

New Accurate Approximations for the Normal Distribution Function with Application in Engineering

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Abstract This article proposes seven accurate closed-form approximations for the standard normal distribution function and related functions, including the complementary error function, the error function, and the quantile function. The approximations refine Hart's (1957) [24] approximation by fitting an exponential polynomial correction through nonlinear least squares, and their accuracy is assessed by the maximum absolute error (Max.AE) and the mean absolute error (Mean.AE) over $z \in [0, 7]$. The most accurate proposed approximation attains a maximum absolute error of 8.57×10^{-9} ; the family is up to 13 times more accurate than the approximations it refines, and among recent methods its most accurate member is surpassed only by a logistic approximation carrying a polynomial of degree 16. We further establish the monotonicity of the approximations, analyse their tail behaviour and computational cost, and benchmark their run time, finding them faster than the built-in normal-CDF routine. The applicability of the approximations is demonstrated on two problems in science and engineering, namely evaluation of the incomplete gamma function and bit-error-rate analysis in digital communications, in both cases with quantitative error assessment.

Keywords Error function, Normal distribution, Approximations, Cumulative distribution function, Maximum absolute error, Bit error rate, Incomplete Gamma function

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1. Introduction

The normal distribution and its related functions—the error function, the complementary error function, and the quantile function—arise in many fields. In engineering, the error function appears in transport phenomena, digital and signal processing, noise analysis, and phase-shift-keying modulation [1, 2]. Other applications of the normal distribution include electron-velocity distributions [3], risk assessment in finance [4], and the social sciences [5], among others; see the references therein for further examples.

The standard normal distribution has mean zero and standard deviation one. Its probability density function (PDF) is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

and its cumulative distribution function (CDF), giving the probability that a random variable is at most z , is

$$\Phi(z) = \int_{-\infty}^z \phi(x) dx.$$

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Despite its central role in statistics and applied science, this integral has no closed-form solution, so it must be approximated. Although exact numerical methods are precise, they can be computationally expensive, which motivates closed-form approximations that balance accuracy against computational efficiency. Such approximations are widely used in stochastic modelling, Monte Carlo simulation, and quantum mechanics, where the normal distribution represents randomness, noise, and error terms.

Approximating the normal CDF has a long history. Early contributions include those of Pólya (1949) [7], Cadwell (1951) [9], Chernoff (1952) [11], Hastings (1955) [10], whose polynomial expressions improved accuracy at the cost of greater complexity, Burr (1967) [13], and Ordaz (1991) [12]. Invertible approximations, which are important for Monte Carlo simulation and random-number generation, were advanced by Abramowitz and Stegun (1964) [14] through the error function (Erf), closely related to the CDF by

$$\Phi(z) = 0.5 \left[1 + \operatorname{Erf} \left(\frac{z}{\sqrt{2}} \right) \right], \quad \operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

More recent work has refined these methods. Aludaat and Alodat (2008) [15] gave a sharper Pólya-type upper bound; Bowling et al. (2009) [17] proposed a logistic approximation balancing simplicity and accuracy; and Eidous and Al-Salman (2016) [16] introduced a one-term form minimizing the mean absolute error (MAE). The explicitly invertible approximation of Soranzo and Epure (2014) [20], with an MAE of 1.3×10^{-4} , has been widely adopted, while Hanandeh and Eidous [18, 19] and Eidous and Al-Rawwash [25] developed accurate simple approximations and their inverses, noting that simple forms, though less precise than more complex ones, are effective in most practical applications. Obeidat et al. (2025) further improved single-term [22] and multiple-term [23] approximations, lowering the Max.AE and Mean.AE of earlier proposals by factors of about four to six. Further simple and accurate approximations have been proposed by Edous and Eidous (2018) [29], Eidous and Abu Hawwas (2021) [30], and Eidous and Ananbeh (2022) [31]; the tanh-based approximation of Eidous, Alzoubi and Hanandeh (2026) [38], published in this journal, attains a maximum absolute error of 2.95×10^{-5} .

A closely related and active line of research develops tight upper and lower bounds for $\Phi(z)$ and the associated Q - and error functions; many of these bounds are sharp enough to serve as highly accurate approximations in their own right. Representative recent contributions include the refinements of the error function by Bercu (2020) [32], the bit-error-rate bounds of Trigui et al. (2021) [33], and the tight bounds of Eidous (2023, 2025, 2026) [34, 35, 36] and Ananbeh and Eidous (2025) [37].

Optimization-based methods have also proved valuable, tuning the parameters of an approximation by global optimization or least-squares fitting. Lipoth et al. (2022) [6] used global optimization to obtain an MAE of 2.73×10^{-5} , and Eidous and Abu-Shareefa (2020) [8] proposed nine approximations—some with MAEs as low as 8.63×10^{-9} —that outperformed 45 existing methods in accuracy and speed. In this article we use optimization to improve the Eidous and Abu-Shareefa approximations, which are built on the Hart [24] approximation, and we study their usefulness in science and engineering.

The performance of the proposed approximations is assessed by the maximum absolute error (Max.AE) and the mean absolute error (Mean.AE). For an approximation $\hat{\Phi}(z)$ of $\Phi(z)$,

$$\text{Max.AE} = \max_z \left| \Phi(z) - \hat{\Phi}(z) \right|, \quad \text{Mean.AE} = \frac{\sum \left| \Phi(z_i) - \hat{\Phi}(z_i) \right|}{N},$$

where z_i runs from 0 to 7 in steps of 0.0001 and N is the number of points. The step 0.0001 is fine enough that halving it leaves the reported errors unchanged to the significant figures quoted. The upper bound $z = 7$ is used because $\Phi(7) = 1 - 1.28 \times 10^{-12}$, so that for $z > 7$ the tail probability $1 - \Phi(z)$ falls below the accuracy relevant to virtually all applications and approaches the double-precision limit; this range also matches the related literature. Negative arguments follow from the symmetry $\Phi(-z) = 1 - \Phi(z)$, and the behaviour for $z > 7$ is examined in Section 2.3.

Motivation and contributions. Evaluating $\Phi(z)$ exactly requires numerical integration, which is costly in the large-scale simulations and real-time systems common in engineering; a closed-form approximation that is at once accurate, monotone and inexpensive to evaluate is therefore of practical value. The aim of this work is to obtain such approximations by refining the Hart (1957) form, and to establish—rather than merely assert—their

accuracy and efficiency. The main contributions are: (i) seven closed-form approximations of $\Phi(z)$, obtained by least-squares fitting of an exponential polynomial correction, the most accurate reaching a maximum absolute error of 8.57×10^{-9} ; (ii) an analysis of their mathematical properties, including a monotonicity condition, the asymptotic and tail behaviour, and the effect of the polynomial degree; (iii) a computational-cost analysis and a measured run-time benchmark showing the approximations to be faster than the built-in normal-CDF routine; (iv) a comparison with the most recent competing approximations over the CDF, its tail, and the bit-error-rate; and (v) two applications—the incomplete gamma function and bit-error-rate analysis—each assessed with quantitative error metrics.

The rest of the article is organized as follows. Section 2 describes the proposed approximations, Section 3 presents two applications in science and engineering, and Section 4 concludes.

2. The Proposed Approximations

In this section, we propose seven improved approximations of the standard normal distribution function. The proposed approximations are built upon the Hart (1957) [24] approximation, which is given by:

$$\Phi^{\text{Hart}}(z) = 1 - \frac{\phi(z)}{z + \sqrt{2/\pi} e^{-0.4z}}.$$

The proposed approximations of $\Phi(z)$ are of the form

$$\Phi^*(z) = 1 - \frac{\phi(z)}{z + c e^{-y(z)z}},$$

where c is a constant and $y(z)$ is a polynomial in z of degree n ; different polynomial degrees yield different approximations. We define $y(z)$ as follows:

$$y(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad n = 0, 1, 2, \dots$$

where $a_0, a_1, \dots, a_n$ are the constants to be determined, obtained with the `nls` function of the R software, which solves the nonlinear least-squares problem by the Gauss–Newton algorithm using R’s default relative-offset convergence criterion (tolerance 10^{-5}); starting values were obtained by differential evolution (DE), a global stochastic search, from 200 random starting points. The robustness of the fitted coefficients to the choice of starting values is examined in Section 2.1. The value of $y(z)$ can thus be evaluated at each $z \geq 0$. For negative values of z we use the symmetry property $\Phi(-z) = 1 - \Phi(z)$ of the Gaussian density. Below are the seven improved approximations obtained by refining the existing approximations for $\Phi(z)$:

1. $\Phi_1^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_1)z}}$
where $c = 0.7965703$ and $y_1 = 0.4294738$
2. $\Phi_2^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_2)z}}$
where $c = 0.79791037$ and $y_2 = 0.45655546 - 0.03807792z$
3. $\Phi_3^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_3)z}}$
where $c = 0.797899283$ and $y_3 = 0.456143757 - 0.036653752z - 0.001019927z^2$
4. $\Phi_4^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_4)z}}$
where $c = \sqrt{2/\pi}$ and $y_4 = 0.4555728 - 0.03392489z - 0.004674953z^2 + 0.001357412z^3$
5. $\Phi_5^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_5)z}}$
where $c = \sqrt{2/\pi}$ and $y_5 = 0.4554407 - 0.03301093z - 0.00651859z^2 + 0.002721178z^3 - 0.0003252627z^4$

$$6. \Phi_6^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_6)z}}$$

$$\text{where } c = \sqrt{2/\pi} \quad \text{and} \quad y_6 = 0.4554275 - 0.03288509z - 0.006874648z^2 + 0.003136104z^3 - 0.0005324791z^4 + 0.00003677024z^5$$

$$7. \Phi_7^*(z) = 1 - \frac{\phi(z)}{z + c e^{-(y_7)z}}$$

$$\text{where } c = \sqrt{2/\pi} \quad \text{and} \quad y_7 = 0.4554289 - 0.03290387z - 0.006796240z^2 + 0.002989056z^3 - 0.0003920266z^4 - 0.00003362461z^5 + 0.00001749899z^6 - 0.000001689979z^7$$

In Table 1, we report the Max.AE and Mean.AE of the older approximations (those of Eidous and Abu-Shareefa (2020) [8]) and the newly proposed improvements to the Hart approximation.

Table 1. Summary of Max.AE and Mean.AE for old approximations and their newly proposed improvements.

Proposed Approximations	Max.AE (Old)	Mean.AE (Old)	Max.AE (New)	Mean.AE (New)
$\Phi_1^*(z)$	9.51×10^{-4}	2.93×10^{-4}	8.10×10^{-4}	1.80×10^{-4}
$\Phi_2^*(z)$	8.64×10^{-5}	1.53×10^{-5}	1.56×10^{-5}	3.13×10^{-6}
$\Phi_3^*(z)$	6.37×10^{-5}	1.00×10^{-5}	8.85×10^{-6}	2.64×10^{-6}
$\Phi_4^*(z)$	2.59×10^{-5}	4.31×10^{-6}	2.54×10^{-6}	4.93×10^{-7}
$\Phi_5^*(z)$	2.46×10^{-6}	5.86×10^{-7}	1.84×10^{-7}	5.63×10^{-8}
$\Phi_6^*(z)$	1.28×10^{-7}	2.84×10^{-8}	2.27×10^{-8}	2.43×10^{-9}
$\Phi_7^*(z)$	4.23×10^{-8}	8.63×10^{-9}	8.57×10^{-9}	9.97×10^{-10}

We observe that the improved approximations are highly accurate, showing an improvement in accuracy ranging from 1.17 times to more than 13 times compared to the older approximations.

Figures 1 to 7 show the error function, which is the difference between $\Phi(z)$ and the proposed approximation $\Phi^*(z)$, for each of the seven proposed approximations.

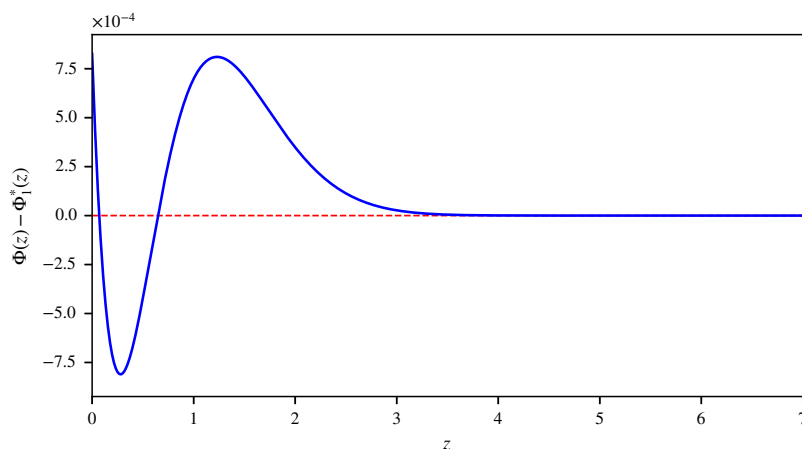
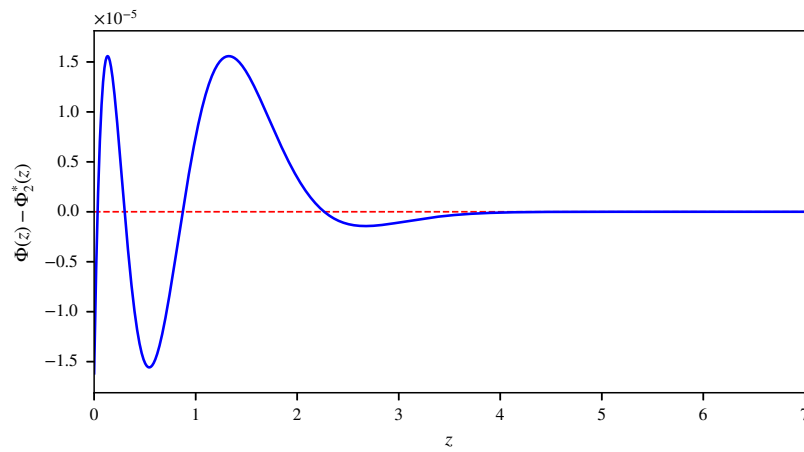
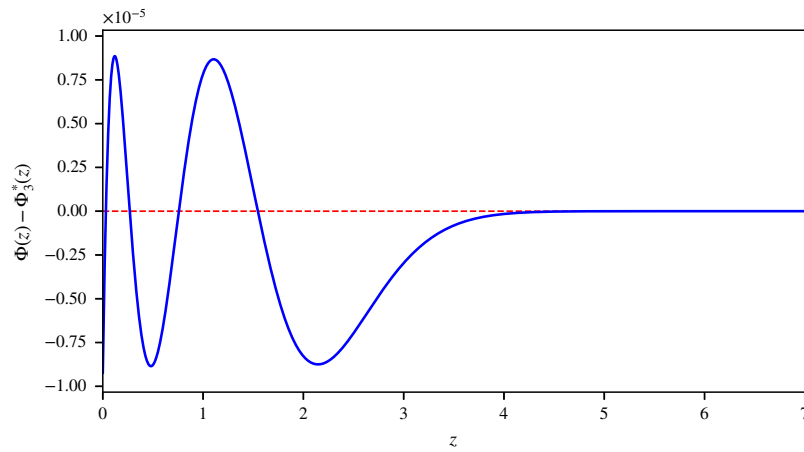
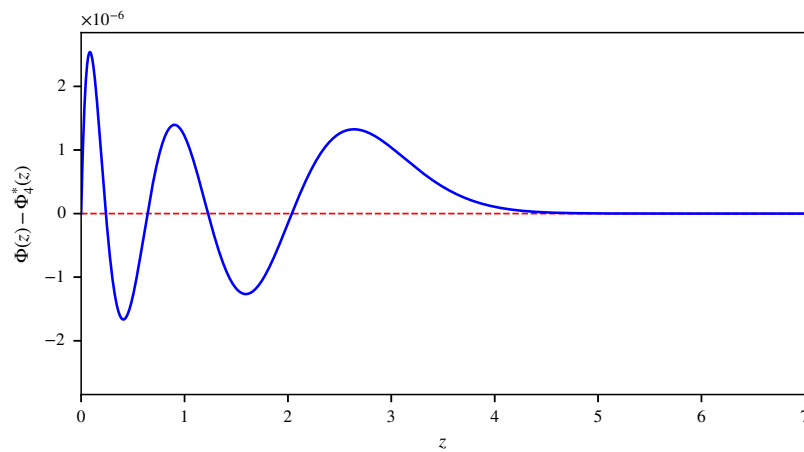


Figure 1. The difference between $\Phi(z)$ and $\Phi_1^*(z)$.

Figure 2. The difference between $\Phi(z)$ and $\Phi_2^*(z)$.Figure 3. The difference between $\Phi(z)$ and $\Phi_3^*(z)$.Figure 4. The difference between $\Phi(z)$ and $\Phi_4^*(z)$.

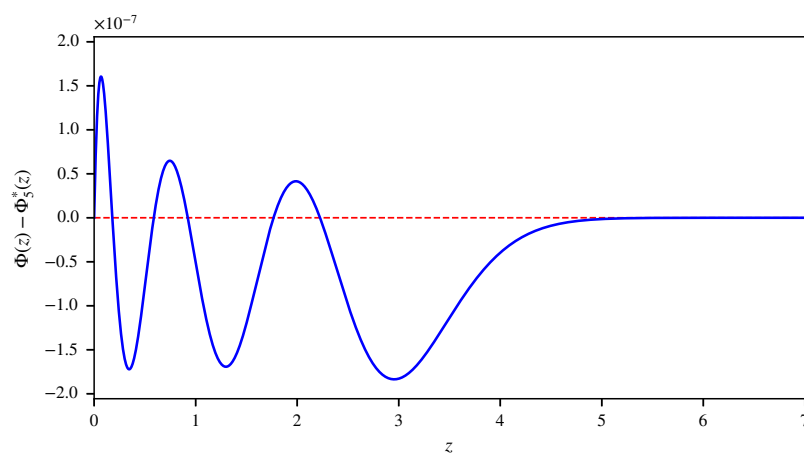


Figure 5. The difference between $\Phi(z)$ and $\Phi_5^*(z)$.

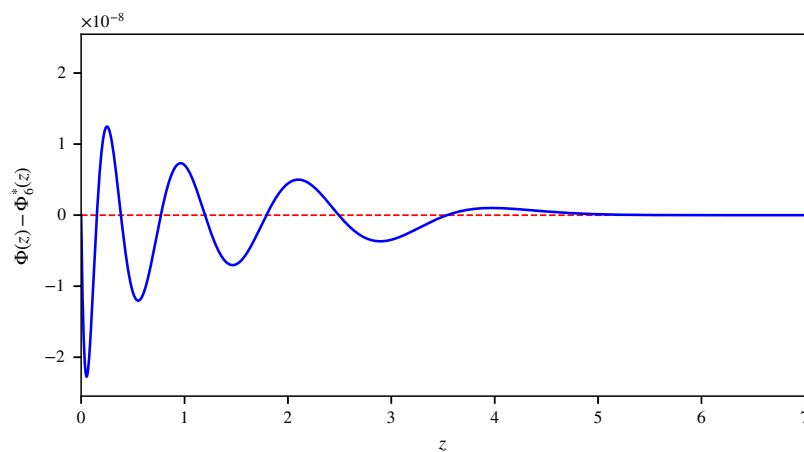


Figure 6. The difference between $\Phi(z)$ and $\Phi_6^*(z)$.

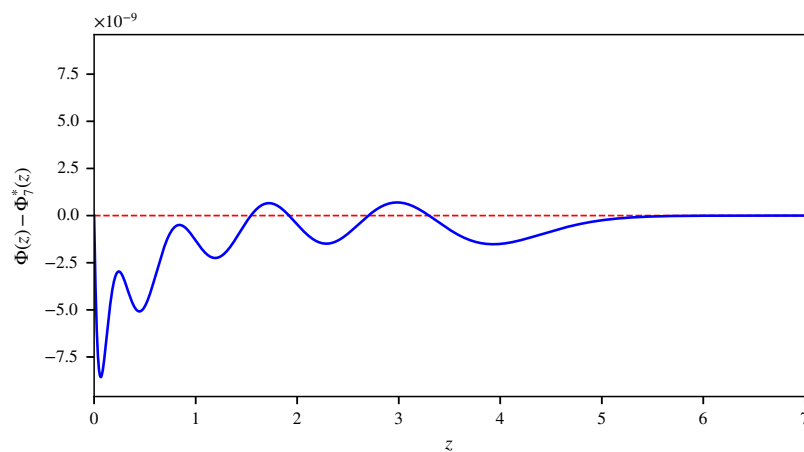


Figure 7. The difference between $\Phi(z)$ and $\Phi_7^*(z)$.

To place the proposed approximations in the context of the most recent literature, Table 2 compares them with several recent approximations cited above, using the same error measures on $z \in [0, 7]$ with step 0.0001. The proposed Φ_7^* is more accurate than all of these methods except the logistic form of Eidous and Al-Rawwash (2025), which is about 19 times more accurate but requires a polynomial of degree 16, more than twice the degree of that in Φ_7^* .

Table 2. Comparison of the proposed approximations with recent approximations from the literature (Max.AE and Mean.AE on $z \in [0, 7]$, step 0.0001).

Approximation	Max.AE	Mean.AE
Obeidat et al. (2025) [22]	9.38×10^{-4}	1.78×10^{-4}
Hanandeh–Eidous (2022) [19]	9.46×10^{-4}	2.09×10^{-4}
Lipoth et al. (2022) [6]	2.72×10^{-5}	8.68×10^{-6}
Eidous–Alzoubi–Hanandeh (2026) [38]	2.95×10^{-5}	9.25×10^{-6}
Eidous–Al-Rawwash (2025) [†] [25]	4.43×10^{-10}	8.86×10^{-11}
Eidous–Abu-Shareefa (2019), #9 [8]	4.23×10^{-8}	6.14×10^{-9}
Proposed $\Phi_5^*(z)$	1.84×10^{-7}	5.63×10^{-8}
Proposed $\Phi_6^*(z)$	2.27×10^{-8}	2.43×10^{-9}
Proposed $\Phi_7^*(z)$	8.57×10^{-9}	9.97×10^{-10}

[†]Computed with $k_3 = 0.072670769$, $k_8 = -3.0263611 \times 10^{-4}$ and $k_{14} = -1.09550613 \times 10^{-6}$ in $a(z)$, with $k_5 = -5.3498 \times 10^{-5}$ as in the displayed formula of [25]. The corresponding values printed there (0.72670769 , $-3.0263611 \times 10^{-3}$ and $+1.09550613 \times 10^{-6}$) appear to be misprints: the corrected set reproduces the published maximum absolute error, 4.43×10^{-10} at $z \approx 0.795$, and the corrected k_3 agrees with the Taylor coefficient $-\frac{2}{3}\phi(0) + \frac{16}{3}\phi(0)^3 = 0.0726712$ of z^2 in $a(z)$.

Table 3. Maximum absolute error of the proposed and competing approximations over several disjoint intervals of z . The column n is the degree of the polynomial appearing in each approximation ($y(z)$ for the proposed and the Eidous-type forms, the exponent polynomial for the Hart-type competitors, and $a(z)$ for the Eidous–Al-Rawwash and Eidous–Alzoubi–Hanandeh forms; Lipoth et al. use a parametric softplus form with no polynomial). The Eidous–Abu-Shareefa (2019) approximations #1, #8 and #9 are the methods refined in this article.

Approximation	n	[0, 2]	[2, 4]	[4, 7]	[7, 14]
Φ_1^*	0	8.25×10^{-4}	3.48×10^{-4}	6.32×10^{-7}	1.78×10^{-14}
Φ_2^*	1	1.62×10^{-5}	3.43×10^{-6}	7.98×10^{-8}	1.30×10^{-14}
Φ_3^*	2	9.23×10^{-6}	8.75×10^{-6}	1.59×10^{-7}	2.50×10^{-14}
Φ_4^*	3	2.54×10^{-6}	1.33×10^{-6}	1.06×10^{-7}	1.90×10^{-14}
Φ_5^*	4	1.72×10^{-7}	1.84×10^{-7}	3.97×10^{-8}	6.73×10^{-14}
Φ_6^*	5	2.27×10^{-8}	5.01×10^{-9}	1.01×10^{-9}	7.62×10^{-15}
Φ_7^*	7	8.57×10^{-9}	1.52×10^{-9}	1.50×10^{-9}	5.57×10^{-14}
Obeidat et al. (2025)	0	9.38×10^{-4}	3.45×10^{-4}	6.30×10^{-7}	1.78×10^{-14}
Hanandeh–Eidous (2022)	0	9.46×10^{-4}	3.75×10^{-4}	6.52×10^{-7}	1.81×10^{-14}
Lipoth et al. (2022)	–	2.72×10^{-5}	2.32×10^{-5}	8.81×10^{-6}	2.31×10^{-10}
Eidous–Alzoubi–Hanandeh (2026)	5	2.64×10^{-5}	2.95×10^{-5}	4.99×10^{-6}	1.19×10^{-12}
Eidous–Al-Rawwash (2025)*	16	4.43×10^{-10}	1.50×10^{-10}	1.46×10^{-10}	1.28×10^{-12}
Eidous–Abu-Shareefa (2019), #1	0	9.51×10^{-4}	3.76×10^{-4}	6.52×10^{-7}	1.81×10^{-14}
Eidous–Abu-Shareefa (2019), #8	5	1.27×10^{-7}	2.20×10^{-8}	3.25×10^{-10}	6.19×10^{-15}
Eidous–Abu-Shareefa (2019), #9	8	4.23×10^{-8}	8.09×10^{-9}	1.65×10^{-9}	1.13×10^{-14}

*Computed with the corrected coefficients given in the footnote to Table 2.

Table 3 compares the proposed approximations with several recent methods over disjoint intervals of z . The maximum absolute error is concentrated in the low-to-moderate range and becomes negligible (of order 10^{-14}) on $[7, 14]$ for all methods; absolute error therefore does not distinguish the approximations in the tail, where the relative error studied in Section 2.3 is the appropriate measure. Within the fitting range, the proposed Φ_k^* match or improve upon the recent competitors at equal or lower polynomial degree. The one method more accurate than every proposed member is the logistic form of Eidous and Al-Rawwash (2025), whose error stays near 10^{-10} on all of $[0, 7]$; it attains this with a polynomial of degree 16, against degree 7 for Φ_7^* . The degree-0 forms of Obeidat et al. (2025) and Hanandeh–Eidous (2022) lie at the accuracy level of Φ_1^* , while the five-parameter softplus form of Lipoth et al. (2022) and the tanh form of Eidous–Alzoubi–Hanandeh (2026), which carries a degree-5 polynomial, are comparable to Φ_2^* ; at the same polynomial degree, Φ_6^* is about three orders of magnitude more accurate than the latter. Most directly, at the degrees they refine the proposed approximations reduce the error of the Eidous–Abu-Shareefa (2019) approximations by roughly a factor of five — Φ_6^* against #8 at degree 5, and Φ_7^* against #9 at degree 7–8 — the latter achieving higher accuracy with a lower-degree polynomial.

2.1. Rationale, asymptotic behaviour and monotonicity

The proposed family is a rational–exponential approximation of the Mills ratio $R(z) = (1 - \Phi(z))/\phi(z)$. Since $1 - \Phi(z) = \phi(z)R(z)$, approximating Φ is equivalent to approximating $1/R(z)$ by the denominator $D(z) = z + ce^{-y(z)z}$. The leading term z reproduces the exact first-order asymptotic $1/R(z) \sim z$ as $z \rightarrow \infty$, so $\Phi^*(z) \rightarrow 1$ and the absolute error vanishes in the tail, while the correction $ce^{-y(z)z}$ captures the next-order behaviour of the Mills-ratio expansion. This explains why the form is effective: it is asymptotically consistent by construction, and the polynomial $y(z)$ only refines the moderate- z region. At the origin, $\Phi^*(0) = 1 - \frac{1}{c\sqrt{2\pi}}$, which equals $\frac{1}{2}$ exactly when $c = \sqrt{2/\pi}$; this motivates the choice $c = \sqrt{2/\pi}$ in $\Phi_4^* - \Phi_7^*$. Increasing the degree n of $y(z)$ systematically reduces both error measures (Table 1), so the family converges to Φ as n grows.

A CDF approximation should be monotonically increasing. Differentiating $\Phi^*(z) = 1 - \phi(z)/D(z)$ and using $\phi'(z) = -z\phi(z)$ gives

$$\Phi^{*'}(z) = \frac{\phi(z) [D'(z) + z D(z)]}{D(z)^2}.$$

Since $\phi(z) > 0$ and $D(z) > 0$ for $z \geq 0$, Φ^* is increasing if and only if $D'(z) + z D(z) \geq 0$. We verified this inequality numerically for all seven approximations on $[0, 7]$ (the minimum of $D'(z) + z D(z)$ exceeds 0.63 in every case), and it holds asymptotically since $D(z) \sim z$; monotonicity over $z < 0$ then follows from the symmetry $\Phi^*(-z) = 1 - \Phi^*(z)$.

The Gauss–Newton iterations of nls are local and cannot by themselves guarantee a global optimum. To reduce this risk, their starting values were obtained by differential evolution, a global stochastic search, from 200 random starting points, so that the local refinement begins near the best solution found over the whole search region rather than from an arbitrary guess. Global optimality cannot be proved for such a procedure, but combining a global search with local refinement makes convergence to a poor local optimum unlikely.

2.2. Computational cost

We compare the per-evaluation cost of the proposed approximations with that of the competing methods considered in this article. All of the Hart-type forms — the proposed Φ_k^* , Obeidat et al. (2025) [22], Hanandeh–Eidous (2022) [19] and the Eidous–Abu-Shareefa (2019) family [8] — share the same structure: one evaluation of ϕ , one further exponential $e^{-y(z)z}$, one division, and a Horner evaluation of the degree- n polynomial $y(z)$. They therefore use two exponentials and $O(n)$ arithmetic operations, with no iteration or series truncation. The logistic form of Eidous–Al-Rawwash (2025) [25] requires only a single exponential but carries a degree-16 polynomial, and the tanh form of Eidous–Alzoubi–Hanandeh (2026) [38] likewise needs a single exponential (through the hyperbolic tangent) with a degree-5 polynomial, whereas the softplus form of Lipoth et al. (2022) [6] replaces the polynomial by one logarithm and two real powers (each evaluated as $x^a = \exp(a \ln x)$), making it the most transcendental-heavy per call; it is, however, the only one of these forms that is explicitly invertible in closed form. Table 4 summarises the counts.

Overall, the Hart-type forms occupy a favourable middle ground: two exponentials and a short polynomial. The proposed Φ_k^* thus attain their accuracy at a cost comparable to the simplest competitors of the same structure, far below the degree-16 polynomial of the logistic form and the power/logarithm calls of the softplus form.

Table 4. Per-evaluation operation count for the proposed and competing approximations: n is the degree of the polynomial, and exp, log and pow count the exponential, logarithm and real-power calls. Each evaluation additionally uses one division; via Horner's rule the polynomial costs n multiplications and n additions, and a real power x^a is itself evaluated as $\exp(a \ln x)$; a hyperbolic tangent is counted as one exponential.

Approximation	n	exp	log	pow
Φ_1^*	0	2	0	0
Φ_2^*	1	2	0	0
Φ_3^*	2	2	0	0
Φ_4^*	3	2	0	0
Φ_5^*	4	2	0	0
Φ_6^*	5	2	0	0
Φ_7^*	7	2	0	0
Obeidat et al. (2025)	0	2	0	0
Hanandeh–Eidous (2022)	0	2	0	0
Lipoth et al. (2022)	–	1	1	2
Eidous–Alzoubi–Hanandeh (2026)	5	1	0	0
Eidous–Al-Rawwash (2025)	16	1	0	0
Eidous–Abu-Shareefa (2019), #1	0	2	0	0
Eidous–Abu-Shareefa (2019), #8	5	2	0	0
Eidous–Abu-Shareefa (2019), #9	8	2	0	0

Table 5. Wall-clock time for 10^7 evaluations in vectorised R (R 4.3.3, Windows 10, x86-64; median of seven runs), with the time per evaluation and the ratio to the built-in `pnorm`.

Approximation	Time (s)	ns/eval	ratio to <code>pnorm</code>
<code>pnorm</code> (baseline)	1.70	170	1.00
Φ_1^*	1.13	113	0.66
Φ_2^*	1.22	122	0.72
Φ_3^*	1.21	121	0.71
Φ_4^*	1.25	125	0.74
Φ_5^*	1.27	127	0.75
Φ_6^*	1.28	128	0.75
Φ_7^*	1.37	137	0.81
Obeidat et al. (2025)	1.12	112	0.66
Hanandeh–Eidous (2022)	1.11	111	0.66
Lipoth et al. (2022)	3.59	359	2.11
Eidous–Al-Rawwash (2025)	0.99	99	0.58
Eidous–Abu-Shareefa (2019), #1	1.08	108	0.64
Eidous–Abu-Shareefa (2019), #8	1.33	133	0.78
Eidous–Abu-Shareefa (2019), #9	1.40	140	0.82

To substantiate the efficiency claim, Table 5 reports the wall-clock time for 10^7 evaluations of each method in vectorised R (median of seven runs on R 4.3.3 under Windows 10, x86-64). All of the proposed approximations are faster than R's built-in `pnorm`: the constant-exponent member Φ_1^* runs at 0.66 times the cost of `pnorm` (113 ns per evaluation against 170 ns), and even the most accurate member Φ_7^* at 0.81 times. Within the Hart-type family the time grows only mildly with the polynomial degree, from about 113 ns at degree 0 to 140 ns at degree 8, consistent with the $O(n)$ cost of Table 4. The dominant factor is the number of transcendental calls rather than the polynomial: the single-exponential logistic form of Eidous–Al-Rawwash (2025) is the fastest at 99 ns ($0.58\times$), the

two-exponential Hart-type forms cluster between 108 and 140 ns, and the softplus form of Lipoth et al. (2022) — whose two real powers each expand into an exponential and a logarithm — is the only method slower than `pnorm`, at 359 ns ($2.11\times$). The proposed Φ_k^* therefore attain their accuracy at a cost below that of the highly optimised built-in routine and comparable to the lightest competing approximations, while remaining explicit and portable, which confirms their suitability for large-scale Monte Carlo simulation and real-time implementation.

2.3. Behaviour in the tail (large z)

The error grid used above is restricted to $z \in [0, 7]$. Because some applications require tail probabilities at larger arguments, we examine the behaviour of the approximations for $z > 7$. In the far tail both $\Phi(z)$ and $\Phi_k^*(z)$ approach 1, so the absolute error is bounded by the vanishing tail $1 - \Phi(z)$ and is no longer an informative measure: as Table 6 shows, the Max.AE of every approximation on $[7, 14]$ is of order 10^{-14} , irrespective of degree. The quantity that matters for tail probabilities is the relative error

$$r_k(z) = \frac{|\Phi_k^*(z) - \Phi(z)|}{1 - \Phi(z)}.$$

Table 6 reports both measures over $z \in [7, 14]$, with the Max.AE on the fitting range $[0, 7]$ shown for reference. The lower-order members extrapolate reliably: the relative tail error of Φ_1^* and Φ_4^* stays near 1.5%. By contrast, the polynomial $y(z)$ of the higher-degree members is fitted only on $[0, 7]$ and is unconstrained beyond it; some of these members (notably Φ_5^* and Φ_7^*) therefore lose all relative tail accuracy once z exceeds about 8–9. Tail extrapolation is thus governed by the behaviour of $y(z)$ outside the fitting interval rather than by the in-range accuracy, and it is not monotone in the degree n . Consequently, for deep-tail probabilities we recommend the simpler members Φ_1^* or Φ_4^* , or refitting $y(z)$ over the wider range demanded by the application.

Table 6. Tail behaviour of the proposed approximations. The Max.AE on the fitting range $[0, 7]$ is reproduced from Table 1 for reference; the tail columns are computed on $z \in [7, 14]$ with step 0.001.

Approximation	Max.AE $[0, 7]$	Max.AE $[7, 14]$	Max. rel. error $[7, 14]$
Φ_1^*	8.10×10^{-4}	1.78×10^{-14}	1.39×10^{-2}
Φ_2^*	1.56×10^{-5}	1.30×10^{-14}	1.38×10^{-1}
Φ_3^*	8.85×10^{-6}	2.50×10^{-14}	6.73×10^{-1}
Φ_4^*	2.54×10^{-6}	1.90×10^{-14}	1.49×10^{-2}
Φ_5^*	1.84×10^{-7}	6.73×10^{-14}	1.00
Φ_6^*	2.27×10^{-8}	7.62×10^{-15}	1.16×10^{-2}
Φ_7^*	8.57×10^{-9}	5.57×10^{-14}	1.00

Table 7 compares the relative error of the tail probability $1 - \Phi(z)$ over sub-intervals of z . Two distinct behaviours emerge. The approximations that share the Mills-ratio (Hart-type) structure keep the relative tail error bounded, of order 1% or below, throughout the tail, provided the exponent polynomial extrapolates sensibly: this is the case for the degree-0 forms Φ_1^* , Obeidat et al. (2025), Hanandeh–Eidous (2022) and Eidous–Abu-Shareefa #1, for Φ_4^* and Φ_6^* , and for the Eidous–Abu-Shareefa approximations #8 and #9. However, because the polynomial $y(z)$ is fitted only on $[0, 7]$ and is otherwise unconstrained, some higher-degree proposed members extrapolate poorly: Φ_2^* and Φ_3^* deteriorate gradually, while Φ_5^* and Φ_7^* lose all relative tail accuracy beyond $z \approx 8$. For deep-tail probabilities the simpler members (Φ_1^* , Φ_4^* , Φ_6^*) or a refitting of $y(z)$ over the required range are therefore recommended, as noted above. By contrast, the softplus form of Lipoth et al. (2022) and the tanh form of Eidous–Alzoubi–Hanandeh (2026) possess a tail that decays at the wrong rate; their relative error is already large near $z = 5$ and grows without bound (Lipoth) or saturates at unity (Eidous–Alzoubi–Hanandeh) deeper in the tail, so these forms are unsuitable for tail probabilities despite their high in-range accuracy. The logistic form of Eidous–Al-Rawwash (2025) behaves differently: its relative tail error is only about 10^{-4} up to $z = 5$, but because its

absolute error, of order 10^{-10} , eventually exceeds the tail probability itself, the relative error passes 1% at $z \approx 5.9$ and approaches unity beyond $z \approx 6.6$.

Table 7. Maximum relative error of the tail probability, $|\Phi^*(z) - \Phi(z)|/(1 - \Phi(z))$, over sub-intervals of z , for the proposed approximations and recent methods from the literature. The column n is the degree of the polynomial in each approximation (Lipoth et al. use a parametric softplus form with no polynomial).

Approximation	n	[3, 5]	[5, 7]	[7, 10]	[10, 14]
Φ_1^*	0	2.02×10^{-2}	1.84×10^{-2}	1.39×10^{-2}	8.71×10^{-3}
Φ_2^*	1	4.65×10^{-3}	1.02×10^{-2}	2.65×10^{-2}	1.38×10^{-1}
Φ_3^*	2	8.58×10^{-3}	1.95×10^{-2}	7.34×10^{-2}	6.71×10^{-1}
Φ_4^*	3	7.69×10^{-3}	1.48×10^{-2}	1.49×10^{-2}	9.81×10^{-3}
Φ_5^*	4	4.99×10^{-3}	5.19×10^{-2}	1.00	1.00
Φ_6^*	5	4.48×10^{-4}	5.91×10^{-3}	1.16×10^{-2}	9.80×10^{-3}
Φ_7^*	7	8.47×10^{-4}	4.26×10^{-2}	1.00	1.00
Obeidat et al. (2025)	0	2.01×10^{-2}	1.83×10^{-2}	1.39×10^{-2}	8.71×10^{-3}
Hanandeh–Eidous (2022)	0	2.09×10^{-2}	1.88×10^{-2}	1.41×10^{-2}	8.76×10^{-3}
Lipoth et al. (2022)	–	1.80	1.75×10^2	1.23×10^8	7.89×10^{21}
Eidous–Alzoubi–Hanandeh (2026)	5	4.34×10^{-1}	9.32×10^{-1}	1.00	1.00
Eidous–Al-Rawwash (2025)*	16	1.41×10^{-4}	9.97×10^{-1}	1.00	1.00
Eidous–Abu-Shareefa (2019), #1	0	2.10×10^{-2}	1.88×10^{-2}	1.41×10^{-2}	8.76×10^{-3}
Eidous–Abu-Shareefa (2019), #8	5	2.80×10^{-4}	4.79×10^{-3}	1.11×10^{-2}	9.79×10^{-3}
Eidous–Abu-Shareefa (2019), #9	8	3.83×10^{-4}	8.75×10^{-3}	1.43×10^{-2}	9.81×10^{-3}

*Computed with the corrected coefficients given in the footnote to Table 2.

3. Application

3.1. Case 1: Incomplete Gamma function

The lower incomplete gamma function plays a key role in science and technology. In science, for example, it appears in the cumulative distribution functions of important probability models such as the gamma and chi-square distributions, allowing the description of partial and time-dependent processes. In engineering, it is widely used in reliability analysis to model time-to-failure data and compute cumulative failure probabilities, supporting the estimation of survival and hazard functions. It also arises in communication systems and physical models such as diffusion and heat transfer. Recent studies highlight its continued relevance in developing flexible statistical models and new distributions for real-world data analysis and reliability applications. For more details see [26], [27], and [28].

The incomplete Gamma function is defined as

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt. \quad (3.1)$$

Three values of a are considered: $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$. These are chosen because, through the identity $P(\chi_k^2 \leq y) = \gamma(k/2, y/2)/\Gamma(k/2)$, they correspond to chi-square distributions with $k = 1, 3, 5$ degrees of freedom, and they are precisely the half-integer cases that can be written in closed form in terms of the error function; they therefore directly exercise the proposed Φ_k^* -based approximation. Using the relation $\text{Erf}(z) = 2\Phi(z\sqrt{2}) - 1$, the corresponding approximation is $\widetilde{\text{Erf}}(z) = 2\Phi_k^*(z\sqrt{2}) - 1$, which is substituted for Erf in the expressions below. The true values of $\gamma(a, x)$ at these values of a are, respectively,

$$\gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi} \text{Erf}(\sqrt{x}), \quad (3.2)$$

$$\gamma\left(\frac{3}{2}, x\right) = \frac{\sqrt{\pi}}{2} \operatorname{Erf}(\sqrt{x}) - \sqrt{x} e^{-x}, \quad (3.3)$$

and

$$\gamma\left(\frac{5}{2}, x\right) = \frac{3\sqrt{\pi}}{4} \operatorname{Erf}(\sqrt{x}) - e^{-x} \sqrt{x} \left(x + \frac{3}{2}\right). \quad (3.4)$$

Figures 8–10 show the performance of the proposed approximations Φ_1^* , Φ_4^* , and Φ_7^* against the true values of the incomplete Gamma function, showing close agreement. These three approximations are chosen to span the accuracy/cost spectrum: Φ_1^* (simplest, constant y), Φ_4^* (intermediate, cubic y with $c = \sqrt{2/\pi}$), and Φ_7^* (most accurate). To quantify the agreement, Table 8 reports the maximum absolute error between the exact $\gamma(a, x)$ and the Φ_k^* -based approximation over $x \in [0, 14]$.

Table 8. Maximum absolute error of the Φ_k^* -based incomplete Gamma approximation over $x \in [0, 14]$.

Approximation	$a = \frac{1}{2}$	$a = \frac{3}{2}$	$a = \frac{5}{2}$
Φ_1^*	2.87×10^{-3}	1.44×10^{-3}	2.15×10^{-3}
Φ_4^*	7.37×10^{-6}	3.69×10^{-6}	5.53×10^{-6}
Φ_7^*	2.84×10^{-8}	1.42×10^{-8}	2.13×10^{-8}

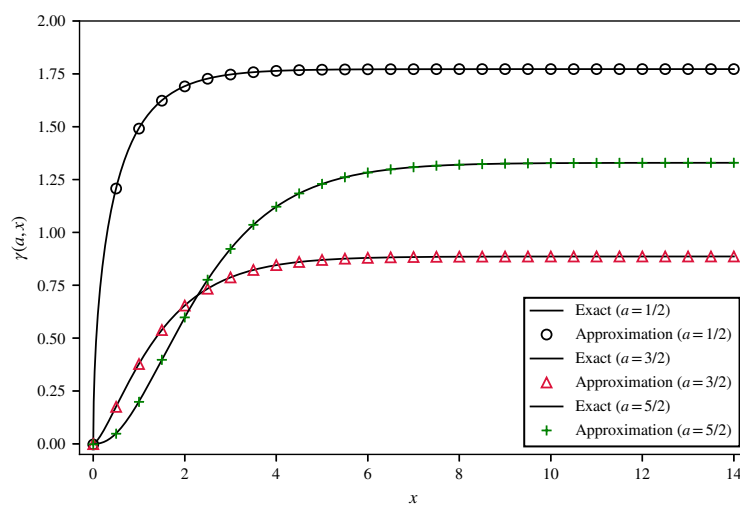
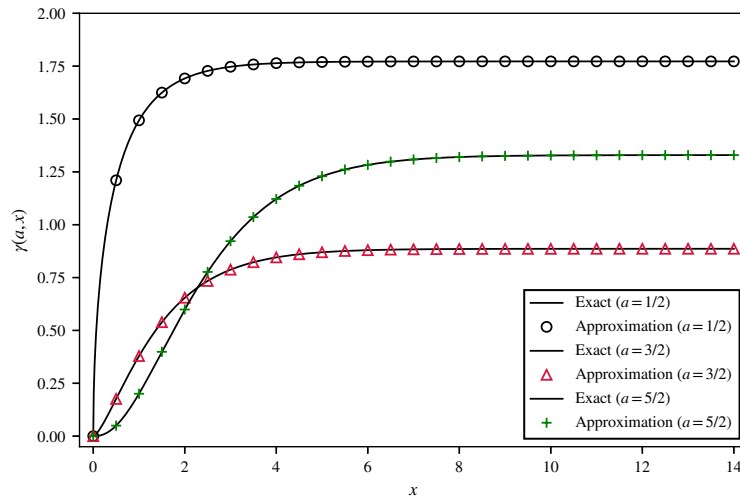
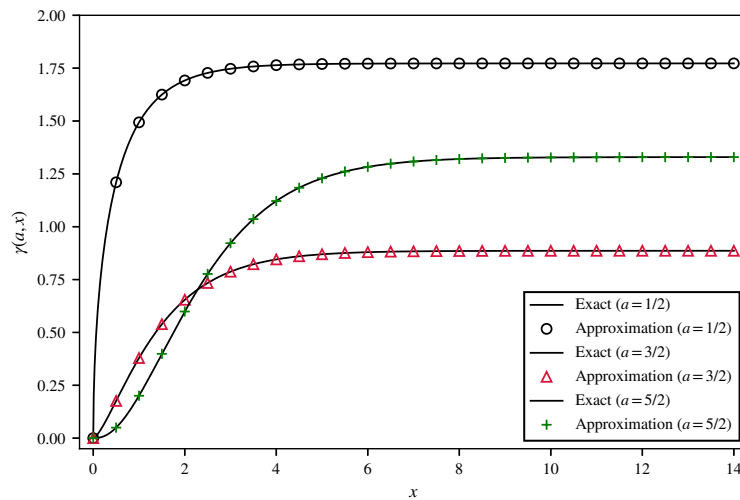


Figure 8. True Incomplete Gamma function with approximation Φ_1^* .

Figure 9. True Incomplete Gamma function with approximation Φ_4^* .Figure 10. True Incomplete Gamma function with approximation Φ_7^* .

3.2. Case 2: Application in Digital Communications

Bandwidth efficiency in Phase Shift Keying (PSK) modulation can be improved by increasing the modulation order M , leading to the general M -PSK signaling scheme. The transmitted signals are given by

$$S_m(t) = \Re \{ g(t) e^{j\theta_m} e^{j2\pi f_c t} \}, \quad m = 1, 2, \dots, M, \quad (3.5)$$

where $\Re\{\cdot\}$ denotes the real part, $j = \sqrt{-1}$ is the imaginary unit, $g(t)$ is the pulse-shaping function, f_c is the carrier frequency, $m = 1, \dots, M$ indexes the M constellation points, and $\theta_m = \frac{2\pi(m-1)}{M}$ is the phase of the m -th symbol. In the particular case $M = 2$, the scheme reduces to Binary Phase Shift Keying (BPSK).

A fundamental performance metric in digital communications is the Bit Error Rate (BER), defined as the probability of incorrect bit detection. For coherent BPSK transmission over an Additive White Gaussian Noise

(AWGN) channel, the BER admits the well-known equivalent representations

$$P_b = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 1 - \Phi\left(\sqrt{\frac{2E_b}{N_0}}\right), \quad (3.6)$$

where E_b/N_0 denotes the signal-to-noise ratio per bit, $Q(\cdot)$ is the Gaussian Q-function, and $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution.

These equivalent formulations highlight that BER evaluation depends on Gaussian tail probabilities. As E_b/N_0 increases, the separation between the signal constellation points grows, leading to a rapid decay in the error probability.

To improve computational efficiency, especially in large-scale simulations or real-time implementations, the BER can be evaluated within the power series extender method (PSEM) framework of Sandoval-Hernandez et al. (2019) [21], a technique for constructing handy closed-form approximations of the Gaussian integrals. In this application we adopt that framework and substitute our proposed approximations of the standard normal CDF (and the associated error function) to demonstrate their accuracy. Substituting the CDF approximation yields

$$\tilde{P}_b = 1 - \tilde{\Phi}\left(\sqrt{\frac{2E_b}{N_0}}\right), \quad (3.7)$$

or equivalently, using the error function representation,

$$\tilde{P}_b = \frac{1}{2} \left[1 - \tilde{\operatorname{erf}}\left(\sqrt{\frac{E_b}{N_0}}\right) \right], \quad (3.8)$$

where $\tilde{\Phi}$ and $\tilde{\operatorname{erf}}$ denote the proposed Φ_k^* -based approximations.

This substitution preserves the probabilistic interpretation of the BER while replacing the special functions by elementary, closed-form expressions. The approximation also retains the correct asymptotic behaviour, ensuring that

$$\tilde{P}_b \rightarrow 0 \quad \text{as} \quad \frac{E_b}{N_0} \rightarrow \infty, \quad \tilde{P}_b \rightarrow \frac{1}{2} \quad \text{as} \quad \frac{E_b}{N_0} \rightarrow 0. \quad (3.9)$$

We evaluate the approximate BER over the range $E_b/N_0 \in [-3, 10]$ dB, which spans error probabilities from about 10^{-1} down to 10^{-5} and covers both the moderate-SNR region and the high-SNR tail relevant to communication system design. Figures 11–13 show the BER at integer values of E_b/N_0 (step 1 dB), whereas the error tables are computed on finer grids, of step 0.05 dB for Table 9 and 0.02 dB for Table 10; in every case the exact and the approximate BER are evaluated on the same grid. The figures (Figures 11–13) display Φ_1^* , Φ_4^* , and Φ_7^* , chosen to span the accuracy/cost spectrum (simplest, intermediate, and most accurate), while Table 9 reports the maximum relative error in $\tilde{P}_b$ over the full range for all seven proposed approximations together with four established approximations from the literature: the base Hart approximation [24], the Pólya approximation [7], the Aludaat–Alodat approximation [15], and the Bowling et al. logistic approximation [17].

The comparison highlights a feature specific to BER evaluation. The BER depends on the Gaussian tail $1 - \Phi(\cdot)$, so an approximation must reproduce the correct tail decay, not merely a small overall error. The Pólya, Aludaat–Alodat, and Bowling approximations have small maximum absolute CDF errors (of order 10^{-3}) yet incur relative BER errors of 60–80%, because their tails decay at the wrong rate (for example, the Pólya tail decays as $e^{-2z^2/\pi}$ rather than the exact $e^{-z^2/2}$). By contrast, the proposed approximations inherit the Mills-ratio structure of the Hart form and therefore respect the true tail, achieving relative BER errors between 2×10^{-2} (Φ_1^*) and about 10^{-4} (Φ_6^* , Φ_7^*) — several orders of magnitude better than the classical sigmoid- and Gaussian-blend approximations.

Table 9. Maximum relative error of the BER approximation over $E_b/N_0 \in [-3, 10]$ dB, for the proposed approximations and four established approximations from the literature.

Approximation	Max. relative BER error
Proposed Φ_1^*	2.02×10^{-2}
Proposed Φ_2^*	3.44×10^{-3}
Proposed Φ_3^*	6.53×10^{-3}
Proposed Φ_4^*	5.14×10^{-3}
Proposed Φ_5^*	2.47×10^{-3}
Proposed Φ_6^*	1.33×10^{-4}
Proposed Φ_7^*	2.11×10^{-4}
Hart (1957) [24]	1.56×10^{-2}
Pólya (1949) [7]	8.04×10^{-1}
Aludaat–Alodat (2008) [15]	7.62×10^{-1}
Bowling et al. (2009) [17]	6.19×10^{-1}

Table 10. Maximum relative error of the BER approximation $\tilde{P}_b$ over sub-intervals of E_b/N_0 (in dB), for the proposed approximations and recent methods from the literature. The column n is the degree of the polynomial in each approximation (Lipoth et al. use a parametric softplus form with no polynomial).

Approximation	n	$[-3, 0]$	$[0, 3]$	$[3, 6]$	$[6, 10]$
Φ_1^*	0	9.63×10^{-3}	1.53×10^{-2}	1.93×10^{-2}	2.02×10^{-2}
Φ_2^*	1	1.92×10^{-4}	2.21×10^{-4}	5.52×10^{-4}	3.47×10^{-3}
Φ_3^*	2	7.04×10^{-5}	3.56×10^{-4}	1.77×10^{-3}	6.59×10^{-3}
Φ_4^*	3	1.18×10^{-5}	2.63×10^{-5}	5.10×10^{-4}	5.21×10^{-3}
Φ_5^*	4	1.93×10^{-6}	1.93×10^{-6}	7.21×10^{-5}	2.52×10^{-3}
Φ_6^*	5	8.48×10^{-8}	1.89×10^{-7}	1.45×10^{-6}	1.39×10^{-4}
Φ_7^*	7	2.04×10^{-8}	1.86×10^{-8}	1.81×10^{-7}	2.21×10^{-4}
Obeidat et al. (2025)	0	9.48×10^{-3}	1.51×10^{-2}	1.92×10^{-2}	2.01×10^{-2}
Hanandeh–Eidous (2022)	0	1.09×10^{-2}	1.64×10^{-2}	2.03×10^{-2}	2.09×10^{-2}
Lipoth et al. (2022)	–	2.32×10^{-4}	7.41×10^{-4}	5.60×10^{-3}	6.89×10^{-1}
Eidous–Alzoubi–Hanandeh (2026)	5	2.00×10^{-4}	7.15×10^{-4}	1.21×10^{-2}	2.73×10^{-1}
Eidous–Al-Rawwash (2025)*	16	3.75×10^{-9}	4.71×10^{-9}	2.58×10^{-8}	2.55×10^{-5}
Eidous–Abu-Shareefa (2019), #1	0	1.09×10^{-2}	1.65×10^{-2}	2.03×10^{-2}	2.10×10^{-2}
Eidous–Abu-Shareefa (2019), #8	5	9.12×10^{-7}	9.61×10^{-7}	1.33×10^{-6}	6.85×10^{-5}
Eidous–Abu-Shareefa (2019), #9	8	1.54×10^{-7}	1.20×10^{-7}	3.26×10^{-6}	1.34×10^{-4}

*Computed with the corrected coefficients given in the footnote to Table 2.

Figures 11–13 illustrate the true BER against the approximations Φ_1^* , Φ_4^* , and Φ_7^* ; the figures again show close agreement between the exact and approximate solutions.

Table 10 reports the maximum relative error of the BER approximation over sub-intervals of E_b/N_0 . Because the BER depends on the Gaussian tail $1 - \Phi(\cdot)$, evaluated at $z = \sqrt{2E_b/N_0}$, the decisive property is the accuracy of the tail rather than the overall absolute error. The approximations that share the Mills-ratio (Hart-type) structure — the proposed Φ_k^* , Obeidat et al. (2025), Hanandeh–Eidous (2022) and the Eidous–Abu-Shareefa (2019) family — reproduce the correct tail decay and therefore remain accurate across all sub-intervals, their accuracy improving with the polynomial degree: Φ_6^* , Φ_7^* and the Eidous–Abu-Shareefa approximations #8 and #9 keep the relative BER error near 10^{-4} even in the high-SNR band $[6, 10]$ dB. By contrast, the softplus form of Lipoth et al. (2022) and the tanh form of Eidous–Alzoubi–Hanandeh (2026), although accurate as overall CDF approximations (maximum absolute errors near 3×10^{-5}), employ a tail that departs from the Gaussian rate; their relative BER error grows

sharply at high SNR, reaching about 69% and 27%, respectively, in $[6, 10]$ dB. The logistic form of Eidous–Al-Rawwash (2025) is the most accurate method in every band, with relative error at most 2.6×10^{-5} : its absolute error, of order 10^{-10} , remains far below the BER over this range, whose largest argument is $z = \sqrt{20} \approx 4.47$. These results show that a small overall CDF error does not by itself guarantee BER accuracy, and that the proposed approximations are well suited to digital-communication applications at modest computational cost.

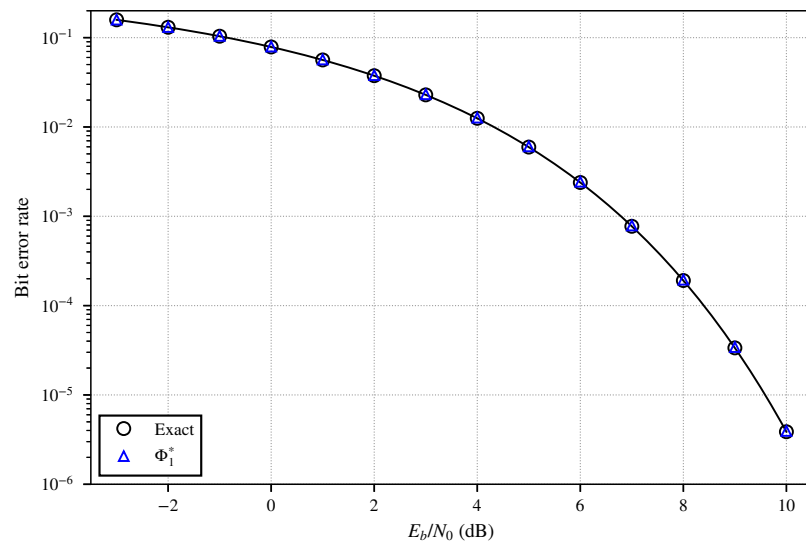


Figure 11. True Bit error probability curve for Phase Shift Keying modulation vs Φ_1^* .

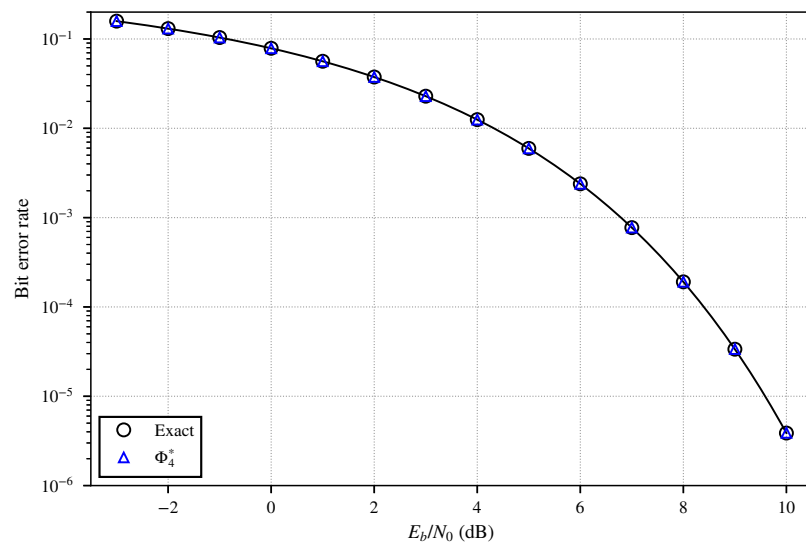


Figure 12. True Bit error probability curve for Phase Shift Keying modulation vs Φ_4^* .

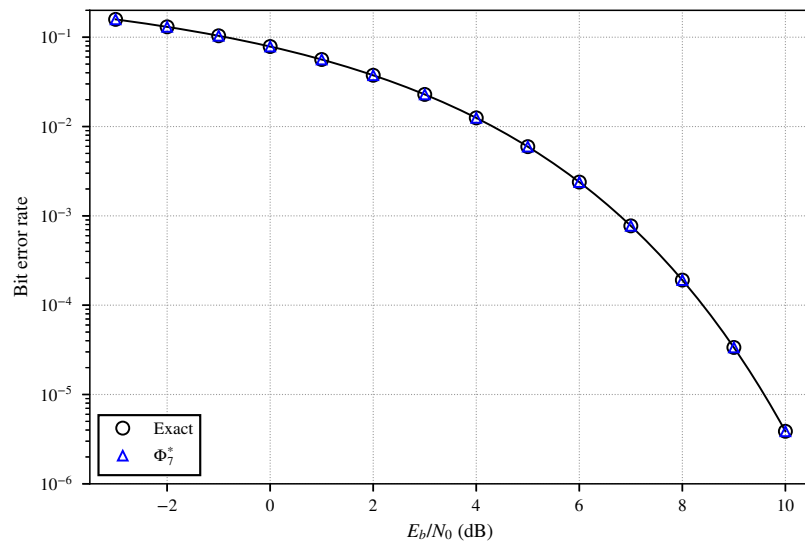


Figure 13. True Bit error probability curve for Phase Shift Keying modulation vs Φ_7^* .

4. Conclusion

In this work we developed seven approximations of the standard normal distribution function $\Phi(z)$ which can be easily applied to other related functions such as the error function. It is observed that as z increases, the absolute error tends to zero. The proposed approximations are monotonically increasing, handy, and easy to program without needing numerical methods. Furthermore, we compared our proposed methods with other existing approximations for a wide range of true values of z ranging from 0 to 7. It is observed that the proposed approximations are more accurate than the competing methods of comparable complexity; only the logistic approximation of Eidous and Al-Rawwash (2025), which uses a polynomial of degree 16, is more accurate. Finally, we applied the proposed approximations to two real-world problems in science and engineering, which allow us to observe that the proposed approximations are accurate and have strong potential to be used for the solution of different practical problems.

The contribution of this work is not the Hart form itself but the systematic least-squares refinement of its exponent by polynomials of increasing degree, together with the accompanying analysis of monotonicity, asymptotic and tail behaviour, computational cost and run time, and the quantitative validation of the two applications. The main limitation is that the higher-degree members are optimised on $[0, 7]$ and should be replaced by the low-degree members, or refitted, for deep-tail use (Section 2.3); the simplest members Φ_1^* and Φ_4^* are recommended when robustness across the whole real line is required, and the higher-degree members Φ_6^* and Φ_7^* when maximum in-range accuracy is the priority.

REFERENCES

1. Bird R.B. (2002). Transport phenomena, *Appl. Mech. Rev.*, 55(1), R1–R4.
2. Proakis J.G. (2001). Digital communications: fourth edition, *McGraw-Hill Book Co.*, New York.
3. Gachev, I. G., Glyavin, M. Y., Manuilov, V. N., Morozkin, M. V., Zavolsky, N. A. (2010). The Influence of Initial Electron Velocities Distribution on the Energy Spectra of the Spent Electron Beam in Gyrotron, *J. Infrared Millim. Te.*, 31, 1109–1114.
4. Conrick, C., Hanson, S. (2013). Normal distribution, probability, and modern financial theory. In: *Vertical Option Spreads*, USA: John Wiley and Sons.
5. Preacher K.J., Curran P.J., Bauer D.J. (2006). Computational tools for probing interactions in multiple linear regression, multilevel modeling, and latent curve analysis, *J. Educ. Behav. Stat.*, 31(4), 437–448.
6. Lipoth, J., Tereda, Y., Papalexioiu, S., & Spiteri, R. J. (2022). A New Very Simply Explicitly Invertible Approximation for the Standard Normal CDF. *AIMS Mathematics*, 7(7), 11635–11646. DOI: 10.3934/math.2022648.

7. Pólya, G. (1949). Remarks on computing the probability integral in one and two dimensions. In Proceedings of the [First] Berkeley Symposium on Mathematical Statistics and Probability (Vol. 1, 63-79). University of California Press.
8. Eidous, O. M., & Abu-Shareefa, R. (2020). New approximations for standard normal distribution function. *Communications in Statistics-Theory and Methods*, 49(6), 1357-1374. DOI: 10.1080/03610926.2018.1563166.
9. Cadwell, J. H. (1951). The bivariate normal integral. *Biometrika*, 38(3/4), 475-479.
10. Hastings, C. (1955). Approximations for digital computers. Princeton, NJ: Princeton University Press.
11. Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics*, 493-507.
12. Ordaz, M. (1991). A simple approximation to the Gaussian distribution. *Structural safety*, 9(4), 315-318.
13. Burr, I. W. (1967). A useful approximation to the normal distribution function, with application to simulation. *Technometrics*, 9(4), 647-651.
14. Abramowitz, M., & Stegun, I. A. (1965). Handbook of mathematical functions: with formulas, graphs, and mathematical tables (Vol. 55). Courier Corporation.
15. Aludaat, K. M., & Alodat, M. T. (2008). A note on approximating the normal distribution function. *Applied Mathematical Sciences*, 2(9), 425-429.
16. Eidous, O., & Al-Salman, S. (2016). One-term approximation for normal distribution function. *Mathematics and Statistics*, 4(1), 15-18.
17. Bowling, S. R., Khasawneh, M. T., Kaewkuekool, S., & Cho, B. R. (2009). A logistic approximation to the cumulative normal distribution. *Journal of industrial engineering and management*, 2(1), 114-127. DOI: 10.3926/jiem.2009.v2n1.p114-127.
18. Hanandeh, A., & Eidous, O. (2021). A new one-term approximation to the standard normal distribution. *Pakistan Journal of Statistics and Operation Research*, 17(2) 381-385. DOI: 10.18187/pjsor.v17i2.3556.
19. Hanandeh, A. & Eidous, O. (2022). Some improvements for existing simple approximations of the Normal Distribution Function. *Pakistan Journal of Statistics and Operation Research*, 18(3), 555-559. DOI: 10.18187/pjsor.v18i3.4007.
20. Soranzo, A., & Epure, E. (2014). Very simply explicitly invertible approximations of normal cumulative and normal quantile function. *Applied Mathematical Sciences*, 8(87), 4323-4341.
21. Sandoval-Hernandez, M. A., Vazquez-Leal, H., Filobello-Nino, U., & Hernandez-Martinez, L. (2019). New handy and accurate approximation for the Gaussian integrals with applications to science and engineering. *Open Mathematics*, 17(1), 1774-1793. DOI: 10.1515/math-2019-0131.
22. Obeidat, M., Al-Jamal, R., & Hanandeh, A. A. (2025). New highly accurate improvements for single-term approximations of the standard normal distribution function. *Pakistan Journal of Statistics and Operation Research*, 21(2), 119-128. DOI: 10.18187/pjsor.v21i2.4949.
23. Obeidat, M., Al-Jamal, R., & Hanandeh, A. A. (2025). Improvements of Some Normal Distribution Function Approximations. *Mathematics and Statistics*, 13 (5), 297 - 301. DOI: 10.13189/ms.2025.130504.
24. Hart, R. G. (1957). A formula for the approximation of definite integrals of the normal distribution function. *Mathematical Tables and Other Aids to Computation*, 11(60), 265-266.
25. Eidous, O. and Al-Rawwash, M. (2025). Approximations for standard normal distribution function and its invertible. *Journal of Algorithms and Computational Technology*, 19(1), 1-5. DOI: 10.1177/17483026251322100.
26. Reyes, J., Marchant, C., Santoro, K. I., and Iriarte, Y. A. (2025). A versatile distribution based on the incomplete gamma function: characterization and applications. *Mathematics*, 13(11), 1749.
27. Mythily, S., Premila, S. C., Manigandan, P., Geetha, K., Pachiyappan, D., and Radhika, A. (2025). A comprehensive study of lifetime distributions in reliability analysis. *Reliability: Theory and Applications*, 20(2), 677-684.
28. Bhatler, S., Jangid, K., Kumawat, S., Baleanu, D., Suthar, D. L., and Purohit, S. D. (2023). Analysis of the family of integral equation involving incomplete types of I and I-functions. *Applied Mathematics in Science and Engineering*, 31(1), 2165280.
29. Edous, M., & Eidous, O. (2018). A simple approximation for the normal distribution function. *Mathematics and Statistics*, 6(4), 47-49. DOI: 10.13189/ms.2018.060401.
30. Eidous, O., & Abu Hawwas, J. (2021). An accurate approximation for the standard normal distribution function. *Journal of Information and Optimization Sciences*, 42(1), 17-27. DOI: 10.1080/02522667.2019.1661642.
31. Eidous, O., & Ananbeh, E. (2022). Approximations for the cumulative distribution function of the standard normal. *Journal of Statistics and Management Systems*, 25(3), 541-547. DOI: 10.1080/09720510.2021.1892278.
32. Bercu, G. (2020). New refinements for the error function with applications in diffusion theory. *Symmetry*, 12(12), Article 2017. DOI: 10.3390/sym12122017.
33. Trigui, I., Agbogla, E., Benjillali, M., Ajib, W., & Zhu, W. (2021). Bit error rate analysis for reconfigurable intelligent surfaces with phase errors. *IEEE Communications Letters*, 25(7), 2176-2180. DOI: 10.1109/LCOMM.2021.3071433.
34. Eidous, O. (2023). New tightness lower and upper bounds for the standard normal distribution function and related functions. *Mathematical Methods in the Applied Sciences*, 46(14), 15011-15019. DOI: 10.1002/mma.9358.
35. Eidous, O. (2025). Lower and upper bounds for the Q-function and its related functions. *Communications in Statistics – Simulation and Computation*, 1-8. DOI: 10.1080/03610918.2025.2515188.
36. Eidous, O. M. (2026). A highly accurate upper bound for the cumulative standard normal distribution function and related functions. *Statistics, Optimization & Information Computing*, 16, 2426-2435. DOI: 10.19139/soic-2310-5070-3391.
37. Ananbeh, E., & Eidous, O. (2025). New simple bounds for the standard normal distribution function. *Communications in Statistics – Simulation and Computation*, 54(7), 2762-2769. DOI: 10.1080/03610918.2024.2326596.
38. Eidous, O. M., Alzoubi, L. M., & Hanandeh, A. A. (2026). Enhancing the accuracy of standard normal distribution approximations. *Statistics, Optimization & Information Computing*, 15, 2819-2828. DOI: 10.19139/soic-2310-5070-3116.