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Equilibrium and Fixed Point Problems and Applications via Kirk–Cesàro Hybrid Iterative Schemes

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Abstract Kirk–Cesàro hybrid iterative schemes are developed in this study to approximate solutions of fixed point and equilibrium problems in real Hilbert spaces. The main novelty of this study is the development of new Kirk–Cesàro hybrid iterative schemes that combine the Kirk iterative structure, Cesàro meaning, and hybrid projection techniques for fixed point and equilibrium problems involving asymptotically nonexpansive mappings. Under appropriate conditions, norm convergence of the produced sequences toward the metric projection onto the fixed point set, as well as toward a joint solution of fixed point and equilibrium problems, is established. The derived results are further applied to convex optimization through an equivalent equilibrium formulation. The proposed framework may provide a useful theoretical basis for future applications in optimization and equilibrium modeling. Computational experiments and graphical illustrations are provided to demonstrate the convergence behavior of the proposed schemes.

Keywords Equilibrium problem, Convex optimization, Kirk–Cesàro iteration, Strong convergence

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1. Introduction

As one of the essential branches of nonlinear analysis, fixed point theory has wide-ranging applications in optimization theory, mathematical economics, game theory, and approximation theory. Numerous mathematical models originating from applied sciences can be cast as fixed point problems, equilibrium problems, or variational inequality problems. In particular, equilibrium problems constitute a unifying setting for the analysis of several key mathematical models, among them variational inequalities, convex optimization problems, and feasibility problems [9, 12, 17, 19, 20, 28, 41, 48].

Driven by their wide range of applicability, equilibrium problems and related fixed point formulations have received growing attention in the recent literature. Numerous iterative approximation techniques have been proposed for addressing equilibrium and fixed point problems in Hilbert and Banach spaces [2, 22, 46, 49]. Alongside these developments, projection-based, viscosity-based, and hybrid iterative procedures have been introduced to handle variational inequality and split feasibility problems [7, 16, 31, 35, 45]. Recent studies have also emphasized the role of iterative and resolvent-based techniques in solving fixed-point and variational-type problems. In particular, Alansari et al. [1] investigated iterative algorithms for a generalized system of mixed variational-like inclusion and altering points problems, while Tabchouche [39] proposed a two-step fixed-point iterative scheme for nonlinear matrix equations.

Optimization and convex feasibility problems can frequently be reformulated as equilibrium problems, thereby revealing a deep connection between equilibrium theory and convex optimization [3,4,37,40]. In a related direction,

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Bataihah [6] recently highlighted the connection between fixed point principles and minimization arguments through a minimizing-sequence approach to the Banach fixed point theorem. A number of recent investigations have additionally examined convergence characteristics and robustness for equilibrium-type problems and asymptotically non-expansive mappings [14, 15, 43].

Baillon [5] first demonstrated that, under Cesàro averaging, a non-expansive mapping in a Hilbert space converges weakly to a fixed point through the following iteration sequence.

$$\mathcal{S}_n u = \frac{1}{n+1}\sum_{j=0}^{n} \mathcal{S}^j u.$$

The above iterative scheme was examined in real Hilbert spaces by Shimizu and Takahashi [36], who obtained strong convergence results for asymptotically non-expansive mappings. As Genel and Lindenstrauss [18] demonstrated, the Mann iteration [26] achieves only weak convergence in infinite-dimensional Hilbert spaces. To overcome this limitation, Nakajo and Takahashi [29] introduced a modified Mann iteration for non-expansive mappings in Hilbert spaces:

$$\begin{cases} u_0 \in C \text{ is arbitrary}, \\ v_n = \alpha_n u_n + (1-\alpha_n) S u_n, \\ C_n = \{w \in C : \|v_n - w\| \le \|u_n - w\|\}, \\ Q_n = \{w \in C : \langle u_n - w,\ u_0 - u_n\rangle \ge 0\}, \\ u_{n+1} = P_{C_n \cap Q_n}(u_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{1}$$

where, $\mathcal{H}$ be a real Hilbert space, $\varnothing \neq \mathcal{K} \subset \mathcal{H}$ is a closed convex subset, and $\mathcal{P}_{\mathcal{K}}$ denotes the metric projection from $\mathcal{H}$ onto $\mathcal{K}$. This approach is commonly known as the $\mathcal{CQ}$ method or the hybrid technique.

A generalization of iteration (1) to Hilbert spaces was subsequently carried out by Kim and Xu [25], who formulated the following scheme for asymptotically non-expansive mappings:

$$\begin{cases} u_0 \in C \text{ is arbitrary}, \\ v_n = \alpha_n u_n + (1-\alpha_n) S^n u_n, \\ C_n = \left\{w \in C : \|v_n - w\|^2 \le \|u_n - w\|^2 + \theta_n\right\}, \\ Q_n = \{w \in C : \langle u_n - w,\ u_0 - u_n\rangle \ge 0\}, \\ u_{n+1} = P_{C_n \cap Q_n}(u_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{2}$$

as $n \to \infty$ iken $\theta_n = (1-\alpha_n)\left(t_n^2 - 1\right)(diam\mathcal{C})^2 \to 0$.They showed that, under certain conditions, the sequence $\{u_n\}$ converges in norm to $\mathcal{P}_{F(\mathcal{S})}(u_0)$. In subsequent studies, it has been observed that the convergence of the hybrid method has been investigated [30, 44].

Zhang and Cui [47] proposed the following iteration scheme: $\mathcal{C}$ represents a closed, convex, and bounded subset of a real Hilbert space $\mathcal{H}$, and S: $\mathcal{C} \to \mathcal{C}$ be an asymptotically non-expansive mapping defined along with t_n, denote $\mathcal{L}_n = \frac{1}{n+1}\sum_{j=0}^{n} t_j$

$$\begin{cases} u_0 \in C \text{ is arbitrary}, \\ v_n = \alpha_n u_n + (1-\alpha_n)\left(\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} S^j u_n\right), \\ C_n = \left\{w \in C : \|v_n - w\|^2 \le \|u_n - w\|^2 + \theta_n\right\}, \\ Q_n = \{w \in C : \langle u_n - w,\ u_0 - u_n\rangle \ge 0\}, \\ u_{n+1} = P_{C_n \cap Q_n}(u_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{3}$$

where $\theta_n = (1-\alpha_n)\left(\mathcal{L}_n^2 - 1\right)(diam\mathcal{C})^2 \to 0$ as $n \to \infty$.

Motivated by the regularizing and stabilizing role of Cesàro meaning techniques in iterative approximation processes, particularly for asymptotically non-expansive mappings where direct iterative methods may exhibit oscillatory behavior, and inspired by the results mentioned above, we introduce the following Kirk–Cesàro hybrid iterative schemes. Let $\mathcal{H}$ be a real Hilbert space and $\varnothing \neq \mathcal{C} \subset \mathcal{H}$ be a closed, and convex subset. $\mathcal{S}: \mathcal{C} \to \mathcal{C}$ be an asymptotically non-expansive mapping defined along with k_n, denote $\mathcal{L}_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j$. Let $k \geq 1$ and $0 \leq n \leq i,\ i \geq 1$ satisfy the conditions $\alpha_n \geq 0,\ \ \alpha_1 > 0,\ \ \sum_{n=0}^{i} \alpha_n = 1$.

$$\begin{cases} u_0 \in C \text{ is arbitrary}, \\ v_n = \displaystyle\sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1} \sum_{j=0}^{n} S^j u_n \right), \\ C_n = \left\{ w \in C : \|v_n - w\|^2 \leq \|u_n - w\|^2 + \theta_n \right\}, \\ Q_n = \{ w \in C : \langle u_n - w,\ u_0 - u_n \rangle \geq 0 \}, \\ u_{n+1} = P_{C_n \cap Q_n}(u_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{4}$$

where $\theta_n = \left(\mathcal{L}_n^2 - 1\right)(diam\mathcal{C})^2 \to 0$ as $n \to \infty$. In this case, we will prove that there exists a sequence $\{u_n\}_{n=0}^{\infty}$ in $\mathcal{C}$ that converges strongly to $\mathcal{P}_{F(\mathcal{S})}(u_0)$.

Additionally, let $F(\mathcal{S}) \cap EP(f) \neq \varnothing$ and $f\colon \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a functional. The iterative scheme is defined as follows for $0 \leq n \leq i,\ i \geq 1$ satisfy the conditions $\alpha_n \geq 0,\ \ \alpha_1 > 0,\ \ \sum_{n=0}^{i} \alpha_n = 1$.

$$\begin{cases} u_0 \in C \text{ is arbitrary}, \\ v_n = \displaystyle\sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1} \sum_{j=0}^{n} S^j u_n \right), \\ x_n \in C, \text{ such that} \quad f(x_n, y) + \dfrac{1}{r_n}\langle y - x_n,\ x_n - v_n \rangle \geq 0, \quad \forall\, v \in C, \\ C_n = \left\{ w \in C : \|x_n - w\|^2 \leq \|u_n - w\|^2 + \theta_n \right\}, \\ Q_n = \{ w \in C : \langle u_n - w,\ u_0 - u_n \rangle \geq 0 \}, \\ u_{n+1} = P_{C_n \cap Q_n}(u_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{5}$$

where $\theta_n = \left(\mathcal{L}_n^2 - 1\right)(diam\mathcal{C})^2 \to 0$ as $n \to \infty$. The sequence $\{u_n\}_{n=0}^{\infty}$ is defined in $\mathcal{C}$ using this iterative scheme. In this case, we will prove that the iteration sequence $\{u_n\}$ strongly converges to $\mathcal{P}_{F(\mathcal{S}) \cap EP(f)}(u_0)$.

Building upon the Kirk–Cesàro hybrid iterative schemes defined in (4) and (5), we establish strong convergence results for asymptotically nonexpansive mappings and equilibrium problems in real Hilbert spaces under suitable control conditions. The proposed framework combines Kirk iteration, Cesàro meaning, and hybrid projection techniques within a unified iterative structure. The proposed Kirk–Cesàro hybrid framework is not restricted to abstract fixed point settings. Through its equilibrium formulation, the method can be applied to engineering optimization, resource allocation, decision-making systems, signal processing, and computational science problems where equilibrium and optimization models arise naturally. Therefore, the present work contributes to the development of mathematical tools that support transdisciplinary research across engineering, applied sciences, and optimization-based decision systems. Recent studies in transdisciplinary engineering and applied science have emphasized the importance of optimization, computational modeling, and systems-oriented methodologies for addressing complex real-world problems [23,24,33,42]. Motivated by this perspective, the present study contributes a rigorous mathematical framework for equilibrium and optimization-based models through the development of Kirk–Cesàro hybrid iterative schemes. The proposed methodology provides convergence guarantees that may support computational approaches arising in engineering optimization, decision-making systems, and other application-oriented domains.

The paper proceeds as follows. Section 2 collects the essential definitions, auxiliary lemmas, and preliminary results needed for the subsequent analysis. Strong convergence theorems for the proposed iterative schemes are

established in Section 3, where numerical illustrations are also provided to support the theoretical findings. Section 4 presents an application of the developed iterative framework to convex optimization problems via an equilibrium formulation.

2. Preliminary

Let $\mathcal{X}$ be a Banach space. In this paper, the notation, $u_n \longrightarrow u$ indicates that the sequence $\{u_n\}$ converges strongly to u, whereas $u_n \rightharpoonup u$ denotes weak convergence in $\mathcal{X}$.

Let $\mathcal{H}$ be a real Hilbert space, let $\mathcal{C}$ be a non-empty closed convex subset of $\mathcal{H}$, and let $f\colon \mathcal{C}\times\mathcal{C}\to\mathbb{R}$ be a bifunction, where $\mathbb{R}$ denotes the set of real numbers. The equilibrium problem is to find $\widehat{u}\in\mathcal{C}$ such that for all $\forall v\in\mathcal{C}$

$$f(\widehat{u},v)\geq 0.$$

The solution set of this inequality is denoted by $EP(f)$ [9].

Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{C}$ be a non-empty closed convex subset of $\mathcal{H}$. Then every element $u\in H$ has a unique nearest point in C. This nearest point is denoted by $\mathcal{P}_{\mathcal{C}}(u)$ and for all $v\in\mathcal{C}$ fulfills the condition:

$$\|u-\mathcal{P}_{\mathcal{C}}(u)\|\leq\|u-v\|\ ,\ \ \forall v\in\mathcal{C}.$$

Such a mapping $\mathcal{P}_{\mathcal{C}}$ is referred to as the metric projection of $\mathcal{H}$ onto $\mathcal{C}$. It is well proved that $\mathcal{P}_{\mathcal{C}}$ is non-expansive. Furthermore, for $\forall u\in\mathcal{H}$ and $w\in\mathcal{C}$, we have

$$w=\mathcal{P}_{\mathcal{C}}(u)\ \Leftrightarrow\ \langle u-w,\ w-v\rangle\ \geq 0,\quad \forall v\in\mathcal{C}$$

[11].

Let $\mathcal{X}$ be a real Banach space and $\mathcal{X}^*$ its dual space. The normalized duality mapping $J\colon\ \mathcal{X}\to 2^{\mathcal{X}^*}$ is defined by $J(x)=\left\{f\in\mathcal{X}^*\ :\ \langle x,f\rangle=\ \|x\|^2=\|f\|^2\right\}$. If $\mathcal{X}$ is a Hilbert space, then J becomes the identity mapping [10].

Definition 2.1. Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $\mathcal{C}$ be a non-empty closed convex subset of $\mathcal{H}$, and let $\mathcal{S}:\mathcal{C}\to\mathcal{C}$ be a mapping. Suppose that, for every $n\geq 1$, there exists a sequence $(k_n)\subset[1,\infty)$ satisfying

$$\|\mathcal{S}^n u-\mathcal{S}^n v\|\leq k_n\|u-v\|\ ;\ \ \forall u,v\in\mathcal{C}$$

with $k_n\to 1$ as $n\to\infty$. Then $\mathcal{S}$ is said to be asymptotically non-expansive. When $k_n\equiv 1$, the mapping $\mathcal{S}$ is referred to as a non-expansive mapping [21].

Remark 2.1. Let $\mathcal{H}$ be a real Hilbert space. If $\mathcal{S}:\mathcal{H}\to\mathcal{H}$ is an asymptotically non-expansive mapping, then the fixed point set $F(\mathcal{S}):=\{u\in C:\ \mathcal{S}u=u\}$ is nonempty and convex [21].

By exploiting the geometric properties of Hilbert spaces, Opial [32] introduced the following concept:

Definition 2.2. Let $\mathcal{X}$ be a Banach space, and let $\{u_n\}$ be any sequence in $\mathcal{X}$ such that $u_n\rightharpoonup u\in\mathcal{X}$. If for any $v\neq u$, the inequality

$$\liminf_{n\to\infty}\|u_n-u\|<\liminf_{n\to\infty}\|u_n-v\|$$

is satisfied, then $\mathcal{X}$ is said to satisfy the Opial condition. It is a classical fact that every Hilbert space fulfills Opial's condition [32].

Definition 2.3. A Banach space $\mathcal{X}$ is said to possess the Kadec-Klee property if for every sequence $\{u_n\}\subset\mathcal{X}$, $u_n\rightharpoonup u$ and $\|u_n\|\to\|u\|$ imply that $u_n\to u$. It is well known that every Hilbert space fulfills the Kadec–Klee property [34].

Several auxiliary lemmas that are central to the subsequent analysis are recalled below.

Lemma 2.1. Let $\mathcal{H}$ be a real Hilbert space. For arbitrary elements $u,v\in\mathcal{H}$, the following identities are satisfied:

$$\|u-v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u-v,\ v\rangle \ ,$$

[27].

Lemma 2.2. Let $\mathcal{H}$ be a real Hilbert space. Thus, the following equalities hold for all $u, v\in \mathcal{H}$ and $\lambda \in [0,1]$

$$\|\lambda u + (1-\lambda)v\|^2 + \lambda(1-\lambda)\|u-v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 ,$$

[8].

Lemma 2.3. Let $\mathcal{H}$ be a real Hilbert space and $\varnothing \neq \mathcal{C} \subset \mathcal{H}$ be a closed, bounded, convex subset. Let $\mathcal{S}$ be an asymptotically non-expansive mapping defined on $\mathcal{C}$. Assume that a sequence $\{u_n\}$ in $\mathcal{C}$ and $w\in \mathcal{H}$ such that $u_n \rightharpoonup w$ and $\mathcal{S}u_n - u_n \to 0$ as $n\to\infty$. Then, $w \in F(\mathcal{S})$ [10].

Lemma 2.4. Let $\mathcal{H}$ be a real Hilbert space and $\varnothing \neq \mathcal{C} \subset \mathcal{H}$ be a bounded subset. Assume that $\mathcal{S}: \mathcal{C}\to\mathcal{C}$ is an asymptotically non-expansive mapping with $F(\mathcal{S}) \neq \varnothing$. Then, for each $\varepsilon > 0$ and every $u\in\mathcal{C}$, there exists an integer l_ε such that whenever $l \geq l_\varepsilon$, one can choose an integer n_l satisfying

$$\left\| \frac{1}{n+1}\sum_{j=0}^{n}\mathcal{S}^j u - \mathcal{S}^l\left(\frac{1}{n+1}\sum_{j=0}^{n}\mathcal{S}^j u\right)\right\| < \varepsilon \ , \quad \forall n \geq n_l \ ,$$

[36].

The class of equilibrium problems can be solved if the bifunction f fulfills the following essential conditions [9, 13]:

(A1) $f(u,u) = 0$ for $\forall u\in\mathcal{C}$,

(A2) $f(u,v) + f(v,u) \leq 0$ for $\forall u, v\in\mathcal{C}$; i.e. f is monotone,

(A3) $\limsup_{t\to 0^+} f(tw + (1-t)u,\ v) \leq f(u,v)$ for $\forall u, v, w\in\mathcal{C}$,

(A4) $f(u,\cdot)$ is convex and lower semi-continuous for all $u \in C$.

Lemma 2.5. Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{C}$ be a non-empty closed convex subset of $\mathcal{H}$. Suppose that the bifunction $f\colon\ \mathcal{C}\times\mathcal{C}\to\mathbb{R}$ fulfills conditions (A1)–(A4). Given $r > 0$ and $u\in\mathcal{H}$, there exists an element $w\in\mathcal{C}$ such that

$$f(w,v) + \frac{1}{r}\langle v-w,\ Jw - Ju\rangle \geq 0 \quad , \quad \forall v\in\mathcal{C} ,$$

[9].

Lemma 2.6. Let $f\colon\ \mathcal{C}\times\mathcal{C}\to\mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). For each $r > 0$, define the operator $\mathcal{S}_r\colon \mathcal{X}\to\mathcal{C}$ by

$$\mathcal{S}_r(u) = \left\{ w\in\mathcal{C} : f(w,v) + \frac{1}{r}\langle v-w,\ Jw - Ju\rangle \geq 0 \ , \forall v\in\mathcal{C} \right\} \quad \forall u\in\mathcal{H}.$$

The following properties hold under the stated conditions:

i. $\mathcal{S}_r$ is single valued, For each $u\in\mathcal{H}$, the element $\mathcal{S}_r$ is single valued.

ii. The operator $\mathcal{S}_r$ fulfills the firm nonexpansiveness condition

$$\|S_r u - S_r v\|^2 \leq \langle S_r u - S_r v,\ u - v\rangle, \qquad \forall\, u, v \in H.$$

iii. $F(\mathcal{S}_r) = EP(f)$,

iv. The set $EP(f)$ is a closed and convex,

[38].

Lemma 2.7. Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{C}$ be a non-empty closed convex subset of $\mathcal{C}$. Assume that the bifunction $f\colon\ \mathcal{C}\times\mathcal{C}\to\mathbb{R}$ fulfills conditions (A1)–(A4), and let $r > 0$. Then, for any $u\in\mathcal{H}$ and $q \in F(\mathcal{S}_r)$, the following estimate is satisfied:

$$\|q - \mathcal{S}_r u\|^2 + \|\mathcal{S}_r\, u - u\|^2 \leq \|q-u\|^2 \ ,$$

[38].

3. Main Results

This section consists of two parts. In the first part, the conditions under which an asymptotically non-expansive mapping $\mathcal{S}$ strongly converges to $\mathcal{P}_{F(\mathcal{S})}(u_0)$ using the Kirk iteration based on the Cesàro mean hybrid method are investigated. In the second part, we establish a strong convergence theorem to identify a common element between the zero points of an asymptotically non-expansive mapping $\mathcal{S}$ and the solution set of an equilibrium problem in a Hilbert space.

3.1. Proposed Iteration Scheme and Strong Convergence for Kirk Iteration

Theorem 3.1.1. Let $\mathcal{H}$ be a real Hilbert space. $\varnothing \neq \mathcal{C} \subset \mathcal{H}$ be a non-empty, closed, convex and bounded subset, and $\mathcal{S}: \mathcal{C} \to \mathcal{C}$ be an asymptotically non-expansive mapping defined with k_n. Then, considering the iterative scheme (4), the sequence $\{u_n\}$ strongly converges to $\mathcal{P}_{F(\mathcal{S})}(u_0)$.

Proof: We first note from Remark 2.1 that S has a fixed point in C, that is, $F(S) \neq \emptyset$. The argument proceeds through four main stages.

Step 1: First, we verify that the sets $\mathcal{C}_n$ and Q_n are both closed and convex for every $n \geq 0$. From the definition of $\mathcal{C}_n$ and Q_n, we know that Q_n is closed and convex and $\mathcal{C}_n$ is closed for $n \in \mathbb{N} \cup \{0\}$. Let us prove that $\mathcal{C}_n$ is convex:

Using Lemma 2.1 in the inequality $\|v_n - w\|^2 \leq \|u_n - w\|^2 + \theta_n$, we have

$$\begin{aligned}
\|v_n - w\|^2 &\leq \|u_n - w\|^2 + \theta_n, \\
\|v_n\|^2 - \|w\|^2 - 2\langle v_n - w, w\rangle &\leq \|u_n\|^2 - \|w\|^2 - 2\langle u_n - w, w\rangle + \theta_n, \\
\|v_n\|^2 - 2\langle v_n - w, w\rangle &\leq \|u_n\|^2 - 2\langle u_n - w, w\rangle + \theta_n, \\
2\langle u_n - w, w\rangle - 2\langle v_n - w, w\rangle &\leq \|u_n\|^2 - \|v_n\|^2 + \theta_n, \\
2\langle u_n, w\rangle - 2\langle w, w\rangle - \Big(2\langle v_n, w\rangle - 2\langle w, w\rangle\Big) &\leq \|u_n\|^2 - \|v_n\|^2 + \theta_n, \\
2\langle u_n, w\rangle - 2\langle v_n, w\rangle &\leq \|u_n\|^2 - \|v_n\|^2 + \theta_n.
\end{aligned}$$

Thus:

$$2\langle u_n - v_n,\ w\rangle \leq \|u_n\|^2 - \|v_n\|^2 + \theta_n$$

from this, we conclude that $\mathcal{C}_n$ is convex.

Step 2: We demonstrate that $F(\mathcal{S}) \subset \mathcal{C}_n$ holds for all $n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}
\|v_n - p\|^2 &= \left\| \sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1} \sum_{j=0}^{n} \mathcal{S}^j u_n \right) - p \right\|^2 \\
&= \left\| \sum_{n=0}^{i} \alpha_n \frac{1}{n+1} \left(\sum_{j=0}^{n} \mathcal{S}^j u_n - \mathcal{S}^j p \right) \right\|^2 \\
&= \left(\frac{1}{n+1} \right)^2 \left\| \sum_{n=0}^{i} \alpha_n \right\|^2 \left\| \sum_{j=0}^{n} \left(\mathcal{S}^j u_n - \mathcal{S}^j p \right) \right\|^2 \\
&\leq \left(\frac{1}{n+1} \right)^2 \left(\sum_{j=0}^{n} k_j \|u_n - p\| \right)^2 \\
&= \left(\frac{1}{n+1} \right)^2 \|u_n - p\|^2 \left(\sum_{j=0}^{n} k_j \right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{n+1}\right)^2 \|u_n - p\|^2 \left(\sum_{j=0}^{n}(k_j - 1 + 1)\right)^2 \\
&= \left(\frac{1}{n+1}\right)^2 \|u_n - p\|^2 \Bigg[\left(\sum_{j=0}^{n}(k_j - 1)\right)^2 \\
&\qquad + 2\left(\sum_{j=0}^{n}(k_j - 1)\right)\left(\sum_{j=0}^{n} 1\right) + \left(\sum_{j=0}^{n} 1\right)^2 \Bigg] \\
&= \left(\frac{1}{n+1}\right)^2 \|u_n - p\|^2 \left(\sum_{j=0}^{n}(k_j - 1)\right)^2 \\
&\quad + \frac{2}{n+1}\|u_n - p\|^2 \sum_{j=0}^{n}(k_j - 1) + \|u_n - p\|^2 \\
&= \|u_n - p\|^2 \left(\frac{1}{n+1}\sum_{j=0}^{n}(k_j - 1)\right)^2 \\
&\quad + \frac{2}{n+1}\|u_n - p\|^2 \sum_{j=0}^{n}(k_j - 1) + \|u_n - p\|^2 \\
&= \|u_n - p\|^2(\mathcal{L}_n - 1)^2 + 2\|u_n - p\|^2 \frac{1}{n+1}\sum_{j=0}^{n}(k_j - 1) + \|u_n - p\|^2 \\
&= \|u_n - p\|^2(\mathcal{L}_n - 1)^2 + 2\|u_n - p\|^2(\mathcal{L}_n - 1) + \|u_n - p\|^2 \\
&= \|u_n - p\|^2 \mathcal{L}_n^2 \\
&= \|u_n - p\|^2 \left(\mathcal{L}_n^2 - 1\right) + \|u_n - p\|^2 \\
&\leq \|u_n - p\|^2 + \theta_n.
\end{aligned}
$$

Thus, it is derived that $p \in \mathcal{C}_n$.

Now, we verify that $F(\mathcal{S}) \subset Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. We will prove this by induction: For $n = 0$, $F(\mathcal{S}) \subset \mathcal{C} = Q_0$. Suppose now that $F(\mathcal{S}) \subset Q_n$, then $\varnothing \neq F(\mathcal{S}) \subset \mathcal{C}_n \cap Q_n$, and there exists a unique $u_{n+1} \in \mathcal{C}_n \cap Q_n$ such that $u_{n+1} = \mathcal{P}_{\mathcal{C}_n \cap Q_n}(u_0)$. Therefore, for all $w \in \mathcal{C}_n \cap Q_n$

$$\langle u_{n+1} - w,\ u_0 - u_{n+1} \rangle \geq 0$$

In particular, for all $p \in F(\mathcal{S})$,

$$\langle u_{n+1} - p,\ u_0 - u_{n+1} \rangle \geq 0 \quad .$$

This implies that $F(\mathcal{S}) \subset Q_{n+1}$. Therefore, $F(\mathcal{S}) \subset Q_n$ for every n. So, we have $F(\mathcal{S}) \subset \mathcal{C}_n \cap Q_n$ for every $n \in \mathbb{N} \cup \{0\}$. Consequently, the iterative sequence $\{u_n\}$ can be generated recursively.

Step 3: We demonstrate that $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$. From the definition of Q_n that $u_n = \mathcal{P}_{Q_n}(u_0)$. Thus, for all $w \in Q_n$ and all $n \in \mathbb{N} \cup \{0\}$,

$$\|u_n - u_0\| \leq \|w - u_0\|.$$

Let $w \in F(\mathcal{S}) \subset Q_n$. Then

$$\|u_n - u_0\| \leq \|w - u_0\|$$

On the other hand, since $u_{n+1} = \mathcal{P}_{\mathcal{C}_n \cap Q_n}(u_0) \in Q_n$ for all $n \in \mathbb{N} \cup \{0\}$

$$\|u_n - u_0\| \leq \|u_{n+1} - u_0\|.$$

Thus, the sequence $\{\|u_n - u_0\|\}$ is non-decreasing. Noting that $\mathcal{C}$ is bounded, we conclude that $\lim_{n\to\infty} \|u_n - u_0\|$ exists. This implies that $\{u_n\}$ is bounded. Noting that $u_{n+1} = \mathcal{P}_{\mathcal{C}_n \cap Q_n}(u_0) \in Q_n$ and $u_n = \mathcal{P}_{Q_n}(u_0)$, we have $\langle u_{n+1} - u_n,\ u_n - u_0\rangle \geq 0$. From Lemma 2.1, for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &= \|(u_{n+1} - u_0) - (u_n - u_0)\|^2 \\ &= \|u_{n+1} - u_0\|^2 - \|u_n - u_0\|^2 - 2\langle u_{n+1} - u_n,\ u_n - u_0\rangle \\ &\leq \|u_{n+1} - u_0\|^2 - \|u_n - u_0\|^2.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Since $u_{n+1} = \mathcal{P}_{\mathcal{C}_n \cap Q_n} \in \mathcal{C}_n$, we have $\|v_n - u_{n+1}\|^2 \leq \|u_n - u_{n+1}\|^2 + \theta_n$. Noting that $\lim_{n\to\infty} k_n = 1$, we have $\mathcal{L}_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \to 1$ as $n \to \infty$. Since $\|u_n - u_{n+1}\| + \sqrt{\theta_n} \to 0$, we have

$$\|v_n - u_{n+1}\| \leq \|u_n - u_{n+1}\| + \sqrt{\theta_n} \to 0.$$

That is, $\|v_n - u_{n+1}\| \to 0$.

Let $A_n = \frac{1}{n+1}\sum_{j=0}^{n} \mathcal{S}^j$. From $v_n = \sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1}\sum_{j=0}^{n} \mathcal{S}^j u_n\right)$, we get $v_n = A_n u_n$. Hence

$$\|u_n - A_n u_n\| = \|u_n - v_n\| \to 0.$$

By Lemma 2.4, we derive that

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left\|A_n u_n - \mathcal{S}^l(A_n u_n)\right\| \leq \limsup_{l\to\infty} \limsup_{n\to\infty} \sup_{u\in C} \left\|A_n u - \mathcal{S}^l(A_n u)\right\| = 0.$$

Therefore,

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left\|A_n u_n - \mathcal{S}^l\left(A_n u_n\right)\right\| = 0.$$

Let $t_\infty = sup\{t_n : n \geq 1\}$. Thus, we derive for $n, l \to \infty$

$$\begin{aligned}\|u_n - \mathcal{S}^l u_n\| &= \|u_n - A_n u_n + A_n u_n - \mathcal{S}^l(A_n u_n) + \mathcal{S}^l(A_n u_n) - \mathcal{S}^l u_n\| \\ &\leq \|u_n - A_n u_n\| + \|A_n u_n - \mathcal{S}^l(A_n u_n)\| + \|\mathcal{S}^l(A_n u_n) - \mathcal{S}^l u_n\| \\ &\leq \|u_n - A_n u_n\| + \|A_n u_n - \mathcal{S}^l(A_n u_n)\| + t_l\|u_n - A_n u_n\| \\ &\leq (1 + t_\infty)\|u_n - A_n u_n\| + \|A_n u_n - \mathcal{S}^l(A_n u_n)\| \to 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|\mathcal{S}u_n - u_n\| &\leq \|\mathcal{S}u_n - \mathcal{S}^{l+1}u_n\| + \|\mathcal{S}^{l+1}u_n - \mathcal{S}^{l+1}u_{n+1}\| \\ &\quad + \|\mathcal{S}^{l+1}u_{n+1} - u_{n+1}\| + \|u_{n+1} - u_n\| \\ &\leq t_\infty\|u_n - \mathcal{S}^l u_n\| + \|u_{n+1} - \mathcal{S}^l u_{n+1}\| \\ &\quad + (1 + t_\infty)\|u_{n+1} - u_n\| \to 0.\end{aligned}$$

Step 4: Let us show that the sequence $\{u_n\}$ converges strongly to $\mathcal{P}_{F(\mathcal{S})}(u_0)$. Let $\eta = \mathcal{P}_{F(\mathcal{S})}(u_0)$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup \eta'$. By Lemma 2.3 $\eta' \in F(\mathcal{S})$. Since $u_n = \mathcal{P}_{Q_n}(u_0)$, $\eta \in F(\mathcal{S}) \subset Q_n$, it follows that $\|u_n - u_0\| \leq \|\eta - u_0\|$. Because $\eta = P_{F(\mathcal{S})}(u_0)$ and the norm is weakly lower semi-continuous, we derive the following:

$$\|\eta - u_0\| \leq \left\|\eta^{'} - u_0\right\| \leq \liminf_{k\to\infty} \|u_{n_k} - u_0\| \leq \limsup_{k\to\infty} \|u_{n_k} - u_0\| \leq \|\eta - u_0\|.$$

Thus, we derive that $\lim_{k\to\infty} \|u_{n_k} - u_0\| = \left\|\eta^{'} - u_0\right\| = \|\eta - u_0\|$. By the Kadec–Klee property of H, we obtain

$$\lim_{k\to\infty} u_{n_k} = \eta^{'} = \eta.$$

Because $\{u_{n_k}\}$ is a subsequence of $\{u_n\}$, we deduce that the sequence $\{u_n\}$ strongly converges to $\mathcal{P}_{F(\mathcal{S})}(u_0)$.

3.2. Convergence Analysis for Equilibrium Problems

Theorem 3.2.1. Let $\mathcal{H}$ be a real Hilbert space and $\varnothing \neq \mathcal{C} \subset \mathcal{H}$ be a closed, convex and bounded subset. Let $\mathcal{S}$: $\mathcal{C} \to \mathcal{C}$ be an asymptotically non-expansive mapping defined with k_n, denote $\mathcal{L}_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j$. Also, let f : $\mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a functional defined by conditions (A1)-(A4). Suppose that $\{r_n\} \subset (0,\infty)$ fulfills $\liminf_{n\to\infty} r_n > 0$. Then, under the consideration of the iterative scheme (5), the sequence $\{u_n\}$ strongly converges to $\mathcal{P}_{F(\mathcal{S})\cap EP(f)}(u_0)$.

Proof. For clarity, the argument follows the same stepwise structure as the proof of Theorem 3.1.1.

Step 1: As in Step 1 of Theorem 3.1.1, the sets C_n and Q_n are closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

Step 2: We aim to demonstrate that $F(\mathcal{S}) \cap EP(f) \subset \mathcal{C}_n \cap Q_n$ for each $n \in \mathbb{N} \cup \{0\}$. Let $p \in F(\mathcal{S}) \cap EP(f)$. We get $x_n = \mathcal{S}_{r_n} v_n$ for $\forall n \in \mathbb{N} \cup \{0\}$, we derive that S_{r_n} is a relatively non-expansive mapping from (2) of Lemma 2.6. Observe that every relatively non-expansive mapping becomes non-expansive in a Hilbert space. Thus, we can use the definitions of fixed points and asymptotically non-expansive mappings for each $n \in \mathbb{N} \cup \{0\}$

$$\|x_n - p\|^2 = \|\mathcal{S}_{r_n} v_n - p\|^2 = \|\mathcal{S}_{r_n} v_n - \mathcal{S}_{r_n} p\|^2 \leq \|v_n - p\|^2.$$

By applying the estimates derived in Step 2 of Theorem 3.1.1, we arrive at $\|x_n - p\|^2 \leq \|u_n - p\|^2 + \theta_n$ which implies that $p \in \mathcal{C}_n$. Consequently, $F(\mathcal{S}) \cap EP(f) \subset \mathcal{C}_n$. Using a similar argument, one also obtains $F(\mathcal{S}) \cap EP(f) \subset Q_n$. So, we derive $F(\mathcal{S}) \cap EP(f) \subset \mathcal{C}_n \cap Q_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus, it follows that the sequence $\{u_n\}$ is well defined.

Step 3: We now show that $\{x_n\}$ is bounded and that $\lim_{n\to\infty} \|u_n - Su_n\| = 0$. Arguing as in Step 3 of Theorem 3.1.1, it follows that $\{u_n\}$ is bounded and

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{6}$$

Since $u_{n+1} = \mathcal{P}_{\mathcal{C}_n \cap Q_n} \in \mathcal{C}_n$, $\|x_n - u_{n+1}\|^2 \leq \|u_n - u_{n+1}\|^2 + \theta_n$. Considering $\theta_n \to 0$, we have

$$\|x_n - u_{n+1}\| \leq \|u_n - u_{n+1}\| + \sqrt{\theta_n} \to 0.$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} \|u_n - x_n\| &= \lim_{n\to\infty} \|u_n - u_{n+1} + u_{n+1} + x_n\| \\ &\leq \lim_{n\to\infty} \|u_n - u_{n+1}\| + \lim_{n\to\infty} \|u_{n+1} - x_n\| \to 0. \end{aligned} \tag{7}$$

Since $p \in F(\mathcal{S}) \cap EP(f) \subset \mathcal{C}_n$, we have

$$\|x_n - p\|^2 \leq \|u_n - p\|^2 + \theta_n. \tag{8}$$

For $x_n = S_{r_n} v_n$, applying Lemma 2.7 and (8), we obtain

$$\begin{aligned} \|x_n - v_n\|^2 &= \|S_{r_n} v_n - v_n\|^2 \\ &\leq \|v_n - p\|^2 - \|S_{r_n} v_n - p\|^2 \\ &\leq \|u_n - p\|^2 + \theta_n - \|x_n - p\|^2 \longrightarrow 0. \end{aligned} \tag{9}$$

Let $A_n = \frac{1}{n+1}\sum_{j=0}^{n}\mathcal{S}^j$. Thus $v_n = A_n u_n$. We have by (7) and (9),

$$\begin{aligned}\|u_n - A_n u_n\| &= \|u_n - v_n\| \\ &\leq \|u_n - x_n\| + \|x_n - v_n\| \to 0.\end{aligned} \tag{10}$$

An application of Lemma 2.4 yields the following:

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\left\|A_n u_n - \mathcal{S}^l(A_n u_n)\right\| \leq \limsup_{l\to\infty}\limsup_{n\to\infty}\sup_{u\in\mathcal{C}}\left\|A_n u - \mathcal{S}^l(A_n u)\right\| = 0.$$

Therefore,

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\left\|A_n u_n - \mathcal{S}^l(A_n u_n)\right\| = 0. \tag{11}$$

Let $k_\infty = \sup\{k_n : n \geq 1\} < \infty$. Then, as $n, l \to \infty$,

$$\begin{aligned}\|u_n - \mathcal{S}^l u_n\| &= \|u_n - A_n u_n + A_n u_n - \mathcal{S}^l(A_n u_n) + \mathcal{S}^l(A_n u_n) - \mathcal{S}^l u_n\| \\ &\leq \|u_n - A_n u_n\| + \|A_n u_n - \mathcal{S}^l(A_n u_n)\| + \|\mathcal{S}^l(A_n u_n) - \mathcal{S}^l u_n\| \\ &\leq \|u_n - A_n u_n\| + \|A_n u_n - \mathcal{S}^l(A_n u_n)\| + t_l\|u_n - A_n u_n\| \\ &\leq (1 + t_\infty)\|u_n - A_n u_n\| + \|A_n u_n - \mathcal{S}^l(A_n u_n)\| \to 0.\end{aligned}$$

Thus, by (7) and (9), we have

$$\|u_n - \mathcal{S}u_n\| \leq \|u_n - x_n\| + \|x_n - v_n\| + \|v_n - \mathcal{S}u_n\| \to 0. \tag{12}$$

Combining (12) and (6) yields the following:

$$\begin{aligned}\|\mathcal{S}u_n - u_n\| &\leq \|\mathcal{S}u_n - \mathcal{S}^{l+1}u_n\| + \|\mathcal{S}^{l+1}u_n - \mathcal{S}^{l+1}u_{n+1}\| \\ &\quad + \|\mathcal{S}^{l+1}u_{n+1} - u_{n+1}\| + \|u_{n+1} - u_n\| \\ &\leq k_\infty\|u_n - \mathcal{S}^l u_n\| + \|\mathcal{S}^l u_{n+1} - u_{n+1}\| \\ &\quad + (1 + k_\infty)\|u_{n+1} - u_n\| \to 0.\end{aligned}$$

Step 4: We finally establish the strong convergence of the sequence $\{u_n\}$ to $\mathcal{P}_{F(\mathcal{S})\cap EP(f)}(u_0)$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup \eta^{'}$. By Lemma 2.3 we derive $\eta^{'} \in F(S)$. Now we will show that $\eta^{'} \in E\mathcal{P}(f)$. We have $x_{n_k} \rightharpoonup \eta^{'}$, $v_{n_k} \rightharpoonup \eta^{'}$ from (2) and (9). Since $x_n = \mathcal{S}_{r_n} v_n$, we have the inequality for all $v \in \mathcal{C}$

$$f(x_n, v) + \frac{1}{r_n}\langle v - x_n,\, x_n - v_n\rangle \geq 0.$$

Substituting n_k for n in the inequality, we have from condition (A2):

$$\frac{1}{r_{n_k}}\langle v - x_{n_k},\, x_{n_k} - v_{n_k}\rangle \geq -f(x_{n_k}, v) \geq f(v, x_{n_k}), \qquad \forall\, v \in \mathcal{C}.$$

Let $k \to \infty$. Since $\liminf_{n\to\infty} r_n > 0$, from (9) and condition (A4), we derive for all $v \in \mathcal{C}$ $f\left(v, \eta^{'}\right) \leq 0$. For $t \in (0,1)$ and $v \in \mathcal{C}$, let $v_t = tv + (1-t)\eta^{'}$. Then $v_t \in \mathcal{C}$. Thus $f\left(v_t, \eta^{'}\right) \leq 0$. Applying condition (A1) yields

$$0 = f(v_t, v_t) \leq t\, f(v_t, v) + (1-t) f\left(v_t, \eta^{'}\right) \leq t f(v_t, v).$$

After dividing both sides by t, we deduce that $f(v_t, v) \geq 0$, $\forall v \in C$. Passing to the limit as $t \to 0$ and using condition (A3), it follows that $f\left(\eta^{'}, v\right) \geq 0$, $\forall v \in C$. So, $\eta^{'} \in E\mathcal{P}(f)$. Denote $\eta = P_{F(\mathcal{S})\cap EP(f)}(u_0)$. We know

that $u_{n+1} = P_{C_n \cap Q_n}(u_0)$, $\eta \in F(\mathcal{S}) \cap EP(f) \subset C_n \cap Q_n$. Then, we have $\|u_{n+1} - u_0\| \leq \|\eta - u_0\|$. Due to the weak lower semi-continuity of the norm, we obtain

$$\|\eta - u_0\| \leq \left\|\eta^{'} - u_0\right\| \leq \liminf_{k\to\infty} \|u_{n_k} - u_0\| \leq \limsup_{k\to\infty} \|u_{n_k} - u_0\| \leq \|\eta - u_0\| .$$

Thus, we have $\lim_{k\to\infty} \|u_{n_k} - u_0\| = \left\|\eta^{'} - u_0\right\| = \|\eta - u_0\|$. An application of the Kadec-Klee property of H, we deduce that $\lim_{k\to\infty} u_{n_k} = \eta^{'} = \eta$. Since $\{u_{n_k}\}$ is a subsequence of the sequence $\{u_n\}$, it follows that the sequence $\{u_n\}$ strongly converges to $\mathcal{P}_{F(\mathcal{S})\cap EP(f)}(u_0)$.

Example 3.2.1. We consider the Hilbert space $\mathbb{R}^2$ with a closed convex subset:

$$\mathcal{C} = \left\{(u, v) \in \mathbb{R}^2 \ : \ -1 \leq u, v \leq 1\right\}.$$

Consider the mapping $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ introduced below, which is asymptotically non-expansive:

$$\mathcal{S}(u, v) = \left(\frac{v}{2}, -\frac{u}{2}\right) .$$

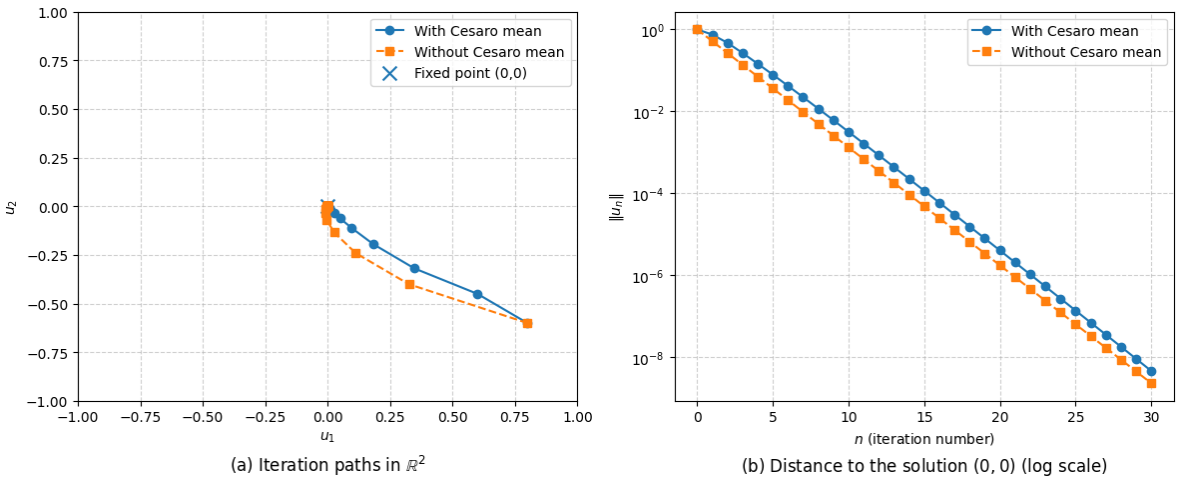
Then $\mathcal{S}$ is non-expansive, hence asymptotically non-expansive. Furthermore, $F(\mathcal{S}) = \{(0, 0)\}$.

Define the bifunction $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ by $f(u, v) = \langle u,\ v - u\rangle$, where $u, v \in \mathcal{C}$. Then f fulfills the standard conditions (A1)–(A4), and the equilibrium problem has the solution set $EP(f) = \{(0, 0)\}$.

Hence, $F(\mathcal{S}) \cap EP(f) = \{(0, 0)\}$. Choose $u_0 = (0.8, -0.6)$ and $r_n = 1$. Using the iterative scheme (5), where the coefficients satisfy $\alpha_i \geq 0$, $\alpha_1 > 0$, $\sum_{n=0}^{i} \alpha_n = 1$, $0 \leq n \leq i$, and $i \geq 1$, the Kirk–Cesàro step is $v_n = \sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1} \sum_{j=0}^{n} \mathcal{S}^j u_n\right)$.

The numerical results show that the Cesàro-based scheme exhibits a smoother and more regular convergence behavior toward the common solution $(0, 0)$. In the left panel of Figure 1, the iteration trajectories in $\mathbb{R}^2$ are displayed for the schemes with and without the Cesàro mean, allowing a direct visual comparison of their paths toward the fixed point.

The right panel of Figure 1 presents the decay of the error $\|u_n\|$ on a logarithmic scale. The decreasing curves confirm the convergence of both schemes to $(0, 0)$, while the Cesàro-based iteration provides a more uniform convergence pattern. These numerical observations are consistent with the theoretical convergence results established in the preceding analysis. Comparison of the proposed iterative scheme with and without the Cesàro mean. The left panel illustrates the iteration trajectories in $\mathbb{R}^2$, while the right panel shows the logarithmic decay of the distance $\|u_n\|$ to the common solution $(0, 0)$, indicating the convergence behavior of the schemes.



(a) Iteration paths in $\mathbb{R}^2$ (b) Distance to the solution $(0, 0)$ (log scale)

Figure 1. Comparison of the proposed iterative scheme with and without Cesàro mean.

4. Application

This section is devoted to an application of the developed Kirk-type hybrid iteration method to convex optimization problems in real Hilbert spaces. Since many optimization problems admit a reformulation as equilibrium problems, the convergence results of Section 3 can be directly applied to approximate minimizers of convex functionals.

Let $\mathcal{C}$ be a non-empty closed convex subset of a real Hilbert space $\mathcal{H}$, and let Φ: $\mathcal{C}\to \mathbb{R}$ be a convex and Fréchet differentiable functional. Consider the convex minimization problem

$$\min_{u\in\mathcal{C}} \Phi(u). \tag{13}$$

A point $u^*\in \mathcal{C}$ is called a minimizer of Φ if

$$\Phi\left(u^*\right) \leq \Phi(v) \ , \quad \forall v \in \mathcal{C}.$$

It is well known that $u^*\in \mathcal{C}$ solves problem (13) if and only if

$$\langle \nabla\Phi(u^*),\, v-u^*\rangle \geq 0, \qquad \forall\, v \in \mathcal{C}. \tag{14}$$

where $\nabla\Phi$ denotes the gradient of Φ, and define the bifunction $f : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$ by

$$f(u,v) = \langle \nabla\Phi(u),\, v-u\rangle\,, \qquad u, v \in \mathcal{C}. \tag{15}$$

Then, the equilibrium problem associated with f becomes equivalent to the convex minimization problem (13). Hence, the developed Kirk-type hybrid iterative method based on Cesàro means can be employed to approximate minimizers of convex optimization problems. On the basis of this equivalence, the following iterative procedure is examined:

$$v_n = \sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1} \sum_{j=0}^{n} \mathcal{S}^j u_n \right), \tag{16}$$

$$x_n = \left\{ w \in \mathcal{C} : \langle \nabla\Phi(w),\, y-w\rangle + \frac{1}{r_n}\langle y-w,\, w-v_n\rangle \geq 0,\ \forall\, y \in \mathcal{C} \right\}, \tag{17}$$

where the coefficients satisfy $\alpha_i \geq 0$, $\alpha_1 > 0$, $\sum_{n=0}^{i} \alpha_n = 1$, $0 \leq n \leq i$, and $i \geq 1$.

The following result is an immediate consequence of Theorem 3.2.1.

Theorem 4.1. Let $\mathcal{C}$ be a non-empty closed convex subset of a real Hilbert space $\mathcal{H}$, and let Φ: $\mathcal{C}\to \mathbb{R}$ be a convex and Fréchet differentiable functional such that $\nabla\Phi$ is inverse strongly monotone. Assume that the minimization problem (13) admits a solution. Suppose that the control conditions of Theorem 3.2.1 are satisfied. Then the sequence $\{u_n\}$ generated by the iterative scheme (16)–(17) converges strongly to a minimizer u^* of the convex optimization problem (13).

Proof. Define the bifunction

$$f(u,v) = \ \langle \nabla\Phi(u),\, v-u\ \rangle\,.$$

Then the equilibrium problem associated with f is equivalent to the convex minimization problem (13). Since $\nabla\Phi$ is inverse strongly monotone, all conditions of Theorem 3.2.1 are satisfied. Therefore, the generated sequence converges strongly to a minimizer of Φ.

A numerical example is provided below to validate the theoretical result.

Example 4.1. Let $\mathcal{H}= \mathbb{R}$ and $\mathcal{C} =[-1, 1]$. Define the convex functional

$$\Phi(u) = u^2\ , \quad u \in \mathcal{C}.$$

Then,

$$\nabla\Phi(u) = 2u.$$

Clearly, Φ is convex and differentiable, and the unique minimizer of Φ is $u^* = 0$. Define the mapping $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ by

$$\mathcal{S}u = \frac{u^2}{2}\ ,\ u \in \mathcal{C}.$$

Choose $u_0 = 1,\ r_n = 1$ and define

$$v_n = \sum_{n=0}^{i} \alpha_n \left(\frac{1}{n+1} \sum_{j=0}^{n} \mathcal{S}^j u_n \right),\ \alpha_i \geq 0\ ,\quad \alpha_1 > 0\ ,\ \sum_{n=0}^{i} \alpha_n = 1,\quad 0 \leq n \leq i,\ i \geq 1\ .$$

The generated sequence converges strongly to the minimizer $u^* = 0$.

Accordingly, the Kirk–Cesàro hybrid framework developed here may also be applicable to broader classes of optimization and variational models arising in nonlinear analysis.

5. Conclusions

This work shows that combining Kirk iteration, Cesàro meaning, and hybrid projection techniques yields an effective approximation approach for equilibrium and fixed point problems in Hilbert spaces. Under appropriate control conditions, the Kirk–Cesàro hybrid iterative schemes introduced here were shown to converge strongly to solutions of the corresponding fixed point and equilibrium problems.

By embedding the Kirk iterative structure into a unified convergence setting, the present approach extends a number of existing Cesàro-type and hybrid iterative methods. Numerical experiments and graphical illustrations further support the theoretical convergence results and illustrate the behavior of the proposed schemes. An application to convex optimization via the equilibrium problem formulation is also included.

The proposed framework may also be adaptable to optimization and equilibrium models arising in various applied disciplines. However, the present study is primarily theoretical, and the investigation of problem-specific applications remains a subject for future research.

Future research may focus on extending the proposed Kirk–Cesàro hybrid framework to Banach spaces and CAT(0) spaces, as well as to settings based on Bregman distances. Further investigations may also incorporate inertial and viscosity-type techniques, adaptive control parameters, and more general classes of equilibrium and fixed point problems.

Authors' Contribution

Lale CONA: Conceptualization, Methodology, Formal Analysis, Investigation, Original Draft Writing, Supervision, Project Administration, and Funding Acquisition. Deniz ŞİMŞEK: Software, Validation, Data Curation, Visualization, and Writing, Review and Editing. Both authors read and approved the published version of the manuscript.

Conflicts of Interest

The authors confirm that there are no conflicts of interest to disclose.

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