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# Bayesian Reliability Inference for the Truncated Cauchy-Power Burr XII Distribution with Progressive Type-II Censoring

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**Abstract** This paper presents a reliability analysis for the Truncated Cauchy-Power Burr XII (TCPBXII) distribution under progressive Type-II censoring with binomial removals. The proposed model provides a flexible framework for analyzing lifetime data characterized by skewness and complex tail behavior, which commonly arise in reliability and survival studies. The main statistical properties of the TCPBXII distribution are derived, including the probability density function, cumulative distribution function, survival function, and hazard rate function. Bayesian estimation of the unknown parameters is developed using the non-informative Jeffreys prior under the squared error loss function (SELF) and LINEX loss function. Since the posterior distributions do not admit closed-form expressions, the Metropolis–Hastings algorithm is implemented to obtain the Bayes estimates.
A comprehensive Monte Carlo simulation study is conducted to assess the performance of the proposed estimation method under different sample sizes and censoring levels. The estimation accuracy is evaluated in terms of the bias and mean squared error (MSE).
Finally, the applicability of the proposed model is demonstrated using a real hydrological dataset of annual river flow measurements. The results indicate that the TCPBXII distribution provides a competitive and flexible model for analyzing lifetime data under progressive censoring schemes with random removals.

**Keywords** Truncated Cauchy distribution; Burr XII distribution; Bayesian estimation; Progressive Type-II censoring; Binomial removals; Reliability analysis.

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## 1. Introduction

Lifetime data analysis plays a fundamental role in various scientific fields such as reliability engineering, survival analysis, and medical sciences [18, 23]. Over the past decades, extensive research has been devoted to developing statistical models capable of describing lifetime phenomena. For instance, Lawless [20] introduced several statistical methodologies for analyzing lifetime data, with emphasis on inferential procedures and model construction, while Nelson [26] provided practical frameworks for reliability data analysis and engineering applications.

In many cases it is not possible to record the complete lifetimes of the experimental units under investigation (e.g. due to the short duration of the experiment or because of the high costs of the experiment). In such cases a number of different censoring schemes are used in reliability experiments. Balakrishnan and Aggarwala [8] studied many of the censoring schemes.progressive Type-II censoring (PTIC) has been widely studied, whereby partially tracing the experiments of time to failure, one can efficiently obtain and analyze failure time data, keeping the time and cost of the experiment to a minimum. A comprehensive treatment of this censoring scheme was

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provided by Balakrishnan [8].

In many real-life reliability experiments, the number of surviving units removed at each failure time may be random rather than fixed in advance. Such removals may occur because of operational, financial, or experimental constraints that cannot be completely predetermined. Under the binomial-removal mechanism, the number of units withdrawn at each stage is treated as a random variable governed by a specified removal probability. Therefore, progressive type- II censoring with binomial removals provides a more realistic and flexible framework than fixed–removal schemes when the withdrawal pattern is subject to random variation during the experiment.

On the other hand, selecting an appropriate probability distribution is a crucial step in modeling lifetime data. The Burr XII distribution, originally introduced by Irving W. Burr and further investigated by Devendra Kumar [19] is one of the most widely used models in reliability analysis due to its flexibility in capturing various hazard rate shapes. However, despite its wide applicability, classical lifetime distributions may fail to adequately model data exhibiting strong skewness and complex tail behavior[28] , particularly under progressive censoring schemes with random removals.

Recent studies have introduced several extensions of the Burr Type XII distribution to enhance its flexibility in modeling complex data. Abdullah et al. developed the Poisson–Topp–Leone Burr Type XII model and demonstrated its applicability through comparisons with several Burr XII extensions [1]. More recently, Khalil et al. investigated the Compound Topp–Leone Burr XII model under censored applications and compared its goodness-of-fit performance with several competing Burr XII extensions [17]. From a Bayesian perspective, Alhamidah et al. studied Bayesian and E-Bayesian estimation for the Burr XII model under different prior specifications and evaluated the estimators through Monte Carlo simulation [4]. These developments highlight the continuing interest in extending the Burr XII family and improving inferential procedures for flexible lifetime models.

To overcome these limitations, several transformation-based families of distributions have been proposed in recent years. Among them, the truncated Cauchy transformation introduced by Aldahlan et al. [5] has proven effective in enhancing model flexibility by incorporating additional shape parameters while preserving tractability. This transformation allows the generating more adaptable distributions capable of capturing diverse data characteristics.

The truncated Cauchy transformation is particularly suitable in the present context because its nonlinear structure provides additional flexibility in controlling the shape of the resulting distribution while retaining the underlying characteristics of the Burr XII baseline model. Moreover, when combined with the power parameter, it offers additional control over distributional shape and tail behavior, which is useful for modeling skewed lifetime data and different hazard-rate patterns [5]. The choice of this transformation is not intended to imply universal superiority over other generators; rather, it provides a flexible mechanism for extending the Burr XII model within the reliability framework considered in this study.

Recently, Elgarhy et al. investigated reliability inference for the truncated Cauchy power-inverted Topp–Leone distribution under progressive Type-II censoring with binomial removals [11]. Their study employed maximum likelihood and Bayesian estimation procedures and demonstrated the usefulness of flexible lifetime models under random removal schemes.

Despite these developments, the combination of the Burr Type XII baseline distribution with the truncated Cauchy–power transformation has received limited attention, particularly under progressive Type-II censoring with random binomial removals. Moreover, Bayesian reliability inference for this specific model under such a censoring mechanism has not been sufficiently investigated. This research gap provides the main motivation for the present study.

Motivated by the above considerations, this paper proposes a new distribution, namely the Truncated Cauchy-Power Burr XII (TCPBXII) distribution. The proposed model combines the flexibility of the Burr XII distribution in modeling hazard rate behavior with the adaptability of the truncated Cauchy transformation, along with an additional power parameter, to provide a more comprehensive framework for analyzing complex lifetime data.

The main contributions of this study are: (i) proposing the TCPBXII distribution as a flexible extension of the Burr Type XII model; (ii) deriving its main distributional and reliability properties; (iii) developing maximum likelihood and Bayesian estimation procedures under progressive Type-II censoring with binomial removals; (iv) obtaining Bayesian estimates under SELF and LINEX loss functions using the Metropolis–Hastings algorithm; (v) evaluating the performance of the proposed estimators through Monte Carlo simulation; and (vi) assessing the applicability of the proposed model using a real hydrological dataset.

The parameters of the proposed distribution are estimated using two approaches: maximum likelihood estimation (MLE) and Bayesian estimation based on the non-informative Jeffreys prior under the squared error loss function (SELF) and the LINEX loss function , within the framework of progressive Type-II censoring with binomial removals [12].

To evaluate the performance of the proposed estimation methods, a Monte Carlo simulation study is conducted under different sample sizes and censoring levels. Furthermore, the applicability of the proposed model is illustrated using a real dataset of annual river flow measurements in Canada.

The remainder of this paper is organized as follows. Section 2 introduces the proposed distribution and its statistical properties. Section 3 presents the estimation methods. Section 4 presents the reliability measures. Section 5 provides the simulation study. Section 6 discusses the real data application. Finally, Section 7 concludes the paper.

## 2. The Truncated Cauchy-Power Burr XII Distribution

In this section, a new lifetime distribution, called the Truncated Cauchy-Power Burr XII (TCPBXII) distribution, is proposed by combining the Burr XII distribution with the truncated Cauchy transformation and an additional power parameter. This construction aims to increase the flexibility of the model in accommodating various distributional shapes encountered in reliability and survival data.

Let $F_B(x)$ denote the cumulative distribution function (CDF) of the Burr XII distribution, which is given by [19] :

$$F_B(x) = 1 - (1 + x^c)^{-k}, \quad x > 0,\ c > 0,\ k > 0 \tag{1}$$

Where $c$ and $k$ are positive shape parameters controlling the tail behavior and overall distributional form of the Burr XII model.

The truncated Cauchy transformation is defined on the interval $(0, 1)$ while preserving mathematical tractability and providing additional flexibility through the arctangent function [5] :

$$F_T(x) = \frac{4}{\pi} \arctan(x), \quad 0 < x < 1 \tag{2}$$

To construct the proposed distribution, the Burr Type XII CDF is inserted into the truncated Cauchy transformation and a power parameter $\theta$ is introduced. The resulting cumulative distribution function of the proposed model is given by:

$$G(x; c, k, \theta) = \frac{4}{\pi} \arctan\left(\left[1 - (1 + x^c)^{-k}\right]^{\theta}\right) \qquad x > 0,\ c > 0,\ k > 0,\ \theta > 0 \tag{3}$$

since $0 < F_B(x) < 1$ for all $x > 0$, the argument of the arctangent function remains within the admissible interval, ensuring that $G(x)$ is a valid cumulative distribution function.

By differentiating Equation (3) with respect to $x$ and applying the chain rule, the probability density function corresponding to the TCPBXII distribution is obtained as follows:

$$f(x;c,k,\theta) = \frac{4}{\pi}\frac{\theta c k x^{c-1}\left(1+x^c\right)^{-k-1}\left[1-\left(1+x^c\right)^{-k}\right]^{\theta-1}}{1+\left(\left[1-\left(1+x^c\right)^{-k}\right]^{\theta}\right)^2},\ x>0,\ c>0,\ k>0,\ \theta>0 \tag{4}$$

The density function is capable of exhibiting various shapes depending on the parameter values [25], including right-skewed and unimodal patterns frequently encountered in lifetime data analysis.

The moments of a probability distribution provide important information about its location, dispersion, and shape characteristics [10]. For the proposed TCPBXII distribution ,the r-th raw moment is given by [20]:

$$\grave{\mu}_r = \frac{4}{\pi}\int_0^1 \left[(1-v)^{\frac{-1}{k}}-1\right]^{\frac{r}{c}}\frac{\theta v^{\theta-1}}{1+v^{2\theta}}dv, \quad \text{where } r>0 \tag{5}$$

Due to the complexity of the probability density function, a closed-form expression for the moments is not available. Therefore, the moments can be evaluated numerically for specific parameter values. The first and second moments may be used to obtain important descriptive measures such as the mean and variance of the proposed distribution.

The survival function plays an important role in reliability analysis as it represents the probability that a unit survives beyond time x . It is defined as [20]:

$$S(x;c,k,\theta) = 1-\frac{4}{\pi}\arctan\left(\left[1-\left(1+x^c\right)^{-k}\right]^{\theta}\right) \quad , \quad x>0,\ c>0,\ k>0,\ \theta>0 \tag{6}$$

The hazard rate function describes the instantaneous failure rate[18] and is defined as[26]:

$$h(x;c,k,\theta) = \frac{f(x;c,k,\theta)}{S(x;c,k,\theta)} \tag{7}$$

The proposed TCPBXII distribution exhibits considerable flexibility in modeling different hazard rate behaviors depending on the parameter values.

To investigate the impact of the model parameters on the distributional characteristics, graphical representations of the PDF, CDF, survival function, and hazard rate function are presented for several parameter combinations.

## 3. Estimation Methods

In this section, the parameters of the proposed TCPBXII distribution are estimated using two methods: maximum likelihood estimation (MLE) and Bayesian estimation under progressive Type-II censoring with binomial removals.

### *3.1. Maximum Likelihood Estimation*

Let X be a random variable following the TCPBXII distribution with parameter vector $\psi = (c,k,\theta)$

Under progressive Type-II censoring with binomial removals [8], let $n$ denote the initial sample size, m denote the number of observed failures, let $x_1, x_2, \ldots, x_m$ denote the observed ordered failure times, and let $R_1, R_2, \ldots, R_m$ represent the number of units removed at each stage and $p$ denote the binomial removal probability, for $i = 1, 2, \ldots, m$.

The likelihood function can be expressed as [7]:

$$L(\psi) \propto \prod_{i=1}^{m} f\left(x_i,\psi\right)\left[S\left(x_i,\psi\right)\right]^{R_i} \tag{8}$$

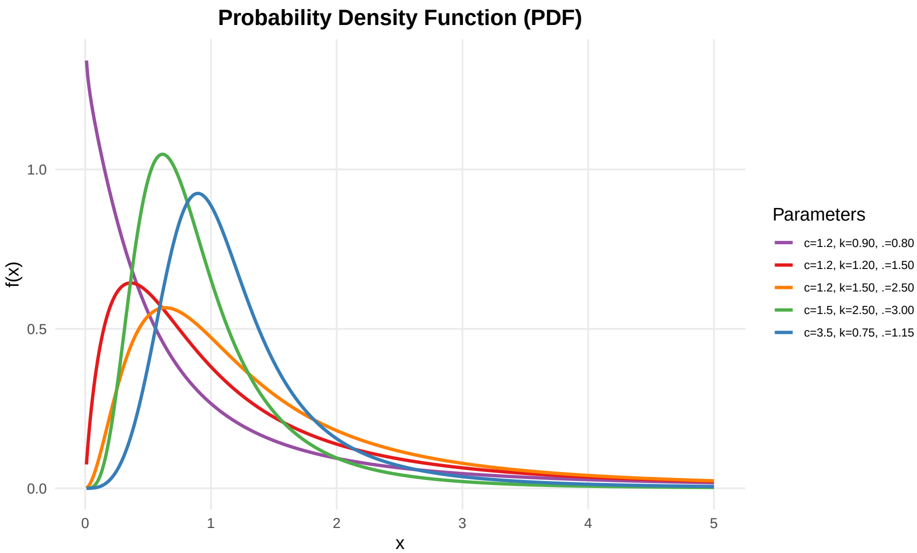



Figure 1. Probability Density Function (PDF) of the TCPBXII distribution

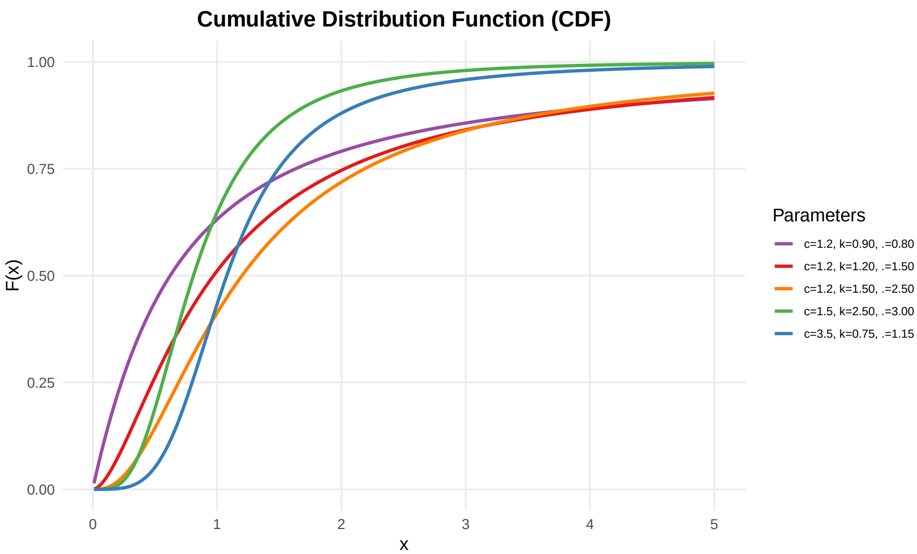



Figure 2. Cumulative Distribution Function (CDF) of the TCPBXII distribution

Substituting Equations (4) and (6) into Equation (8), The likelihood function of the TCPBXII distribution can be written explicitly in terms of the unknown parameters $c, k$, and $\theta$.

Here, $f(x_i, \psi)$ and $S(x_i, \psi)$ denote the probability density function and survival function, respectively.

Taking the natural logarithm of the likelihood function, the log-likelihood is obtained to simplify the optimization process and facilitate numerical estimation. as:

$$\ell(\psi) = \ln L(\psi)$$

The maximum likelihood estimators of the parameters $\mathrm{c}, \mathrm{k}$, and $\theta$ are obtained by differentiating the log-likelihood function with respect to each parameter and equating the resulting equations to zero.

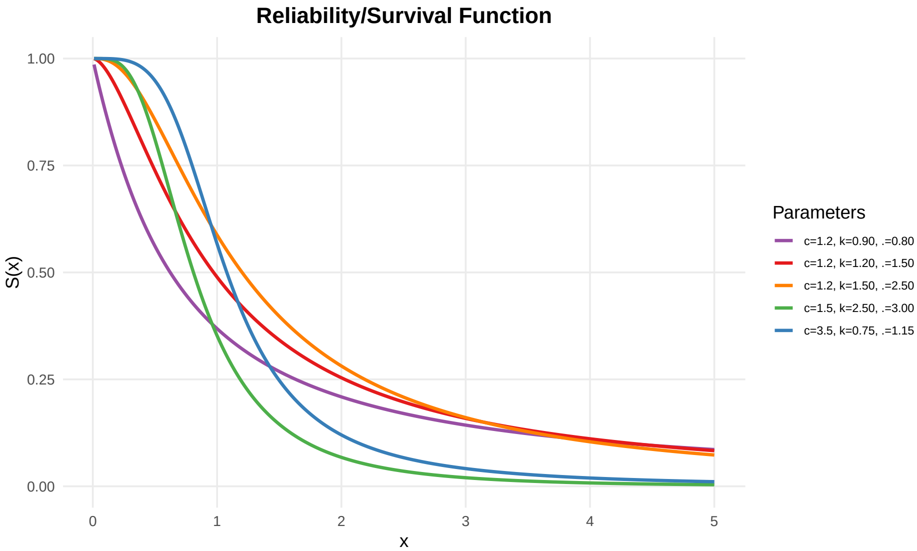



Figure 3. Survival Function of the TCPBXII distribution

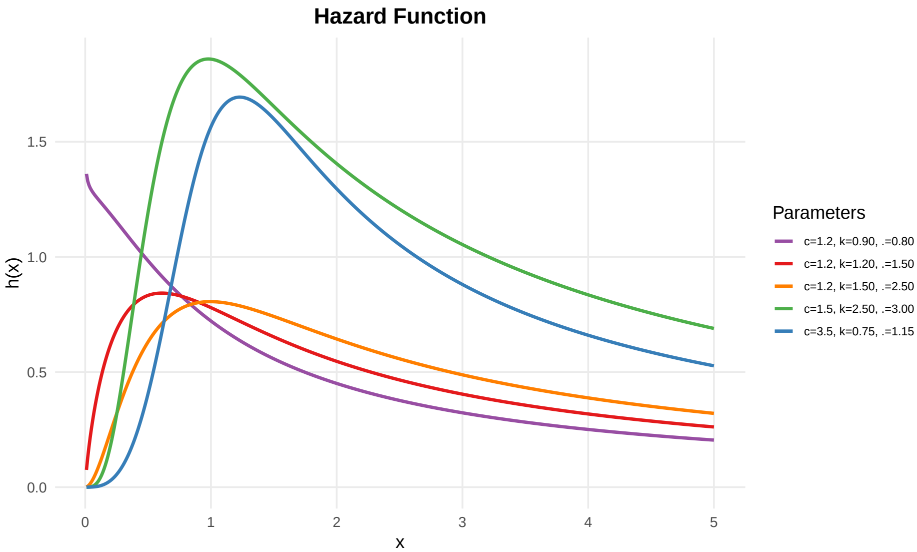



Figure 4. Hazard Rate Function (HRF) of the TCPBXII distribution

Since the resulting likelihood equations are nonlinear and analytically intractable, numerical optimization procedures such as the Newton-Raphson algorithm are employed to obtain the maximum likelihood estimates.

### *3.2. Bayesian Estimation*

In this subsection, Bayesian estimation is considered for estimating the parameters of the proposed TCPBXII distribution [9].

*3.2.1. Jeffreys prior* In the absence of prior information, Jeffreys prior is used as a non-informative prior distribution, defined as [14]:

$$\pi(\psi) \propto \sqrt{|I(\psi)|}$$

where $I(\psi)$ is the Fisher information matrix .

Jeffreys prior is widely used in Bayesian inference [13] because of its invariance property under parameter transformations and its non-informative nature.

The Bayesian analysis in the present study is based on the Jeffreys prior, which was selected to provide a non-informative prior specification and to reduce the influence of subjective prior assumptions on the resulting estimates. Nevertheless, Bayesian estimates may be sensitive to the choice of prior distribution, particularly for small samples or under heavy censoring. Alternative informative priors, such as Gamma priors for the positive model parameters, may therefore lead to different posterior estimates when substantial prior information is available. A formal prior-sensitivity analysis was not conducted in the present study and is acknowledged as a limitation. Investigating the robustness of the Bayesian estimates under alternative prior specifications constitutes an important direction for future research.

*3.2.2. Posterior Distribution* Using Bayes' theorem, the posterior distribution is given by:

$$\pi(\psi \mid x) \propto L(\psi)\pi(\psi)$$

where $L(\psi)$ is the likelihood function and $\pi(\psi)$ is the prior distribution.

*3.2.3. Metropolis-Hastings Algorithm [15]* Since the posterior distribution does not admit a closed-form expression, the Metropolis–Hastings algorithm [27] is employed to generate samples from the posterior distribution [15].

The algorithm proceeds as follows:

1. Select initial values for $c, k$, and $\theta$.
2. Generate candidate values from proposal distributions.
3. Compute the acceptance probability.
4. Accept or reject the proposed values.
5. Repeat the procedure until convergence is achieved.

*3.2.4. Bayesian Estimation under Squared Error Loss Function* To estimate the unknown parameters, the squared error loss function (SELF) is employed. Under this loss function, the Bayes estimator of a parameter is given by the posterior mean.

Similarly, the Bayes estimators for the parameters $c, k$ and $\theta$ are obtained as:

$$\hat{C}_B = E(c \mid x),\ \widehat{K}_B = E(k \mid x),\ \hat{\theta}_B = E(\theta \mid x)$$

Since these posterior expectations do not have closed-form solutions, they are approximated using the posterior samples generated by the Metropolis-Hastings algorithm.

*3.2.5. Bayesian Estimation under LINEX Loss Function* The LINEX loss function [30] is suitable when the costs of overestimation and underestimation are not equal [24].

Under the LINEX loss function, the Bayes estimator of a parameter $\theta$ is given by:

$$\hat{\theta}_{\text{LINEX}} = -\frac{1}{a}\ln\left(E\left[e^{-a\theta}/x\right]\right), \quad \text{a} \neq 0 \tag{9}$$

where a controls the direction and degree of asymmetry of the loss function. Accordingly, the LINEX Bayes estimators of $c, k$ and $\theta$ are obtained by replacing $\theta$ with the corresponding parameter. Since these posterior expectations do not have closed-form solutions, they are approximated using the posterior samples generated by the Metropolis-Hastings algorithm.

## 4. Reliability Measures

Reliability measures [20] constitute an essential component of lifetime data analysis. Once the parameter estimates are obtained, the survival function and hazard rate function can be estimated by replacing the unknown parameters with their corresponding estimates. These measures provide valuable information about the reliability performance and failure behavior of the system over time and are widely used in reliability engineering and survival analysis.

### 4.1. Reliability Estimation under Maximum Likelihood Estimation

Under the maximum likelihood approach, the reliability function can be estimated by substituting the maximum likelihood estimates of the model parameters into the reliability function. Thus, the MLE of the reliability function is given by:

$$\widehat{R}_{MLE}(t) = 1 - \frac{4}{\pi}\arctan\left[\left(1 - \left(1 + t^{\hat{c}_{MLE}}\right)^{-\hat{k}_{MLE}}\right)^{\hat{\theta}_{MLE}}\right], \tag{10}$$

## 5. Simulation Study

In this section, a Monte Carlo simulation study [22] is conducted to evaluate the performance of the proposed estimators for the parameters of the TCPBXII distribution under progressive Type-II censoring with binomial removals. All simulation experiments were carried out using R software.

### 5.1. Simulation Design

To investigate the performance of the estimators under different conditions, two sets of true parameter values are considered.

The first set is:

$$c = 1.2, \quad k = 1.2, \quad \theta = 1.5$$

The set selected to examine the behavior of the estimators under different shapes of the distribution.

The simulation is carried out for three different sample sizes:
n $= 50, 100, 200$

To model the randomness in the removal mechanism, three values of the binomial removal probability are considered:
p $= 0.15, 0.5, 0.85$

These values represent different levels of removal probability, ranging from low to high.

For each sample size , two observed failure numbers $(m)$ were considered under progressive Type-II censoring with binomial removals.

For each simulated dataset, the progressive Type-II censoring scheme with binomial removals is applied, and the parameters are estimated using maximum likelihood estimation (MLE) and Bayesian estimation under the squared error loss function (SELF) and LINEX loss function.

For each combination of $n, m$, and $p$, the simulation experiment is repeated 1000 times to ensure the stability of the results.

### 5.2. Simulation Algorithm

A Monte Carlo simulation study was conducted to evaluate the finite-sample performance of the proposed estimators under different censoring schemes and sample sizes. Random samples were generated from the TCPBXII distribution under progressive Type-II censoring with binomial removals. The simulation algorithm can be summarized as follows:

1. Generate a random sample from the TCPBXII distribution.

2. Apply progressive Type-II censoring.
3. Generate random removals according to the binomial distribution [8].
4. Compute the MLE and Bayes estimators.
5. Repeat the procedure 1000 times.
6. Calculate Bias and MSE.

### *5.3. Performance Measures*

To evaluate the accuracy of the estimators, two performance measures are used: bias and mean squared error (MSE).

The bias of an estimator is defined as:
$\mathrm{Bias}(\widehat{\theta}) = \mathrm{E}(\widehat{\theta} - \theta)$
and the mean squared error is given by:
$\mathrm{MSE}(\hat{\theta}) = \mathrm{E}\left[(\hat{\theta} - \theta)^2\right]$
Smaller values of bias and MSE indicate better estimation performance.

Bias and MSE are among the most commonly used criteria for assessing the performance of statistical estimators, where smaller values indicate higher estimation accuracy and efficiency.

### *5.4. Simulation Results*

The results of the simulation study are summarized in Tables 1.

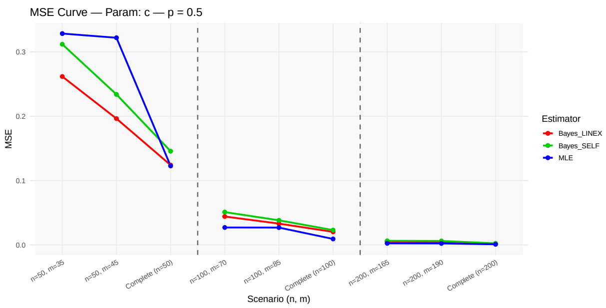
Table 1. Simulation results for the first parameter setting $(\mathrm{c} = 1.2,\ \mathrm{k} = 1.2,\ \theta = 1.5)$

| | | | | MLE | | | | LINEX | | | SELF | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | P | M | Par. | Bias | MSE | Est | Bias | MSE | Est | Bias | MSE | Est |
| 50 | 0.15 | 35 | C | 0.01795 | 0.34581 | 1.21795 | 0.03656 | 0.21223 | 1.23656 | 0.04693 | 0.24648 | 1.24693 |
| | | | K | 0.04855 | 0.21819 | 1.24855 | -0.26226 | 0.10094 | 0.93774 | -0.23634 | 0.14087 | 0.96366 |
| | | | $\theta$ | 0.04779 | 0.24228 | 1.54779 | -0.30968 | 0.11286 | 1.19032 | -0.27762 | 0.13049 | 1.22238 |
| | | 45 | C | -0.00883 | 0.29968 | 1.19117 | 0.04256 | 0.19568 | 1.24256 | 0.05200 | 0.23768 | 1.25200 |
| | | | K | 0.05896 | 0.19333 | 1.25896 | -0.24171 | 0.08170 | 0.95829 | -0.21963 | 0.11483 | 0.98037 |
| | | | $\theta$ | 0.08710 | 0.18171 | 1.58710 | -0.29339 | 0.10234 | 1.20661 | -0.26360 | 0.11934 | 1.23640 |
| | | Comp. | C | -0.01664 | 0.12267 | 1.18336 | 0.04259 | 0.08812 | 1.24259 | 0.05186 | 0.10380 | 1.25186 |
| | | | K | 0.06018 | 0.06939 | 1.26018 | -0.22530 | 0.02870 | 0.97470 | -0.20467 | 0.04042 | 0.99533 |
| | | | $\theta$ | 0.09796 | 0.09528 | 1.59796 | -0.28098 | 0.03765 | 1.21902 | -0.25205 | 0.04374 | 1.24795 |
| | 0.5 | 35 | C | 0.07074 | 0.32844 | 1.27074 | 0.05285 | 0.26171 | 1.25285 | 0.06196 | 0.31182 | 1.26196 |
| | | | K | 0.02652 | 0.18931 | 1.22652 | -0.24411 | 0.08994 | 0.95589 | -0.22134 | 0.12793 | 0.97866 |
| | | | $\theta$ | 0.04727 | 0.27536 | 1.54727 | -0.27323 | 0.09574 | 1.22677 | -0.24359 | 0.11183 | 1.25641 |
| | | 45 | C | 0.06817 | 0.32196 | 1.26817 | 0.05985 | 0.19628 | 1.25985 | 0.06815 | 0.23386 | 1.26815 |
| | | | K | 0.03584 | 0.16091 | 1.23584 | -0.19850 | 0.06545 | 1.00150 | -0.17889 | 0.09314 | 1.02111 |
| | | | $\theta$ | 0.05246 | 0.24691 | 1.55246 | -0.24321 | 0.07596 | 1.25679 | -0.21546 | 0.08759 | 1.28454 |
| | | Comp. | C | 0.07176 | 0.12273 | 1.27176 | 0.06263 | 0.12407 | 1.26263 | 0.07056 | 0.14562 | 1.27056 |
| | | | K | 0.01319 | 0.07059 | 1.21319 | -0.19915 | 0.02592 | 1.00085 | -0.18070 | 0.03740 | 1.01930 |
| | | | $\theta$ | 0.03770 | 0.10022 | 1.53770 | -0.23976 | 0.02928 | 1.26024 | -0.21290 | 0.03386 | 1.28710 |
| | 0.85 | 35 | C | 0.05223 | 0.37907 | 1.25223 | 0.04008 | 0.22463 | 1.24008 | 0.05014 | 0.26124 | 1.25014 |
| | | | K | 0.01436 | 0.26859 | 1.21436 | -0.26020 | 0.10142 | 0.93980 | -0.23582 | 0.14344 | 0.96418 |
| | | | $\theta$ | 0.02299 | 0.28838 | 1.52299 | -0.30285 | 0.11052 | 1.19715 | -0.27092 | 0.12834 | 1.22908 |
| | | 45 | C | 0.03105 | 0.33061 | 1.23105 | 0.04642 | 0.16847 | 1.24642 | 0.05562 | 0.19593 | 1.25562 |
| | | | K | 0.03998 | 0.21927 | 1.23998 | -0.22105 | 0.07485 | 0.97895 | -0.19957 | 0.10552 | 1.00043 |
| | | | $\theta$ | 0.04435 | 0.23944 | 1.54435 | -0.27995 | 0.09690 | 1.22005 | -0.25063 | 0.11287 | 1.24937 |
| | | Comp. | C | 0.03967 | 0.13096 | 1.23967 | 0.05233 | 0.11509 | 1.25233 | 0.06107 | 0.13247 | 1.26107 |
| | | | K | 0.04317 | 0.08456 | 1.24317 | -0.20323 | 0.02660 | 0.99677 | -0.18293 | 0.03748 | 1.01707 |
| | | | $\theta$ | 0.04195 | 0.09708 | 1.54195 | -0.26720 | 0.03663 | 1.23280 | -0.23900 | 0.04282 | 1.26100 |

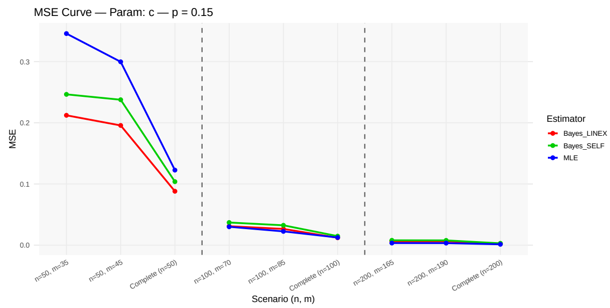
| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 0.15 | 70 | C | -0.00752 | 0.03005 | 1.19248 | 0.04135 | 0.03099 | 1.24135 | 0.04966 | 0.03701 | 1.24966 |
| | | | K | 0.00100 | 0.02075 | 1.20100 | -0.19076 | 0.04925 | 1.00924 | -0.17377 | 0.05756 | 1.02623 |
| | | | $\theta$ | 0.04325 | 0.06486 | 1.54325 | -0.21073 | 0.07676 | 1.28927 | -0.18645 | 0.08541 | 1.31355 |
| | | 85 | C | 0.00651 | 0.02254 | 1.20651 | 0.03919 | 0.02657 | 1.23919 | 0.04734 | 0.03247 | 1.24734 |
| | | | K | -0.01696 | 0.01556 | 1.18304 | -0.17384 | 0.03895 | 1.02616 | -0.15789 | 0.04495 | 1.04211 |
| | | | $\theta$ | 0.02319 | 0.04865 | 1.52319 | -0.19809 | 0.05757 | 1.30191 | -0.17431 | 0.07841 | 1.32569 |
| | | Comp. | C | 0.02389 | 0.01252 | 1.22389 | 0.04511 | 0.01211 | 1.24511 | 0.05291 | 0.01477 | 1.25291 |
| | | | K | -0.04088 | 0.01037 | 1.15912 | -0.17325 | 0.01558 | 1.02675 | -0.15853 | 0.01823 | 1.04147 |
| | | | $\theta$ | -0.00594 | 0.03266 | 1.49406 | -0.19876 | 0.02824 | 1.30124 | -0.17614 | 0.03281 | 1.32386 |
| | 0.5 | 70 | C | 0.00291 | 0.02707 | 1.20291 | 0.05293 | 0.04413 | 1.25293 | 0.06061 | 0.05098 | 1.26061 |
| | | | K | 0.04875 | 0.02346 | 1.24875 | -0.15615 | 0.03669 | 1.04385 | -0.14030 | 0.04242 | 1.05970 |
| | | | $\theta$ | 0.08630 | 0.06384 | 1.58630 | -0.18395 | 0.05697 | 1.31605 | -0.16098 | 0.07774 | 1.33902 |
| | | 85 | C | 0.03603 | 0.02694 | 1.23603 | 0.06077 | 0.03309 | 1.26077 | 0.06778 | 0.03824 | 1.26778 |
| | | | K | 0.01811 | 0.01760 | 1.21811 | -0.14253 | 0.03272 | 1.05747 | -0.12794 | 0.03821 | 1.07206 |
| | | | $\theta$ | 0.03576 | 0.04788 | 1.53576 | -0.18017 | 0.04273 | 1.31983 | -0.15868 | 0.07414 | 1.34132 |
| | | Comp. | C | 0.01786 | 0.00925 | 1.21786 | 0.05782 | 0.02025 | 1.25782 | 0.06479 | 0.02301 | 1.26479 |
| | | | K | 0.03326 | 0.01035 | 1.23326 | -0.13213 | 0.01117 | 1.06787 | -0.11831 | 0.01302 | 1.08169 |
| | | | $\theta$ | 0.06212 | 0.02576 | 1.56212 | -0.16598 | 0.02196 | 1.33402 | -0.14445 | 0.02571 | 1.35555 |
| | 0.85 | 70 | C | 0.03775 | 0.02956 | 1.23775 | 0.05590 | 0.04809 | 1.25590 | 0.06328 | 0.05512 | 1.26328 |
| | | | K | 0.03374 | 0.02545 | 1.23374 | -0.13656 | 0.03346 | 1.06344 | -0.12114 | 0.03900 | 1.07886 |
| | | | $\theta$ | 0.02843 | 0.05436 | 1.52843 | -0.18591 | 0.07267 | 1.31409 | -0.16297 | 0.06696 | 1.33703 |
| | | 85 | C | 0.02664 | 0.02613 | 1.22664 | 0.05452 | 0.04356 | 1.25452 | 0.06176 | 0.05049 | 1.26176 |
| | | | K | 0.04687 | 0.01909 | 1.24687 | -0.12001 | 0.02516 | 1.07999 | -0.10514 | 0.02872 | 1.09486 |
| | | | $\theta$ | 0.03145 | 0.05382 | 1.53145 | -0.18038 | 0.05450 | 1.31962 | -0.15790 | 0.06261 | 1.34210 |
| | | Comp. | C | 0.01918 | 0.01098 | 1.21918 | 0.05895 | 0.02019 | 1.25895 | 0.06578 | 0.02299 | 1.26578 |
| | | | K | 0.05000 | 0.01018 | 1.25000 | -0.11764 | 0.00934 | 1.08236 | -0.10374 | 0.01073 | 1.09626 |
| | | | $\theta$ | 0.04700 | 0.02305 | 1.54700 | -0.17723 | 0.02747 | 1.32277 | -0.15578 | 0.02362 | 1.34422 |
| 200 | 0.15 | 165 | C | 0.05861 | 0.00346 | 1.25861 | 0.06586 | 0.00652 | 1.26586 | 0.07190 | 0.00795 | 1.27190 |
| | | | K | -0.01450 | 0.00906 | 1.18550 | -0.10254 | 0.02173 | 1.09746 | -0.09110 | 0.02132 | 1.10890 |
| | | | $\theta$ | -0.01492 | 0.01765 | 1.48508 | -0.14285 | 0.04318 | 1.35715 | -0.12478 | 0.06904 | 1.37522 |
| | | 190 | C | 0.05303 | 0.00336 | 1.25303 | 0.06591 | 0.00640 | 1.26591 | 0.07192 | 0.00783 | 1.27192 |
| | | | K | -0.01656 | 0.00679 | 1.18344 | -0.10715 | 0.01629 | 1.09285 | -0.09613 | 0.01599 | 1.10387 |
| | | | $\theta$ | -0.00338 | 0.01324 | 1.49662 | -0.13872 | 0.03238 | 1.36128 | -0.12074 | 0.06769 | 1.37926 |
| | | Comp. | C | 0.05896 | 0.00140 | 1.25896 | 0.06242 | 0.00249 | 1.26242 | 0.06844 | 0.00303 | 1.26844 |
| | | | K | -0.02911 | 0.00386 | 1.17089 | -0.10655 | 0.00844 | 1.09345 | -0.09530 | 0.00826 | 1.10470 |
| | | | $\theta$ | -0.01833 | 0.00736 | 1.48167 | -0.13706 | 0.02526 | 1.36294 | -0.11862 | 0.02427 | 1.38138 |
| | 0.5 | 165 | C | 0.01697 | 0.00242 | 1.21697 | 0.05944 | 0.00507 | 1.25944 | 0.06585 | 0.00638 | 1.26585 |
| | | | K | 0.00427 | 0.00901 | 1.20427 | -0.12664 | 0.02667 | 1.07336 | -0.11441 | 0.02619 | 1.08559 |
| | | | $\theta$ | 0.01784 | 0.01420 | 1.51784 | -0.16293 | 0.03205 | 1.33707 | -0.14386 | 0.05561 | 1.35614 |
| | | 190 | C | 0.03031 | 0.00229 | 1.23031 | 0.05537 | 0.00492 | 1.25537 | 0.06169 | 0.00615 | 1.26169 |
| | | | K | -0.01227 | 0.00715 | 1.18773 | -0.10928 | 0.01942 | 1.09072 | -0.09761 | 0.01864 | 1.10239 |
| | | | $\theta$ | -0.01606 | 0.01351 | 1.48394 | -0.15378 | 0.02404 | 1.34622 | -0.13506 | 0.04171 | 1.36494 |
| | | Comp. | C | 0.01748 | 0.00092 | 1.21748 | 0.05596 | 0.00192 | 1.25596 | 0.06224 | 0.00241 | 1.26224 |
| | | | K | 0.00029 | 0.00316 | 1.20029 | -0.11428 | 0.00881 | 1.08572 | -0.10268 | 0.00862 | 1.09732 |
| | | | $\theta$ | 0.00487 | 0.00559 | 1.50487 | -0.15787 | 0.01647 | 1.34213 | -0.13911 | 0.01928 | 1.36089 |
| | 0.85 | 165 | c | -0.03050 | 0.00272 | 1.16950 | 0.05119 | 0.00449 | 1.25119 | 0.05788 | 0.00562 | 1.25788 |
| | | | K | 0.05191 | 0.00749 | 1.25191 | -0.12865 | 0.01887 | 1.07135 | -0.11626 | 0.02595 | 1.08374 |
| | | | $\theta$ | 0.11533 | 0.01530 | 1.61533 | -0.14199 | 0.04088 | 1.35801 | -0.12207 | 0.04695 | 1.37793 |
| | | 190 | C | -0.02522 | 0.00204 | 1.17478 | 0.04974 | 0.00447 | 1.24974 | 0.05613 | 0.00557 | 1.25613 |
| | | | K | 0.05239 | 0.00562 | 1.25239 | -0.11465 | 0.01415 | 1.08535 | -0.10316 | 0.02155 | 1.09684 |
| | | | $\theta$ | 0.10386 | 0.01148 | 1.60386 | -0.13859 | 0.03066 | 1.36141 | -0.11977 | 0.03522 | 1.38023 |
| | | Comp. | C | -0.02088 | 0.00117 | 1.17912 | 0.05008 | 0.00163 | 1.25008 | 0.05655 | 0.00207 | 1.25655 |
| | | | K | 0.03807 | 0.00342 | 1.23807 | -0.12577 | 0.00701 | 1.07423 | -0.11370 | 0.01002 | 1.08630 |
| | | | $\theta$ | 0.09383 | 0.00708 | 1.59383 | -0.14473 | 0.02690 | 1.35527 | -0.12520 | 0.01772 | 1.37480 |

It can be observed that the Bias and MSE values generally decrease as the sample size increases, indicating improved estimation accuracy. Furthermore, higher removal probabilities may lead to increased estimation errors due to the reduction in available information. The Bayesian estimators based on SELF and LINEX loss functions frequently provide competitive and, in many cases, more accurate estimates than the corresponding MLEs.

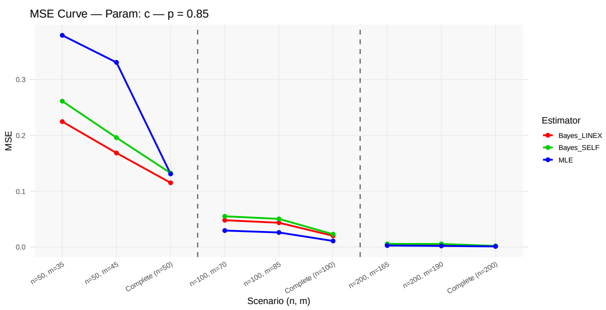
Table 2 presents the simulation results for estimating the reliability function using the maximum likelihood and Bayesian methods under SELF and LINEX loss functions. The results show that the estimated reliability values are generally close to the corresponding true reliability values. Moreover, the estimation accuracy tends to improve as the sample size increases, as reflected by the reduction in Bias and MSE in many cases. The Bayesian estimators under SELF and LINEX provide competitive reliability estimates compared with the MLE. In general, these findings support the effectiveness of the proposed methods for estimating the reliability function of the TCPBXII distribution under progressive Type-II censoring with binomial removals.



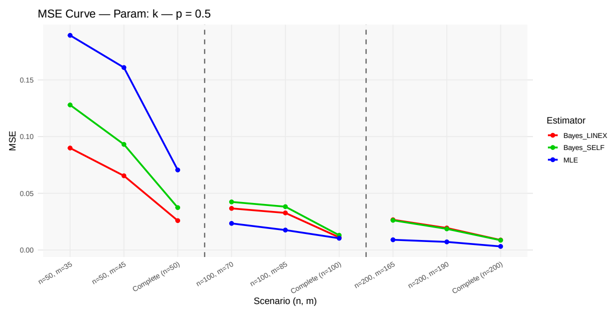

(a)




(b)




(c)




(d)

Table 2. Simulation results for the estimation of the reliability function

| R = 0.641554547 | | | MLE | | | LINEX | | | SELF | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N | P | M | Bias | MSE | Est | Bias | MSE | Est | Bias | MSE | Est |
| 50 | 0.15 | 35 | -0.01686 | 0.00218 | 0.62469 | -0.00009 | 0.00170 | 0.64146 | 0.00074 | 0.00166 | 0.64229 |
| | | 45 | -0.01496 | 0.00170 | 0.62659 | -0.00099 | 0.00140 | 0.64056 | 0.00044 | 0.00137 | 0.64199 |
| | | Comp. | -0.00325 | 0.00036 | 0.63830 | 0.00004 | 0.00037 | 0.64159 | 0.00009 | 0.00034 | 0.64164 |
| | 0.5 | 35 | 0.00023 | 0.00273 | 0.64178 | 0.00875 | 0.00205 | 0.65030 | 0.00886 | 0.00197 | 0.65042 |
| | | 45 | 0.00055 | 0.00231 | 0.64210 | -0.00019 | 0.00180 | 0.64137 | 0.00785 | 0.00183 | 0.64940 |
| | | Comp. | 0.00453 | 0.00043 | 0.64608 | -0.00104 | 0.00043 | 0.64052 | 0.00322 | 0.00036 | 0.64478 |
| | 0.85 | 35 | -0.01211 | 0.00260 | 0.62945 | 0.00000 | 0.00199 | 0.64156 | 0.00118 | 0.00198 | 0.64274 |
| | | 45 | -0.01656 | 0.00207 | 0.62500 | -0.00186 | 0.00161 | 0.63969 | -0.00076 | 0.00144 | 0.64080 |
| | | Comp. | -0.00770 | 0.00045 | 0.63386 | -0.00491 | 0.00043 | 0.63665 | -0.00028 | 0.00042 | 0.64128 |
| 100 | 0.15 | 35 | 0.00304 | 0.00031 | 0.64459 | 0.00373 | 0.00031 | 0.64529 | 0.00371 | 0.00030 | 0.64527 |
| | | 45 | 0.00285 | 0.00027 | 0.64441 | 0.00335 | 0.00026 | 0.64490 | 0.00340 | 0.00026 | 0.64495 |
| | | Comp. | 0.00263 | 0.00023 | 0.64418 | 0.00320 | 0.00023 | 0.64475 | 0.00320 | 0.00022 | 0.64476 |
| | 0.5 | 35 | 0.00313 | 0.00037 | 0.64469 | 0.00314 | 0.00037 | 0.64470 | 0.00299 | 0.00031 | 0.64455 |
| | | 45 | 0.00056 | 0.00032 | 0.64212 | -0.00065 | 0.00031 | 0.64090 | 0.00280 | 0.00026 | 0.64435 |
| | | Comp. | 0.00359 | 0.00027 | 0.64514 | -0.00064 | 0.00027 | 0.64091 | 0.00253 | 0.00022 | 0.64408 |
| | 0.85 | 35 | -0.00365 | 0.00039 | 0.63790 | -0.00047 | 0.00037 | 0.64109 | 0.00015 | 0.00038 | 0.64171 |
| | | 45 | -0.00660 | 0.00033 | 0.63495 | -0.00399 | 0.00032 | 0.63756 | -0.00351 | 0.00032 | 0.63804 |
| | | Comp. | -0.00613 | 0.00029 | 0.63543 | -0.00348 | 0.00028 | 0.63808 | -0.00319 | 0.00028 | 0.63836 |
| 200 | 0.15 | 35 | -0.00236 | 0.00018 | 0.63919 | -0.00252 | 0.00020 | 0.63903 | -0.00255 | 0.00019 | 0.63900 |
| | | 45 | -0.00001 | 0.00017 | 0.64154 | 0.00249 | 0.00017 | 0.64404 | 0.00404 | 0.00016 | 0.64559 |
| | | Comp. | 0.00210 | 0.00014 | 0.64366 | 0.00225 | 0.00015 | 0.64381 | 0.00376 | 0.00014 | 0.64532 |
| | 0.5 | 35 | -0.00344 | 0.00024 | 0.63812 | -0.00070 | 0.00023 | 0.64086 | 0.00232 | 0.00019 | 0.64387 |
| | | 45 | -0.00532 | 0.00021 | 0.63623 | -0.00320 | 0.00019 | 0.63835 | -0.00267 | 0.00017 | 0.63889 |
| | | Comp. | -0.00490 | 0.00018 | 0.63665 | -0.00299 | 0.00017 | 0.63856 | -0.00035 | 0.00015 | 0.64121 |
| | 0.85 | 35 | 0.00497 | 0.00024 | 0.64653 | 0.00307 | 0.00024 | 0.64462 | 0.00483 | 0.00024 | 0.64639 |
| | | 45 | 0.00458 | 0.00021 | 0.64614 | 0.00267 | 0.00021 | 0.64423 | 0.00268 | 0.00021 | 0.64423 |
| | | Comp. | 0.00426 | 0.00018 | 0.64581 | 0.00260 | 0.00017 | 0.64415 | 0.00420 | 0.00018 | 0.64576 |

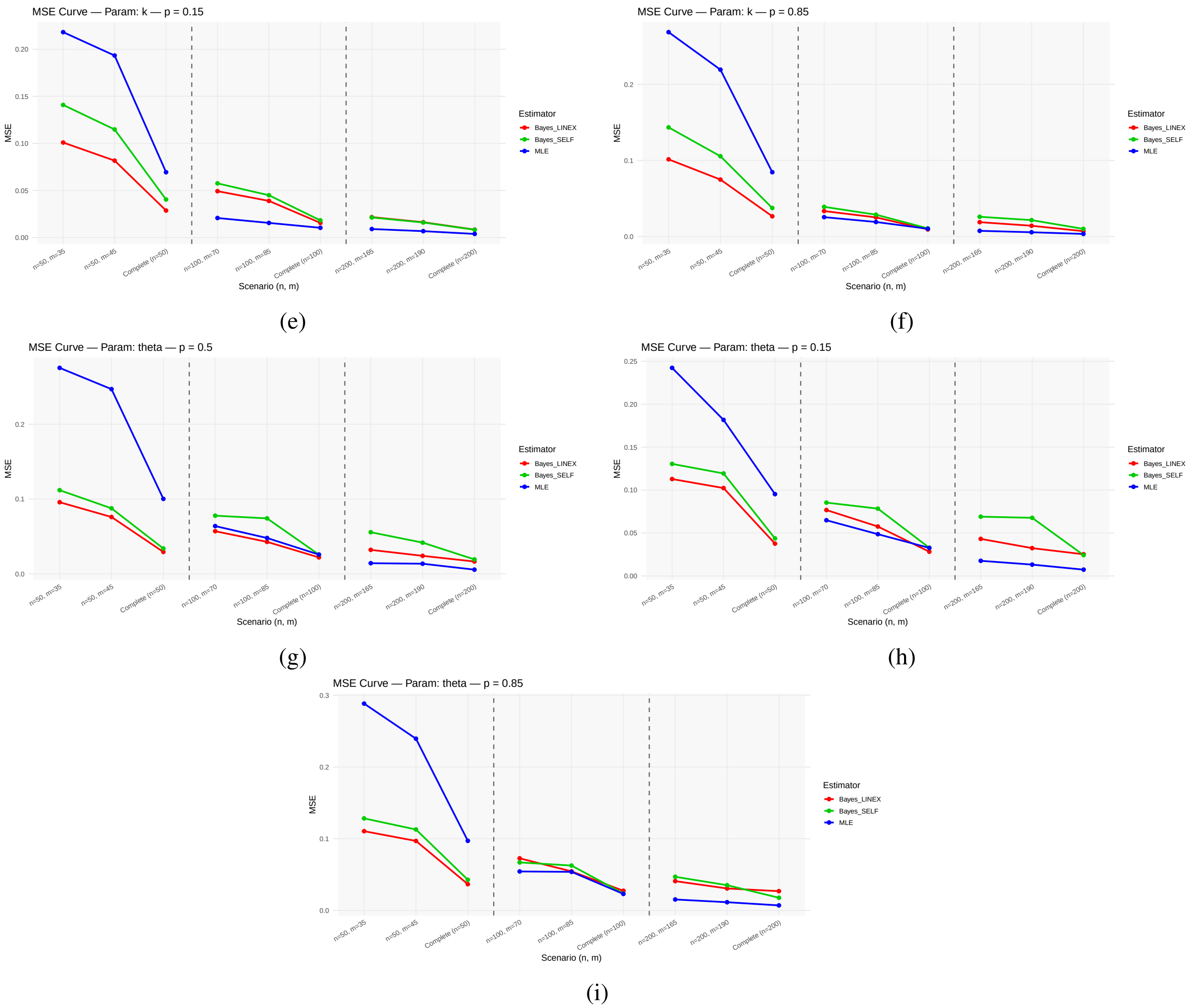


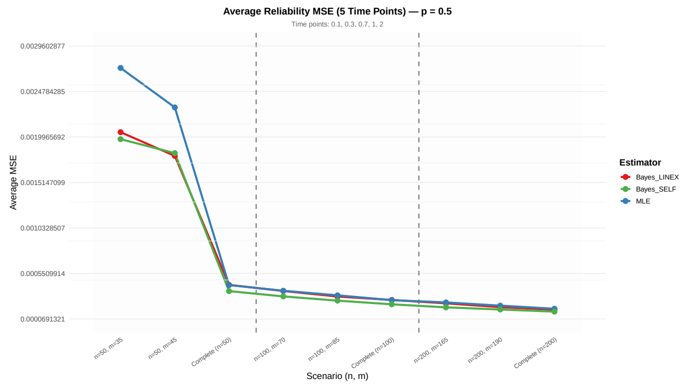

Figure 5. Simulation plots for parameter estimation of the TCPBXII distribution under progressive Type-II censoring with binomial removals.

### 5.5. *MCMC Convergence Diagnostics*

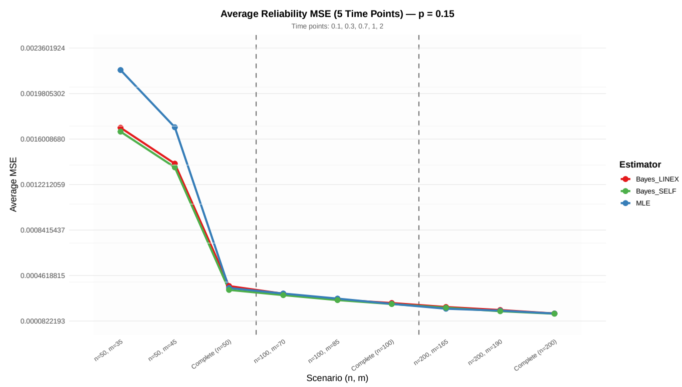
To assess the convergence of the MCMC algorithm used for Bayesian estimation, three independent chains with different initial values were considered for the model parameters c, k, and $\theta$. Convergence was evaluated using the Gelman-Rubin diagnostic $(\widehat{R})$, posterior density plots, and trace plots.

Table 3. Gelman-Rubin convergence diagnostic for the MCMC chains

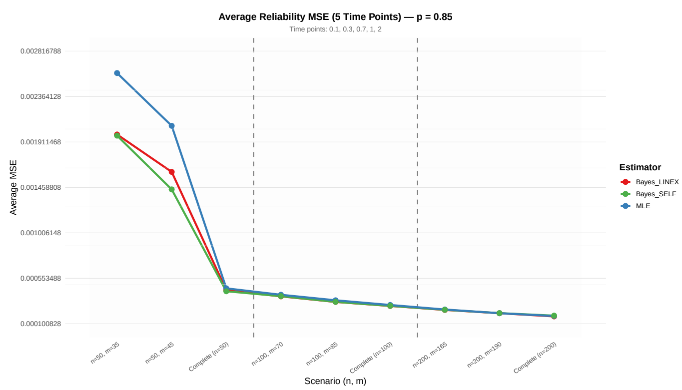
| Parameter | $\widehat{R}$ | Upper 95% CI |
|---|---|---|
| C | 1.007 | 1.023 |
| K | 1.079 | 1.221 |
| $\theta$ | 1.032 | 1.081 |




(a)




(b)




(c)

Figure 6. Simulation plots for reliability estimation of the TCPBXII distribution under progressive Type-II censoring with binomial removals.

The Gelman-Rubin diagnostic values are close to one for all model parameters, with $\widehat{R} = 1.007, 1.079$, and $1.032$ for $\mathrm{c}, \mathrm{k}$, and $\theta$, respectively. These results indicate satisfactory convergence of the MCMC chains. Although the value for k is relatively higher, it remains reasonably close to one, supporting acceptable convergence.

### 5.5.1. Posterior Density Plots

The posterior density plots for the three independent MCMC chains are presented in Figure 7.

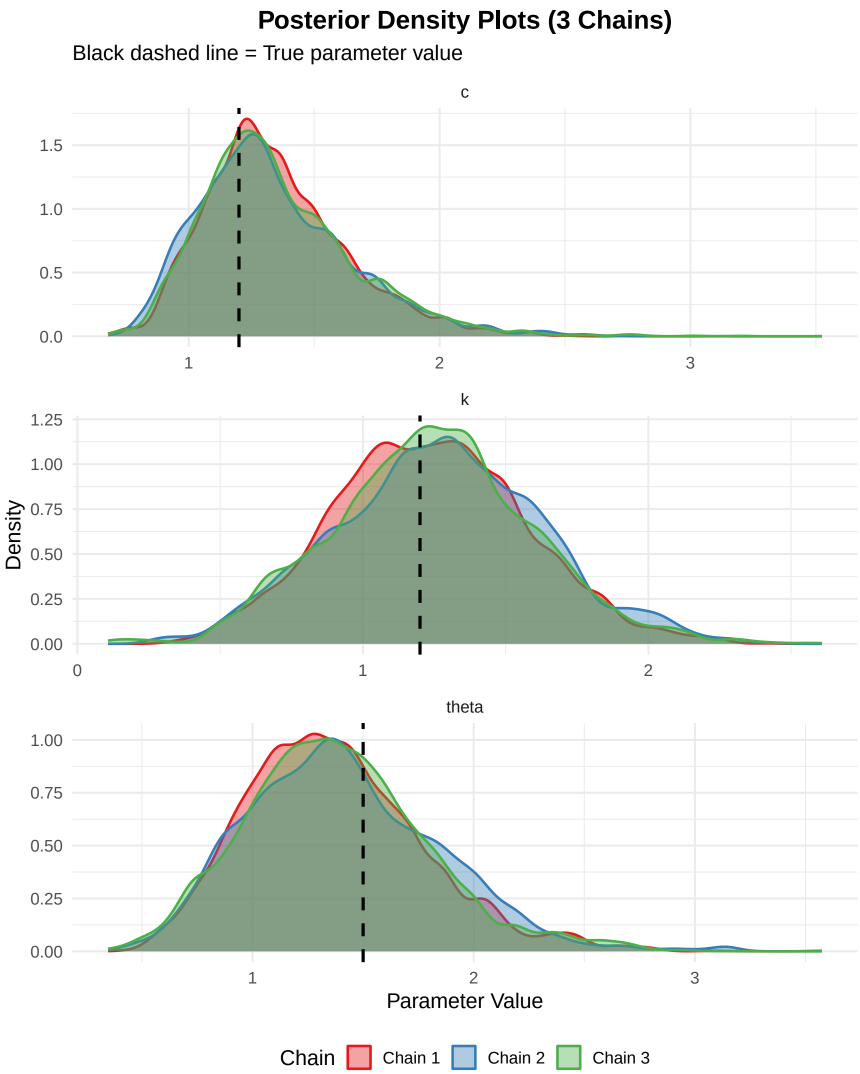



Figure 7. Posterior density plots for the parameters c, k, and $\theta$ based on three MCMC chains

Figure 7 shows substantial overlap among the posterior densities of the three MCMC chains for c, k, and $\theta$. This provides visual evidence of satisfactory convergence and stability of the MCMC chains.

### 5.5.2. Trace Plots

The trace plots of the three MCMC chains are presented in Figure 8.

Figure 8 indicates stable mixing of the three chains without noticeable trends, supporting satisfactory MCMC convergence.

From a practical perspective, the simulation results suggest that larger sample sizes generally lead to improved estimation accuracy, as reflected by lower MSE values. The Bayesian estimators under SELF and LINEX loss functions showed competitive performance and, in many cases, yielded lower MSE values than the MLE. Although computational time was not explicitly recorded in the present study, the Bayesian procedure generally requires

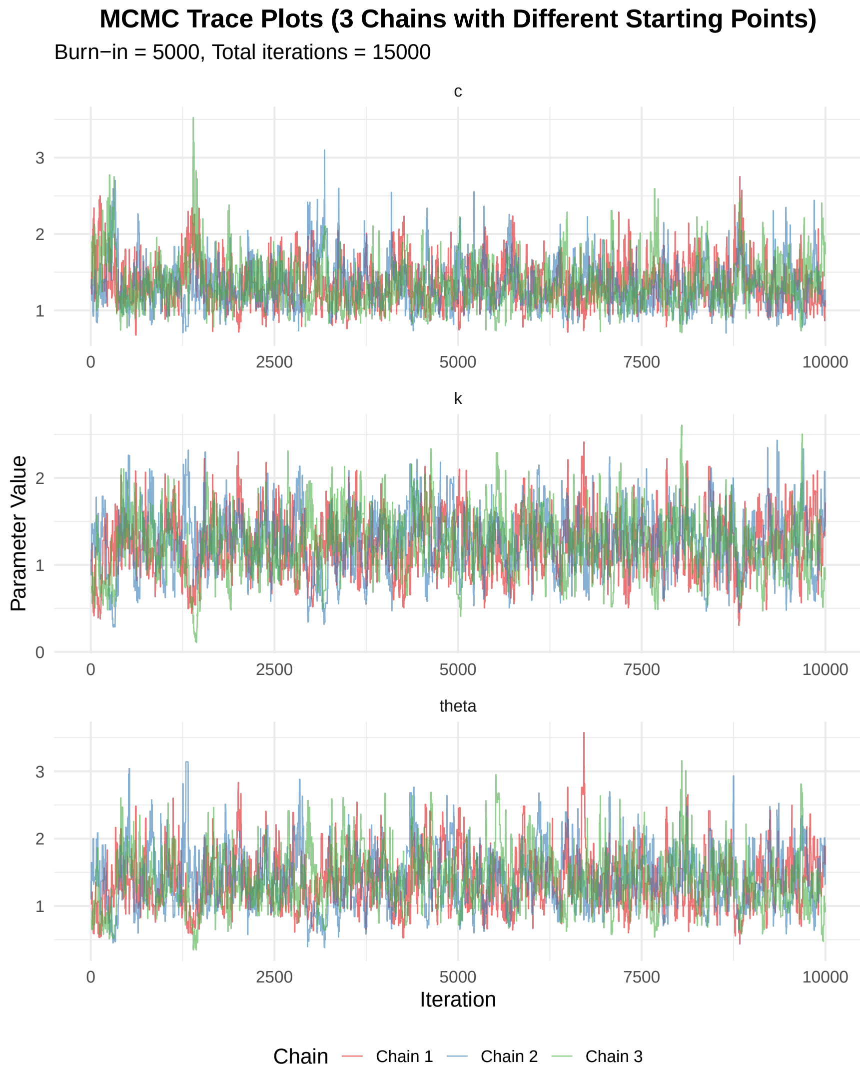



Figure 8. Trace plots for the parameters c, k, and $\theta$ based on three MCMC chains.

greater computational effort because it relies on Metropolis-Hastings sampling. Therefore, the observed gains in estimation accuracy should be weighed against this additional computational requirement in practical applications. Regarding parameter interpretation, the parameters c, k and $\theta$ primarily act as shape parameters controlling the distributional form. In particular, the power parameter $\theta$ provides additional flexibility in controlling skewness and tail behavior and should be interpreted as a statistical shape parameter rather than as a parameter representing a specific physical mechanism. Sensitivity to the removal probability was partially examined by considering three substantially different values, $p = 0.15, 0.50$, and $0.85$, representing low, moderate, and high removal levels. The simulation results indicate that the estimators remain applicable across these removal settings, although estimation accuracy generally deteriorates as the removal probability increases because less information remains available. It should be noted, however, that the present simulation assumes that the removal probability used in the censoring mechanism is correctly specified. A dedicated robustness analysis under misspecified removal probabilities is beyond the scope of the present study and constitutes a useful direction for future investigation.

## 6. Real Data Application

To illustrate the practical applicability of the proposed TCPBXII distribution, a real hydrological dataset is analyzed in this section. The dataset consists of annual river flow measurements recorded for the North Saskatchewan River at Edmonton, Canada [29]. These data have been widely used in the statistical literature for modeling skewed environmental and reliability-type data.

The dataset contains 48 observations representing annual flow measurements. Such hydrological data typically exhibit right-skewed behavior [2] and considerable variability, making them suitable for modeling using flexible lifetime distributions.

### *6.1. Descriptive Statistics*

Before fitting the proposed model, basic descriptive statistics of the dataset are computed. These statistics provide preliminary information about the nature of the data.

Table 4. Descriptive statistics of the North Saskatchewan River dataset

| Measure | Value |
|---|---|
| Mean | 51.49541667 |
| Median | 40.4 |
| Variance | 1048.250208 |
| Min | 19.89 |
| Max | 185.56 |
| Skewness | 2.068605757 |
| Kurtosis (Excess) | 4.950717643 |

the descriptive statistics reported in Table 4 indicate that the dataset exhibits substantial positive skewness, as evidenced by the skewness coefficient of 2.069. Furthermore, the mean exceeds the median, confirming the asymmetric nature of the data. The relatively large variance and excess kurtosis value (4.951) suggest considerable variability and departure from normality. These characteristics support the use of flexible lifetime distributions, such as the proposed TCPBXII model, for modeling the observed river flow measurements.

In addition, graphical tools such as histograms and boxplots are used to visualize the distributional shape of the data.

Based on these preliminary findings, the proposed TCPBXII distribution and several competing models are fitted to the dataset and compared using various goodness-of-fit measures.

### *6.2. Model Fitting and Comparison*

To evaluate the performance of the proposed distribution, the TCPBXII model is fitted to the dataset and compared with several well-known lifetime distributions, including :

Burr XII distribution, Lognormal distribution, Gamma distribution, Weibull distribution.

These models are commonly used in reliability and environmental data analysis. The goodness-of-fit of the competing models is assessed using several statistical criteria, including

Akaike Information Criterion (AIC)[31], Bayesian Information Criterion (BIC)[31], Kolmogorov–Smirnov statistic (KS) [21], Cramér–von Mises statistic (CVM)[3], Anderson–Darling statistic (AD)[16].

smaller values of AIC, BIC, KS, CVM, and AD indicate a better fit to the observed data.

The goodness-of-fit results indicate that the proposed TCPBXII distribution provides the best fit among all competing models.

From a hydrological perspective, the improved fit of the TCPBXII model may be attributed to its greater flexibility in accommodating the pronounced skewness and tail behavior observed in the river-flow data. The

Table 5. Goodness-of-fit measures for the competing distributions.

| Model | LogLik | AIC | BIC | KS | KS_pvalue | CVM | AD |
|---|---|---|---|---|---|---|---|
| TCPBXII | -39.09603894 | 84.19207787 | 89.8056809 | 0.071560898 | 0.966575052 | 0.043435753 | 0.319246921 |
| BurrXII | -40.15822668 | 84.31645336 | 88.05885538 | 0.092300993 | 0.808108633 | 0.073860573 | 0.466110202 |
| Lognormal | -40.31115179 | 84.62230358 | 88.3647056 | 0.099460092 | 0.729329364 | 0.08438122 | 0.567933902 |
| Gamma | -43.97125389 | 91.94250777 | 95.6849098 | 0.135603822 | 0.340569489 | 0.185658868 | 1.166619484 |
| Weibull | -48.16254203 | 100.3250841 | 104.0674861 | 0.140340883 | 0.300891816 | 0.276851365 | 1.772849504 |

additional power parameter $\theta$ acts as a statistical shape parameter that increases the flexibility of the distribution. However, it is not interpreted here as representing a specific physical hydrological phenomenon. Rather, its role is to allow the model to adapt more effectively to the distributional characteristics of the observed data.

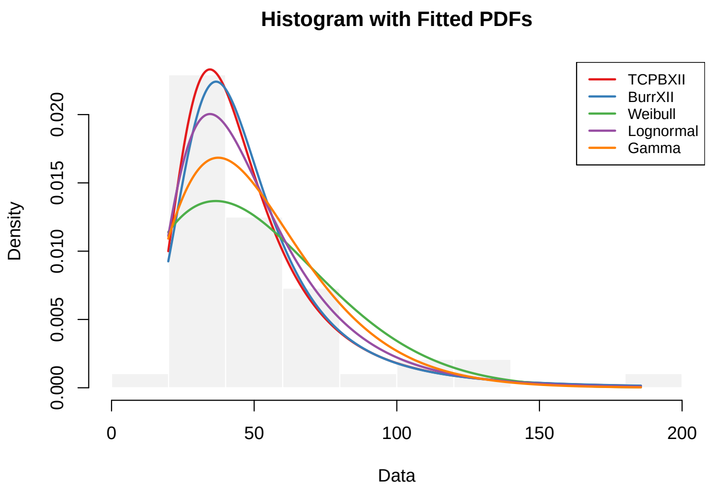



Figure 9. Histogram of the dataset with fitted probability density functions

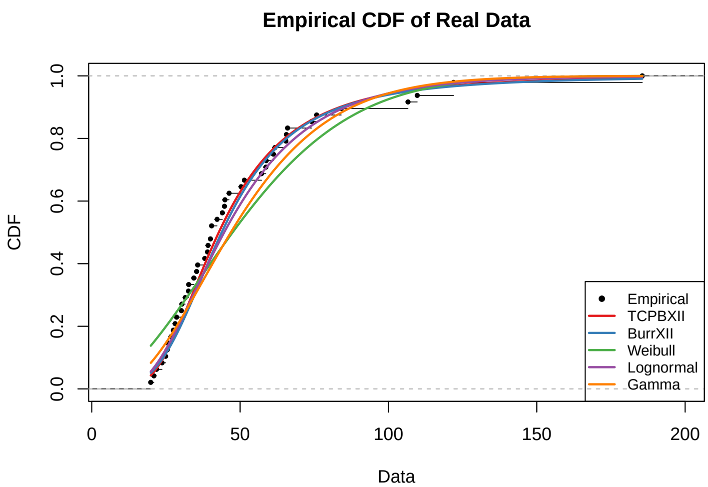



Figure 10. Empirical cumulative distribution function (CDF).

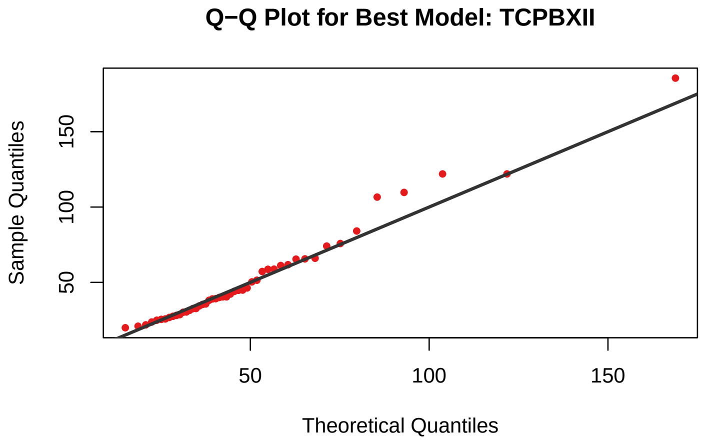



Figure 11. Q-Q plot for the fitted TCPBXII distribution

Table 6. parameter estimates of the TCPBXII model based on the real dataset

| Parameter | MLE | Bayes_SELF | Bayes LINEX |
|---|---|---|---|
| C | 1.551153699 | 3.829879284 | 3.473089182 |
| K | 2.465179023 | 0.752924292 | 0.706690968 |
| O | 2.901258364 | 1.154031671 | 1.067093099 |

Table 6 presents the parameter estimates of the TCPBXII model obtained using maximum likelihood estimation (MLE) and Bayesian estimation under SELF and LINEX loss functions. The estimates vary across the three estimation procedures, reflecting the differences between the likelihood-based and Bayesian approaches. The parameters c, k, and $\theta$ primarily act as shape parameters controlling the distributional form. In particular, the power parameter $\theta$ provides additional flexibility in controlling the shape and tail behavior of the distribution. These estimated parameters are subsequently used to obtain the reliability estimates presented in the following subsection.

#### 6.2.1. *Reliability Estimation*

To provide a practical reliability assessment of the fitted TCPBXII model, the reliability function was estimated at selected values using the maximum likelihood and Bayesian methods under SELF and LINEX loss function. The resulting reliability estimates are presented in Table 7.

Table 7. Reliability estimates for the TCPBXII model based on the real dataset

| Time (t) | MLE | Bayes_ SELF | Bayes_LINEX |
|---|---|---|---|
| 20 | 0.955922195 | 0.944892049 | 0.944581799 |
| 40 | 0.559410124 | 0.560801838 | 0.559161329 |
| 60 | 0.244784053 | 0.24766967 | 0.246419975 |
| 80 | 0.113545856 | 0.120691625 | 0.120006107 |
| 100 | 0.058389544 | 0.067217973 | 0.066845036 |
| 120 | 0.03276816 | 0.04148605 | 0.0412699 |
| 160 | 0.01254125 | 0.019499983 | 0.019412206 |

Table 7 shows that the estimated reliability decreases steadily as the observed value increases. The three estimation methods provide very similar reliability estimates, particularly at lower values, indicating consistency between the maximum likelihood and Bayesian procedures. At t=20, the estimated reliability is approximately 0.956 using MLE and 0.945 using both Bayesian approaches, whereas at t = 100, it decreases to approximately 0.058, 0.067, 0.067 under MLE, Bayes-SELF, and Bayes-LINEX, respectively. This decreasing pattern reflects the declining survival probability associated with increasing river-flow levels under the fitted TCPBXII model.

### *6.3. Results and Discussion*

The goodness-of-fit results reported in Table 7 demonstrate that the proposed TCPBXII distribution provides the best overall fit to the hydrological dataset among all competing models. In particular, the TCPBXII model achieves the smallest values of AIC, BIC, KS, CVM, and AD statistics, together with the largest KS p-value, indicating superior agreement with the observed data.

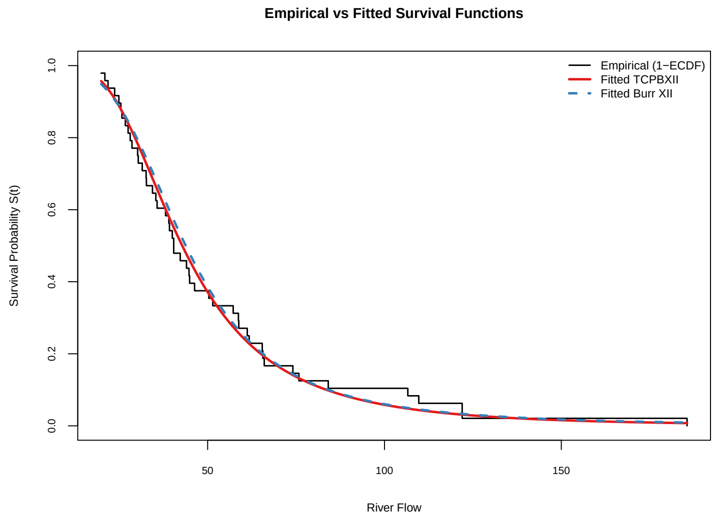



Figure 12. Empirical and fitted survival functions for the North Saskatchewan River dataset.

The empirical survival function is closely followed by the fitted TCPBXII survival curve over most of the observed range. The fitted Burr XII model also shows reasonable agreement with the empirical survival function. Overall, this graphical comparison provides additional support for the adequacy of the proposed TCPBXII model in describing the observed river-flow data.

The quantile residuals Q-Q plot in Figure 13 shows that most residuals lie close to the theoretical normal reference line, with only slight deviations at the extreme tails. This pattern indicates that the fitted TCPBXII model provides an adequate representation of the observed river-flow data and does not exhibit substantial systematic lack of fit.

Furthermore, the graphical diagnostics presented support the adequacy of the proposed model. The fitted density curve closely follows the empirical distribution of the data, while the empirical cumulative distribution function and Q-Q plot indicate satisfactory agreement between the fitted model and the observed data.

The reliability estimates reported in Table 6 further support the practical applicability of the proposed TCPBXII model. The reliability function decreases steadily as the observed river-flow level increases, while the MLE, Bayes-SELF, and Bayes-LINEX methods yield closely comparable estimates. This agreement indicates that the reliability assessment is stable across the considered estimation procedures.

For a hydrological perspective, the river-flow data exhibit pronounced right-skewness and substantial variability, with a relatively long upper tail. These characteristics require a flexible model capable of accommodating

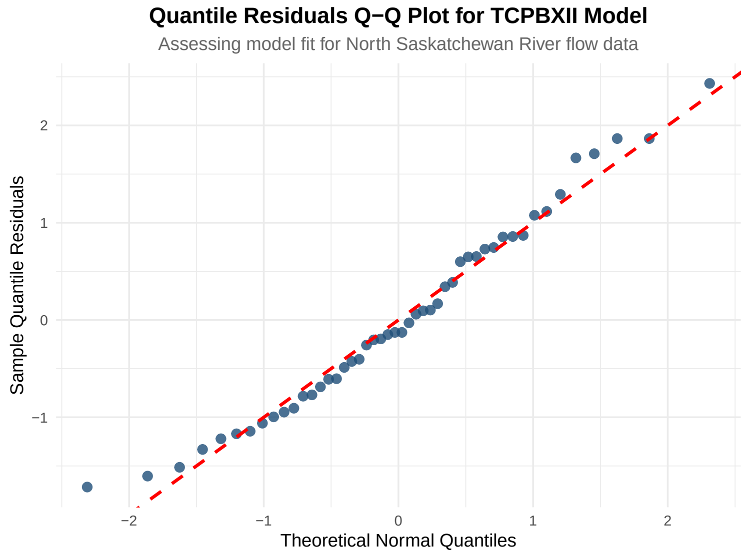



Figure 13. Quantile residuals Q-Q plot for the fitted TCPBXII model.

asymmetric behavior and extreme flow observations. The Burr XII distribution provides a flexible baseline for positive and right-skewed data, while the truncated Cauchy-power transformation introduces additional shape flexibility, allowing the proposed TCPBXII model to adapt more closely to the observed distributional pattern. The superior fit is supported by the goodness-of-fit measures and graphical diagnostics reported above. The power parameter, however, should not be interpreted as representing a specific physical hydrological phenomenon; rather, it acts as an additional statistical shape parameter that enhances the flexibility of the proposed distribution.

From a practical water-resources perspective, the fitted TCPBXII model can be used to quantify the probability of river-flow levels exceeding specified thresholds though its reliability function. Such probability estimates may support hydrological risk assessment by providing information on the likelihood of relatively high-flow events. Consequently, the model can serve as a statistical tool for planning and risk evaluation in water-resource applications, particularly when the observed data are positively skewed and contain extreme values. These interpretations should be viewed as statistical assessments based on the fitted flow distribution rather than as direct predictions of specific flood events.

Overall, these findings suggest that the additional flexibility introduced through the truncated Cauchy transformation enhances the modeling capability of the Burr XII distribution and makes the TCPBXII model a useful alternative for analyzing skewed environmental and reliability-related datasets.

## 7. Conclusion

In this paper , a reliability analysis of the Truncated Cauchy-Power Burr XII (TCPBXII) distribution under progressive Type-II censoring with binomial removals has been presented. The proposed distribution was constructed by combining the Burr XII distribution with the truncated Cauchy transformation and an additional power parameter, resulting in a flexible model suitable for lifetime data analysis.

The statistical properties of the proposed distribution, including the probability density function, cumulative distribution function, survival function, and hazard rate function, were derived and discussed. Parameter estimation was carried out using both maximum likelihood estimation (MLE) and Bayesian estimation under SELF and LINEX loss functions. Since the posterior distributions do not admit closed-form solutions, the Metropolis–Hastings algorithm was employed to obtain the Bayes estimates.

A Monte Carlo simulation study was conducted under different sample sizes and censoring levels. The results indicated that estimation accuracy improves with increasing sample size , whereas higher censoring levels generally lead to larger estimation errors. The Bayesian estimators under SELF and LINEX loss functions generally showed competitive performance and, in many cases, provided more accurate estimates than the corresponding maximum likelihood estimators.

The same real data set is also analyzed by using the herein proposed distribution within the framework of lifetime and reliability data with progressive censoring including binomial removals. In order to compare the fitted TCPBXII model with other competing lifetime distribution models, which include some new ones, several well-known goodness-of-fit statistics/criteria have been employed. The findings obtained from the application of the aforementioned criteria, such as AIC, BIC, KS-statistic, CVM-statistic and AD-statistic, clearly confirm that the newly introduced TCPBXII distribution model outperforms the other models within the framework of the corresponding real data set. The fitted model has also been graphically verified. In addition, reliability estimates obtained using MLE, Bayes-SELF, and Bayes-LINEX showed a decreasing pattern with increasing river-flow levels. The close estimates obtained by the three methods also indicate consistent reliability assessment for the fitted TCPBXII model.

Despite its flexibility and satisfactory performance, the proposed TCPBXII model has some limitations. Its additional power parameter increases model complexity and may require greater computational effort, particularly under Bayesian estimation using MCMC. Moreover, the performance of the model may depend on the sample size and censoring level, especially under heavy censoring or small samples. The present analysis is also limited to the considered simulation settings and hydrological dataset; therefore, further investigation using different censoring schemes, parameter settings, removal probabilities and their possible misspecification, as well as other real datasets is recommended to assess the robustness and broader applicability of the proposed model.

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