0,j),(i^*,j+1)} = (1-\theta(j)) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{i^*}}{i^*!}\, dS(x)$$

and

$$P^{\theta}_{(0,j),(i^*,0)} = \theta(j) \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{i^*},$$

for $1 \le i^* \le N-1$. For $0 < i \le N-1$, we have

$$P^{\theta}_{(i,j),(i^*,j^*)} = (1-\theta(j)) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{i^*-i+1}}{(i^*-i+1)!}_{\{i^*-i+1\ge 0\}}\, dS(x)$$

and

$$P^{\theta}_{(i,j),(i^*,0)} = \theta(j) \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{i^*-i+1}_{\{i^*-i+1\ge 0\}}.$$

$$P^{\theta}_{(0,j1),(i2,j1+1)} = (1-\theta(j1)) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{i2}}{i2!}\, dS(x)$$

and

$$P^{\theta}_{(0,j1),(i2,0)} = \theta(j1) \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{i2},$$

for $1 \leq i2 \leq N-1$. For $0 < i1 \leq N-1$, we have

$$P^{\theta}_{(i1,j1),(i2,j2)} = (1-\theta(j1)) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{i2-i1+1}}{(i2-i1+1)!}\, dS(x)$$

and

$$P^{\theta}_{(i1,j1),(i2,0)} = \theta(j1) \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{i2-i1+1}_{\{i2-i1+1 \geq 0\}}.$$

As pointed out in Section 2, very few papers have studied epistemic uncertainty quantification for queuing models and, to the best of our knowledge, its application to queuing models with dependent breakdowns is relatively unexplored. In particular, it is worth noting that no study has been done on the M/G/1/N queueing model with dependent breakdowns; and some queueing models (without dependance) discussed in the literature can be obtained as special cases of our model. For example, if we ignore the dependency evoked in the model associated with $\theta(j)$, i.e. $\theta(j) = \theta$ the model reduces to M/G/1/N queue with classical breakdowns which has been studied in [6] for a different train of thought and another aim.

Regarding the model associated with $\theta(j)$, one can identify the following models for the dependence of the breakdown probability on the number of successive services.

Geometric decay: For $\theta \in (0,1)$ we let

$$\theta(j) = \theta^j.$$

Linear decay: For $\theta \in (0,1)$ we let

$$\theta(j) = \left(1 - \frac{1}{(1+j)^b}\right)\theta,$$

for some constant $0 < b \leq 1$.

Stepwise decay: Let $M_k \in \mathbb{N}$, $1 \leq k \leq K$, such that

$$0 = M_0 < M_1 < M_2 < \cdots < M_K < M_{K+1} = \infty.$$

For $\theta \in (0,1)$ we let $\theta(j) = \theta^k$ for $M_{k-1} \leq j < M_k$.

For our analysis, we will only consider the model corresponding to the geometric decay. It may be remarked that for fixed θ, this model has an algorithmically tractable solution. In this case, one interesting feature of this kind of models is the stationary distribution, which can be obtained numerically by solving the system of equations: $\pi_\theta P_\theta = \pi_\theta$ and $\pi_\theta e = 1$, where e denotes the unit vector. Nevertheless, incorporating uncertainty in structural input parameter θ usually makes the model more realistic but also more complex to analyze. In the sequel, we develop a new computational algorithm that allows us to provide an efficient way of characterizing statistically the uncertain performance measures of the M/G/1/N queueing model with dependent breakdowns.

4. The uncertainty analysis of the M/G/1/N queueing model with dependent breakdown

In this section we sketch more details of our basic approach, and propose an efficient computational algorithm to uncertainty analysis of the M/G/1/N queue with dependent breakdowns. In uncertainty quantification approaches, the uncertain model parameters can be considered as random variables, which allows us to take into account the effect that this randomness of the input parameters has on the performance measures of the queueing model. Thereby, in order to describe the underlying randomness of the uncertain parameters, we introduce a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω denotes the event space equipped with σ-algebra $\mathcal{F}$ and probability measure $\mathbf{P}$. As outlined in the introduction, the probability of a server breakdown is harsh to assess using historical data. Typically, we are confronted with uncertainty concerning the true value of the parameters defining the distributions of the time between occurrence of breakdowns, in our case the probability of a server breakdown θ. Thereby, we assume that the parameter epistemic uncertainty will be represented by the rectangular distribution on $[a, b]$, in writing, $\theta(\omega) \sim \mathcal{U}[a, b]$, where $\omega \in \Omega$. In the sequel, the focus is on uncertainty analysis of the M/G/1/N queue with dependent breakdowns with respect to epistemic uncertainty inflicted on the breakdowns probability.

The developed algorithm for uncertainty quantification of our queueing model is described in Section 4.1. The uncertainty analysis of the considered models is provided in Section 4.2.

INPUTS : Fix the queue capacity: N;
Fix the arrival rate: λ;
Fix the size sample: N_{MC};
Fix the number of replications: N_R.

BEGIN

STEP 0
  Define the service density function: $s(x)$; and **set** its parameters;
END
STEP 1
  Choose a specific epistemic uncertainty distribution associated to the random variable $\theta(\omega)$;
END
STEP 2
  Generate a sample for the random variable $\theta(\omega)$ of size N_{MC} according to the chosen epistemic uncertainty distribution;
END
STEP 3
  for $n = 1 : N_{MC}$ **do**
    Calculate the transition probability matrix P_{θ_n} at each point θ_n of the sample associated with $\theta(\omega)$;
    Solve the system of equations: $\pi_{\theta_n} P_{\theta_n} = \pi_{\theta_n}$ and $\pi_{\theta_n} e = 1$;
  end
  Acquire the sample associated with π_θ of size N_{MC};
  Compute the main quantities of interest related to the obtained sample associated with $\pi_{\theta(\omega)}$;
END
STEP 4
  Repeat N_R times the procedures of **STEPS 2-3**; then **estimate** the final quantities of interest of the induced results.
END
OUTPUTS: The expected value: $\mathbb{E}\left[perf(\omega)\right]$;
The variance: $\mathbb{V}ar\left[perf(\omega)\right]$;
The skewness coefficient: $\mathbb{S}kew\left[perf(\omega)\right]$;
The kurtosis coefficient: $\mathbb{K}urt\left[perf(\omega)\right]$.

END

Algorithm 1: Uncertainty quantification algorithm.

4.1. Uncertainty quantification algorithm

In this section we present the main steps of a computational algorithm that permits us to understand thoroughly the procedure of obtaining the quantities of interest, that specify certain characteristics of the uncertain performance measures of our queueing model. This performance will be denoted by *perf*(ω) and it can be the stationary distribution, the blocking probability, the average number of customers, the average waiting time of an arbitrary customer in the system or any other performance. The whole computational procedures of our basic approach are summarized in the algorithm 1.

The choice of the probability distribution, raised in Step 1, for expressing the parameter epistemic uncertainty is of importance. While we are dealing with an uncertain input parameter $\theta(\omega)$, which is the probability of a server breakdown, so we will typically choose a particular distribution that takes its values on a support $[a, b] \subseteq [0, 1]$. In this sense the most suitable choice is that of the rectangular distribution $\mathcal{U}[a, b]$. Nevertheless, considering another uncertainty distribution by taking into account this specificity of the random variable $\theta(\omega)$ and based on available statistical data also makes sense. For example, one can think to the triangular distribution.

Remark 1
It is worth noting that the system of equations evoked in Step 3 serves to propagate the epistemic uncertainty from the input parameter θ to the stationary distribution π_θ. This system represents a functional linear relationship expressed as $\pi_{\theta(\omega)} = g(\theta(\omega))$ which links the input parameter to the output measures, where g is a linear function. Solving this system of equations at fixed points $\{\theta_n\}$, with, say, $a \leq \theta_n \leq b$, then one can obtain a set of solutions $\{\pi_{\theta_n}\}$ which constitutes a sample for the uncertain stationary distribution $\pi_{\theta(\omega)}$, in notation $\pi(\omega)$. This approach is suited for queueing models in which the stationary distribution can be numerically solved, so then one can insert randomness after the numerical evaluation.

It is well known that the service process plays a major role in the performance evaluation of queueing systems. An appropriate way for defining the service density function, mentioned in Step 0, is to choose distributions with different coefficients of variation (CV). In the following we illustrate the uncertainty analysis of the M/G/1/N queue with dependent breakdowns with the help of four examples.

4.2. Uncertainty analysis

In this section we focus on the numerical evaluation, under epistemic uncertainty, of the key uncertain performance measures of the M/G/1/N queueing model with dependent breakdowns. We are typically interested in statistical characterization of the stationary distribution, the blocking probability, the average number of customers and the average waiting time of an arbitrary customer in the system. More specifically, the main important performance measure for a finite buffer queue is the blocking probability; see (**??**), which can be obtained as follows

$$PBL(\omega) := \frac{\pi_0(\omega) + \rho - 1}{\pi_0(\omega) + \rho}, \tag{1}$$

where $\pi_0(\omega)$ is the first component of the uncertain stationary distribution $\pi(\omega)$.

Let $L_{\theta(\omega)}$ the number of customers in the system which is also a transformation of the random variable $\theta(\omega)$, in notation $L(\omega)$. This useful measure is defined as follows

$$L(\omega) := \sum_{k=1}^{N} k\ \pi_k(\omega), \tag{2}$$

where $\pi_k(\omega)$ is the k-th component of the stationary distribution $\pi(\omega)$.

In the same vein, the average waiting time of an arbitrary customer in the system can be deduced by means of Little's rule, which is given by

$$W(\omega) := \frac{L(\omega)}{\hat{\lambda}(\omega)}, \tag{3}$$

where $\hat{\lambda}(\omega) = \lambda(1 - PBL(\omega))$ is the effective arrival rate.

In this setting, the performance measures $PBL(\omega)$, $L(\omega)$ and $W(\omega)$ depend on the values of the random variable $\theta(\omega)$, which means that themselves are random variables. In other words, we know $PBL(\omega)$, $L(\omega)$ and $W(\omega)$ only conditionally on the values of random variable $\theta(\omega)$, thereby we have

$$PBL(\omega)\big|_{\theta(\omega)=\theta} = \frac{\pi_0+\rho-1}{\pi_0+\rho},\ L(\omega)\big|_{\theta(\omega)=\theta} = \sum_{k=1}^{N} k\ \pi_k,\ W(\omega)\big|_{\theta(\omega)=\theta} = \frac{L}{\hat{\lambda}}, \tag{4}$$

where $\hat{\lambda} = \lambda(1-PBL)$.

Taking into account this randomness, this brings us to estimate the statistical characteristics of these performances such as the expectation value, the variance, the skewness and the kurtosis. The procedure of obtaining these quantities of interest consists primary in using the resulting sample relating to the random variable $\pi(\omega)$ obtained by our algorithm with the help of the definitions given in (4). This gives us other samples that characterizing the uncertain performances $PBL(\omega)$, $L(\omega)$ and $W(\omega)$. Then, it suffices to utilize the induced samples to estimate their quantities of interest.

Remark 2
It is worth noting that the expectation and the variance of $L(\omega)$ and $W(\omega)$ can be straightforwardly estimated as follows

$$\mathbb{E}\left[L(\omega)\right] = \sum_{k=0}^{N} k\ \mathbb{E}\left[\pi_k(\omega)\right], \tag{5}$$

and

$$\begin{aligned}\mathbb{E}\left[W(\omega)\right] &= \int \frac{L}{\hat{\lambda}}\, f_{\theta(\omega)}(\theta) d\theta \\ &= \frac{1}{b-a}\int_a^b \frac{L}{\hat{\lambda}}\ {}_{\{\theta\in[a,b]\}}(\theta)\, d\theta,\end{aligned} \tag{6}$$

where $f_{\theta(\omega)}$ is probability density function of the random variable $\theta(\omega)$ and ${}_{\{.\}}$ denotes the indicator function.

Similarly, following the same lines, one can also assess

$$\mathbb{V}ar\left[L(\omega)\right] = \sum_{k=1}^{N} k^2\ \mathbb{V}ar\left[\pi_k(\omega)\right] + \sum_{k=1}^{N}\sum_{l=1,k\neq l}^{N} k\, l\ \mathbb{C}ov(\pi_k(\omega), \pi_l(\omega)), \tag{7}$$

where

$$\mathbb{C}ov(\pi_k(\omega), \pi_l(\omega)) = \mathbb{E}\left[\left(\pi_k(\omega) - \mathbb{E}\left[\pi_k(\omega)\right]\right)\left(\pi_l(\omega) - \mathbb{E}\left[\pi_l(\omega)\right]\right)\right], \tag{8}$$

and

$$\mathbb{V}ar\left[W(\omega)\right] = \mathbb{E}\left[\left(W(\omega) - \mathbb{E}\left[W(\omega)\right]\right)^2\right]. \tag{9}$$

In addition to estimation of the expected value and the variance of the uncertain performance measure $perf(\omega)$, one can also estimate other statistical characteristics such as the skewness and the kurtosis coefficient. Thereby, the skewness coefficient of the uncertain performance measure $perf(\omega)$, that measures the asymmetry of probability distribution around its mean can be straightforwardly assessed as follows

$$\mathbb{S}kew\left[perf(\omega)\right] := \frac{\mathbb{E}\left[(perf(\omega) - \mathbb{E}[perf(\omega)])^3\right]}{\mathbb{V}ar\left[perf(\omega)\right]^{3/2}}. \tag{10}$$

In the similar way, the kurtosis coefficient describing the shape of the probability distribution of $perf(\omega)$ can also evaluated as follows

$$\mathbb{K}urt\left[perf(\omega)\right] := \frac{\mathbb{E}\left[(perf(\omega) - \mathbb{E}[perf(\omega)])^4\right]}{\left(\mathbb{E}\left[(perf(\omega) - \mathbb{E}[perf(\omega)])^2\right]\right)^2}. \tag{11}$$

Thereby, for example, the skewness coefficient of the uncertain stationary distribution $\pi(\omega)$, that measures the asymmetry of probability distribution around its mean can be assessed as follows

$$\mathbb{S}kew\left[\pi(\omega)\right] := \frac{\mathbb{E}\left[(\pi_{\theta(\omega)} - \mathbb{E}[\pi_{\theta(\omega)}])^3\right]}{\mathbb{V}ar\left[\pi_{\theta(\omega)}\right]^{3/2}}. \tag{12}$$

In the similar way, the kurtosis coefficient describing the shape of the probability distribution of $\pi(\omega)$ can also evaluated as follows

$$\mathbb{K}urt\left[\pi(\omega)\right] := \frac{\mathbb{E}\left[(\pi(\omega) - \mathbb{E}[\pi(\omega)])^4\right]}{\left(\mathbb{E}\left[(\pi(\omega) - \mathbb{E}[\pi(\omega)])^2\right]\right)^2}. \tag{13}$$

Due to the two-dimensionality of the embedded Markov chain (i, j), describing the state of the M/G/1/N queue with dependent breakdowns, then its stationary distribution $\pi_{(i,j)}(\omega)$ defined as the probability to have in steady-state i customers in the system, and j successfully served customers since the last breakdown. In this case, it may be noted here that $\pi_i(\omega)$, $0 \le i \le N-1$, is actually defined as

$$\pi_i(\omega) := \sum_{j=0}^{N-1} \pi_{(i,j)}(\omega). \tag{14}$$

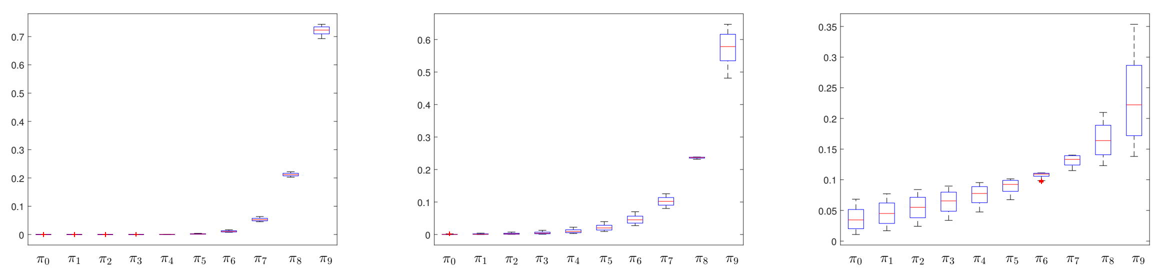
In what follows, we apply our algorithm for quantifying uncertainty in the M/G/1/N queueing model with dependent breakdown, with the purpose of characterizing statistically its performance measures. We also compare our results with those generated by the Taylor series expansion (TSE) method presented in [27].
For that, we provide a variety of numerical results for various performance measures and the results are presented via tables and graphs. All the computations are done on Matlab Software Package and they are reported in seven decimal places. As outlined above, we experiment with four service time distributions to obtain different CV. Specifically, we use the deterministic distribution (CV $= 0$), the two stage generalized Erlang distribution (CV is between $1/\sqrt{2}$ and 1), denoted E_2, the exponential distribution (CV $= 1$), and the hyperexponential distribution (CV ≥ 1), denoted HE_2, which is mixture of two exponential distributions.

4.2.1. The M/D/1/N queueing model with dependent breakdowns In this section we assume that the distribution of the service is deterministic, with d denoting the deterministic service time. By applying our algorithm, we will set out a number of statistical characteristics of interest for the M/D/1/N queue with dependent breakdowns. For that, we fix the buffer space $N = 10$, the arrival rate $\lambda = 1$, the repair rate $r = 1$, and the deterministic service time $d = 1.5$. Concerning the model associated with $\theta(j)$, we always retain the geometric decay model for the dependence of the breakdown probability on the number of successive services. Furthermore, to perform the numerical experiments, we suppose that the breakdown probability θ is uncertain. For that, we consider three scenarios for this uncertain parameter: $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$, $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$ and $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$. For all numerical experiments, we fix the sample size of $\theta(\omega)$ to $N_{MC} = 10000$.

Since the input parameter θ is random, the resulting stationary distribution calculated via Algorithm 1, is a random variable. In Table 1 we provide the computational results of main statistical characteristics of interest for the stationary distribution of M/D/1/10 queue with dependent breakdowns. Typically, we obtain estimated values for its expectation, variance, skewness and kurtosis. In addition, we also visualize the resulting sample for each component of the uncertain stationary distribution $\pi(\omega)$ with boxplots, where other statistical characteristics such as the first quartile, the median and the third quartile are pointed out, The same performance metrics were computed using the TSE approach (see Table 2), revealing very close results. Figure 1 displays boxplots of the obtained sample for the uncertain stationary distribution $\pi(\omega)$ of M/D/1/10 queue with dependent breakdowns. Other statistical characteristics of system performance measures of interest for this queue are also presented in Table 3 and 4.

	$\pi_k(\omega)$	$\mathbb{E}\left[\pi_k(\omega)\right]$	$\mathbb{V}ar\left[\pi_k(\omega)\right]$	$\mathbb{S}kew\left[\pi_k(\omega)\right]$	$\mathbb{K}urt\left[\pi_k(\omega)\right]$
	$\pi_0(\omega)$	0.0000005	0.0000000	1.6676995	4.8102007
	$\pi_1(\omega)$	0.0000028	0.0000000	1.4886544	4.1843850
	$\pi_2(\omega)$	0.0000135	0.0000000	1.2790345	3.5500303
	$\pi_3(\omega)$	0.0000672	0.0000000	1.0472341	2.9685531
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_4(\omega)$	0.0003498	0.0000000	0.8014581	2.4876096
	$\pi_5(\omega)$	0.0020110	0.0000008	0.6010069	2.1875438
	$\pi_6(\omega)$	0.0111488	0.0000065	0.4489939	2.0139977
	$\pi_7(\omega)$	0.0529996	0.0000281	0.3063444	1.8974504
	$\pi_8(\omega)$	0.2119732	0.0000300	0.0658769	1.8015854
	$\pi_9(\omega)$	0.7214331	0.0002117	-0.2726925	1.8872224
	$\pi_0(\omega)$	0.0005760	0.0000003	1.1563585	3.2489394
	$\pi_1(\omega)$	0.0012710	0.0000012	1.0116319	2.9053816
	$\pi_2(\omega)$	0.0025280	0.0000039	0.8717494	2.6184914
	$\pi_3(\omega)$	0.0050340	0.0000111	0.7192051	2.3546129
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_4(\omega)$	0.0102939	0.0000304	0.5587333	2.1309883
	$\pi_5(\omega)$	0.0215560	0.0000762	0.3958777	1.9596992
	$\pi_6(\omega)$	0.0461844	0.0001537	0.2331168	1.8427557
	$\pi_7(\omega)$	0.1021740	0.0001759	0.0446727	1.7703257
	$\pi_8(\omega)$	0.2363439	0.0000036	-0.8004509	2.5463952
	$\pi_9(\omega)$	0.5740383	0.0022702	-0.2438874	1.8675080
	$\pi_0(\omega)$	0.0362347	0.0003040	0.2201196	1.7458652
	$\pi_1(\omega)$	0.0456201	0.0003372	0.0795993	1.6918520
	$\pi_2(\omega)$	0.0546514	0.0003352	-0.0425797	1.6813347
	$\pi_3(\omega)$	0.0640731	0.0002930	-0.1651406	1.7065522
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_4(\omega)$	0.0753341	0.0002136	-0.3196571	1.7838839
	$\pi_5(\omega)$	0.0895218	0.0001073	-0.5793049	2.0149389
	$\pi_6(\omega)$	0.1076248	0.0000156	-1.2677249	3.4340309
	$\pi_7(\omega)$	0.1312002	0.0000673	-0.4762200	1.8098044
	$\pi_8(\omega)$	0.1649384	0.0007011	0.0739019	1.6874898
	$\pi_9(\omega)$	0.2308009	0.0041772	0.2819098	1.7931359

Table 1. Statistical characteristics of interest for the stationary distribution of M/D/1/10 queue with dependent breakdowns via MCS.



(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 1. Boxplots of the uncertain stationary distribution $\pi(\omega)$ of M/D/1/10 queue with dependent breakdowns.

	$\pi_k(\omega)$	$\mathbb{E}[\pi_k(\omega)]$	$\mathbb{V}ar[\pi_k(\omega)]$	$\mathbb{S}kew[\pi_k(\omega)]$	$\mathbb{K}urt[\pi_k(\omega)]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_0(\omega)$	0.0000010	0.0000000	1.6676999	4.8102012
	$\pi_1(\omega)$	0.0000031	0.0000005	1.4886550	4.1843856
	$\pi_2(\omega)$	0.0000141	0.0000011	1.2790348	3.5500310
	$\pi_3(\omega)$	0.0000711	0.0000004	1.0472344	3.1187649
	$\pi_4(\omega)$	0.0003505	0.0000011	0.8014602	2.4876133
	$\pi_5(\omega)$	0.0020114	0.0000012	0.6010115	2.1875504
	$\pi_6(\omega)$	0.0115033	0.0000082	0.45066489	2.0144307
	$\pi_7(\omega)$	0.0530225	0.0000311	0.3063450	2.3206897
	$\pi_8(\omega)$	0.2125640	0.0000287	0.0667795	1.8015868
	$\pi_9(\omega)$	0.7214333	0.0002123	-0.2726920	1.8872232
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_0(\omega)$	0.0005988	0.0000027	1.1564223	3.2489393
	$\pi_1(\omega)$	0.0012721	0.0000011	1.0116401	2.9167706
	$\pi_2(\omega)$	0.0025288	0.0000034	0.8717532	2.6225618
	$\pi_3(\omega)$	0.0050343	0.0000108	0.7192121	2.3546133
	$\pi_4(\omega)$	0.0102939	0.0000304	0.5587333	2.1309883
	$\pi_5(\omega)$	0.0215564	0.0001150	0.4021395	1.9633459
	$\pi_6(\omega)$	0.0461864	0.0001552	0.2331211	1.8438727
	$\pi_7(\omega)$	0.1021751	0.0001857	0.0446733	1.7703114
	$\pi_8(\omega)$	0.2363445	0.0000040	-0.8004531	2.5464020
	$\pi_9(\omega)$	0.5740388	0.0022715	-0.2438887	1.8675089
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_0(\omega)$	0.0362353	0.0003045	0.2201200	1.7458661
	$\pi_1(\omega)$	0.0456209	0.0003474	0.1156707	1.6918633
	$\pi_2(\omega)$	0.0546518	0.0003355	-0.0425813	1.6813356
	$\pi_3(\omega)$	0.06410407	0.0003119	-0.1651404	1.7066355
	$\pi_4(\omega)$	0.0753344	0.0002140	-0.3196568	1.7838843
	$\pi_5(\omega)$	0.0895220	0.0001074	-0.5793116	2.0149394
	$\pi_6(\omega)$	0.1076254	0.0000200	-1.2677251	3.4340311
	$\pi_7(\omega)$	0.1312006	0.0000675	-0.4762102	1.8098048
	$\pi_8(\omega)$	0.1649391	0.0007014	0.0739022	1.68750114
	$\pi_9(\omega)$	0.2308013	0.0041774	0.2819210	1.7931441

Table 2. Statistical characteristics of interest for the stationary distribution of M/D/1/10 queue with dependent breakdowns via TSE.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$PBL(\omega)$	0.3333335	0.0000000	1.6679481	4.8101942
	$L(\omega)$	9.6382611	0.0008473	-0.3723125	1.9587057
	$W(\omega)$	14.4573955	0.0019064	-0.3722811	1.9586689
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$PBL(\omega)$	0.3335891	0.0000000	1.1554825	3.2459587
	$L(\omega)$	9.2192311	0.0305004	-0.4384480	2.0239633
	$W(\omega)$	13.8302743	0.0681949	-0.4361005	2.0212968
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$PBL(\omega)$	0.3489741	0.0000543	0.1964850	1.7331416
	$L(\omega)$	7.0207207	0.3978720	0.0739505	1.7060313
	$W(\omega)$	10.8340198	0.8869829	0.0822939	1.7069835

Table 3. Statistical Analysis of M/D/1/10 Queue Performance with Dependent Breakdowns via Monte Carlo Simulation.

		$\mathbb{E}\,[.]$	$\mathbb{V}ar\,[.]$	$\mathbb{S}kew\,[.]$	$\mathbb{K}urt\,[.]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$PBL(\omega)$	0.2830039	0.0000010	1.6399451	4.7911426
	$L(\omega)$	9.6485601	0.0008673	-0.2922135	2.0846239
	$W(\omega)$	14.6648902	0.0020845	-0.3900432	2.1096743
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$PBL(\omega)$	0.3345620	0.0000000	1.1554825	3.2467944
	$L(\omega)$	9.2195634	0.0306543	-0.4384321	2.0239086
	$W(\omega)$	13.8801246	0.0680345	-0.4674008	2.0245321
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$PBL(\omega)$	0.2996754	0.0000098	0.2005408	1.0598672
	$L(\omega)$	7.5640344	0.4176409	0.0546707	1.6789021
	$W(\omega)$	10.3452901	0.5550897	0.0664532	1.6609874

Table 4. Statistical Analysis of M/D/1/10 Queue Performance with Dependent Breakdowns via Taylor Series Expansions.

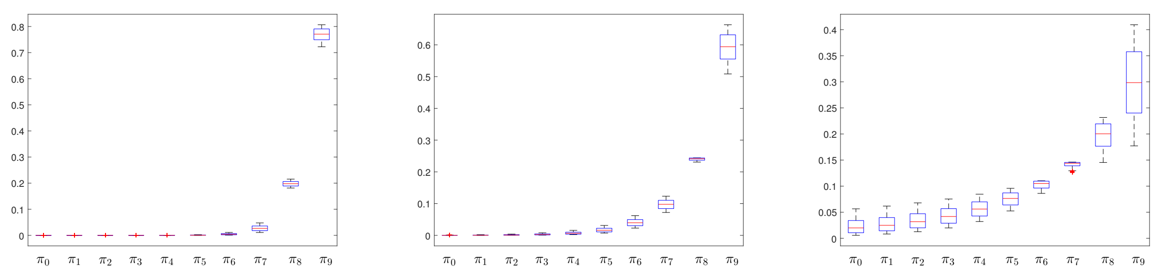
4.2.2. The M/E$_2$/1/N queueing model with dependent breakdowns In this section we are mainly interested in uncertainty quantification in the M/E$_2$/1/N queue with dependent breakdowns. The service times are assumed to follow two stage generalized Erlang distribution whose density function with two parameters μ_1 and μ_2 ($\mu_1 \neq \mu_2$) is given by

$$s(x) = \frac{\mu_1 \mu_2}{\mu_2 - \mu_1} \left(e^{-\mu_1 x} - e^{-\mu_2 x} \right), \, x \geq 0.$$

Note that, the CV of the E$_2$ distribution is given by

$$CV = (\mu_1^2 + \mu_2^2)^{1/2} / (\mu_1 + \mu_2).$$

For numerical illustrations we set the following numerical values: $\lambda = 1$, $r = 1$, $\mu_1 = 2$ and $\mu_2 = 1$ (CV $= 0.7453560$). The other parameters are fixed to the similar values as those given in Section 4.2.1. We now investigate the uncertain performance measures of the M/E$_2$/1/10 queue with dependent breakdowns. As in the previous example, the proposed algorithm allows us to acquire the most relevant statistical characteristics of system performance measures of interest. The corresponding results to the mean, the variance, the skewness and the kurtosis of the stationary distribution $\pi(\omega)$ are listed in Table 5. Similarly, boxplots relative to each component of the obtained sample for the stationary distribution $\pi(\omega)$ of the M/E$_2$/1/10 queue with dependent breakdowns are shown in Figure 2. Furthermore, in Table 6 we give the computational results of the statistical characterization of the main performance metrics of the studied queueing model. Identical performance measures were calculated via the TSE method Table (7 and 8), showing highly similar outcomes.



(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 2. Boxplots of the uncertain stationary distribution $\pi(\omega)$ of M/E$_2$/1/10 queue with dependent breakdowns.

	$\pi_k(\omega)$	$\mathbb{E}[\pi_k(\omega)]$	$\mathbb{V}ar[\pi_k(\omega)]$	$\mathbb{S}kew[\pi_k(\omega)]$	$\mathbb{K}urt[\pi_k(\omega)]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_0(\omega)$	0.0000002	0.0000000	2.0236650	6.3677406
	$\pi_1(\omega)$	0.0000012	0.0000000	1.8211032	5.5102145
	$\pi_2(\omega)$	0.0000059	0.0000000	1.6061326	4.6986224
	$\pi_3(\omega)$	0.0000297	0.0000000	1.3735279	3.9327871
	$\pi_4(\omega)$	0.0001543	0.0000000	1.1173048	3.2231941
	$\pi_5(\omega)$	0.0008275	0.0000004	0.8349678	2.6045360
	$\pi_6(\omega)$	0.0046795	0.0000084	0.5220490	2.1224449
	$\pi_7(\omega)$	0.0273333	0.0001085	0.2024948	1.8512854
	$\pi_8(\omega)$	0.1979023	0.0000944	0.0464819	1.7872770
	$\pi_9(\omega)$	0.7690656	0.0005697	-0.2031709	1.8580329
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_0(\omega)$	0.0003292	0.0000000	1.1562142	3.3622071
	$\pi_1(\omega)$	0.0006590	0.0000003	1.0101432	3.0058941
	$\pi_2(\omega)$	0.0014125	0.0000010	0.8533967	2.6723295
	$\pi_3(\omega)$	0.0031651	0.0000039	0.6943665	2.3865694
	$\pi_4(\omega)$	0.0072717	0.0000140	0.5342558	2.1535444
	$\pi_5(\omega)$	0.0170061	0.0000455	0.3708211	1.9727808
	$\pi_6(\omega)$	0.0404171	0.0001235	0.1996656	1.8451315
	$\pi_7(\omega)$	0.0975873	0.0002160	-0.0015323	1.7730511
	$\pi_8(\omega)$	0.2400220	0.0000164	-0.6622116	2.0524283
	$\pi_9(\omega)$	0.5921294	0.0019207	-0.1477028	1.8349805
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_0(\omega)$	0.0233982	0.0002036	0.6216968	2.2394443
	$\pi_1(\omega)$	0.0280466	0.0002297	0.5298282	2.1147224
	$\pi_2(\omega)$	0.0344168	0.0002489	0.4205208	1.9907253
	$\pi_3(\omega)$	0.0435631	0.0002546	0.2860173	1.8717455
	$\pi_4(\omega)$	0.0567107	0.0002321	0.1159816	1.7721229
	$\pi_5(\omega)$	0.0755584	0.0001642	-0.1265833	1.7247835
	$\pi_6(\omega)$	0.1025798	0.0000554	-0.6335468	2.0006493
	$\pi_7(\omega)$	0.1414871	0.0000283	-1.3181858	3.6701572
	$\pi_8(\omega)$	0.1967377	0.0006151	-0.3707519	1.9359658
	$\pi_9(\omega)$	0.2975012	0.0044418	-0.0630740	1.7822095

Table 5. Statistical characteristics of interest for the stationary distribution of M/E_2/1/10 queue with dependent breakdowns via MCS.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$PBL(\omega)$	0.3333334	0.0000000	2.0267032	6.3677353
	$L(\omega)$	9.7290779	0.0018417	-0.2963033	1.9238469
	$W(\omega)$	14.5936252	0.0041439	-0.2962893	1.9238295
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$PBL(\omega)$	0.3334796	0.0000000	1.1573075	3.3603531
	$L(\omega)$	9.3017243	0.0191304	-0.3279513	1.9602223
	$W(\omega)$	13.9605487	0.0429016	-0.3261434	1.9583519
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$PBL(\omega)$	0.3435156	0.0000378	0.5989019	2.2014312
	$L(\omega)$	7.6661088	0.3386103	-0.3156778	1.9121188
	$W(\omega)$	11.8771449	0.7689653	-0.3079754	1.9058532

Table 6. Statistical characteristics of interest for the main performance measures of M/E_2/1/10 queue with dependent breakdowns via MCS.

	$\pi_k(\omega)$	$\mathbb{E}[\pi_k(\omega)]$	$\mathbb{V}ar[\pi_k(\omega)]$	$\mathbb{S}kew[\pi_k(\omega)]$	$\mathbb{K}urt[\pi_k(\omega)]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_0(\omega)$	0.0000103	0.0000000	2.0236436	6.3688623
	$\pi_1(\omega)$	0.0000115	0.0000004	1.8235244	5.5137885
	$\pi_2(\omega)$	0.0000602	0.0000002	1.6154988	4.7598003
	$\pi_3(\omega)$	0.0000346	0.000000	1.4563752	4.5373287
	$\pi_4(\omega)$	0.0002000	0.0000000	1.1220534	3.2234310
	$\pi_5(\omega)$	0.0020825	0.0000011	1.0567333	0.0453666
	$\pi_6(\omega)$	0.0056079	0.0000100	0.4952200	2.1224459
	$\pi_7(\omega)$	0.0330466	0.0001284	0.2190594	2.8521854
	$\pi_8(\omega)$	0.3977043	0.0002014	0.5509829	2.0457872
	$\pi_9(\omega)$	0.7690656	0.0005697	-0.2031709	2.2288032
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_0(\omega)$	0.0004102	0.0000003	1.2162432	3.5320074
	$\pi_1(\omega)$	0.0006950	0.0000019	1.01019870	3.0233209
	$\pi_2(\omega)$	0.0014215	0.0000017	2.00536177	3.2674673
	$\pi_3(\omega)$	0.0032655	0.0000046	1.25287503	2.55749022
	$\pi_4(\omega)$	0.0170521	0.0000144	0.8554258	2.1153566
	$\pi_5(\omega)$	0.0210561	0.0000565	0.4008611	2.0927008
	$\pi_6(\omega)$	0.0441791	0.0001239	0.2552656	2.1844451
	$\pi_7(\omega)$	0.117874	0.0002220	-0.0019343	2.0730611
	$\pi_8(\omega)$	0.2400220	0.0000164	-0.6622116	2.1082483
	$\pi_9(\omega)$	0.7052124	0.0024214	-0.1257138	2.4863398
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_0(\omega)$	0.0265382	0.0002112	0.8006918	2.2539224
	$\pi_1(\omega)$	0.0280466	0.0002297	0.5298282	2.1472124
	$\pi_2(\omega)$	0.0344226	0.0004284	0.7405820	2.0907625
	$\pi_3(\omega)$	0.0443535	0.0003056	0.3092601	2.2118717
	$\pi_4(\omega)$	0.100756	0.0002553	0.1268159	2.0046772
	$\pi_5(\omega)$	0.1075958	0.0002462	-0.1006533	2.1387247
	$\pi_6(\omega)$	0.1113570	0.0001009	-0.5635446	2.0011694
	$\pi_7(\omega)$	0.1415413	0.0000331	-1.1134781	3.9107527
	$\pi_8(\omega)$	0.2386777	0.0013461	-0.3519005	2.3433965
	$\pi_9(\omega)$	0.2975012	0.0044418	-0.0630740	2.5437220

Table 7. Statistical characteristics of interest for the stationary distribution of M/E_2/1/10 queue with dependent breakdowns via STE.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$PBL(\omega)$	0.3333336	0.0000000	2.0267213	6.3677545
	$L(\omega)$	10.2100722	0.0022678	-0.2244089	2.2645339
	$W(\omega)$	14.1513679	0.0041556	-0.2645539	2.0445692
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$PBL(\omega)$	0.3335786	0.0000012	1.1573977	3.6554836
	$L(\omega)$	9.5567301	0.0220913	-0.2423279	2.3306960
	$W(\omega)$	14.4387966	0.0443792	-0.2205326	2.0334295
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$PBL(\omega)$	0.3466955	0.0000400	1.0032275	2.2014877
	$L(\omega)$	8.2046666	0.3446038	-0.2556403	2.4329125
	$W(\omega)$	12.2155870	1.0833276	-0.3365707	2.0069058

Table 8. Statistical characteristics of interest for the main performance measures of M/E_2/1/10 queue with dependent breakdowns via TSE.

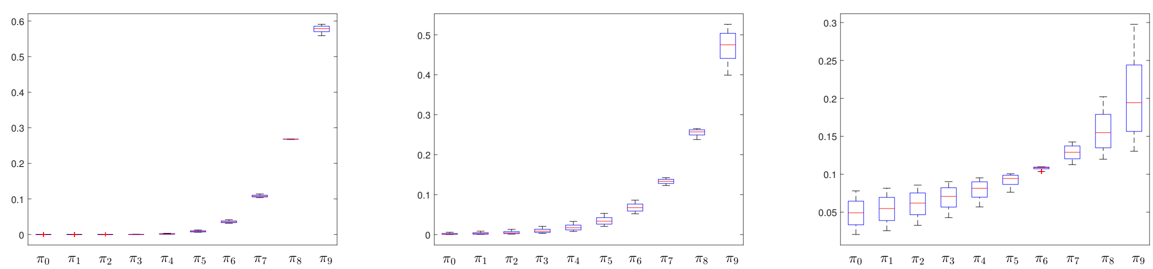
4.2.3. The M/M/1/N queueing model with dependent breakdowns The focus of this section is uncertainty quantification in the M/M/1/N queue with dependent breakdowns. The service time distribution is exponentially distributed with rate μ. Using our algorithm we obtained the results for this queueing model with the following fixed parameters: $\lambda = 1$, $r = 1$ and $\mu = 2/3$. The other parameters remain unchanged. In our numerical computation, the different estimated values for the statistical characteristics relating to the uncertain stationary distribution of the considered queue are presented in Table 9 and in Figure 3. The TSE approach was used to evaluate the same performance metrics (Table 10 with Table 12), yielding nearly identical results. The main performance measures of this model are characterized statistically by the quantities obtained numerically and displayed in Table 11.

	$\pi_k(\omega)$	$\mathbb{E}[\pi_k(\omega)]$	$\mathbb{V}ar[\pi_k(\omega)]$	$\mathbb{S}kew[\pi_k(\omega)]$	$\mathbb{K}urt[\pi_k(\omega)]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_0(\omega)$	0.0000049	0.0000000	1.4966675	4.2185773
	$\pi_1(\omega)$	0.0000187	0.0000000	1.2806544	3.5607375
	$\pi_2(\omega)$	0.0000773	0.0000000	1.0483145	2.9741817
	$\pi_3(\omega)$	0.0003342	0.0000000	0.7985949	2.4833247
	$\pi_4(\omega)$	0.0015304	0.0000007	0.5386169	2.1235316
	$\pi_5(\omega)$	0.0088333	0.0000033	0.4324344	2.0097785
	$\pi_6(\omega)$	0.0358059	0.0000081	0.3689475	1.9514329
	$\pi_7(\omega)$	0.1080922	0.0000088	0.2812677	1.8859566
	$\pi_8(\omega)$	0.2678683	0.0000001	-0.8590647	2.3986875
	$\pi_9(\omega)$	0.5774343	0.0000840	-0.3302236	1.9280852
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_0(\omega)$	0.0018548	0.0000024	0.9365668	2.7395467
	$\pi_1(\omega)$	0.0030097	0.0000050	0.8166762	2.5136658
	$\pi_2(\omega)$	0.0052598	0.0000111	0.6800008	2.2933810
	$\pi_3(\omega)$	0.0096027	0.0000247	0.5368580	2.1050400
	$\pi_4(\omega)$	0.0180281	0.0000518	0.3932518	1.9594249
	$\pi_5(\omega)$	0.0346257	0.0000889	0.2669274	1.8640824
	$\pi_6(\omega)$	0.0678474	0.0001002	0.1472861	1.8003426
	$\pi_7(\omega)$	0.1331034	0.0000345	-0.1048674	1.7437190
	$\pi_8(\omega)$	0.2553157	0.0000612	-0.5570079	2.1236020
	$\pi_9(\omega)$	0.4713523	0.0013411	-0.2709065	1.8780485
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_0(\omega)$	0.0489266	0.0003007	0.0107555	1.7094104
	$\pi_1(\omega)$	0.0541070	0.0002866	-0.0539700	1.7121012
	$\pi_2(\omega)$	0.0607621	0.0002556	-0.1322407	1.7282586
	$\pi_3(\omega)$	0.0690967	0.0002038	-0.2295209	1.7668774
	$\pi_4(\omega)$	0.0794299	0.0001318	-0.3649633	1.8520527
	$\pi_5(\omega)$	0.0921732	0.0000524	-0.6248814	2.1082955
	$\pi_6(\omega)$	0.1080441	0.0000025	-1.0091078	3.2562644
	$\pi_7(\omega)$	0.1286019	0.0000857	-0.1176765	1.6815394
	$\pi_8(\omega)$	0.1572319	0.0006106	0.1887999	1.7521273
	$\pi_9(\omega)$	0.2016260	0.0024860	0.3058635	1.8243202

Table 9. Statistical characteristics of interest for the stationary distribution of M/M/1/10 queue with dependent breakdowns via MCS.

	$\pi_k(\omega)$	$\mathbb{E}\left[\pi_k(\omega)\right]$	$\mathbb{V}ar\left[\pi_k(\omega)\right]$	$\mathbb{S}kew\left[\pi_k(\omega)\right]$	$\mathbb{K}urt\left[\pi_k(\omega)\right]$
	$\pi_0(\omega)$	0.0000051	0.0000001	1.4966679	4.2185889
	$\pi_1(\omega)$	0.0000298	0.0000000	1.2806544	3.5605685
	$\pi_2(\omega)$	0.0002014	0.0000020	1.0483028	2.9777401
	$\pi_3(\omega)$	0.0003432	0.0000000	1.9874487	2.4833247
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_4(\omega)$	0.0015304	0.0000007	0.5386169	2.1235316
	$\pi_5(\omega)$	0.0089922	0.0000037	0.4324349	2.0097789
	$\pi_6(\omega)$	0.0358226	0.0000098	0.3689499	1.9514553
	$\pi_7(\omega)$	0.1084778	0.0000122	0.2813699	1.8880069
	$\pi_8(\omega)$	0.2679973	0.0000008	-0.8591654	2.3986888
	$\pi_9(\omega)$	0.5774339	0.0000844	-0.3302201	1.9280863
	$\pi_0(\omega)$	0.0018566	0.0000024	0.9365697	2.7395679
	$\pi_1(\omega)$	0.0030121	0.0000051	0.8166880	2.5136678
	$\pi_2(\omega)$	0.0052760	0.0000111	0.6800320	2.2934481
	$\pi_3(\omega)$	0.0098803	0.0000258	0.5368584	2.1050566
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_4(\omega)$	0.0180299	0.0000520	0.3932809	1.9594557
	$\pi_5(\omega)$	0.0357829	0.0001994	0.2669670	1.8642341
	$\pi_6(\omega)$	0.0678479	0.0001110	0.1474984	1.8003433
	$\pi_7(\omega)$	0.1331044	0.0003245	-0.1048671	1.7437231
	$\pi_8(\omega)$	0.2553213	0.0001217	-0.5570077	2.1236195
	$\pi_9(\omega)$	0.4713545	0.0013501	-0.2709105	1.8780489
	$\pi_0(\omega)$	0.0489268	0.0003011	0.0107560	1.7094106
	$\pi_1(\omega)$	0.0541114	0.0002872	-0.0539687	1.7121023
	$\pi_2(\omega)$	0.0607630	0.0002574	-0.1322413	1.7282592
	$\pi_3(\omega)$	0.0691974	0.0002051	-0.2295213	1.7668794
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_4(\omega)$	0.0794354	0.0001322	-0.3649638	1.8520529
	$\pi_5(\omega)$	0.0921735	0.0000530	-0.6248811	2.1082974
	$\pi_6(\omega)$	0.1080450	0.0000031	-1.0091115	3.2562656
	$\pi_7(\omega)$	0.1286022	0.0001961	-0.1176766	1.6815411
	$\pi_8(\omega)$	0.1572325	0.0006110	1.0000011	1.7521274
	$\pi_9(\omega)$	0.2016264	0.0025043	0.3058641	1.8243200

Table 10. Statistical characteristics of interest for the stationary distribution of M/M/1/10 queue with dependent breakdowns via TSE.



(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 3. Boxplots of the uncertain stationary distribution $\pi(\omega)$ of M/M/1/10 queue with dependent breakdowns.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
	$PBL(\omega)$	0.3333355	0.0000000	1.4968792	4.2185301
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$L(\omega)$	9.3627970	0.0008236	-0.4174854	2.0018766
	$W(\omega)$	14.0442167	0.0018525	-0.4172505	2.0015944
	$PBL(\omega)$	0.3341559	0.0000004	0.9340156	2.7333606
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$L(\omega)$	8.9092295	0.0374145	-0.4329682	2.0104599
	$W(\omega)$	13.3770156	0.0829787	-0.4288476	2.0061553
	$PBL(\omega)$	0.3543107	0.0000522	-0.0130609	1.7091229
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$L(\omega)$	6.7335498	0.3029902	0.1601259	1.7451878
	$W(\omega)$	10.6019127	0.6945122	0.1657000	1.7471684

Table 11. Statistical characteristics of interest for the main performance measures of M/M/1/10 queue with dependent breakdowns via MCS.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
	$PBL(\omega)$	0.3333355	0.0000000	1.4968793	4.2185305
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$L(\omega)$	9.3627972	0.0008241	-0.417482	2.0018768
	$W(\omega)$	14.0442161	0.0018533	-0.4172500	2.0015948
	$PBL(\omega)$	0.3341600	0.0000004	0.9340159	2.7333614
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$L(\omega)$	8.9092316	0.0374149	-0.4329676	2.0105231
	$W(\omega)$	13.3770163	0.0833654	-0.4288467	2.0061556
	$PBL(\omega)$	0.3543121	0.0000529	-0.0130625	1.7091233
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$L(\omega)$	6.7335500	0.3029920	0.1601270	1.7452025
	$W(\omega)$	10.6019144	0.6945129	0.1657008	1.7472002

Table 12. Statistical characteristics of interest for the main performance measures of M/M/1/10 queue with dependent breakdowns via TSE.

4.2.4. The M/HE$_2$/1/N queueing model with dependent breakdowns This section deals with the uncertainty quantification in the M/HE$_2$/1/N queue with dependent breakdowns. Let us assume that the service times follow an hyperexponential distribution with the following density function

$$s(x) = \gamma\mu_1 e^{-\mu_1 x} + (1-\gamma)\mu_2 e^{-\mu_2 x}, \;\; x \geq 0,$$

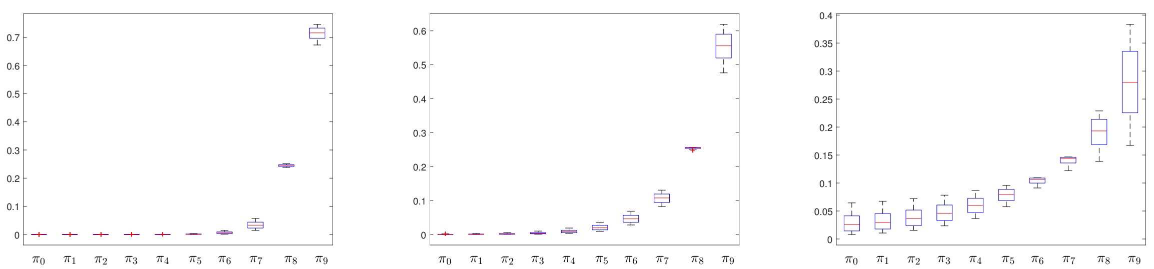
where $0 \leq \gamma \leq 1$. Its expected value $1/\mu$ is equal to $\gamma/\mu_1 + (1-\gamma)/\mu_2$. Note that the CV of this distribution is given by

$$CV = \sqrt{\frac{1+(2\gamma-1)^2}{1-(2\gamma-1)^2}}.$$

We apply Algorithm 1 and obtain the numerical results, relating to the uncertainty analysis of this queue, for the following fixed parameters: $\lambda = 1$, $r = 1$, $\mu_1 = 1.5$, $\mu_2 = 0.6$ and $\gamma = 0.4$ (CV $= 1.0408330$). Other parameters values are same as those given in Section 4.2.1. The quantities of interest specifying statistical characteristics of the uncertain stationary distribution $\pi(\omega)$ of the M/HE$_2$/1/10 queue with dependent breakdowns are given in Table 13. The boxplots represent the median, the 25% and 75% percentiles and the extreme values (outliers) of obtained samples for each component of the stationary distribution are shown in Figure 4. Moreover, the resulting computational results that characterizing statistically certain performance measures of the studied queue are given in Table 15. We applied the TSE method to compute the same measures (see Table 14 and Table 16), observing minimal discrepancies.

	$\pi_k(\omega)$	$\mathbb{E}\left[\pi_k(\omega)\right]$	$\mathbb{V}ar\left[\pi_k(\omega)\right]$	$\mathbb{S}kew\left[\pi_k(\omega)\right]$	$\mathbb{K}urt\left[\pi_k(\omega)\right]$
	$\pi_0(\omega)$	0.0000005	0.0000000	2.0265182	6.4080071
	$\pi_1(\omega)$	0.0000022	0.0000000	1.8120329	5.5009970
	$\pi_2(\omega)$	0.0000101	0.0000000	1.5897673	4.6659606
	$\pi_3(\omega)$	0.0000479	0.0000000	1.3507491	3.8860149
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_4(\omega)$	0.0002338	0.0000000	1.0876993	3.1685197
	$\pi_5(\omega)$	0.0011753	0.0000009	0.8002969	2.5545324
	$\pi_6(\omega)$	0.0062967	0.0000137	0.4777542	2.0820676
	$\pi_7(\omega)$	0.0337697	0.0001450	0.1618580	1.8409120
	$\pi_8(\omega)$	0.2445995	0.0000142	0.0142993	1.7753280
	$\pi_9(\omega)$	0.7138638	0.0004291	-0.2337699	1.8851937
	$\pi_0(\omega)$	0.0005380	0.0000002	1.1091007	3.2501061
	$\pi_1(\omega)$	0.0009651	0.0000006	0.9653364	2.9140667
	$\pi_2(\omega)$	0.0019620	0.0000019	0.8069219	2.5918290
	$\pi_3(\omega)$	0.0041997	0.0000062	0.6485319	2.3220680
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_4(\omega)$	0.0092191	0.0000197	0.4902558	2.1062343
	$\pi_5(\omega)$	0.0205858	0.0000570	0.3287477	1.9418812
	$\pi_6(\omega)$	0.0467059	0.0001340	0.1584235	1.8294237
	$\pi_7(\omega)$	0.1071980	0.0001914	-0.0532777	1.7724088
	$\pi_8(\omega)$	0.2547391	0.0000047	-1.3013673	3.5956560
	$\pi_9(\omega)$	0.5538867	0.0016338	-0.1612796	1.8475316
	$\pi_0(\omega)$	0.0288655	0.0002567	0.5336749	2.1191236
	$\pi_1(\omega)$	0.0325090	0.0002622	0.4675674	2.0433026
	$\pi_2(\omega)$	0.0385110	0.0002630	0.3708342	1.9477906
	$\pi_3(\omega)$	0.0474691	0.0002493	0.2433650	1.8496560
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_4(\omega)$	0.0603408	0.0002077	0.0749373	1.7675086
	$\pi_5(\omega)$	0.0785533	0.0001282	-0.1822830	1.7438430
	$\pi_6(\omega)$	0.1041854	0.0000300	-0.8022598	2.2491909
	$\pi_7(\omega)$	0.1403428	0.0000491	-1.0426969	2.8797655
	$\pi_8(\omega)$	0.1902491	0.0006709	-0.2868373	1.8815601
	$\pi_9(\omega)$	0.2789736	0.0038614	-0.0608873	1.7851678

Table 13. Statistical characteristics of interest for the stationary distribution of M/HE$_2$/1/10 queue with dependent breakdowns via MCS.



(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 4. Boxplots of the uncertain stationary distribution $\pi(\omega)$ of M/HE$_2$/1/10 queue with dependent breakdowns.

	$\pi_k(\omega)$	$\mathbb{E}[\pi_k(\omega)]$	$\mathbb{V}ar[\pi_k(\omega)]$	$\mathbb{S}kew[\pi_k(\omega)]$	$\mathbb{K}urt[\pi_k(\omega)]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$\pi_0(\omega)$	0.0000005	0.0000000	2.0265180	6.4080071
	$\pi_1(\omega)$	0.0000024	0.0000002	1.8120329	5.5009972
	$\pi_2(\omega)$	0.0000103	0.0000002	1.5897674	4.6659609
	$\pi_3(\omega)$	0.0000494	0.0000011	1.3507500	3.8860152
	$\pi_4(\omega)$	0.0002345	0.0000003	1.0876999	3.1685201
	$\pi_5(\omega)$	0.0011757	0.0000012	0.8003045	2.5545328
	$\pi_6(\omega)$	0.0063007	0.0000142	0.4777544	2.0821011
	$\pi_7(\omega)$	0.0346653	0.0001455	0.1618584	1.8409125
	$\pi_8(\omega)$	0.2450443	0.0000147	0.0143155	1.7753330
	$\pi_9(\omega)$	0.7138645	0.0004299	-0.2337731	1.8851942
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$\pi_0(\omega)$	0.0005383	0.0000012	1.1091011	3.2501114
	$\pi_1(\omega)$	0.0009672	0.0000006	0.9653463	2.9140676
	$\pi_2(\omega)$	0.0019624	0.0000022	0.8050336	3.0043559
	$\pi_3(\omega)$	0.0041999	0.0000071	0.6485323	2.3220691
	$\pi_4(\omega)$	0.0092191	0.0000197	0.4902558	2.1062343
	$\pi_5(\omega)$	0.0205861	0.0000573	0.3287480	1.9418814
	$\pi_6(\omega)$	0.0467114	0.0001340	0.1584238	1.8294240
	$\pi_7(\omega)$	0.1072033	0.0002025	-0.0532774	1.7724108
	$\pi_8(\omega)$	0.2547391	0.0000047	-1.3013673	3.6234550
	$\pi_9(\omega)$	0.5538922	0.0018664	-0.1612785	1.8475558
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$\pi_0(\omega)$	0.0288659	0.0002570	0.5336801	2.1191240
	$\pi_1(\omega)$	0.0325112	0.0002631	0.5023354	2.0433031
	$\pi_2(\omega)$	0.0385114	0.0002633	0.3710884	1.9477911
	$\pi_3(\omega)$	0.0474699	0.0002511	0.2433653	1.8496565
	$\pi_4(\omega)$	0.0603408	0.0002077	0.0749373	1.7675086
	$\pi_5(\omega)$	0.0785536	0.0001283	-0.1822827	1.7438432
	$\pi_6(\omega)$	0.1041859	0.0000303	-0.8022600	2.2491914
	$\pi_7(\omega)$	0.1403431	0.0000505	-1.0426964	2.9118797
	$\pi_8(\omega)$	0.1903022	0.0006713	-0.2868374	1.8828856
	$\pi_9(\omega)$	0.2904478	0.0045861	-0.0608869	1.7852011

Table 14. Statistical characteristics of interest for the stationary distribution of M/HE$_2$/1/10 queue with dependent breakdowns via TSE.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$PBL(\omega)$	0.2105266	0.0000000	2.0295591	6.4079942
	$L(\omega)$	9.6627180	0.0019544	-0.3193288	1.9524271
	$W(\omega)$	12.2394441	0.0031357	-0.3193013	1.9523915
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$PBL(\omega)$	0.2108614	0.0000000	1.1095769	3.2468100
	$L(\omega)$	9.2102726	0.0218406	-0.3382493	1.9758514
	$W(\omega)$	11.6728329	0.0348509	-0.3357557	1.9732601
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$PBL(\omega)$	0.2279992	0.0000500	0.5038631	2.0767419
	$L(\omega)$	7.4974712	0.3466073	-0.2842669	1.8891166
	$W(\omega)$	9.8016434	0.5564293	-0.2772504	1.8841237

Table 15. Statistical characteristics of interest for the main performance measures of M/HE$_2$/1/10 queue with dependent breakdowns via MCS.

		$\mathbb{E}[.]$	$\mathbb{V}ar[.]$	$\mathbb{S}kew[.]$	$\mathbb{K}urt[.]$
$\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$	$PBL(\omega)$	0.2105267	0.0000000	2.0295592	6.4079946
	$L(\omega)$	9.6627184	0.0019555	-0.3193284	2.0332459
	$W(\omega)$	12.2394449	0.0031411	-0.3193121	1.9523931
$\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$	$PBL(\omega)$	0.2108632	0.0000020	1.1133456	3.2468105
	$L(\omega)$	9.2102755	0.0218424	-0.3382503	2.2449758
	$W(\omega)$	12.006712	0.0350664	-0.3357550	1.9732751
$\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$	$PBL(\omega)$	0.2388642	0.0000502	0.5038652	2.1135607
	$L(\omega)$	7.5116749	0.3466115	-0.2842666	1.88934519
	$W(\omega)$	10.007661	0.5621642	-0.2772500	1.8841240

Table 16. Statistical characteristics of interest for the main performance measures of M/HE$_2$/1/10 queue with dependent breakdowns via TSE.

5. Risk analysis

In this section, we are interested in developing an approach to risk analysis of the M/G/1/N queueing model with dependent breakdowns. In this work, the notion of *risk* means the probable effect incurred by neglecting or not taking into account parametric uncertainty in the frame of evaluation of performance measures of queueing systems. In classical queueing modeling, it is emphasized that all model parameters are accurately known, and all calculation are carried out with fixed parameters. These parameters are generally considered as the estimated mean values, that resume the essential information on available statistical data. In this present work, we are concerned with the probability of a server breakdown θ. Let $\bar{\theta}$ be the mean value of the random variable $\theta(\omega)$, and $perf_{\bar{\theta}}$ denote a certain performance measure of the M/G/1/N queue with dependent breakdowns, assessing at the fixed parameter $\bar{\theta}$. In our case, the performance measure $perf_{\bar{\theta}}$ can be obtained numerically by using the mean of the sampled random variable $\{\theta_n \,|\, n = 1, \dots, N_{MC}\}$. Thereby, the risk incurred by working with uncertain performance measure $perf_{\theta(\omega)}$ instead that evaluated at fixed parameter $perf_{\bar{\theta}}$ is given by

$$R_{\theta(\omega)} = perf_{\theta(\omega)} - perf_{\bar{\theta}}. \tag{15}$$

The main purpose of risk analysis is to avoid that the current performance $perf_{\theta(\omega)}$ be significantly deviated from the projected performance $perf_{\bar{\theta}}$. This will allow us to prevent for $R_{\theta(\omega)}$ to have large positive values. In order to evaluate the risk in taking $perf_{\theta(\omega)}$ as performance measure rather than $perf_{\bar{\theta}}$, one can estimate

$$\mathbf{P}\left(R_{\theta(\omega)} \geq \kappa\right), \tag{16}$$

where κ is a threshold linked to the problem formulating the deviation from the planned the decision maker is willing to take.

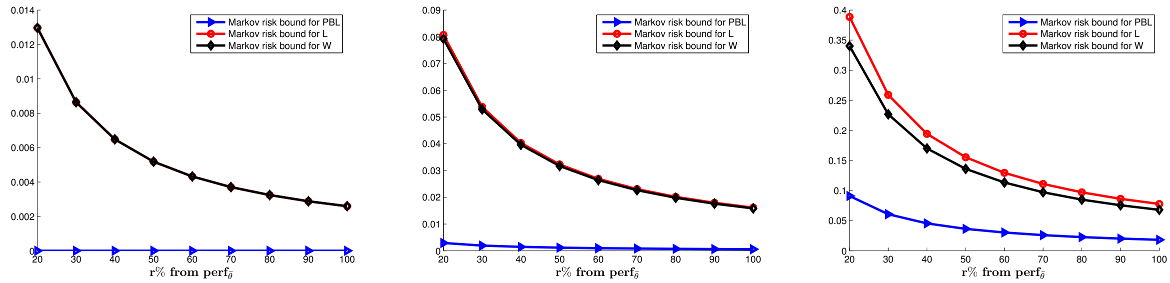
In the sequel we investigate the risk that the actual performance deviates more than, say, r percent from the numerical value $perf_{\bar{\theta}}$. Then one can choose

$$\kappa = \frac{r}{100} perf_{\bar{\theta}}. \tag{17}$$

Thereby to estimate the resulting risk, for fixed value of κ, it is suffices to bound the mentioned quantity in (16). For that we use the following Markov's inequality

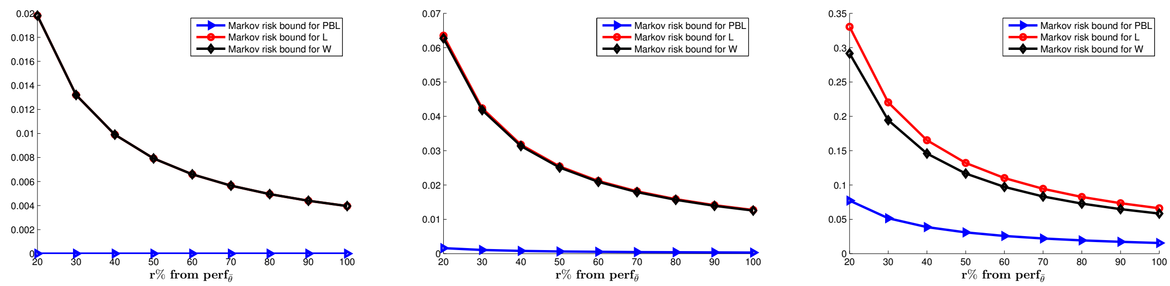
$$\mathbf{P}\left(|R_{\theta(\omega)}| \geq \kappa\right) \leq \frac{\mathbb{E}\left(|R_{\theta(\omega)}|\right)}{\kappa}. \tag{18}$$

The Markov risk bound (18) is numerically assessed for different fixed values of κ. The resulting numerical results for the M/G/1/10 queue with dependent breakdowns are displayed in Figures 5, 6, 7 and 8. We have considered the same scenarios for the uncertain parameter $\theta(\omega)$ and the same probability distributions for the service times. All parameters values are same as those given in the previous sections.



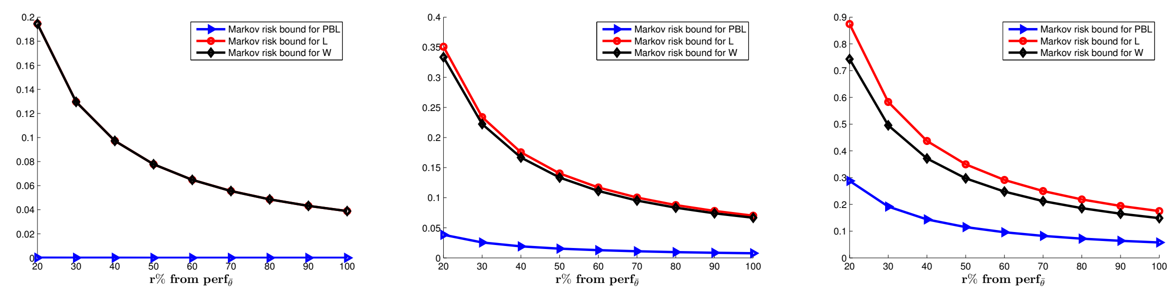
(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 5. Markov risk bounds for the main performance measures of M/D/1/10 queue with dependent breakdowns.



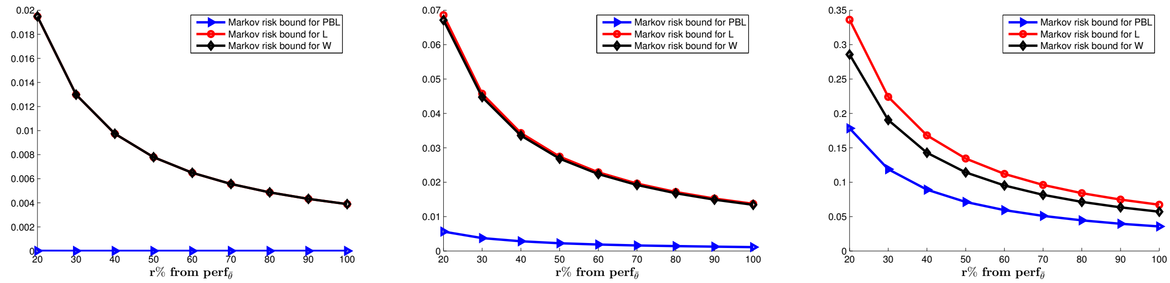
(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 6. Markov risk bounds for the main performance measures of M/E_2/1/10 queue with dependent breakdowns.



(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 7. Markov risk bounds for the main performance measures of M/M/1/10 queue with dependent breakdowns.



(a) $\theta(\omega) \sim \mathcal{U}[0.1, 0.3]$. (b) $\theta(\omega) \sim \mathcal{U}[0.4, 0.6]$. (c) $\theta(\omega) \sim \mathcal{U}[0.7, 0.9]$.

Figure 8. Markov risk bounds for the main performance measures of M/HE$_2$/1/10 queue with dependent breakdowns.

It may be observed from Figures 5, 6, 7 and 8 that varying in the values of the threshold κ, causes varying values of the Markov risk bounds for the main performance measures of M/G/1/10 queue with dependent breakdowns. The main insight provided by this analysis is that the values of κ effect the Markov risk bounds in a monotone increasing way. This allows us to judge the impact of ignoring epistemic-uncertainty in assessment of key performance metrics of the studied queueing model. Furthermore, it is easy to see that, the change of the ranges of values of the random variable $\theta(\omega)$ involves the change of the values of the resulting Markov bounds, which also shows the effect of the dependent breakdowns on computing of the same performance measures of this queueing model.

6. Conclusion

We have proposed a new methodology for uncertainty quantification in M/G/1/N queueing model with dependent breakdowns. We have shown that uncertainty analysis and risk analysis can be integrated into one framework. In this paper we have argued that performance measures of unreliable queueing systems are heavily influenced by server breakdowns. Under the epistemic-uncertainty inflicted on the breakdowns probability, we have developed an efficient computational algorithm that allows us to characterize statistically the key uncertain performance measures of the M/G/1/N queueing model with dependent breakdowns. A relevant statistical characterization is typically provided by estimating some quantities of interest such as the mean, the variance, the skewness and the kurtosis of the underlying performances. We have also revealed the effect of ignoring epistemic-uncertainty in performance evaluation of queueing models with dependent breakdowns. Numerical simulations demonstrate the effectiveness of the proposed algorithm, showing strong agreement with the TSE method.
These promising results suggest exciting research directions, particularly in extending our approach to more complex systems such as stochastic networks with multiple uncertain parameters.

References

1. Antonelli, F., Cortellessa, V., Gribaudo, M., Pinciroli, R., Trivedi, K.S. and Trubiani, C, *Analytical modeling of performance indices under epistemic uncertainty applied to cloud computing systems*, Future Generation Computer Systems **102**: 746–761, 2020.
2. Baccelli, F., Kauffmann, B. and Veitch, D, *Inverse problems in queueing theory and internet probing*, Queueing Systems **63** (1): 59–107,2009.
3. Boxma O.J. and Mandjes M.R.H, *Queueing and risk models dependencies*, Springer **102**: 69–86, 2022.
4. Bachi, K., Abbas, K. and Heidergott, B, *Robust power series algorithm for epistemic uncertainty propagation in Markov chain models*, Stochastic Models **36** (1): 20–47, 2020.
5. Bachi, K., Abbas, K. and Heidergott, B, *Statistical Taylor series expansion : An approach for epistemic uncertainty propagation in Markov reliability models*, RAIRO - Operations Research. DOI: https://doi.org/10.1051/ro/2019091, 2020.
6. Bachi, K., Chauvière, C., Djellout, H. and Abbas, K, *Propagation of epistemic uncertainty in queueing models with unreliable server using chaos expansions*, Communications in Statistics - Simulation and Computation. DOI: https://doi.org/10.1080/03610918.2019.1577966, 2019.
7. Berkhout, J., Heidergott, B., Sommer, J. and Daduna, H, *Robustness analysis of generalized Jackson network*, Computational Management Science **16**: 697–714, 2019

8. Dhople S.V. and Domínguez-García A.D, *A parametric uncertainty analysis method for Markov reliability and reward models*, IEEE Transactions on Reliability **61** (3): 634–648, 2012.
9. Du, X., Venigell, P.K. and Liu, D, *Robust mechanism synthesis with random and interval variables*, Mechanism and Machine Theory **44** (7): 1321–1337, 2009.
10. Fiessler, B., Rackwitz, R., Neumann, H, *Quadratic limit states in structural reliability*, Journal of Engineering Mechanics **105**: 661–676,1979.
11. Grenyer A., Erkoyuncu J.A, Addepalli S. and Zhao Y., *Compound Uncertainty Quantification and Aggregation for Reliability Assessment in Industrial Maintenance*, Diagnosis and Prognostics of Machines: Applications and Advances, **11(5)**, 560; 2023 https://doi.org/10.3390/machines11050560
12. Gross, D. and Harris, C.M, , Fundamentals of Queueing Theory. Wiley, New York, 1985.
13. Haverkort, B. and Meeuwissen, A, *Sensitivity and uncertainty analysis in performance modelling. In: Proceedings* 11*th symposium on reliable distributed systems*, IEEE Computer Society Press, pp 93–102,1992.
14. Henderson, S, *Input model uncertainty: why do we care and what should we do about it?*, In: Chick S, Sanchez PJ, Ferrin D, Morrice DJ (eds) Proceedings of the 2003 winter simulation conference, pp 90–100,2003.
15. Hesamian, G. and Chachi, J , *Interval-valued parametric distribution functions: Methodologies and applications*, Journal of Mathematical Extension **10** (1): 87–105, 2016.
16. Ke, J.C. Huang, H. and Lin, C.H, *A batch-arrival queue with multiple servers and fuzzy parameters: Parametric programming approach*, Mathematical and Computational Applications **11** (3): 181–191, 2006.
17. Ke, J.C. and Lin, C.H, *Fuzzy analysis of queueing systems with an unreliable server: A nonlinear programming approach*, Applied Mathematics and Computation **175** (1): 330–346, 2006.
18. Kumari, B. and Govindarajan, R, *The analysis of performance measures of generalized trapezoidal fuzzy queuing model with an unreliable server*, International Journal of Engineering Research and Application **7** (5): 136–139, 2017.
19. Lee, S.H. and Kwak, B.M, *Response surface augmented moment method for efficient reliability analysis*, Structural Safety **28** (3):261–272, 2006.
20. Lopatatzidis, S., De Bock, J., De Cooman, G. et De Vuyst, S. and Walraevens, J, *Robust queueing theory: an initial study using imprecise probabilities*, Queueing Systems **82**: 75–101, 2016.
21. Ouazine, S. and Abbas, K, *A functional approximation for retrial queues with two way communication*, Annals of Operations Research **247**: 211–227,2016.
22. Papadopoulos, C.E. and Yeung, H, *Uncertainty estimation and Monte Carlo simulation method*, Flow Measurement and Instrumentation **12** (4): 291–298, 2001.
23. Sauer, C. and Daduna, H, *Availability formulas and performance measures for separable degradable networks*, Economic Quality Control **18** (2): 165–194, 2003.
24. Sommer, J., Berkhout, J., Daduna, H. and Heidergott, B, *Analysis of Jackson networks with infinite supply and unreliable nodes*, Queueing Systems **87** (1-2): 181–207,2017.
25. Soufit M., Ouazine S. and Abbas K, *Taylor series expansion approach for epistemic uncertainty propagation in queueing-inventory models*, Mathematical Methods in the Applied Sciences **41** (18): 9164–9175, 2018.
26. Sztrik, J. and Gál, T, *A recursive solution of a queueing model for a multi-terminal system subject to breakdowns*, Performance Evaluation **11** (1): 1–7, 1990.
27. Takhedmit, B. and Abbas, K, *A parametric uncertainty analysis method for queues with vacations*, Journal of Computational and Applied Mathematics **312**: 143–155, 2017.
28. Xiu D., Karniadakis G.E, *Modeling uncertainty in flow simulations via generalized polynomial chaos*, Journal of Computational Physics **187** (1): 137–167,2003.
29. Xu, H. and Rahman, S , *A generalized dimension-reduction method for multidimensional integration in stochastic mechanics*, International Journal for Numerical Methods in Engineering **61** (12): 1992–2019, 2004.
30. Zadeh L.A, *Fuzzy Sets*, Information and Control **8** (3): 338–353, 1965.
31. Boxma,O.J. and Mandjes,M.R.H, *Queueing and risk models dependencies*, Springer **102**: 69–86, 2022.
32. Badila,E.S, Boxma,O.J., Resing,J.A.C and Winands,E.M.M, *Queues and risk models with simultaneous arrivals*, Adv.Appl.Prob**46**: 812–831, 2014.
33. Kalos, M.H. and Whitlock, P.A, *Monte Carlo Methods*, John Wiley & Sons **8**: 2008
34. Ouazine,S and Abbas K, *P-Box uncertainty quantification of queues using the taylor series expansion*, International scientific journal 'Mathematical Modeling': 83–85, 2019.