

Joint Detection of Random Coefficient Variability and ARCH Effects via Optimal Parametric and Pseudo-Gaussian Tests

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Abstract We propose parametric and pseudo-Gaussian tests for jointly detecting random coefficient variability and ARCH(1) effects in linear regression models. Both tests are derived from the ULAN property of the model and are locally asymptotically most powerful unbiased. The pseudo-Gaussian test is valid under a broad class of error distributions, not just the Gaussian. Existing procedures treat these two effects separately; our test handles them simultaneously, at the cost of a modest, quantifiable power loss relative to each marginal test at its own boundary alternative – the price of the equal-power (MSSMP) criterion underlying its construction. Monte Carlo simulations under normal, logistic, Student, and skew-normal errors show that the joint test consistently achieves the highest power when both effects are simultaneously present, at a modest power cost relative to the relevant marginal test when only one effect occurs; we confirm this with three empirical applications: housing prices, Bitcoin returns, and U.S. inflation/interest rate data.

Keywords Uniform local asymptotic normality, optimal tests, pseudo-Gaussian test, heteroskedasticity, RCR, ARCH

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1. Introduction and Notations

1.1. Random coefficients regression and ARCH effects

Financial and economic data often exhibit changing volatility patterns that are difficult to capture using classical regression models [16, 3, 14].

We fill this gap with a unified test for random coefficient variability and ARCH effects, based on the Uniform Local Asymptotic Normality (ULAN) property. In Monte Carlo simulations and real-data applications, this joint test achieves the highest power of the three statistics when both effects are present. When only one effect occurs, it loses some power relative to the corresponding marginal test, a consequence of the equal-power (MSSMP) criterion on which its construction is based.

Classical regression models assume fixed coefficients and constant variance, assumptions that are often unrealistic in practice [7, 8, 10, 13].

Several models have been proposed to handle these limitations. Such as the ARCH models for modeling conditional heteroskedasticity [9] and the Random Coefficient Regression (RCR) models for parameter variability (see, [15, 39, 40, 27, 5, 1, 17, 26]). However, these approaches are generally studied separately, even though both effects may occur simultaneously.

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In this paper, we consider a framework that combines ARCH effects and random coefficient variability [5, 30, 13]. We develop a robust test designed to detect these two effects jointly [12, 1, 17, 26, 27]. The Simulation results suggest that the proposed approach better captures situations where both effects are present.

This work extends previous studies on random coefficient models and ARCH processes ([17]; [25]), and also connects to more recent developments in robust inference and model-based diagnostics for volatility and parameter instability [6, 22, 29, 35, 4, 11]. In particular, our test procedure belongs to a growing class of semi-parametric inference methods that balance efficiency and robustness, as advocated in the statistical computing literature (see, e.g., [23, 18, 2, 3, 11]).

More recently, machine-learning diagnostics (anomaly detection, neural volatility models) have been used to flag parameter instability or heteroskedasticity without imposing a parametric structure. But they are often black-box procedures [33], and standard anomaly-detection methods do not control test size formally [32]; neither offers a local optimality guarantee.

The simulation results indicate an improved performance compared to the existing methods [17, 1, 26, 27] for RCR models and [25, 30] for ARCH models. Indeed, these tests are not designed for contexts in which both random coefficients and conditional heteroskedasticity coexist [5, 13], limiting their applicability in practical situations. In contrast, our test is specifically designed for this setting, offering a more comprehensive and efficient solution [2, 3].

The purpose of this study is to develop an optimal test to jointly detect random coefficient variability and ARCH effects. Specifically, we propose an optimal parametric test statistic for detecting heteroskedasticity and random coefficient variability and evaluate its performance through stochastic simulations.

To achieve this, we consider a random coefficient regression model with ARCH(1) errors, defined as follows (for $i = 1, \dots, n$, where n stands for the number of observations, $n \in \mathbb{N}^*$):

$$Y_i = \mu + (\beta + \xi_i)X_i + \varepsilon_i, \quad (1)$$

where, the error term ε_i follows an ARCH(1) process and is defined as:

$$\varepsilon_i = \sigma_i Z_i = \sqrt{\alpha_0 + \alpha_1 \varepsilon_{i-1}^2} Z_i, \quad (2)$$

here, Z_i is an independent and identically distributed (*i.i.d.*) sequence of random variables, independent of the past values of ε_i , with mean 0 and unit variance, and

- Y_i (or more specifically, a triangular array $Y_i^{(n)}$) denotes the observed scalar response for the individual i ;
- X_i (or more specifically, a triangular array $X_i^{(n)}$, where not all $X_i^{(n)}$ are identical) represents a non-stochastic scalar exogenous regressor;
- μ and β represent scalar parameters (regression parameters);
- The error term (ε_i) follows an ARCH(1) specification, with conditional variance ($\sigma_i^2 > 0$) depending on the past squared errors. Conditional on the past information, its probability density function is given by

$$f : \varepsilon_i \mapsto f(\varepsilon_i) := \frac{1}{\sigma_i} f_1 \left(\frac{\varepsilon_i}{\sigma_i} \right),$$

where ($f_1 \in \mathcal{F}_0$), with ($\mathcal{F}_0$) denoting an appropriate class of standardized densities (see Section 2.1);

- (ξ_i) is an unobserved *i.i.d.* sequence of random perturbations with zero mean, variance $\sigma_\xi^2 \geq 0$, and, whenever $\sigma_\xi^2 > 0$, probability density function:

$$h : \xi \mapsto h(\xi) := \frac{1}{\sigma_\xi} h_1 \left(\frac{\xi}{\sigma_\xi} \right),$$

where $h_1 \in \mathcal{F}_0$;

- α_0 and α_1 are constants such that : $0 \leq \alpha_1 < 1$ and $\alpha_0 > 0$, to guaranty the stability of the process ε_i .

Under the null hypothesis ($\sigma_\xi^2 = \alpha_1 = 0$), the model in equation (1) reduces to the traditional linear regression model:

$$Y_i = \mu + \beta X_i + \varepsilon_i, \quad i = 1, \dots, n. \quad (3)$$

Therefore, it is important to perform a preliminary test of the null hypothesis $\mathcal{H}_0 : \sigma_\xi^2 = \alpha_1 = 0$, with unspecified μ, β, α_0 and f , against the alternative hypothesis $\mathcal{H}_1 : \sigma_\xi^2 > 0$ and $\alpha_1 > 0$, still with unspecified μ, β, α_0, f and h .

If $\alpha_1 = 0$ and $\sigma_\xi^2 > 0$, the error term ε_i simplifies to $\sqrt{\alpha_0}Z_i$ (Z_i is *i.i.d.*, with mean of 0 and variance 1). This reduces the model to a standard random coefficient regression (RCR) model, as studied in [17].

Similarly, when $\sigma_\xi^2 = 0$ and $\alpha_1 \in]0, 1[$, the model describes a regression with ARCH(1) errors, which captures time-varying volatility in the error term, consistent with the analysis in [25].

This paper makes three main contributions. First, we prove the ULAN property for the proposed model. Second, we construct parametric and pseudo-Gaussian tests that are unbiased and locally asymptotically most powerful (LAMP), obtained as a projection weighted by the inverse of the joint asymptotic covariance matrix, following the MSSMP criterion of Schaafsma and Smid; in the pseudo-Gaussian case this covariance matrix is diagonal (Section 3), so the MSSMP weights reduce exactly to an equally-weighted combination of the two already-standardized marginal statistics of [17] and [25], a consequence of asymptotic orthogonality, not an ad hoc averaging.. Third, we evaluate the proposed tests using Monte Carlo simulations under various distributional assumptions and illustrate their practical relevance via a real data application.

Relation to Fihri et al. (2020) and Lee and King (1993). Fihri et al [17] derive a LAMP pseudo-Gaussian test for $\mathcal{H}_0 : \sigma_\xi^2 = 0$ in the RCR model with *i.i.d.* errors of constant variance α_0 . Lee and King [25] propose an optimal test for $\mathcal{H}_0 : \alpha_1 = 0$ in a standard regression with fixed coefficients. Both tests perform well in their respective settings, but are not designed for the case where random coefficients and ARCH errors coexist. The present paper addresses this situation by testing $\sigma_\xi^2 = \alpha_1 = 0$ jointly. The statistic T_{Al} reduces to T_F when $\alpha_1 = 0$, and to the Lee-King statistic when $\sigma_\xi^2 = 0$. The simulation tables in Section 4 reflect this: P_F is strongest in the first row, P_{LK} in the first column, and P_{Al} dominates in the remaining entries.

Joint treatment of parameter instability and conditional heteroskedasticity has also been considered outside this specific model, notably in structural-break and macroeconomic settings [21, 31]. These approaches typically address unknown break dates in a structural-change framework under heteroskedastic (often GARCH-type) errors, whereas the present paper considers a fixed random-coefficient (rather than break-point) alternative and derives a locally and asymptotically optimal ULAN-based test. To our knowledge, no unified ULAN-based test has been developed for the random coefficient regression model with ARCH(1) errors specifically. The present framework is restricted to ARCH(1) errors and a single regressor; extending it to $GARCH(p, q)$ errors or multiple regressors remains an open problem, which we discuss further in Section 6.

1.2. Outline of paper

To construct locally and asymptotically optimal tests of the null hypothesis that the regression coefficients are non-random, Section 2 establishes the ULAN property of the model in a neighborhood of $\sigma_\xi^2 = 0$ and $\alpha_1 = 0$, following the approach of [41, 39, 1, 26] under regularity conditions on f_1 (Assumptions $\mathcal{A}$ and $\mathcal{C}$). Section 3 constructs parametrically efficient tests for a given density f_1 , and establishes their local asymptotic optimality after projecting out the nuisance parameters (μ, β, α_0) from the information matrix. Section 4 reports Monte Carlo power results under several error distributions. Section 5 applies the three tests to the housing dataset of [42] ($n = 88$ observations, price on assessment), to weekly Bitcoin returns (2020–2024), and to U.S. inflation and interest rate data [28], and discusses the conclusions.

2. Uniform Local Asymptotic Normality (ULAN)

2.1. Notation and basic assumptions

Consider the sequence of observed responses: $\mathbf{Y}^{(n)} = (Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)})$, where $n \in \mathbb{N}^*$ represents the sample size used in the model (1). The probability distribution of this sequence is denoted as $P_{\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1; f_1, h_1}^{(n)}$, or, more compactly, $P_{\theta; f_1, h_1}^{(n)}$, with the parameter vector $\theta = (\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1)'$, and f_1 and h_1 representing the standardized densities of ε and ξ .

Under the null hypothesis ($\sigma_\xi^2 = \alpha_1 = 0$), the distribution simplified to $P_{\mu, \beta, \alpha_0, 0, 0; f_1}^{(n)}$, where h_1 no longer has an effect.

To ensure the robustness of the testing procedures while avoiding strong assumptions on higher-order moments, an identification constraint is imposed on the density f_1 . Unlike the usual constraint based on the mean and standard deviation, we consider here a family of standardized densities $\mathcal{F}_0$, defined by:

$$\mathcal{F}_0 := \left\{ f_1 : \int_{-1}^1 f_1(z) dz = 0.5 = \int_{-\infty}^0 f_1(z) dz \right\};$$

which corresponds to densities with a zero median and a median absolute deviation equal to one. This normalization, inspired by [17], avoids moment existence conditions while ensuring identifiability. Common examples of standardized densities that belong to $\mathcal{F}_0$ include standardized Laplace and symmetric logistic distributions. In particular, the standard normal distribution ($\mathcal{N}_1$) and the standard logistic distribution (l_1) are examples, with parameters $a \approx 0.4549$ and $b \approx 1.207$, respectively. These distributions are given by the following formulas:

$$\mathcal{N}_1(z) = \sqrt{\frac{a}{2\pi}} \exp\left(\frac{-az^2}{2}\right), \quad l_1(z) = \sqrt{b} \exp(-\sqrt{b}z) / (1 + \exp(-\sqrt{b}z))^2.$$

The primary technical tool utilized here is Uniform Local Asymptotic Normality (ULAN) at θ of the form $(\mu, \beta, \alpha_0, 0, 0)'$ and with respect to $(\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1)'$, applied to parametric families of distributions:

$$\mathcal{P}_{f_1, h_1}^{(n)} = \left\{ P_{\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1; f_1, h_1}^{(n)} : (\mu, \beta)' \in \mathbb{R}^2, \alpha_0 > 0, \sigma_\xi^2 \geq 0 \text{ and } \alpha_1 \in [0, 1] \right\}.$$

The Uniform Local Asymptotic Normality (ULAN) property extends the concept of Local Asymptotic Normality (LAN) to the model under the null hypothesis $\sigma_\xi^2 = \alpha_1 = 0$. In this context, the log-likelihood derivatives with respect to σ_ξ^2 and α_1 vanish near zero, a non-linearity that complicates the derivation of linear-quadratic approximations typically obtained from Taylor expansions.

Ensuring the ULAN property requires imposing certain regularity conditions, some of which go back to [19], such as (A.3) and (C.2). Similar conditions have been studied in the works of [2], [4], and [17]. Based on these foundations, we assume the following condition holds:

Assumption A:

The density f_1 is such that

(A.1) f_1 belongs to $\mathcal{F}_0$;

(A.2) $f_1(z) > 0$ for all $z \in \mathbb{R}$;

(A.3) the function $z \mapsto f_1(z)$ has two continuous derivatives, with the first and second derivatives denoted $\dot{f}_1$ and $\ddot{f}_1$, respectively. Denote

$$\varphi_{f_1}(x) := -\dot{f}_1(x)/f_1(x), \quad \psi_{1, f_1}(x) := \ddot{f}_1(x)/f_1(x) \text{ and } \psi_{2, f_1}(x) := -x\dot{f}_1(x)/f_1(x) - 1 = x\varphi_{f_1}(x) - 1.$$

Assume the following integrals are finite:

$$\mathcal{I}_\varphi(f_1) := \int_{\mathbb{R}} \varphi_{f_1}^2(z) f_1(z) dz, \quad \mathcal{I}_{\psi_1}(f_1) := \int_{\mathbb{R}} \psi_{1, f_1}^2(z) f_1(z) dz, \quad \mathcal{I}_{\psi_2}(f_1) := \int_{\mathbb{R}} \psi_{2, f_1}^2(z) f_1(z) dz \quad \text{and} \quad \mathcal{K}_{\varphi\varphi}(f_1) := \int_{\mathbb{R}} z \varphi_{f_1}^2(z) f_1(z) dz;$$

(A.4) the matrices $\Gamma_{f_1}^{(n)}(\theta)$ and $\Gamma_{f_1}(\theta)$, defined in equation (8) below, must be of full rank.

Suppose that $\mathcal{F}_A$ denotes the set of all densities that satisfy assumptions (A.2)-(A.4), then for any function f_1 within $\mathcal{F}_A$,

1. $\mathcal{I}_{\varphi\psi_1}(f_1) = \int_{\mathbb{R}} \varphi_{f_1}(z) \psi_{1,f_1}(z) f_1(z) dz$ and $\mathcal{K}_{\varphi\psi_1}(f_1) = \int_{\mathbb{R}} z \varphi_{f_1}(z) \psi_{1,f_1}(z) f_1(z) dz$ are finite;
2. $\int \varphi_{f_1}(z) f_1(z) dz = \int z^2 \varphi_{f_1}(z) f_1(z) dz = \int \psi_{1,f_1}(z) f_1(z) dz = \int z \psi_{1,f_1}(z) f_1(z) dz = \int \psi_{2,f_1}(z) f_1(z) dz = 0$;
3. $\int z \varphi_{f_1}(z) f_1(z) dz = 1$ and $\int z^2 \psi_{1,f_1}(z) f_1(z) dz = 2$;
4. the function $\theta \mapsto \Gamma_{f_1}(\theta)$ is continuous for each value of $\theta \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times [0, 1[$.

Remark 1 (On Assumption $\mathcal{A}$)

The normalization in (A.1) uses the median and median absolute deviation rather than the mean and standard deviation. This sidesteps moment existence conditions, which matter when working with heavy-tailed distributions. Assumption (A.3) asks for two continuous derivatives of f_1 ; the normal, logistic, Student- t , and skew-normal densities all qualify. Assumption (A.4) rules out degenerate information matrices; in practice, this can be checked numerically once f_1 is specified.

In particular, for $f_1 \in \mathcal{F}_0$, the quantity $\mu + (\beta + \xi_i)X_i$ is to be interpreted as the (conditional) median of Y_i given ξ_i and X_i , rather than its mean. If f has finite expectation, the conditional mean also exists and differs from the median by a constant depending on f ; when f is symmetric, the two coincide. So Assumption (A.1) needs no moment condition on f_1 .

Assumption $\mathcal{B}$:

The covariates $X_i = X_i^{(n)}$ for $i = 1, \dots, n$, are defined such that, considering

$$M_j^{(n)} = \frac{1}{n} \sum_{i=1}^n \left(X_i^{(n)} \right)^j, \quad j = 1, \dots, 4;$$

(B.1) the classic Noether condition, that is, when $\bar{X}^{(n)} = M_1^{(n)}$,

$$\lim_{n \rightarrow \infty} \left[\max_{1 \leq i \leq n} (X_i^{(n)} - \bar{X}^{(n)})^2 / \sum_{i=1}^n (X_i^{(n)} - \bar{X}^{(n)})^2 \right] = 0;$$

(B.2) the sequence $M_4^{(n)}$ remains bounded as n approaches ∞ .

Assumption (B.2) ensures that any infinite subsequence of $M_j^{(n)}$ contains a further subsequence in which:

(B.3) the sequence $M_j^{(n)}$ converges to specific values μ_j^X for $j = 1, \dots, 4$; more precisely, $\bar{X}^{(n)}$ converges to μ_1^X . Note that the asymptotic results hold under assumptions (B.1) and (B.2) as $n \rightarrow \infty$. However, those results that require the convergence of the local experiment to establish the ULAN property only hold as $n \rightarrow \infty$ under assumption (B.3).

Remark 2 (On Assumption $\mathcal{B}$)

Assumption (B.1) is the Noether condition: no single observation should dominate the regression sum of squares. This holds whenever the X_i are i.i.d. or bounded, so it is not restrictive in practice. Assumption (B.2) bounds the fourth empirical moment of the covariates, which is needed to control remainder terms in the ULAN expansion. Both conditions are standard in the semi-parametric regression literature; see [17] and [1] for similar setups.

Assumption $\mathcal{C}$:

It concerns density (normalized) h_1 of random coefficient:

(C.1) the following conditions hold: $\int_{\mathbb{R}} \xi h_1(\xi) d\xi = 0$ and $\int_{\mathbb{R}} \xi^2 h_1(\xi) d\xi = 1$;

(C.2) define:

$$\mathcal{I}_{\psi_1\psi_1}^x(f_1; y) = \begin{cases} \frac{1}{y^2} \int_{z=-\infty}^{\infty} \frac{\left[\int_{w=0}^y \int \ddot{f}_1(z-xvw) x^2 v^2 h_1(v) dv dw \right]^2}{x^4 \mathcal{I}_{\psi_1}(f_1)} dz & \text{if } y > 0 \\ x^4 \mathcal{I}_{\psi_1}(f_1) & \text{if } y = 0 \end{cases}.$$

Alternatively:

$$\mathcal{I}_{\psi_1\psi_1}^x(f_1; y) = \begin{cases} \frac{1}{y^2} \int_{z=-\infty}^{\infty} \left[\frac{\int f_1(z-xyv)xvh_1(v)dv}{[\int f_1(z-xyv)h_1(v)dv]^{1/2}} \right]^2 dz & \text{if } y > 0 \\ x^4 \mathcal{I}_{\psi_1}(f_1) & \text{if } y = 0 \end{cases}.$$

The function $y \mapsto \mathcal{I}_{\psi_1\psi_1}^x(f_1; y)$ is right-continuous at $y = 0$ for all x .

It's important to note that Assumption (C.2) involves the pair of densities (f_1, h_1) rather than h_1 alone. Specifically, for all $f_1 \in \mathcal{F}_A$, define:

$$\mathcal{F}_{C|f_1} = \left\{ h_1 \mid h_1 \text{ satisfies (C.1) and } (f_1, h_1) \text{ satisfy (C.2)} \right\}.$$

Remark 3 (On Assumption C)

Assumption (C.1) centers and scales h_1 , standard when σ_ξ^2 is already a free parameter. Assumption (C.2) is less intuitive: it requires right-continuity at zero of a Fisher information functional that mixes f_1 and h_1 . For a given application, it can be checked by verifying that $y \mapsto \mathcal{I}_{\psi_1\psi_1}^x(f_1; y)$ has no jump at $y = 0$, which in practice amounts to a simple numerical integration.

The density h_1 of the random coefficient is used only to write the likelihood, and hence to establish the ULAN property. It does not appear in the central sequence, nor in the test statistics used in practice, so it can be chosen freely. The best choice is the Gaussian density $h_1 = \mathcal{N}_1$, which is the one used throughout the paper.

2.2. ULAN

Consider the null hypothesis $\theta = (\mu, \beta, \alpha_0, \sigma_\xi^2 = 0, \alpha_1 = 0)'$ and local alternatives defined by small perturbations of the form $\theta + n^{-1/2} \nu^{(n)} \tau^{(n)}$, where

$$\nu^{(n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (\frac{1}{n} \sum_{i=1}^n X_i^2)^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (\frac{1}{n} \sum_{i=1}^n X_i^4)^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tau^{(n)} = \begin{pmatrix} \tau_1^{(n)} \\ \tau_2^{(n)} \\ \tau_3^{(n)} \\ \tau_4^{(n)} \\ \tau_5^{(n)} \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \quad (4)$$

and $(\tau^{(n)})' \tau^{(n)}$ remains uniformly bounded as n approaches infinity ($\sup_n (\tau^{(n)})' \tau^{(n)} < \infty$).

Define the standardized residual $Z_i^{(n)}$ by

$$Z_i^{(n)}(\mu, \beta, \alpha_0) = \alpha_0^{-1/2} (Y_i - \mu - \beta X_i), \quad (5)$$

and note that, under $P_{\mu, \beta, \alpha_0, 0, 0; f_1}^{(n)}$, $Z_i^{(n)}$ coincides with $\varepsilon_i / \sqrt{\alpha_0}$. We then have the following proposition (denote $\nu_2^{(n)} = (nM_2^{(n)})^{-1/2} = \left(\sum_{i=1}^n X_i^2 \right)^{-1/2}$ and $\nu_4^{(n)} = (nM_4^{(n)})^{-1/2} = \left(\sum_{i=1}^n X_i^4 \right)^{-1/2}$).

Proposition 2.1 (ULAN)

Assume that Assumptions (B.1)–(B.2) and C are satisfied. Let $f_1 \in \mathcal{F}_A$ and $h_1 \in \mathcal{F}_{C|f_1}$. Then, the family

$$\mathcal{P}_{f_1, h_1}^{(n)} = \left\{ P_{\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1; f_1, h_1}^{(n)} \mid (\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1)' \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times [0, 1] \right\}, \quad (6)$$

is uniformly locally asymptotically normal (ULAN) at any $\theta = (\mu, \beta, \alpha_0, 0, 0)'$, with central sequence

$$\Delta_{f_1}^{(n)}(\theta) := \begin{pmatrix} \Delta_{f_1;1}^{(n)}(\theta) \\ \Delta_{f_1;2}^{(n)}(\theta) \\ \Delta_{f_1;3}^{(n)}(\theta) \\ \Delta_{f_1;4}^{(n)}(\theta) \\ \Delta_{f_1;5}^{(n)}(\theta) \end{pmatrix} = \begin{pmatrix} \frac{n^{-1/2}}{\sqrt{\alpha_0}} \sum_{i=1}^n \varphi_{f_1}(Z_i) \\ \frac{\nu_2^{(n)}}{\sqrt{\alpha_0}} \sum_{i=1}^n \varphi_{f_1}(Z_i) X_i \\ \frac{n^{-1/2}}{2\alpha_0} \sum_{i=1}^n \psi_{2,f_1}(Z_i) \\ \frac{\nu_4^{(n)}}{2\alpha_0} \sum_{i=1}^n \psi_{1,f_1}(Z_i) X_i^2 \\ \frac{n^{-1/2}}{2} \sum_{i=1}^n Z_{i-1}^2 \psi_{2,f_1}(Z_i) \end{pmatrix}. \quad (7)$$

Remark 4 (Second-order nature of the central sequence)

Unlike μ , β , and α_0 , whose central-sequence components in (7) behave in a standard way under a routine Taylor expansion, the log-likelihood derivatives with respect to σ_ξ^2 and α_1 vanish identically at the null point $\sigma_\xi^2 = \alpha_1 = 0$, so the usual linear-quadratic log-likelihood approximation required for LAN does not follow directly (see Fihri et al., 2020, Section 1.2, for the same phenomenon in the simpler RCR-only case). Establishing ULAN here therefore requires expanding the log-likelihood one order further, which is why $\Delta_{f_1;4}^{(n)}$ and $\Delta_{f_1;5}^{(n)}$ in (7) involve the second-order functions ψ_{1,f_1} and ψ_{2,f_1} rather than φ_{f_1} alone. Intuitively, detecting a small amount of random-coefficient or ARCH variability requires probing the curvature of the log-density near the null, not just its slope.

and information matrix

$$\Gamma_{f_1}^{(n)}(\theta) := \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} \\ \Gamma_{14} & \Gamma_{24} & \Gamma_{34} & \Gamma_{44} & \Gamma_{45} \\ \Gamma_{15} & \Gamma_{25} & \Gamma_{35} & \Gamma_{45} & \Gamma_{55} \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha_0} \mathcal{I}_\varphi(f_1) & \frac{\bar{X}^{(n)}}{\alpha_0 (M_2^{(n)})^{1/2}} \mathcal{I}_\varphi(f_1) & \frac{1}{2\alpha_0^{3/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{M_2^{(n)}}{2\alpha_0^{3/2} (M_2^{(n)})^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{1}{2\sqrt{\alpha_0}} \mathcal{K}_{\varphi\varphi}(f_1) \\ \frac{\bar{X}^{(n)}}{\alpha_0 (M_2^{(n)})^{1/2}} \mathcal{I}_\varphi(f_1) & \frac{1}{\alpha_0} \mathcal{I}_\varphi(f_1) & \frac{1}{2\alpha_0^{3/2} (M_2^{(n)})^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{M_3^{(n)}}{2\alpha_0^{3/2} (M_2^{(n)})^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{\bar{X}^{(n)}}{2\sqrt{\alpha_0} (M_2^{(n)})^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) \\ \frac{1}{2\alpha_0^{3/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{\bar{X}^{(n)}}{2\alpha_0^{3/2} (M_2^{(n)})^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{1}{4\alpha_0^2} \mathcal{I}_{\psi_2}(f_1) & \frac{M_2^{(n)}}{4\alpha_0^2 (M_4^{(n)})^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) & \frac{1}{4\alpha_0} \mathcal{I}_{\psi_2}(f_1) \\ \frac{M_2^{(n)}}{2\alpha_0^{3/2} (M_4^{(n)})^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{M_3^{(n)}}{2\alpha_0^{3/2} (M_2^{(n)})^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{M_2^{(n)}}{4\alpha_0^2 (M_4^{(n)})^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) & \frac{1}{4\alpha_0^2} \mathcal{I}_{\psi_1}(f_1) & \frac{M_2^{(n)}}{4\alpha_0 (M_4^{(n)})^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) \\ \frac{1}{2\sqrt{\alpha_0}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{\bar{X}^{(n)}}{2\sqrt{\alpha_0} (M_2^{(n)})^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{1}{4\alpha_0} \mathcal{I}_{\psi_2}(f_1) & \frac{M_2^{(n)}}{4\alpha_0 (M_4^{(n)})^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) & \frac{\mu_4(f_1)}{4} \mathcal{I}_{\psi_2}(f_1) \end{pmatrix}. \quad (8)$$

Specifically, for any parameter vector $\theta^{(n)} := (\mu^{(n)}, \beta^{(n)}, \alpha_0^{(n)}, 0, 0)'$, where

$$n^{1/2}(\mu^{(n)} - \mu), (\nu_2^{(n)})^{-1}(\beta^{(n)} - \beta), \text{ and } n^{1/2}(\alpha_0^{(n)} - \alpha_0), \quad (9)$$

are of order $O(1)$, and for any bounded sequence $\tau^{(n)} \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, under $P_{\theta^{(n)};f_1}^{(n)}$, as $n \rightarrow \infty$, we have:

$$\begin{aligned} \Lambda_{\theta^{(n)} + n^{-1/2} \nu^{(n)} \tau^{(n)} / \theta^{(n)}; f_1, h_1}^{(n)} &:= \log \left(\frac{dP_{\theta^{(n)} + n^{-1/2} \nu^{(n)} \tau^{(n)}; f_1, h_1}^{(n)}}{dP_{\theta^{(n)}; f_1}^{(n)}} \right), \\ &= \tau^{(n)'} \Delta_{f_1}^{(n)}(\theta^{(n)}) - \frac{1}{2} \tau^{(n)'} \Gamma_{f_1}^{(n)}(\theta) \tau^{(n)} + o_p(1). \end{aligned}$$

Additionally, along any subsequence where $|\Gamma_{f_1}^{(n)}(\theta)| > 0$,

$$\left(\Gamma_{f_1}^{(n)}(\theta) \right)^{-1/2} \Delta_{f_1}^{(n)}(\theta^{(n)}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I). \quad (10)$$

Along subsequences satisfying (B.3), $\Gamma_{f_1}^{(n)}(\theta)$ converges to

$$\Gamma_{f_1}(\theta) := \begin{pmatrix} \frac{1}{\alpha_0} \mathcal{I}_\varphi(f_1) & \frac{\mu_1^X}{\alpha_0(\mu_2^X)^{1/2}} \mathcal{I}_\varphi(f_1) & \frac{1}{2\alpha_0^{3/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{\mu_2^X}{2\alpha_0^{3/2}(\mu_4^X)^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{1}{2\sqrt{\alpha_0}} \mathcal{K}_{\varphi\varphi}(f_1) \\ \frac{\mu_1^X}{\alpha_0(\mu_2^X)^{1/2}} \mathcal{I}_\varphi(f_1) & \frac{1}{\alpha_0} \mathcal{I}_\varphi(f_1) & \frac{\mu_1^X}{2\alpha_0^{3/2}(\mu_2^X)^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{\mu_3^X}{2\alpha_0^{3/2}(\mu_2^X \mu_4^X)^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{\mu_1^X}{2\sqrt{\alpha_0}(\mu_2^X)^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) \\ \frac{1}{2\alpha_0^{3/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{\mu_1^X}{2\alpha_0^{3/2}(\mu_2^X)^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{1}{4\alpha_0^2} \mathcal{I}_{\psi_2}(f_1) & \frac{\mu_2^X}{4\alpha_0^2(\mu_4^X)^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) & \frac{1}{4\alpha_0} \mathcal{I}_{\psi_2}(f_1) \\ \frac{\mu_2^X}{2\alpha_0^{3/2}(\mu_4^X)^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{\mu_3^X}{2\alpha_0^{3/2}(\mu_2^X \mu_4^X)^{1/2}} \mathcal{I}_{\varphi\psi_1}(f_1) & \frac{\mu_2^X}{4\alpha_0^2(\mu_4^X)^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) & \frac{1}{4\alpha_0^2} \mathcal{I}_{\psi_1}(f_1) & \frac{\mu_2^X}{4\alpha_0(\mu_4^X)^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) \\ \frac{1}{2\sqrt{\alpha_0}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{\mu_1^X}{2\sqrt{\alpha_0}(\mu_2^X)^{1/2}} \mathcal{K}_{\varphi\varphi}(f_1) & \frac{1}{4\alpha_0} \mathcal{I}_{\psi_2}(f_1) & \frac{\mu_2^X}{4\alpha_0(\mu_4^X)^{1/2}} \mathcal{K}_{\varphi\psi_1}(f_1) & \frac{\mu_4(f_1)}{4} \mathcal{I}_{\psi_2}(f_1) \end{pmatrix}. \quad (11)$$

Applying Le Cam's Third Lemma under $P_{\theta+n^{-1/2}\nu^{(n)}\tau;f_1,h_1}^{(n)}$ yields:

$$\Delta_{f_1}(\theta) \xrightarrow{\mathcal{L}} \mathcal{N}(\Gamma_{f_1}(\theta)\tau, \Gamma_{f_1}(\theta)). \quad (12)$$

The central sequences satisfy the following asymptotic linearity property:

$$\Delta_{f_1}^{(n)}(\theta + n^{-1/2}\nu^{(n)}\tau) - \Delta_{f_1}^{(n)}(\theta) = -\Gamma_{f_1}^{(n)}(\theta)\tau + o_p(1), \quad (13)$$

for all τ under $P_{\theta;f_1}^{(n)}$, as $n \rightarrow \infty$.

Moreover, ULAN implies LAN and guaranties the asymptotic linearity of the central sequences.

For subsequences satisfying (B.3), $\Gamma_{f_1}(\theta)$ can replace $\Gamma_{f_1}^{(n)}(\theta)$. The non-diagonal form of $\Gamma_{f_1}^{(n)}(\theta)$ in (8) suggests that the nuisance parameters μ , β and α_0 are generally not orthogonal to the parameters of interest σ_ξ^2 and α_1 . To address this, it is necessary to project $\Delta_{f_1;4}^{(n)}$ and $\Delta_{f_1;5}^{(n)}$ appropriately when designing optimal test statistics for $\mathcal{H}_0$. Additionally, density h_1 does not appear in the central sequence (7) or in the information matrix (8), and thus will not be incorporated into the optimal test statistics for $\mathcal{H}_0$.

Since h_1 enters the model only through the likelihood used to establish the ULAN property, and not through the resulting central sequence or test statistics, we now fix h_1 to the standard normal density $\mathcal{N}_1$, as in the Monte Carlo design of Section 4. The general condition on the pair (f_1, h_1) in Assumption C is kept only to specify the semiparametric alternative class $\mathcal{H}_1^{(n)}(f_1)$; it plays no role in the simulations or applications reported here.

3. Optimal parametric and pseudo-Gaussian tests

The null hypothesis $\sigma_\xi^2 = \alpha_1 = 0$ of the non-randomness of the regression coefficient and the absence of ARCH(1) effects in (1) can formally be written as:

$$\mathcal{H}_0^{(n)} := \bigcup_{g_1 \in \mathcal{F}_0} \mathcal{H}_0^{(n)}(g_1) := \bigcup_{g_1 \in \mathcal{F}_0} \bigcup_{\mu \in \mathbb{R}} \bigcup_{\beta \in \mathbb{R}} \bigcup_{\alpha_0 > 0} \left\{ P_{\mu, \beta, \alpha_0, 0, 0; g_1}^{(n)} \right\},$$

where the (standardized) density of the noise remains unspecified— a semiparametric hypothesis, thus. Parametric alternatives will be considered, of the form, for some fixed density $f_1 \in \mathcal{F}_A$,

$$\mathcal{H}_1^{(n)}(f_1) := \bigcup_{\mu \in \mathbb{R}} \bigcup_{\beta \in \mathbb{R}} \bigcup_{\alpha_0 > 0} \bigcup_{\sigma_\xi^2 > 0} \bigcup_{\alpha_1 \in]0, 1[} \bigcup_{h_1 \in \mathcal{F}_{C|f_1}} \left\{ P_{\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1; f_1, h_1}^{(n)} \right\}.$$

Before studying the semiparametric hypothesis $\mathcal{H}_0^{(n)}$, we first consider the parametric problem of testing $\mathcal{H}_0^{(n)}(f_1)$ against $\mathcal{H}_1^{(n)}(f_1)$.

3.1. Optimal parametric test

Assuming that the innovation density f_1 is known, an important consequence of the ULAN property is that the local experiments are

$$\left\{ P_{\theta+n^{-1/2}\nu^{(n)}\tau;f_1,h_1}^{(n)} / \tau \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}^+, h_1 \in \mathcal{F}_{\mathcal{C}|f_1} \right\}, \quad (14)$$

with $\theta = (\mu, \beta, \alpha_0, 0, 0)'$, converges to the Gaussian shift experiment given by:

$$\left\{ \mathcal{N}(\Gamma_{f_1}(\theta)\tau; \Gamma_{f_1}(\theta)) / \tau \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \right\}. \quad (15)$$

Therefore, the locally optimal test is based on the residual obtained by projecting $(\Delta_4, \Delta_5)'$ onto the nuisance components $(\Delta_1, \Delta_2, \Delta_3)'$ based on Γ (see, e.g., Chapter 11 in [24]). That residual takes the form

$$\begin{pmatrix} \Delta_{f_1;4}^{*(n)}(\theta) \\ \Delta_{f_1;5}^{*(n)}(\theta) \end{pmatrix} := \begin{pmatrix} \Delta_{f_1;4}^{(n)}(\theta) \\ \Delta_{f_1;5}^{(n)}(\theta) \end{pmatrix} - \begin{pmatrix} \Gamma_{f_1;14} & \Gamma_{f_1;24} & \Gamma_{f_1;34} \\ \Gamma_{f_1;15} & \Gamma_{f_1;25} & \Gamma_{f_1;35} \end{pmatrix} \begin{pmatrix} \Gamma_{f_1;11} & \Gamma_{f_1;12} & \Gamma_{f_1;13} \\ \Gamma_{f_1;12} & \Gamma_{f_1;22} & \Gamma_{f_1;23} \\ \Gamma_{f_1;13} & \Gamma_{f_1;23} & \Gamma_{f_1;33} \end{pmatrix}^{-1} \begin{pmatrix} \Delta_{f_1;1}^{(n)}(\theta) \\ \Delta_{f_1;2}^{(n)}(\theta) \\ \Delta_{f_1;3}^{(n)}(\theta) \end{pmatrix}. \quad (16)$$

$(\Delta_{f_1;4}^{*(n)}(\theta), \Delta_{f_1;5}^{*(n)}(\theta))'$ is asymptotically normal under $P_{\theta;f_1}^{(n)}$ with mean zero and asymptotic variance

$$\Gamma_{f_1}^*(\theta) := \begin{pmatrix} \Gamma_{f_1;44} & \Gamma_{f_1;45} \\ \Gamma_{f_1;45} & \Gamma_{f_1;55} \end{pmatrix} - \begin{pmatrix} \Gamma_{f_1;14} & \Gamma_{f_1;24} & \Gamma_{f_1;34} \\ \Gamma_{f_1;15} & \Gamma_{f_1;25} & \Gamma_{f_1;35} \end{pmatrix} \begin{pmatrix} \Gamma_{f_1;11} & \Gamma_{f_1;12} & \Gamma_{f_1;13} \\ \Gamma_{f_1;12} & \Gamma_{f_1;22} & \Gamma_{f_1;23} \\ \Gamma_{f_1;13} & \Gamma_{f_1;23} & \Gamma_{f_1;33} \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_{f_1;14} & \Gamma_{f_1;15} \\ \Gamma_{f_1;24} & \Gamma_{f_1;25} \\ \Gamma_{f_1;34} & \Gamma_{f_1;35} \end{pmatrix}. \quad (17)$$

We have the following:

$$\begin{aligned} \begin{pmatrix} \Delta_{f_1;4}^{*(n)}(\theta) \\ \Delta_{f_1;5}^{*(n)}(\theta) \end{pmatrix} &\stackrel{\mathcal{H}_0}{\sim} \mathcal{N}(0; \Gamma_{f_1}^*(\theta)) \\ &\stackrel{\mathcal{H}_1}{\sim} \mathcal{N}(\Gamma_{f_1}^*(\theta)\tau_{4,5}; \Gamma_{f_1}^*(\theta)), \text{ where } \tau_{4,5} = (\tau_4, \tau_5)'. \end{aligned} \quad (18)$$

When a single parameter of interest is involved (e.g., σ_ξ^2 alone, as in Fihri et al., 2020, or α_1 alone, as in Lee and King, 1993), the residual $\Delta_{f_1;k}^{*(n)}(\theta)$ obtained by projecting out the nuisance scores is already scalar, and the resulting test statistic follows directly by standardisation. In the present joint setting, however, two parameters of interest are involved simultaneously, so the projected residual $(\Delta_{f_1;4}^{*(n)}, \Delta_{f_1;5}^{*(n)})'$ in (16) is bivariate. No test based on this residual can be uniformly most powerful against the full two-dimensional alternative $(\sigma_\xi^2, \alpha_1) \in \mathbb{R}^+ \times]0, 1[$, since optimal directions differ across alternatives. Following Schaafsma and Smid (1966) [35], we therefore construct the scalar statistic $T_{f_1}^{*(n)}(\theta)$ in (19) as the projection of $(\Delta_{f_1;4}^{*(n)}, \Delta_{f_1;5}^{*(n)})'$ onto a fixed direction δ , calibrated by the inverse of the joint asymptotic covariance $\Gamma_{f_1}^*(\theta)$ so as to weight each component by its own information content, rather than averaging them.

Therefore, a test that is locally uniformly asymptotically most powerful for the sequence $\{P_{\mu,\beta,\alpha_0,0,0;f_1}^{(n)}\}$ of the null hypotheses, against the alternatives $\{P_{\mu,\beta,\alpha_0,\sigma_\xi^2,\alpha_1;f_1,h_1}^{(n)} | \sigma_\xi^2 > 0, \alpha_1 \in]0, 1[, h_1 \in \mathcal{F}_{\mathcal{C}|f_1}\}$, is based on (see [36]):

$$T_{f_1}^{*(n)}(\theta) := \frac{\delta'(\Gamma_{f_1}^*(\theta))^{-1} \begin{pmatrix} \Delta_{f_1;4}^{*(n)}(\theta) \\ \Delta_{f_1;5}^{*(n)}(\theta) \end{pmatrix}}{\sqrt{\delta'(\Gamma_{f_1}^*(\theta))^{-1}\delta}}, \quad (19)$$

where, $\delta = (\delta_1, \delta_2)'$ and $\delta_i (i = 1, 2)$ denotes the square root of the i^{th} diagonal element of $\Gamma_{f_1}^*(\theta)$.

Remark 5 (Scalar nature of the test statistic)

Note that $T_{f_1}^{*(n)}(\theta)$, defined in (19), is a scalar quantity rather than a bivariate statistic. Here, $\delta = (\delta_1, \delta_2)'$ is a fixed weight vector in $\mathbb{R}^2$, where δ_i denotes the square root of the i -th diagonal element of $\Gamma_{f_1}^*(\theta)$. The numerator $\delta'(\Gamma_{f_1}^*(\theta))^{-1} \begin{pmatrix} \Delta_{f_1;4}^{*(n)}(\theta) \\ \Delta_{f_1;5}^{*(n)}(\theta) \end{pmatrix} \in \mathbb{R}$ is a linear functional mapping the bivariate vector $(\Delta_{f_1;4}^{*(n)}, \Delta_{f_1;5}^{*(n)})'$ to $\mathbb{R}$, and the denominator $\sqrt{\delta'(\Gamma_{f_1}^*(\theta))^{-1}\delta}$ is a positive normalizing constant. Hence, $T_{f_1}^{*(n)}(\theta)$ is scalar-valued, which justifies comparison with the standard normal quantile z_α .

The vector δ represents the direction of the alternative hypothesis, following the idea of the most stringent testing anywhere most powerful (MSSMP) of Schaafsma & Smid [35] and Shi & Kudo [36]. Specifically, the choice $\delta_i = \sqrt{[\Gamma_{f_1}^*(\theta)]_{ii}}$ ensures that the test is calibrated to give equal weight to the two components of interest, the random coefficient variance σ_ξ^2 and the ARCH parameter α_1 relative to their respective Fisher information. Under $\mathcal{H}_0^{(n)}(f_1)$, from (18) that $T_{f_1}^{*(n)}(\theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$, as $n \rightarrow \infty$, which confirms that the asymptotic level α is achieved by rejecting $\mathcal{H}_0^{(n)}(f_1)$ whenever $T_{f_1}^{*(n)}(\hat{\theta}^{(n)}) > z_\alpha$.

The weight vector $\delta = (\sqrt{\Gamma_{f_1;44}^*}, \sqrt{\Gamma_{f_1;55}^*})'$ used in the definition of $T_{f_1}^{*(n)}$ is not chosen arbitrarily. Among all fixed-direction tests of the form ϕ_δ based on $\delta'(\Gamma_{f_1}^*)^{-1}(\Delta_4^*, \Delta_5^*)' / \sqrt{\delta'(\Gamma_{f_1}^*)^{-1}\delta}$, this is precisely the direction singled out as optimal by the most stringent somewhere most powerful (MSSMP) criterion of Schaafsma and Smid [35] (see also Shi and Kudo [36] for the multivariate normal case with orthant-restricted alternative): it is the unique choice, up to scale, that equalises the test's local power against the two boundary alternatives $(\tau_4, 0)$ and $(0, \tau_5)$ taken separately. This is because no fixed direction δ can be simultaneously optimal against both boundary alternatives: the most powerful test against $(\tau_4, 0)$ rejects for large values of $\delta'(\Gamma_{f_1}^*)^{-1}(\Delta_4^*, \Delta_5^*)'$ with δ proportional to the first column of $\Gamma_{f_1}^*$, and symmetrically for $(0, \tau_5)$. The MSSMP construction is therefore the natural way to combine the two marginal tests without favouring either source of heterogeneity.

Since θ is unknown, it is replaced by an estimator $\hat{\theta}^{(n)} = (\hat{\mu}^{(n)}, \hat{\beta}^{(n)}, \hat{\alpha}_0^{(n)}, 0, 0)'$ satisfying the following assumptions. We then show that:

$$T_{f_1}^{*(n)}(\theta) - T_{f_1}^{*(n)}(\hat{\theta}^{(n)}) = o_p(1) \text{ under } P_{\theta, f_1}^{(n)}, \text{ as } n \rightarrow \infty. \quad (20)$$

Assumption D:

(D.1) $n^{1/2}\nu^{(n)-1}(\hat{\theta}^{(n)} - \theta) = o_p(1)$ under $P_{\mu, \beta, \alpha_0, 0, 0; f_1}^{(n)}$, as $n \rightarrow \infty$;

(D.2) the estimator $\hat{\theta}^{(n)}$ is locally asymptotically discrete, which means that within a ball of radius r centered at the origin, the number of possible values for $\hat{\theta}^{(n)}$ in shrinking ellipsoids of the form $\theta + \nu^{(n)}\mathcal{B}(r)$ is finite as n tends to infinity.

Assumption (D.1) ensures consistency at the required convergence rate under the null hypothesis. Similarly, assumption (D.2) is standard in asymptotic theory. In fact, any estimator that satisfies assumption (D.1) can be converted into one that satisfies both (D.1) – (D.2) by discretizing its three components in mesh grids $cn^{-1/2}$, $c\nu_2^{(n)}$, and $cn^{-1/2}$, respectively (where $c > 0$ is arbitrary).

Under $\mathcal{H}_0 : \sigma_\xi^2 = \alpha_1 = 0$, the model reduces to the linear regression $Y_i = \mu + \beta X_i + \varepsilon_i$ with i.i.d. errors of variance α_0 . We therefore estimate (μ, β) by ordinary least squares, compute the residuals $\hat{\varepsilon}_i = Y_i - \hat{\mu} - \hat{\beta}X_i$, and estimate α_0 by $\hat{\alpha}_0 = n^{-1} \sum_{i=1}^n \hat{\varepsilon}_i^2$. The standardized residuals used in the test statistics are then $\hat{Z}_i = \hat{\alpha}_0^{-1/2} \hat{\varepsilon}_i$. These estimators are $\sqrt{n}$ -consistent under $\mathcal{H}_0$, so they satisfy (D.1), and after the discretization described above they also satisfy (D.2).

Given assumption (D.1), in the definition of $T_{f_1}^{*(n)}$, the continuous mapping theorem implies that replacing $\Gamma_{f_1}^{*(n)}(\theta)$ with $\Gamma_{f_1}^{*(n)}(\hat{\theta}^{(n)})$ only results in an $o_p(1)$ impact.

Recall that ULAN implies the asymptotic relative linearity in (13) for all $\tau^{(n)}$, due to assumptions (D.1) and (D.2), which leads to the following result under $P_{\theta;f_1}^{(n)}$, as $n \rightarrow \infty$:

$$\begin{aligned}\Delta_{f_1}^{(n)}(\theta) - \Delta_{f_1}^{(n)}(\hat{\theta}^{(n)}) &= \Gamma_{f_1}(\theta)\tau^{(n)} + o_p(1) \\ &= \Gamma_{f_1}(\theta)n^{1/2}\nu^{(n)-1}(\hat{\theta}^{(n)} - \theta) + o_p(1).\end{aligned}\quad (21)$$

Specifically, this results in:

$$\begin{aligned}\Delta_{f_1}^{(n)}(\theta) - \Delta_{f_1}^{(n)}(\hat{\theta}^{(n)}) &= \Gamma_{f_1}(\theta)n^{1/2}\nu^{(n)-1} \begin{pmatrix} \hat{\mu}^{(n)} - \mu \\ \hat{\beta}^{(n)} - \beta \\ \hat{\alpha}_0 - \alpha_0 \\ 0 \\ 0 \end{pmatrix} + o_p(1), \\ &= n^{1/2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \\ \Gamma_{14} & \Gamma_{24} & \Gamma_{34} \\ \Gamma_{15} & \Gamma_{25} & \Gamma_{35} \end{pmatrix} \begin{pmatrix} \hat{\mu}^{(n)} - \mu \\ (M_2^{(n)})^{1/2}(\hat{\beta}^{(n)} - \beta) \\ \hat{\alpha}_0 - \alpha_0 \end{pmatrix} + o_p(1),\end{aligned}\quad (22)$$

and similarly:

$$\begin{pmatrix} \Delta_{f_1;1}^{(n)}(\theta) \\ \Delta_{f_1;2}^{(n)}(\theta) \\ \Delta_{f_1;3}^{(n)}(\theta) \end{pmatrix} - \begin{pmatrix} \Delta_{f_1;1}^{(n)}(\hat{\theta}) \\ \Delta_{f_1;2}^{(n)}(\hat{\theta}) \\ \Delta_{f_1;3}^{(n)}(\hat{\theta}) \end{pmatrix} = n^{1/2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{pmatrix} \begin{pmatrix} \hat{\mu}^{(n)} - \mu \\ (M_2^{(n)})^{1/2}(\hat{\beta}^{(n)} - \beta) \\ \hat{\alpha}_0 - \alpha_0 \end{pmatrix} + o_p(1),$$

and

$$\begin{pmatrix} \Delta_{f_1;4}^{(n)}(\theta) \\ \Delta_{f_1;5}^{(n)}(\theta) \end{pmatrix} - \begin{pmatrix} \Delta_{f_1;4}^{(n)}(\hat{\theta}) \\ \Delta_{f_1;5}^{(n)}(\hat{\theta}) \end{pmatrix} = n^{1/2} \begin{pmatrix} \Gamma_{14} & \Gamma_{24} & \Gamma_{34} \\ \Gamma_{15} & \Gamma_{25} & \Gamma_{35} \end{pmatrix} \begin{pmatrix} \hat{\mu}^{(n)} - \mu \\ (M_2^{(n)})^{1/2}(\hat{\beta}^{(n)} - \beta) \\ \hat{\alpha}_0 - \alpha_0 \end{pmatrix} + o_p(1).$$

For the test statistic $T_{f_1}^{*(n)}(\theta)$, using the linearity result above, we have:

$$T_{f_1}^{*(n)}(\theta) - T_{f_1}^{*(n)}(\hat{\theta}) = \frac{1}{\sqrt{\delta'(\Gamma_{f_1}^*)^{-1}\delta}} \delta'(\Gamma_{f_1}^*)^{-1} \left[\begin{pmatrix} \Delta_{f_1;4}^*(\theta) \\ \Delta_{f_1;5}^*(\theta) \end{pmatrix} - \begin{pmatrix} \Delta_{f_1;4}^*(\hat{\theta}) \\ \Delta_{f_1;5}^*(\hat{\theta}) \end{pmatrix} \right].$$

Substituting the definition of $\Delta_{f_1}^*$ in terms of Δ_{f_1} and the projected nuisance components, this becomes

$$\begin{aligned}T_{f_1}^{*(n)}(\theta) - T_{f_1}^{*(n)}(\hat{\theta}) &= \frac{1}{\sqrt{\delta'(\Gamma_{f_1}^*)^{-1}\delta}} \delta'(\Gamma_{f_1}^*)^{-1} \left[\begin{pmatrix} \Delta_{f_1;4}(\theta) \\ \Delta_{f_1;5}(\theta) \end{pmatrix} - \begin{pmatrix} \Delta_{f_1;4}(\hat{\theta}) \\ \Delta_{f_1;5}(\hat{\theta}) \end{pmatrix} \right. \\ &\quad \left. - \begin{pmatrix} \Gamma_{14} & \Gamma_{24} & \Gamma_{34} \\ \Gamma_{15} & \Gamma_{25} & \Gamma_{35} \end{pmatrix} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{pmatrix}^{-1} \left[\begin{pmatrix} \Delta_{f_1;1}(\theta) \\ \Delta_{f_1;2}(\theta) \\ \Delta_{f_1;3}(\theta) \end{pmatrix} - \begin{pmatrix} \Delta_{f_1;1}(\hat{\theta}) \\ \Delta_{f_1;2}(\hat{\theta}) \\ \Delta_{f_1;3}(\hat{\theta}) \end{pmatrix} \right] \right].\end{aligned}$$

Using the linear expansions of $\Delta_{f_1;j}(\boldsymbol{\theta}) - \Delta_{f_1;j}(\hat{\boldsymbol{\theta}})$ established above, this reduces to

$$\begin{aligned} T_{f_1}^{*(n)}(\boldsymbol{\theta}) - T_{f_1}^{*(n)}(\hat{\boldsymbol{\theta}}) &= \frac{n^{1/2}}{\sqrt{\boldsymbol{\delta}'(\boldsymbol{\Gamma}_{f_1}^*)^{-1}\boldsymbol{\delta}}} \boldsymbol{\delta}'(\boldsymbol{\Gamma}_{f_1}^*)^{-1} \left[\begin{pmatrix} \Gamma_{14} & \Gamma_{24} & \Gamma_{34} \\ \Gamma_{15} & \Gamma_{25} & \Gamma_{35} \end{pmatrix} \begin{pmatrix} \hat{\mu} - \mu \\ (M_2^{(n)})^{1/2}(\hat{\beta} - \beta) \\ \hat{\alpha}_0 - \alpha_0 \end{pmatrix} \right. \\ &\quad \left. - \begin{pmatrix} \Gamma_{14} & \Gamma_{24} & \Gamma_{34} \\ \Gamma_{15} & \Gamma_{25} & \Gamma_{35} \end{pmatrix} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{pmatrix} \begin{pmatrix} \hat{\mu} - \mu \\ (M_2^{(n)})^{1/2}(\hat{\beta} - \beta) \\ \hat{\alpha}_0 - \alpha_0 \end{pmatrix} + o_p(1) \right]. \end{aligned}$$

Since the two matrix products inside the brackets cancel exactly, we obtain

$$T_{f_1}^{*(n)}(\boldsymbol{\theta}) - T_{f_1}^{*(n)}(\hat{\boldsymbol{\theta}}) = 0 + o_p(1).$$

Applying Le Cam's Third Lemma yields the following result.

Proposition 3.1

Suppose that Assumptions (B.1)–(B.2) and (C.1)–(C.2) hold, and suppose that the estimators $\hat{\boldsymbol{\theta}}^{(n)}$ satisfy Assumption (D.2), and fix $f_1 \in \mathcal{F}_A$. Then, the sequence of tests that rejects the null hypothesis $\mathcal{H}_0^{(n)}(f_1)$ whenever $T_{f_1}^{*(n)}(\hat{\boldsymbol{\theta}}^{(n)})$ exceeds the $(1 - \alpha)$ quantile of the standard normal distribution, denoted by z_α :

1. has an asymptotic level of α ;
2. is locally asymptotically most powerful unbiased (LAMPUB) in the sense of Le Cam [24, Chapter 11], at asymptotic level α , along (B.3)-subsequences, in a shrinking $n^{-1/2}$ -neighborhood of $\mathcal{H}_0^{(n)}(f_1)$, against alternatives of the form

$$\bigcup_{\mu \in \mathbb{R}} \bigcup_{\beta \in \mathbb{R}} \bigcup_{\alpha_0 > 0} \bigcup_{\sigma_\xi^2 > 0} \bigcup_{\alpha_1 \in [0, 1]} \bigcup_{h_1 \in \mathcal{F}_C|f_1} \left\{ P_{\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1; f_1, h_1}^{(n)} \right\}; \quad (23)$$

3. has asymptotic power given by $1 - \Phi\left(z_\alpha - \frac{\boldsymbol{\delta}'}{\sqrt{\boldsymbol{\delta}'(\boldsymbol{\Gamma}_{f_1}^*)^{-1}\boldsymbol{\delta}}} \tau_{4,5}\right)$ (still along (B.3)-subsequences), where

Φ denotes the standard normal cumulative distribution function, against alternatives of the form $P_{\boldsymbol{\theta} + n^{-1/2}\boldsymbol{\nu}^{(n)}\boldsymbol{\tau}; f_1, h_1}^{(n)}$.

Proof of Proposition 3.1

Step 1: (Asymptotic level). Under $\mathcal{H}_0^{(n)}(f_1)$, the statistic $T_{f_1}^{*(n)}$ is a scalar, and asymptotically standard normal: $\boldsymbol{\delta}'(\boldsymbol{\Gamma}_{f_1}^*)^{-1}$ maps the bivariate vector $(\Delta_{f_1;4}^{*(n)}, \Delta_{f_1;5}^{*(n)})'$ to $\mathbb{R}$, and the denominator normalizes the variance to one. The asymptotic normality $T_{f_1}^{*(n)}(\hat{\boldsymbol{\theta}}^{(n)}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$ then follows from (18) and the ULAN expansion in Proposition 2.1, combined with the rate condition (20). The asymptotic level α follows immediately.

Step 2: LAMPUB property. The residual in $(\Delta_{f_1;4}^{*(n)}, \Delta_{f_1;5}^{*(n)})'$ in (16) is the component of $(\Delta_{f_1;4}^{(n)}, \Delta_{f_1;5}^{(n)})'$ orthogonal to the nuisance scores $(\Delta_{f_1;1}^{(n)}, \Delta_{f_1;2}^{(n)}, \Delta_{f_1;3}^{(n)})'$. In the limiting Gaussian experiment, the optimal unbiased level- α test for $\tau_4 = \tau_5 = 0$ against $\tau_4 > 0, \tau_5 > 0$ is based on the one-sided projection $\boldsymbol{\delta}'(\boldsymbol{\Gamma}_{f_1}^*)^{-1}(\Delta_{f_1;4}^{*(n)}, \Delta_{f_1;5}^{*(n)})'$ in the direction $\boldsymbol{\delta}$; see [24, Chapter 11] and [35]. Local asymptotic equivalence extends this optimality to the original experiment.

Step 3: (Asymptotic power). Under contiguous alternatives, Le Cam's Third Lemma modifies the limiting distribution as follows:

$$T_{f_1}^{*(n)}(\hat{\boldsymbol{\theta}}^{(n)}) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\frac{\boldsymbol{\delta}'\tau_{4,5}}{\sqrt{\boldsymbol{\delta}'(\boldsymbol{\Gamma}_{f_1}^*)^{-1}\boldsymbol{\delta}}}, 1\right) \quad \text{under } P_{\boldsymbol{\theta} + n^{-1/2}\boldsymbol{\nu}^{(n)}\boldsymbol{\tau}; f_1, h_1}^{(n)},$$

which yields the asymptotic power expression in point 3. $\square$

Remark 6

Proposition 3.1 does not claim the existence of a uniformly most powerful test against the composite alternative (23), which is known not to exist in general. The optimality considered here is local and asymptotic, which is the appropriate notion under the ULAN framework, as commonly adopted in the ULAN literature [17, 1, 4]

An important special case is the Gaussian version ($f_1 = \mathcal{N}_1$), equations (7) and (11) reduce to:

$$\Delta_{\mathcal{N}_1}^{(n)}(\theta) = \begin{pmatrix} \frac{an^{-1/2}}{\sqrt{\alpha_0}} \sum_{i=1}^n Z_i \\ \frac{a\nu_2^{(n)}}{\sqrt{\alpha_0}} \sum_{i=1}^n Z_i X_i \\ \frac{an^{-1/2}}{2\alpha_0} \sum_{i=1}^n (Z_i^2 - \frac{1}{a}) \\ \frac{a^2\nu_4^{(n)}}{2\alpha_0} \sum_{i=1}^n (Z_i^2 - \frac{1}{a}) X_i^2 \\ \frac{an^{-1/2}}{2} \sum_{i=1}^n Z_{i-1}^2 (Z_i^2 - \frac{1}{a}) \end{pmatrix}, \quad (24)$$

and

$$\Gamma_{\mathcal{N}_1}^{(n)}(\theta) = \begin{pmatrix} \frac{a}{\alpha_0} & \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & 0 & 0 & 0 \\ \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & \frac{a}{\alpha_0} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\alpha_0^2} & \frac{aM_2^{(n)}}{2\alpha_0^2(M_4^{(n)})^{1/2}} & \frac{1}{2\alpha_0} \\ 0 & 0 & \frac{aM_2^{(n)}}{2\alpha_0^2(M_4^{(n)})^{1/2}} & \frac{a^2}{2\alpha_0^2} & \frac{aM_2^{(n)}}{2\alpha_0(M_4^{(n)})^{1/2}} \\ 0 & 0 & \frac{1}{2\alpha_0} & \frac{aM_2^{(n)}}{2\alpha_0(M_4^{(n)})^{1/2}} & \frac{3}{2a^2} \end{pmatrix}. \quad (25)$$

3.2. Pseudo-Gaussian test

The Gaussian versions of equations (16) and (17) (in addition to the calculus of (24) and (25)), are given by:

$$\Delta_{\mathcal{N}_1}^{*(n)}(\theta) = \begin{pmatrix} \Delta_{\mathcal{N}_{1;4}}^{*(n)}(\theta) \\ \Delta_{\mathcal{N}_{1;5}}^{*(n)}(\theta) \end{pmatrix} = \begin{pmatrix} \frac{a^2\nu_4^{(n)}}{2\alpha_0} \sum_{i=1}^n Z_i^2 (X_i^2 - M_2^{(n)}) \\ \frac{an^{-1/2}}{2} \sum_{i=1}^n (Z_{i-1}^2 - \frac{1}{a})(Z_i^2 - \frac{1}{a}) \end{pmatrix}, \quad (26)$$

and

$$\Gamma_{\mathcal{N}_1}^{*(n)}(\theta) = \begin{pmatrix} \frac{a^2}{2\alpha_0^2} \left(1 - \frac{M_2^{(n)^2}}{M_4^{(n)}}\right) & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix}. \quad (27)$$

Remark 7 (Orthogonality and reduction to an equally-weighted combination)

Note that $\Gamma_{\mathcal{N}_1}^{*(n)}(\theta)$ in (27) is *diagonal*: at the null, the random-coefficient score $\Delta_{\mathcal{N}_{1;4}}^{*(n)}$ and the ARCH score $\Delta_{\mathcal{N}_{1;5}}^{*(n)}$ are asymptotically orthogonal under the Gaussian reference density. Substituting a diagonal Γ^* and $\delta_i = \sqrt{[\Gamma^*]_{ii}}$ into the general MSSMP statistic (19) gives, after simplification,

$$T_{\mathcal{N}_1}^{*(n)}(\theta) = \frac{1}{\sqrt{2}} \left(\frac{\Delta_{\mathcal{N}_{1;4}}^{*(n)}(\theta)}{\sqrt{\Gamma_{\mathcal{N}_{1;44}}^{*(n)}(\theta)}} + \frac{\Delta_{\mathcal{N}_{1;5}}^{*(n)}(\theta)}{\sqrt{\Gamma_{\mathcal{N}_{1;55}}^{*(n)}(\theta)}} \right),$$

i.e., an equally-weighted sum of the two already-standardized (unit asymptotic variance) marginal statistics. This is not an ad hoc simplification of the MSSMP direction: it is its exact form whenever Γ^* is diagonal. Equal

weighting works here because orthogonality removes any information asymmetry between the two components that the MSSMP weights would otherwise have to correct for. When $\mathbf{\Gamma}^*$ is not diagonal, as for a general f_1 in the fully parametric test of Section 3.1, the weights differ from $1/\sqrt{2}$ and the resulting statistic is a genuinely non-trivial weighted projection rather than a simple average.

The Gaussian central sequence $\Delta_{\mathcal{N}_1}^{*(n)}(\theta)$ in (26) produces optimal asymptotic tests under $g_1 = \mathcal{N}_1$. Define

$$\begin{aligned}\bar{Z}^{(n)} &= \frac{1}{n} \sum_{i=1}^n Z_i^{(n)}; \mu_2(g_1) = E_{g_1}(Z_i^2) = \int_{\mathbb{R}} z^2 g_1(z) dz; \mu_4(g_1) = E_{g_1}(Z_i^4) = \int_{\mathbb{R}} z^4 g_1(z) dz; \\ m_2^{(n)} &= \frac{1}{n} \sum_{i=1}^n Z_i^2 = \hat{\mu}_2(g_1); m_4^{(n)} = \frac{1}{n} \sum_{i=1}^n Z_i^4 = \hat{\mu}_4(g_1).\end{aligned}$$

We show that the Gaussian optimal test remains valid beyond the Gaussian assumption. This is obtained through a simple modification in (26) leading to a pseudo-Gaussian test, which remains valid when the actual density $g_1 \in \mathcal{F}_A^2$, where $\mathcal{F}_A^2$ denotes the class of all densities $g_1 \in \mathcal{F}_A$ such that $\mu_4(g_1) < \infty$. Define

$$\begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{\bullet(n)}(\theta) \\ \Delta_{\mathcal{N}_1;5}^{\bullet(n)}(\theta) \end{pmatrix} = \begin{pmatrix} \frac{a^2 \nu_4^{(n)}}{2\alpha_0} \sum_{i=1}^n (Z_i^2 - m_2^{(n)}) (X_i^2 - M_2^{(n)}) \\ \frac{an^{-1/2}}{2} \sum_{i=1}^n (Z_{i-1}^2 - m_2^{(n)}) (Z_i^2 - m_2^{(n)}) \end{pmatrix}. \quad (28)$$

Using $m_2^{(n)} = \frac{1}{a} + o_p(1)$ and $m_4^{(n)} = \frac{3}{a^2} + o_p(1)$, the variance of $\begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{\bullet(n)}(\theta) \\ \Delta_{\mathcal{N}_1;5}^{\bullet(n)}(\theta) \end{pmatrix}$ is consistently estimated under $P_{\theta;g_1}^{(n)}$ by

$$\mathbf{\Gamma}_{\mathcal{N}_1}^{\bullet(n)}(\theta) := \begin{pmatrix} \frac{a^4}{4\alpha_0^2} \left(1 - \frac{(M_2^{(n)})^2}{M_4^{(n)}}\right) [m_4^{(n)} - (m_2^{(n)})^2] & 0 \\ 0 & \frac{a^2}{4} [m_4^{(n)} - (m_2^{(n)})^2]^2 \end{pmatrix}, \quad (29)$$

which no longer depends on the unknown density g_1 and, along the subsequences of (B.3), converges to

$$\mathbf{\Gamma}_{\mathcal{N}_1,g_1}^{\bullet}(\theta) := \frac{a^2}{4} (\mu_4(g_1) - \mu_2^2(g_1)) \begin{pmatrix} \frac{a^2}{\alpha_0^2} \left(1 - \frac{(\mu_2^X)^2}{\mu_4^X}\right) & 0 \\ 0 & (\mu_4(g_1) - (\mu_2(g_1))^2) \end{pmatrix}. \quad (30)$$

Let $\Sigma_{\mathcal{N}_1,g_1}$ denote the covariance matrix between $(\Delta_4^{\bullet(n)}, \Delta_5^{\bullet(n)})$ and $(\Delta_{g_1;4}^{(n)}, \Delta_{g_1;5}^{(n)})$, given by

$$\begin{aligned}\Sigma_{\mathcal{N}_1,g_1}(\theta) &= \begin{pmatrix} E_{g_1}(\Delta_{\mathcal{N}_1;4}^{\bullet(n)} \Delta_{g_1;4}^{(n)}) & E_{g_1}(\Delta_{\mathcal{N}_1;4}^{\bullet(n)} \Delta_{g_1;5}^{(n)}) \\ E_{g_1}(\Delta_{\mathcal{N}_1;5}^{\bullet(n)} \Delta_{g_1;4}^{(n)}) & E_{g_1}(\Delta_{\mathcal{N}_1;5}^{\bullet(n)} \Delta_{g_1;5}^{(n)}) \end{pmatrix}, \\ &= \begin{pmatrix} \frac{a^2}{2\alpha_0^2} \left(1 - \frac{(M_2^{(n)})^2}{M_4^{(n)}}\right) & 0 \\ 0 & \frac{a}{4} (\mu_4(g_1) - \mu_2^2(g_1)) [E_{g_1}(Z_i^3 \varphi_{g_1}(Z_i)) - \mu_2(g_1)] \end{pmatrix}, \\ &= \begin{pmatrix} \frac{a^2}{2\alpha_0^2} \left(1 - \frac{(M_2^{(n)})^2}{M_4^{(n)}}\right) & 0 \\ 0 & \frac{a}{2} \mu_2(g_1) [\mu_4(g_1) - \mu_2^2(g_1)] \end{pmatrix},\end{aligned} \quad (31)$$

since $E_{g_1}(Z_i^3 \varphi_{g_1}(Z_i)) = \int x^3 g_1'(x) dx = 3 \int x^2 g_1(x) dx = 3\mu_2(g_1)$.

Consequently under $P_{\theta;g_1}^{(n)}$, as $n \rightarrow \infty$, for any $g_1 \in \mathcal{F}_A^2$, let

$$T_{\mathcal{N}_1,g_1}^{\bullet(n)}(\theta) := \frac{\delta^{\bullet'} (\Sigma_{\mathcal{N}_1,g_1}(\theta))^{-1} \begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{\bullet(n)}(\theta) \\ \Delta_{\mathcal{N}_1;5}^{\bullet(n)}(\theta) \end{pmatrix}}{\sqrt{\delta^{\bullet'} \mathbf{\Lambda}^{\bullet} \delta^{\bullet}}}, \quad (32)$$

with $\Lambda^\bullet = (\Sigma_{\mathcal{N}_1, g_1}(\theta))^{-1} \Gamma_{\mathcal{N}_1}^\bullet(\theta) (\Sigma_{\mathcal{N}_1, g_1}^{-1}(\theta))^{-1}$ where, $\delta^\bullet = (\delta_1^\bullet, \delta_2^\bullet)'$, and $\delta_i^\bullet (i = 1, 2)$ denotes the square root of the i^{th} diagonal element of $\Lambda^{\bullet-1}$.

The test statistic under $P_{\theta; g_1}^{(n)}$ is given by the following expression:

$$T_{\mathcal{N}_1, g_1}^{\bullet(n)}(\theta) = \frac{\nu_4^{(n)}}{\sqrt{2[\mu_4(g_1) - \mu_2^2(g_1)] \left(1 - \frac{M_2^{(n)2}}{M_4^{(n)}}\right)}} \sum_{i=1}^n Z_i^2 (X_i^2 - M_2^{(n)}) + \frac{n^{-1/2} \sum_{i=1}^n (Z_i^2 - m_2^{(n)}) (Z_{i-1}^2 - m_2^{(n)})}{\sqrt{2} [\mu_4(g_1) - \mu_2^2(g_1)]}.$$

It is estimated by the following expression:

$$\hat{T}^{\bullet(n)}(\theta) = \frac{\nu_4^{(n)}}{\sqrt{2[m_4^{(n)} - m_2^{(n)2}] \left(1 - \frac{M_2^{(n)2}}{M_4^{(n)}}\right)}} \sum_{i=1}^n Z_i^2 (X_i^2 - M_2^{(n)}) + \frac{n^{-1/2} \sum_{i=1}^n (Z_i^2 - m_2^{(n)}) (Z_{i-1}^2 - m_2^{(n)})}{\sqrt{2} [m_4^{(n)} - m_2^{(n)2}]}$$

Since θ is unknown under the null hypothesis, it is replaced by its estimator $\hat{\theta}$. therefore, under $P_{\theta; g_1}^{(n)}$, for any $g_1 \in \mathcal{F}_{\mathcal{A}}^2$, $T_{\mathcal{N}_1}^{\bullet(n)}(\hat{\theta}^{(n)}) - T_{\mathcal{N}_1}^{\bullet(n)}(\theta)$ is $o_p(1)$. To establish this result, it suffices to show that $\Delta_{\mathcal{N}_1}^{*(n)}(\hat{\theta}^{(n)}) - \Delta_{\mathcal{N}_1}^{*(n)}(\theta)$ is $o_p(1)$.

Using the asymptotic linearity of $\Delta_{\mathcal{N}_1}^{(n)}(\theta)$ under $P_{\theta; g_1}^{(n)}$, for any $\tau = (\tau_1, \tau_2, \tau_3, 0, 0)'$ and any $g_1 \in \mathcal{F}_{\mathcal{A}}^2$, we obtain:

$$\Delta_{\mathcal{N}_1}^{(n)}(\theta + n^{-1/2} \nu^{(n)} \tau) - \Delta_{\mathcal{N}_1}^{(n)}(\theta) = -\Gamma_{\mathcal{N}_1; g_1}^{(n)}(\theta) \tau + o_p(1), \quad (33)$$

with

$$\Gamma_{\mathcal{N}_1; g_1}^{(n)}(\theta) = E_{g_1} [\Delta_{\mathcal{N}_1}(\theta) \Delta_{g_1}'(\theta)], \quad (34)$$

which gives the following matrix:

$$\Gamma_{\mathcal{N}_1; g_1}^{(n)}(\theta) = \begin{pmatrix} \frac{a}{\alpha_0} & \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & -\frac{a\mu_1}{2\alpha_0^{3/2}} & 0 & -\frac{a\mu_1\mu_2}{2\sqrt{\alpha_0}} \\ \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & \frac{a}{\alpha_0} & -\frac{a\mu_1\bar{X}^{(n)}}{2\alpha_0^{3/2}(M_2^{(n)})^{1/2}} & 0 & -\frac{a\mu_1\mu_2\bar{X}^{(n)}}{2\sqrt{\alpha_0}(M_2^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2}{2\alpha_0^2} & \frac{aM_2^{(n)}}{2\alpha_0^2(M_4^{(n)})^{1/2}} & \frac{a\mu_2^2}{2\alpha_0} \\ 0 & 0 & \frac{a^2\mu_2M_2^{(n)}}{2\alpha_0^2(M_4^{(n)})^{1/2}} & \frac{a^2}{2\alpha_0^2} & \frac{a\mu_2^2M_2^{(n)}}{2\alpha_0(M_4^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2^2}{2\alpha_0} & \frac{a\mu_2M_2^{(n)}}{2\alpha_0(M_4^{(n)})^{1/2}} & \frac{a\mu_2\mu_4}{2} \end{pmatrix}. \quad (35)$$

This implies that:

$$\Delta_{\mathcal{N}_1}^{(n)}(\hat{\theta}) - \Delta_{\mathcal{N}_1}^{(n)}(\theta) = - \begin{pmatrix} \frac{a}{\alpha_0} & \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & -\frac{a\mu_1}{2\alpha_0^{3/2}} \\ \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & \frac{a}{\alpha_0} & -\frac{a\mu_1\bar{X}^{(n)}}{2\alpha_0^{3/2}(M_2^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2}{2\alpha_0^2} \\ 0 & 0 & \frac{a^2\mu_2M_2^{(n)}}{2\alpha_0^2(M_4^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2^2}{2\alpha_0} \end{pmatrix} \begin{pmatrix} n^{1/2}(\hat{\mu} - \mu) \\ (\sum_{i=1}^n X_i^2)^{1/2}(\hat{\beta} - \beta) \\ n^{1/2}(\hat{\alpha}_0 - \alpha_0) \end{pmatrix} + o_p(1).$$

By isolating the components, we get:

$$\begin{pmatrix} \Delta_{\mathcal{N}_1;1}^{(n)}(\hat{\theta}) \\ \Delta_{\mathcal{N}_1;2}^{(n)}(\hat{\theta}) \\ \Delta_{\mathcal{N}_1;3}^{(n)}(\hat{\theta}) \end{pmatrix} - \begin{pmatrix} \Delta_{\mathcal{N}_1;1}^{(n)}(\theta) \\ \Delta_{\mathcal{N}_1;2}^{(n)}(\theta) \\ \Delta_{\mathcal{N}_1;3}^{(n)}(\theta) \end{pmatrix} = - \begin{pmatrix} \frac{a}{\alpha_0} & \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & -\frac{a\mu_1}{2\alpha_0^{3/2}} \\ \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & \frac{a}{\alpha_0} & -\frac{a\mu_1\bar{X}^{(n)}}{2\alpha_0^{3/2}(M_2^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2}{2\alpha_0^2} \end{pmatrix} \begin{pmatrix} n^{1/2}(\hat{\mu} - \mu) \\ (\sum_{i=1}^n X_i^2)^{1/2}(\hat{\beta} - \beta) \\ n^{1/2}(\hat{\alpha}_0 - \alpha_0) \end{pmatrix} + o_p(1).$$

Similarly, for the last two components:

$$\begin{aligned} \begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{*(n)}(\hat{\theta}) \\ \Delta_{\mathcal{N}_1;5}^{*(n)}(\hat{\theta}) \end{pmatrix} - \begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{*(n)}(\theta) \\ \Delta_{\mathcal{N}_1;5}^{*(n)}(\theta) \end{pmatrix} &= - \begin{pmatrix} 0 & 0 & \frac{a^2 \mu_2 M_2^{(n)}}{2\alpha_0^2 (M_4^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2}{2\alpha_0} \end{pmatrix} \begin{pmatrix} n^{1/2}(\hat{\mu} - \mu) \\ (\sum_{i=1}^n X_i^2)^{1/2}(\hat{\beta} - \beta) \\ n^{1/2}(\hat{\alpha}_0 - \alpha_0) \end{pmatrix} + o_p(1) \\ &= - \frac{a\mu_2}{2\alpha_0^2} \begin{pmatrix} \frac{aM_2^{(n)}}{(M_4^{(n)})^{1/2}} \\ \alpha_0 \mu_2 \end{pmatrix} n^{1/2}(\hat{\alpha}_0 - \alpha_0) + o_p(1). \end{aligned} \quad (36)$$

We now aim to show: $\Delta_{\mathcal{N}_1}^{*(n)}(\hat{\theta}) - \Delta_{\mathcal{N}_1}^{*(n)}(\theta) = o_p(1)$. For that, calculate:

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & \Gamma_{34} \\ 0 & 0 & \Gamma_{35} \end{pmatrix} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & 0 \\ \Gamma_{12} & \Gamma_{22} & 0 \\ 0 & 0 & \Gamma_{33} \end{pmatrix}^{-1} \left[\begin{pmatrix} \Delta_{\mathcal{N}_1;1}(\hat{\theta}) \\ \Delta_{\mathcal{N}_1;2}(\hat{\theta}) \\ \Delta_{\mathcal{N}_1;3}(\hat{\theta}) \end{pmatrix} - \begin{pmatrix} \Delta_{\mathcal{N}_1;1}(\theta) \\ \Delta_{\mathcal{N}_1;2}(\theta) \\ \Delta_{\mathcal{N}_1;3}(\theta) \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & 0 & \Gamma_{34}\Gamma_{33}^{-1} \\ 0 & 0 & \Gamma_{35}\Gamma_{33}^{-1} \end{pmatrix} \begin{pmatrix} \frac{a}{\alpha_0} & \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & \frac{-a\mu_1}{2\alpha_0^{3/2}} \\ \frac{a\bar{X}^{(n)}}{\alpha_0(M_2^{(n)})^{1/2}} & \frac{a}{\alpha_0} & \frac{-a\mu_1\bar{X}^{(n)}}{2\alpha_0^{3/2}(M_2^{(n)})^{1/2}} \\ 0 & 0 & \frac{a\mu_2}{2\alpha_0^2} \end{pmatrix} \begin{pmatrix} n^{1/2}(\hat{\mu} - \mu) \\ (\sum_{i=1}^n X_i^2)^{1/2}(\hat{\beta} - \beta) \\ n^{1/2}(\hat{\alpha}_0 - \alpha_0) \end{pmatrix} + o_p(1) \\ &= \frac{a\mu_2}{2\alpha_0^2} \begin{pmatrix} \frac{aM_2^{(n)}}{2\alpha_0^2(M_4^{(n)})^{1/2}} \times 2\alpha_0^2 \\ \frac{\mu_2}{2\alpha_0} \times 2\alpha_0^2 \end{pmatrix} n^{1/2}(\hat{\alpha}_0 - \alpha_0) + o_p(1) \\ &= \frac{a\mu_2}{2\alpha_0^2} \begin{pmatrix} \frac{aM_2^{(n)}}{(M_4^{(n)})^{1/2}} \\ \mu_2 \alpha_0 \end{pmatrix} n^{1/2}(\hat{\alpha}_0 - \alpha_0) + o_p(1). \end{aligned} \quad (37)$$

Since the expressions in equation (37) and (36) coincide, we can conclude that:

$$\begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{*(n)}(\hat{\theta}^{(n)}) \\ \Delta_{\mathcal{N}_1;5}^{*(n)}(\hat{\theta}^{(n)}) \end{pmatrix} - \begin{pmatrix} \Delta_{\mathcal{N}_1;4}^{*(n)}(\theta) \\ \Delta_{\mathcal{N}_1;5}^{*(n)}(\theta) \end{pmatrix} = o_p(1).$$

This implies that

$$T_{\mathcal{N}_1;g_1}^{\bullet(n)}(\hat{\theta}^{(n)}) - T_{\mathcal{N}_1;g_1}^{\bullet(n)}(\theta) = o_p(1). \quad (38)$$

In summary, we obtain the following result.

Proposition 3.2

Let $\hat{\theta}^{(n)}$ satisfy Assumptions (D), and assume that Assumptions (B) and (C) hold for any $g_1 \in \mathcal{F}_A^2$. Then,

1. for any $\theta = (\mu, \beta, \alpha_0, 0, 0)'$, the test statistic $T_{\mathcal{N}_1}^{\bullet(n)}(\hat{\theta}^{(n)})$ is asymptotically normal, with mean zero under $P_{\theta;g_1}^{(n)}$, mean $\frac{1}{\sqrt{2}}\delta^{\bullet'}\tau_{4,5}$ under $P_{\theta+n^{-1/2}\nu^{(n)}\tau;g_1,h_1}^{(n)}$, and variance one in both cases;
2. the sequence of tests that reject the null hypothesis $\mathcal{H}_A^{(n)2} := \bigcup_{g_1 \in \mathcal{F}_A^2} \mathcal{H}_0^{(n)}(g_1)$ whenever $T_{\mathcal{N}_1,g_1}^{\bullet(n)}(\hat{\theta}^{(n)})$ exceeds the $(1 - \alpha)$ quantile of the standard normal distribution is locally asymptotically most powerful and unbiased at the asymptotic level α . This holds for $\mathcal{H}_A^{(n)2}$ against alternatives of the form

$$\bigcup_{\mu \in \mathbb{R}} \bigcup_{\beta \in \mathbb{R}} \bigcup_{\alpha_0 > 0} \bigcup_{\sigma_\xi^2 > 0} \bigcup_{\alpha_1 \in]0,1[} \bigcup_{h_1 \in \mathcal{F}_{C|\mathcal{N}_1}} \left\{ P_{\mu,\beta,\alpha_0,\sigma_\xi^2,\alpha_1;\mathcal{N}_1,h_1}^{(n)} \right\};$$

3. still along (B.3)-subsequences, this test has asymptotic power

$$1 - \Phi\left(z_\alpha - \frac{1}{\alpha_0} \left(\frac{1 - (\mu_2^X)^2 / \mu_4^X}{2(\mu_4(g_1) - (\mu_2(g_1))^2)} \right)^{1/2} \tau_4 - \frac{1}{\sqrt{2}} \mu_2(g_1) \tau_5 \right)$$

where, as usual, Φ denotes the standard normal distribution function, against alternatives of the form $P_{\theta+n^{-1/2}\nu^{(n)}\tau;g_1,h_1}^{(n)}$.

If $\alpha_1 = 0$, the model simplifies to a random coefficient regression model:

$$Y_i = \mu + (\beta + \xi_i)X_i + \varepsilon_i,$$

where ε_i , for $i = 1, \dots, n$, is an i.i.d. sequence of unobserved error terms with variance $\alpha_0 > 0$. This model was introduced by [17]. For the null hypothesis $\sigma_\xi^2 = 0$ and the alternative $\sigma_\xi^2 > 0$, the test statistic for this model is given by:

$$T_F^{(n)}(\hat{\theta}) = \frac{\nu_4^{(n)}}{\sqrt{(m_4^{(n)} - m_2^{(n)2})\left(1 - \frac{M_2^{(n)2}}{M_4^{(n)}}\right)}} \sum_{i=1}^n Z_i^2 (X_i^2 - M_2^{(n)}).$$

Conversely, if $\sigma_\xi^2 = 0$, the model reduces to a regression with ARCH(1) errors:

$$Y_i = \mu + \beta X_i + \varepsilon_i,$$

where ε_i , for $i = 1, \dots, n$, follows an ARCH(1) process defined as $\varepsilon_i = \sigma_i Z_i$ with conditional variance

$$\sigma_i^2 = \alpha_0 + \alpha_1 \varepsilon_{i-1}^2.$$

This model was studied by [25]. For the null hypothesis $\alpha_1 = 0$ and the alternative $\alpha_1 > 0$, the test statistic is given by:

$$T_{LK}^{(n)}(\hat{\theta}^{(n)}) = \frac{n^{-1/2}}{(m_4^{(n)} - m_2^{(n)2})} \sum_{i=1}^n (Z_i^2 - m_2^{(n)})(Z_{i-1}^2 - m_2^{(n)}).$$

The statistics $T_F^{(n)}$ and $T_{LK}^{(n)}$ above are precisely the marginal components $\Delta_{\mathcal{N}_1;4}^{\bullet(n)}/\sqrt{\Gamma_{\mathcal{N}_1;11}^{\bullet}}$ and $\Delta_{\mathcal{N}_1;5}^{\bullet(n)}/\sqrt{\Gamma_{\mathcal{N}_1;22}^{\bullet}}$ of the pseudo-Gaussian statistic $T_{\mathcal{N}_1,g_1}^{\bullet(n)}$ derived in Section 3.2, evaluated under $\alpha_1 = 0$ and $\sigma_\xi^2 = 0$ respectively. Since $\Gamma_{\mathcal{N}_1}^{\bullet}(\theta)$ (and its limit $\Gamma_{\mathcal{N}_1,g_1}^{\bullet}(\theta)$) is diagonal, Remark 7 applies directly: the MSSMP-optimal combination of these two orthogonal directions collapses exactly to their equally-weighted sum. We therefore define the joint test statistic as

$$T_{Al}(\hat{\theta}^{(n)}) = \frac{1}{\sqrt{2}} \left(T_F^{(n)}(\hat{\theta}^{(n)}) + T_{LK}^{(n)}(\hat{\theta}^{(n)}) \right),$$

This is the statistic used throughout the rest of the paper for jointly testing both random coefficient variability and ARCH effects.

Remark 8 (Computational cost)

Computing $T_{Al}^{(n)}$ adds essentially no cost beyond computing $T_F^{(n)}$ and $T_{LK}^{(n)}$ separately: both share the same OLS residuals and empirical moments, and $T_{Al}^{(n)}$ is obtained from them by a single addition and rescaling (Remark 7), with no matrix inversion required since $\Gamma_{\mathcal{N}_1}^{\bullet}(\theta)$ in Eq. (27) is diagonal and known in closed form. The joint test therefore remains $O(n)$, the same order as either marginal test.

4. Simulation

The objective of this section is to evaluate the performance of the proposed tests at an asymptotic significance level of 5%. To this end, we conduct a Monte Carlo simulation using the R programming language, based on 2500 repetitions, generating independent samples, each consisting of 500 observations. The data are simulated based on the random coefficient regression model with ARCH(1) errors, given by:

$$Y_i = \mu + (\beta + \xi_i)X_i + \varepsilon_i, i = 1, \dots, 500$$

where ε_i follows an ARCH(1) process defined as $\varepsilon_i = \sigma_i Z_i$ and $\sigma_i^2 = \alpha_0 + \alpha_1 \varepsilon_{i-1}^2$.

We compute and analyze the statistical power of the tests under different scenarios by considering:

- the power of the test for the random coefficient regression model with ARCH(1) errors, denoted P_{Al} ;
- the power of the test for the regression model with ARCH(1) errors, corresponding to the case where $\sigma_\xi^2 = 0$, denoted P_{LK} ;
- the power of the test for the random coefficient regression model obtained by setting $\alpha_1 = 0$, denoted P_F .

The simulation parameters are set as follows:

1. $\mu = 3$, $\beta = 10$ and $\alpha_0 = 0.1$;
2. the explanatory variables X_i are independently and identically distributed (i.i.d.) as $\mathcal{N}(0, 1)$;
3. the ARCH parameter α_1 takes values 0 (null hypothesis), 0.025, 0.05, 0.1, 0.2 for alternative hypotheses ;
4. the random coefficient terms ξ_i are i.i.d. Gaussian with mean zero and variance σ_ξ^2 , where $\sigma_\xi^2 = 0$ under the null hypothesis and takes values 0.025, 0.05, 0.1, 0.2 in alternative scenarios;
5. the error terms ε_i are generated from different distributions:
 - symmetric densities: normal $\mathcal{N}(0, 1/a)$ ($a = 0.4549$), logistic ($l_1, b = 1.207$), and Student's t-distribution with 8 degrees of freedom (t_8);
 - asymmetric densities: Skewed normal distributions.

In each replication, μ , β and α_0 are unknown and are estimated under $\mathcal{H}_0$: $(\hat{\mu}, \hat{\beta})$ by OLS, the residuals being $\hat{\varepsilon}_i = Y_i - \hat{\mu} - \hat{\beta}X_i$, and $\hat{\alpha}_0 = n^{-1} \sum_i \hat{\varepsilon}_i^2$. The test statistics are computed from these residuals.

The tables below present the powers of the test statistics for each scenario.

α_1	Test	σ_ξ^2				
		0	0.025	0.05	0.1	0.2
0	P_{Al}	0.0544	0.096	0.1584	0.3484	0.7288
	P_F	0.056	0.1336	0.2488	0.5764	0.924
	P_{LK}	0.0548	0.0472	0.0472	0.0588	0.0552
0.025	P_{Al}	0.21	0.3076	0.3764	0.572	0.8184
	P_F	0.052	0.1268	0.2472	0.5472	0.9036
	P_{LK}	0.31	0.3112	0.2624	0.2344	0.162
0.05	P_{Al}	0.4836	0.564	0.614	0.7584	0.8836
	P_F	0.054	0.1168	0.2292	0.4964	0.876
	P_{LK}	0.6812	0.6296	0.5868	0.5116	0.3844
0.1	P_{Al}	0.8684	0.8884	0.9096	0.9436	0.9716
	P_F	0.0472	0.11	0.1876	0.4128	0.8092
	P_{LK}	0.9692	0.9512	0.9384	0.9004	0.7976
0.2	P_{Al}	0.9948	0.9964	0.998	0.9992	0.998
	P_F	0.0588	0.9964	0.1288	0.2352	0.528
	P_{LK}	1	0.9996	1	0.9992	0.9944

Table 1. Power of the test statistics under the normal distribution $\mathcal{N}(0; 1/a)$ where $a = 0.4549$

α_1	Test	σ_ξ^2				
		0	0.025	0.05	0.1	0.2
0	P_{Al}	0.0504	0.0716	0.0964	0.152	0.3112
	P_F	0.0536	0.0896	0.1244	0.2612	0.5368
	P_{LK}	0.0532	0.0472	0.048	0.0468	0.0544
0.025	P_{Al}	0.4092	0.4532	0.5	0.5816	0.7044
	P_F	0.0488	0.0908	0.124	0.2208	0.482
	P_{LK}	0.5792	0.5408	0.5508	0.5212	0.4412
0.05	P_{Al}	0.7888	0.7904	0.8248	0.8452	0.8952
	P_F	0.0556	0.0724	0.1012	0.1852	0.3816
	P_{LK}	0.9168	0.9	0.8948	0.8812	0.8432
0.1	P_{Al}	0.986	0.9848	0.9904	0.9836	0.9908
	P_F	0.048	0.0632	0.0948	0.124	0.2436
	P_{LK}	0.9992	0.998	0.9972	0.9956	0.9916
0.2	P_{Al}	0.996	0.996	0.9972	0.9936	0.9988
	P_F	0.0612	0.0612	0.0764	0.082	0.102
	P_{LK}	0.9996	0.9996	0.9996	0.9988	1

Table 2. Power of the test statistics under the logistic distribution $rlogis(n, \sqrt{1.207})$.

α_1	Test	σ_ξ^2				
		0	0.025	0.05	0.1	0.2
0	P_{Al}	0.048	0.122	0.1984	0.4836	0.8628
	P_F	0.0512	0.1764	0.3676	0.754	0.9808
	P_{LK}	0.0556	0.0588	0.042	0.0544	0.0532
0.025	P_{Al}	0.1376	0.2372	0.3444	0.6016	0.9124
	P_F	0.0556	0.1712	0.356	0.724	0.9816
	P_{LK}	0.1812	0.1768	0.146	0.1356	0.1176
0.05	P_{Al}	0.258	0.3572	0.4836	0.696	0.9224
	P_F	0.0604	0.164	0.3316	0.6964	0.968
	P_{LK}	0.354	0.3208	0.3152	0.2644	0.1956
0.1	P_{Al}	0.5184	0.6228	0.7036	0.8352	0.956
	P_F	0.0492	0.1492	0.2972	0.6436	0.9552
	P_{LK}	0.6828	0.6436	0.6096	0.5192	0.4024
0.2	P_{Al}	0.8492	0.8832	0.9152	0.9508	0.9852
	P_F	0.0584	0.1244	0.2472	0.5096	0.8968
	P_{LK}	0.9548	0.9436	0.9184	0.8808	0.776

Table 3. Power of the test statistics under the Student's t -distribution with $df = 8$.

α_1	Test	σ_ξ^2				
		0	0.025	0.05	0.1	0.2
0	P_{Al}	0.0464	0.1816	0.39	0.7696	0.9864
	P_F	0.0484	0.3024	0.6268	0.9632	1
	P_{LK}	0.0548	0.0528	0.0508	0.0548	0.0516
0.025	P_{Al}	0.1092	0.274	0.4608	0.8076	0.982
	P_F	0.0552	0.28	0.5972	0.9456	0.9996
	P_{LK}	0.1324	0.1128	0.0996	0.0924	0.0788
0.05	P_{Al}	0.1916	0.3888	0.576	0.8464	0.9872
	P_F	0.0548	0.2688	0.5904	0.9368	0.9996
	P_{LK}	0.2904	0.2372	0.204	0.16	0.0924
0.1	P_{Al}	0.4236	0.5968	0.7448	0.9108	0.9904
	P_F	0.0456	0.2548	0.566	0.916	0.9992
	P_{LK}	0.6056	0.5172	0.4536	0.312	0.1856
0.2	P_{Al}	0.8196	0.8776	0.9372	0.9708	0.9968
	P_F	0.0536	0.2096	0.4792	0.8716	0.9964
	P_{LK}	0.946	0.8964	0.8596	0.7076	0.4728

Table 4. Power of the test statistics under the skew-normal distribution.

Interpretation

According to the previous tables, we observe that for the three considered models (random coefficient regression with ARCH errors, random coefficient regression, and regression with ARCH errors) and across all studied distributions (normal, logistic, Student-t, and skew-normal), the test power remains close to the significance level $\alpha = 5\%$ under the null hypothesis $\mathcal{H}_0 : \sigma_\xi^2 = \alpha_1 = 0$. This situation corresponds to the absence of random variability in the regression coefficient and the absence of ARCH(1) effects. This result is expected, as under the null hypothesis, the probability of rejection should match the pre-specified risk level.

The first row of each table, which corresponds to the case $\alpha_1 = 0$ and $\sigma_\xi^2 > 0$, highlights the superiority of the Fihri statistic: P_F consistently displays the highest power, making it the most suitable test in this scenario. Conversely, the first column, representing the case $\sigma_\xi^2 = 0$ and $\alpha_1 > 0$, favors the Lee statistic, as P_{LK} outperforms the others in terms of power, thus constituting the best choice in this context.

Finally, in the central part of the tables, corresponding to the alternative hypothesis $\mathcal{H}_1$ where both parameters $\sigma_\xi^2 > 0$ and $\alpha_1 > 0$, the statistic proposed in this article stands out clearly. Indeed, the P_{Al} test consistently exhibits the highest power, confirming its effectiveness in detecting the joint violation of both null conditions. In other words, our proposed statistic proves to be the most powerful tool when the regression model simultaneously incorporates random coefficient heterogeneity and ARCH-type errors.

5. Real data analysis

5.1. Housing price data (Wooldridge, 2016)

For this study, we used a dataset from [42] consisting of 88 houses. The data includes *price* (property prices) as the dependent variable (Y) and *assess* (property tax assessment) as the explanatory variable (X). We analyzed three distinct regression models.

The test statistics were computed using *R* software, ensuring precise and reproducible estimates. We evaluated the significance of the parameters in each model under the null hypothesis ($\mathcal{H}_0$). The results are summarized in Table 5.

Model	Test statistic value	p-value	Decision
RCR with ARCH(1) errors (joint)	$T_{Al} = 1.653834$	0.04908059	Reject $\mathcal{H}_0$ ($p < 0.05$)
Random coefficient regression	$T_F = 1.637973$	0.05071362	Fail to reject $\mathcal{H}_0$ ($p > 0.05$)
Regression with ARCH(1) errors	$T_{LK} = 0.7009016$	0.2416822	Fail to reject $\mathcal{H}_0$ ($p > 0.05$)

Table 5. Results of Statistical Tests and p-values.

The results in Table 5 deserve careful interpretation. While the joint test T_{Al} rejects $\mathcal{H}_0$ at the 5% level ($p = 0.049$), the marginal test T_F fails to reject at the same threshold ($p = 0.051$). These p-values are admittedly close, and we do not claim a sharp qualitative difference between the two models based on this application alone. Rather, the key observation is that *only the joint test* simultaneously accounts for both random coefficient variability and ARCH effects, suggesting that the data contain a signal that is detectable only when both features are modeled together. The practical implication is that ignoring either source of heterogeneity leads to a loss of power, consistent with the simulation results in Section 4. We acknowledge that a single dataset with $n = 88$ observations provides limited statistical power, and a more conclusive empirical illustration would benefit from larger or more heterogeneous datasets.[†]

Decomposing the joint statistic into its marginal components clarifies which effect drives the rejection. The random-coefficient test, $T_F = 1.637973$ ($p = 0.0507$), sits right at the significance boundary, while the ARCH test, $T_{LK} = 0.7009$ ($p = 0.2417$), is far from significant. So the joint test's rejection ($p = 0.0491$) comes mostly from the random-coefficient effect, with the ARCH signal adding only a small push, enough to tip p from 0.0507 to below 0.05.

Figure 1 shows this directly. Panel (a) plots $\log(\hat{\varepsilon}_i^2)$ against $\log(X_i^2)$: a clear upward trend (Spearman $\rho = 0.243$, $p = 0.023$), matching the random-coefficient signal. Panel (b) plots $\log(\hat{\varepsilon}_i^2)$ against its own lag: no trend at all ($\rho = -0.030$, $p = 0.785$). Only the random-coefficient effect leaves a visible mark on the residuals.

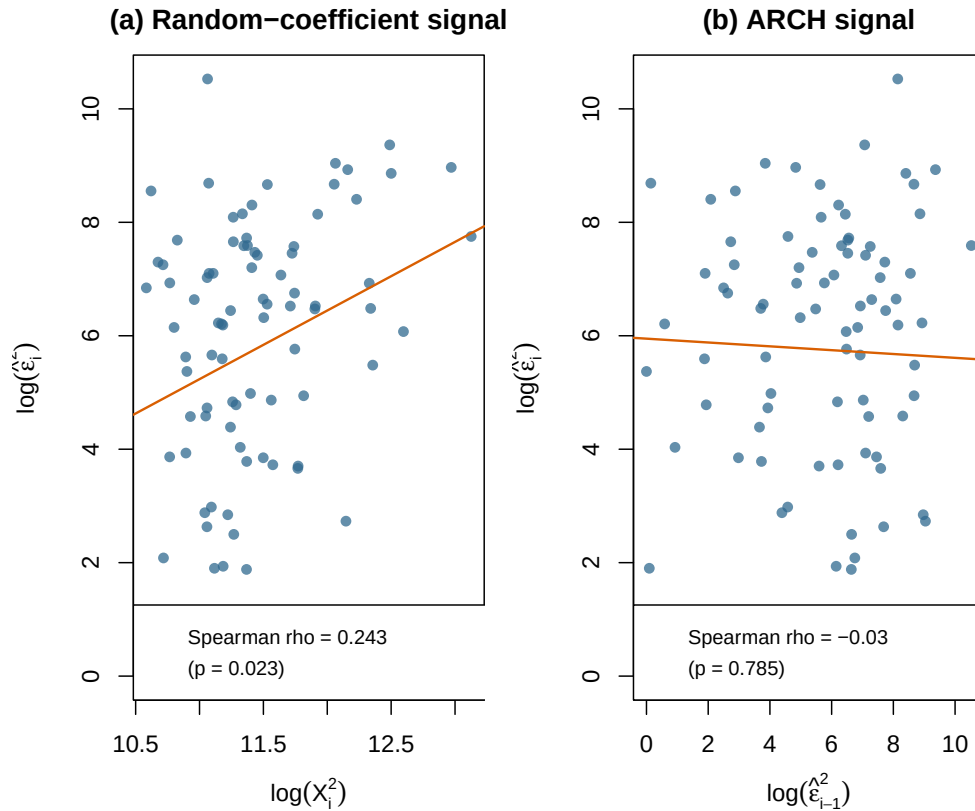


Figure 1. Residual diagnostics for the housing data. (a) OLS squared residuals against X_i^2 (random-coefficient signal), Spearman $\rho = 0.243$, $p = 0.023$. (b) OLS squared residuals against their own first lag (ARCH signal), $\rho = -0.030$, $p = 0.785$. Orange lines: linear fit, log-log scale.

5.2. Bitcoin weekly returns (2020-2024)

As a second empirical illustration, we consider weekly Bitcoin (BTC-USD) log-returns over the period January 2020 to December 2024, a turbulent episode that includes the COVID-19 crash (March 2020), two major bull runs (late 2020 and late 2021), a prolonged bear market

[†]The dataset of Wooldridge (2016)[42] contains only $n = 88$ observations, which limits the power of all three tests. The near-borderline p-values should therefore be interpreted with caution, and the simulation results in Section 4 provide stronger evidence of the superiority of the joint test.

(2022–2023), and a renewed rally in 2024. This makes it a natural candidate for jointly testing random coefficient variability and ARCH effects.

The data consist of $n = 260$ weekly observations, retrieved from Yahoo Finance via the `quantmod` package [34].[‡]

The nuisance parameters are estimated under $\mathcal{H}_0$ by OLS ($\hat{\mu}$, $\hat{\beta}$) and by the residual variance $\hat{\sigma}_0^2 = n^{-1} \sum_{i=1}^n \hat{\varepsilon}_i^2$. The three test statistics are computed as described in Section 3, with results summarized in Table 6.

Model	Test statistic	p-value	Decision
RCR with ARCH(1) errors (joint)	$T_{Al} = 4.1980$	0.000013	Reject $\mathcal{H}_0$ ($p < 0.05$)
Random coefficient regression only	$T_F = 4.1012$	0.000021	Reject $\mathcal{H}_0$ ($p < 0.05$)
Regression with ARCH(1) errors only	$T_{LK} = 1.8357$	0.0332	Reject $\mathcal{H}_0$ ($p < 0.05$)

Table 6. Results of statistical tests on Bitcoin weekly log-returns (2020–2024), with Y_i = log-return and X_i = log-trading volume, $n = 260$ observations. Data source: Yahoo Finance, retrieved via the `quantmod` package [34].

The results in Table 6 are particularly striking. All three tests reject $\mathcal{H}_0$ at the 5% significance level, which is consistent with the well-documented stylized facts of cryptocurrency markets [38]: strong volatility clustering (captured by T_{LK}) and significant instability in the relationship between returns and trading volume (captured by T_F). The joint statistic T_{Al} yields the largest value (4.198, $p \approx 0$), confirming that the two phenomena coexist and that modeling them jointly provides the most powerful detection. In contrast to the housing data example of Section 5, where the signal was borderline, the Bitcoin dataset provides a clear-cut rejection under all three tests, thereby further validating the relevance of the proposed joint framework in financial time series applications characterized by simultaneous parameter instability and conditional heteroskedasticity.

5.3. U.S. inflation and interest rates (Newbold and Bos, 1985)

As a third empirical illustration, we consider the quarterly U.S. inflation rate and Treasury bill interest rate series originally used by [28] to illustrate their stochastic parameter regression model, covering 1953:Q1–1980:Q2 ($n = 110$); the series are available via the `qinfl` and `qintr` datasets of the `astsa` R package [37]. We model the interest rate as the response variable Y_i and the inflation rate as the regressor X_i . This choice is particularly fitting given that the Gaussian test of Newbold and Bos (1985)[28] is nested within, and generalized by, our pseudo-Gaussian statistic T_{Al} (see Section 3).

The three test statistics, computed as described in Section 3, are summarized in Table 7.

Model	Test statistic value	p-value	Decision
RCR with ARCH(1) errors (joint)	$T_{Al} = 3.535427$	0.0002036	Reject $\mathcal{H}_0$ ($p < 0.001$)
Random coefficient regression	$T_F = 2.015631$	0.021919	Reject $\mathcal{H}_0$ ($p < 0.05$)
Regression with ARCH(1) errors	$T_{LK} = 2.984218$	0.001422	Reject $\mathcal{H}_0$ ($p < 0.01$)

Table 7. Results of statistical tests and p -values, U.S. inflation and interest rate data (Newbold and Bos, 1985).

Unlike the borderline result on the Housing dataset, here all three tests reject $\mathcal{H}_0$, confirming the presence of both random coefficient variability and ARCH effects in this relationship. More importantly, the joint statistic is considerably more significant than either marginal test taken alone ($p = 0.0002$ versus $p = 0.022$ and $p = 0.0014$), illustrating that when both sources of heterogeneity are genuinely present, combining them into a single test yields a sharper rejection than either marginal procedure, consistent with the power gains observed in the simulation study of Section 4.

6. Conclusions and perspectives

In this paper, we studied how to test for a random regression coefficient when the errors follow an ARCH(1) process. We proved that the model satisfies the ULAN property near the null hypothesis. Everything else in the paper rests on this result.

Using ULAN, we obtained the central sequence and the information matrix. We then removed the nuisance parameters: the intercept, the regression coefficient and the error scale. This gave a parametric test that is locally and asymptotically optimal for a given error density.

[‡]The data were downloaded using the following R command: `getSymbols("BTC-USD", src = "yahoo", from = "2020-01-01", to = "2024-12-31", periodicity = "weekly")`. The log-returns are computed as $Y_i = \log(P_i/P_{i-1})$, where P_i is the weekly closing price, and the explanatory variable is $X_i = \log(\text{Volume}_i)$. After differencing, the final sample consists of $n = 260$ complete weekly observations over the period January 2020 to December 2024.

We also built a pseudo-Gaussian version. It does not require Gaussian errors. The two parameters of interest are asymptotically orthogonal, so the joint test simply adds the two marginal statistics with equal weights. When one of the two effects is absent, the joint test becomes the test of Fihri et al. (2020) [17] or the test of Lee and King (1993) [25].

Our main point is to test the two effects together. They are usually studied one at a time. In the simulations, we used 2500 replications, samples of 500 observations and four error distributions. The joint test works well when both effects are present. We then applied the tests to housing prices, weekly Bitcoin returns, and U.S. inflation and interest rates. The housing result is borderline, so we do not draw a strong conclusion from it. For Bitcoin, all tests reject the null hypothesis.

Some work remains. We fixed the error density in advance, but it could be estimated from the data. This would give an adaptive test. Rank-based versions of the test are also possible. The model is limited to ARCH(1) errors and one regressor, so GARCH(p, q) errors and several regressors are still open. Finally, the tests should be tried on larger data sets.

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Appendix: Auxiliary technical results

The following lemma is used in the proof of Proposition 2.1 (ULAN property).

Lemma .1 (Asymptotic relative linearity)

Under Assumptions (A), (B.1)–(B.2), and (C), for all $\tau \in \mathbb{R}^2 \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$\Delta_{f_1}^{(n)}(\theta + n^{-1/2}\nu^{(n)}\tau) - \Delta_{f_1}^{(n)}(\theta) = -\Gamma_{f_1}^{(n)}(\theta)\tau + o_p(1)$$

under $P_{\theta; f_1}^{(n)}$, as $n \rightarrow \infty$.

This result is a consequence of the quadratic mean differentiability of the log-likelihood and is established following the approach of Hajek (1972) [19] and Akharif (2003) [2]. The ULAN expansion in Proposition 2.1 follows by combining this lemma with the convergence of $\Gamma_{f_1}^{(n)}(\theta)$ along subsequences satisfying Assumption (B.3).

Proof of Proposition 2.1 (ULAN)

The proof checks Swensen's (1985) six conditions (Conditions 1.2–1.7 in his Lemma 1) uniformly in a neighborhood of $(\mu, \beta, \alpha_0, 0, 0)$. Conditions other than 1.2 are routine; we detail Condition 1.2, the quadratic mean differentiability of $(\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1) \mapsto q_{\mu, \beta, \alpha_0, \sigma_\xi^2, \alpha_1; f_1}^{1/2}(y)$, in three steps. Note that, exactly at $\alpha_1 = 0$, $\sigma_i^2 = \alpha_0 + \alpha_1 \varepsilon_{i-1}^2$ reduces to the constant α_0 , so that the ε_i are i.i.d. with density f_1 at this point, and the classical (non-conditional) quadratic mean differentiability theory applies.

Step 1 (σ_ξ^2 -direction). When $\alpha_1 = 0$, the model reduces exactly to the random coefficient regression model of Fihri et al. (2020), with $(\mu, \beta, \alpha_0, \sigma_\xi^2)$ playing the role of their $(\mu, \beta, \sigma^2, \sigma_\xi^2)$. Quadratic mean differentiability in the σ_ξ^2 -direction, jointly with (μ, β, α_0) , therefore follows directly from their Lemma A.1 (parts (i)–(iii)), which establishes this property along the corresponding subsequence.

Step 2 (α_1 -direction). At $\alpha_1 = 0$, $\sigma_i^2 \equiv \alpha_0$ does not depend on ε_{i-1} , so the map $\alpha_1 \mapsto q_{\mu, \beta, \alpha_0, 0, \alpha_1; f_1}^{1/2}(y)$ is, at this point, a classical (non-conditional) quadratic mean differentiability problem, of the same type as Step 1: writing $\sigma_i^2(\alpha_1) = \alpha_0 + \alpha_1 \varepsilon_{i-1}^2$ and treating ε_{i-1}^2 as playing the role of X_i^2 in Fihri et al.'s Lemma A.1 (part (i)), the same three-step argument ((a)–(c) therein) yields quadratic mean differentiability in α_1 , jointly with (μ, β, α_0) , at $\alpha_1 = 0$.

Step 3 (joint differentiability, σ_ξ^2 and α_1 together). Write, for $r^2 := \sigma_\xi^2$, $s^2 := \alpha_1$,

$$q_{r^2, s^2}^{1/2}(y) - q_{0,0}^{1/2}(y) = \underbrace{\left[q_{r^2, s^2}^{1/2}(y) - q_{0, s^2}^{1/2}(y) \right]}_{=: A(r, s)(y)} + \underbrace{\left[q_{0, s^2}^{1/2}(y) - q_{0,0}^{1/2}(y) \right]}_{=: B(s)(y)}$$

(arguments $\mu, \beta, \alpha_0, f_1$ suppressed). By Step 2, $\int B(s)(y)^2 dy = o(s^4)$ as $s \rightarrow 0$. For $A(r, s)$, Step 1 gives $\int A(r, 0)(y)^2 dy = o(r^4)$; extending this bound to $A(r, s)$ for s in a neighborhood of 0 requires that the constant in Step 1's argument, which depends on α_1 only through the (now s^2 -dependent) conditional variance $\sigma_i^2 = \alpha_0 + s^2 \varepsilon_{i-1}^2$ of ε_i , remains bounded as $s \rightarrow 0$; this holds since $\sigma_i^2 \rightarrow \alpha_0$ continuously and the relevant information functionals in Assumption A are continuous in the scale parameter. Combining, $\int [A(r, s)(y) + B(s)(y)]^2 dy \leq 2 \int A(r, s)(y)^2 dy + 2 \int B(s)(y)^2 dy = o(r^4) + o(s^4)$, establishing joint quadratic mean differentiability at $(\mu, \beta, \alpha_0, 0, 0)$. $\square$

Remark 9 (Remark on uniformity.)

Swensen's Condition 1.2 must hold *uniformly* in a neighborhood of $(\mu, \beta, \alpha_0, 0, 0)$, not merely at the point itself. Steps 1–3 establish differentiability exactly at $\alpha_1 = 0$, where (ε_i) is i.i.d.; extending this to a full neighborhood, where $\alpha_1 \neq 0$ and σ_i^2 depends on ε_{i-1} ,

requires controlling the conditional (martingale) structure of the ARCH(1) process away from the null, along the lines of Hall and Heyde's (1980) martingale central limit theorem under a fourth-moment condition on ε_{i-1} . We do not carry out this extension in detail here; it would follow the same strategy, combined with a martingale central limit theorem argument for the ARCH(1) component away from the null, and is left for future work.

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