STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–5.
Published online in International Academic Press (www.IAPress.org)

Dynamics of M-hypercyclic C_0-semigroups: Spectral conditions and orbital behavior

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Abstract This paper investigates new developments and implications regarding M-hypercyclic C_0-semigroups acting on a separable, infinite-dimensional complex Banach space $\mathcal{X}$. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup of bounded linear operators with generator $\mathcal{A}$, and let M be a non-zero subspace of $\mathcal{X}$. We establish that for an M-hypercyclic semigroup $\mathcal{T}$, any element a^* in the dual space $\mathcal{X}^*$ possessing a bounded orbit under the adjoint semigroup $\mathcal{T}^*$ must necessarily belong to the annihilator $M^\perp$. Moreover, it is shown that the condition $\ker(\mathcal{A}^* - \lambda I) \subset M^\perp$ holds for every $\lambda \in \mathbb{C}$. Finally, the study explores and identifies several criteria that ensure the M-hypercyclicity of $\mathcal{T}$.

Keywords Hypercyclicity, M-transitivity, M-Hypercyclicity, C_0-semigroup, Point spectrum.

AMS 2010 subject classifications 47A10, 47A16, 47D03, 47D09.

DOI: 10.19139/soic-2310-5070-4152

1. Introduction

In recent years, the study of chaotic and hypercyclic operators on infinite-dimensional Banach spaces has emerged as a prominent area of functional analysis. A continuous linear operator T on a separable Banach space $\mathcal{X}$ is classified as hypercyclic if there exists an element $x \in \mathcal{X}$ such that its orbit, $orb(T, x) := \{T^n x; n \geq 0\}$, is dense in $\mathcal{X}$. The vector x in this context is identified as a hypercyclic vector for T. The historical foundations of this theory date back to 1929, when Birkhoff [6] demonstrated that the translation operator $h \mapsto h(. + \lambda)$ on $H(\mathbb{C})$ serves as the first instance of a hypercyclic operator. Through the application of Baire's Category Theorem, it is well-established that for operators on separable Banach spaces, hypercyclicity is equivalent to topological transitivity. Rolewicz [16] further advanced the field by proving that every separable infinite-dimensional Banach space supports a hypercyclic operator, specifically by showing that the weighted backward shift βS on $\mathcal{L}^p$ is hypercyclic provided that $|\beta| > 1$. Detailed expositions and fundamental results can be found in Kitai's seminal thesis [13], as well as in [1, 4, 12].

The transition from discrete operators to continuous families was formalized in 1997 by Desch, Schappacher, and Webb [8], who introduced the theory of hypercyclic C_0-semigroups. A C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on $\mathcal{X}$ is regarded as hypercyclic if it possesses a dense orbit $orb(\tau, x) := \{T_t x, t \geq 0\}$. In [8], the authors proved that such a semigroup is hypercyclic if it is topologically transitive, meaning that for any two non-empty open sets $A, B \subset \mathcal{X}$, there exists a time $s > 0$ for which $T_s(A) \cap B \neq \emptyset$. Furthermore, the hypercyclicity of a single operator T_s for any fixed $s > 0$ is a sufficient condition for the hypercyclicity of the entire semigroup. Comprehensive discussions on these semigroups are available in [3, 5, 9, 11, 12, 15].

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ISSN 2310-5070 (online) ISSN 2311-004X (print)
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Recall that a family of bounded linear operators $\mathcal{T} = (T_t)_{t\geq 0}$ is defined as a strongly continuous C_0 semigroup if it satisfies the semigroup property ($T_0 = I$ and $T_{s+t} = T_sT_t$) and the strong continuity condition (the map $t \mapsto T_tx$ is continuous for all x). Every such semigroup is associated with a unique infinitesimal generator $\mathcal{A}$ defined by $\mathcal{A}(x) = \lim_{t\to 0} \frac{T_tx-x}{t}$ on its domain $D(\mathcal{A})$. In the specific case where $\mathcal{A}$ is a bounded operator, the semigroup is uniformly continuous and takes the form $T_t = e^{t\mathcal{A}}$. To analyze the asymptotic behavior of $\mathcal{T}$, we utilize the following sets:

- $\mathcal{H}^0(\mathcal{T}) := \{x \in \mathcal{X} \mid \lim\limits_{t\to+\infty} T_tx = 0\}$.
- $\mathcal{H}^\infty(\mathcal{T}) = \{x \in \mathcal{X}/\forall\epsilon > 0, \exists y \in \mathcal{X}, \exists t > 0$ such that $\|y\| < \epsilon$ and $||T_tx - y|| < \epsilon\}$.
- $\mathcal{H}^{\infty,0}(\mathcal{T}) := \{x \in \mathcal{X} \mid \forall\epsilon > 0, \exists y \in \mathcal{X}, \exists s > 0 : \|y\| < \epsilon$ and $T_sy = x\}$.

A vital density criterion was established in [3]:

Theorem 1.1
Suppose that $\mathcal{T} = (T_t)_{t\geq 0}$ is a C_0-semigroup on $\mathcal{X}$, a separable Banach space. If the density of both $\mathcal{H}^0(\mathcal{T})$ and $\mathcal{H}^\infty(\mathcal{T})$ in $\mathcal{X}$ is satisfied, then the hypercyclicity of $\mathcal{T}$ follows.

It is important to observe that while hypercyclicity necessitates that $\mathcal{H}^\infty(\mathcal{T}) = \mathcal{X}$, the density of $\mathcal{H}^0(\mathcal{T})$ is not a requirement; indeed, there exist hypercyclic semigroups where $\mathcal{H}^0(\mathcal{T})$ is trivial [3]. Additionally, Desch et al. [8] established a connection between hypercyclicity and the spectral attributes of the generator:

Theorem 1.2 ([8])
Let $\mathcal{X}$ be a complex separable Banach space and $\mathcal{A}$ be the generator of $\mathcal{T}$. If there exists a connected open domain $D \subset \mathbb{C}$ intersecting the imaginary axis and an analytic function $h : D \to \mathcal{X}$ such that $h(\alpha) \in \ker(\mathcal{A} - \alpha I)$, and if the span of its image is dense in $\mathcal{X}$, then $\mathcal{T}$ is hypercyclic.

Conversely, a necessary condition for a C_0-semigroup to be hypercyclic is that the point spectrum of the adjoint generator is empty, i.e., $\sigma_p(\mathcal{A}^*) = \emptyset$ [8]. More recently, the framework of M-hypercyclicity was proposed in [17, 18], focusing on the density of orbits relative to a specific subspace M of $\mathcal{X}$. This paper continues the investigation of these properties first introduced in 2015.
The notion of M-hypercyclicity for C_0-semigroups was introduced in [17],[18], where the basic properties were established, including the necessary condition $ker(T_t^* - \lambda I) \subset M^\perp$. The SVEP of the generator was related to M-hypercyclicity in [19].
Long-time behavior of evolution problems is a central theme in analysis and applied mathematics. For linear C_0-semigroups, hypercyclicity describes a different, chaotic kind of asymptotic behavior, in which a single orbit is dense in the space. Dynamical systems with the semigroup property have also been studied in more general metric settings, together with recurrence and entropy-type invariants [14]. In contrast with these works, the present paper concerns the linear dynamics of C_0-semigroups, and more precisely the spectral and orbital obstructions to M-hypercyclicity. The present paper goes further in three directions. First, we replace the eigenvector condition of [18] by a condition on bounded adjoint orbits and translate it to the generator, obtaining $ker(A^* - \lambda I) \subset M^\perp$ for every $\lambda \in \mathbb{C}$. Second, we give a transfer principle through intertwining operators, which produces new M-hypercyclic semigroups from hypercyclic ones. Third, we show that two hypercyclic semigroups can be handled simultaneously by an invertible operator that maps one subspace onto the other.

2. Main Results

Throughout this study, we consider M to be a non-trivial closed subspace of a complex, separable, infinite-dimensional Banach space $\mathcal{X}$. Let $B(\mathcal{X})$ denote the algebra of all bounded linear operators on $\mathcal{X}$. We begin by reviewing the fundamental properties of M-hypercyclic C_0-semigroups as established in [17], and further extend the results presented in [19] regarding this framework.

The notion of M-hypercyclicity for C_0-semigroups was first proposed by the authors in [17].

Definition 2.1 ([17]). We say that a C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on $\mathcal{X}$ is M-hypercyclic if some vector $x \in \mathcal{X}$ exists such that the set of points $\{T_tx : t \geq 0\} \cap M$ is dense within M.

Example 2.2 ([17]). Let $\mathcal{X}$ be an infinite-dimensional separable Banach space supporting a hypercyclic C_0-semigroup $(A_t)_{t\geq 0}$. The semigroup defined by $T_t := A_t \oplus I$ for all $t \geq 0$ is M-hypercyclic with respect to $M = \mathcal{X} \oplus \{0\}$, although it does not satisfy the criteria for global hypercyclicity on $\mathcal{X} \oplus \mathcal{X}$.

The next result shows that M-hypercyclicity can be produced from classical hypercyclicity by means of an intertwining operator. Recall that if $B\mathcal{T}_t = \mathcal{S}_t B$, then B carries orbits of $\mathcal{T}$ onto orbits of $\mathcal{S} = (S_t)_{t\leq 0}$. Since B is continuous and $\mathcal{T}$ has a dense orbit, the image of this orbit is dense in the range of B, although it need not be dense in the whole space $\mathcal{X}$. Consequently, the semigroup $\mathcal{S}$ is in general not hypercyclic, but it is M-hypercyclic for $M = R(B)$. This gives a systematic way of constructing M-hypercyclic semigroups from known hypercyclic ones.

Proposition 2.3
Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a hypercyclic C_0-semigroup on $\mathcal{X}$ with hypercyclic vector x. Suppose $B \in B(\mathcal{X})$ is a non-zero operator with a closed range $M = R(B)$. If a C_0-semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ satisfies the intertwining relation $BT_t = S_t B$ for all $t \geq 0$, then $\mathcal{S}$ is M-hypercyclic, where Bx serves as the M-hypercyclic vector.

Proof: Given that x is hypercyclic for $\mathcal{T}$, its orbit $orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$. Utilizing the intertwining property, we observe that:

$$\begin{aligned} \mathrm{orb}(\mathcal{S}, Bx) &= \{S_t Bx : t \geq 0\} \\ &= \{BT_t x : t \geq 0\} \\ &= B(\mathrm{orb}(\mathcal{T}, x)) \end{aligned}$$

which is clearly a subset of $R(B) = M$. To prove density in M, let U be a non-empty relatively open subset of M. There exists an open set $V \subset \mathcal{X}$ such that $U = V \cap M$. Since B is continuous, $B^{-1}(V)$ is a non-empty open set. The density of $orb(\mathcal{T}, x)$ implies that $orb(\mathcal{T}, x) \cap B^{-1}(V)$ is non-empty, which directly leads to $B(orb(\mathcal{T}, x)) \cap U = orb(\mathcal{S}, Bx) \cap U \neq \emptyset$. Therefore, Bx is the required M-hypercyclic vector. ■

Remark 2.4. Proposition 2.3 is formulated in terms of the semigroup operators T_t and S_t: the intertwining relation $BT_t = S_t B$ is required for every $t \geq 0$. In practice, however, a C_0-semigroup is most often specified through its infinitesimal generator. The following corollary therefore translates this condition to the level of generators.

Corollary 2.5
Let $\mathcal{T}$ and $\mathcal{R}$ be C_0-semigroups with generators $\mathcal{A}$ and R, respectively. If $\mathcal{T}$ is hypercyclic and there exists a non-zero operator $B \in B(\mathcal{X})$ with closed range $M = R(B)$ such that $B\mathcal{A} = RB$ on $D(\mathcal{A})$ with $B(D(\mathcal{A})) \subset D(R)$, then $\mathcal{R}$ is M-hypercyclic.

Proof: It is sufficient to show that the generator relation $B\mathcal{A} = RB$ implies the semigroup relation $BT_t = R_t B$. For any $x \in D(\mathcal{A})$, define $w_s = R_{t-s}BT_s x$. Differentiating with respect to s gives $\frac{d}{ds}w_s = -RR_{t-s}BT_s x + R_{t-s}B\mathcal{A}T_s x$. Since $RB = B\mathcal{A}$, the derivative is zero, meaning w_s is constant. Consequently, $w_0 = w_t$, or $R_t Bx = BT_t x$. By the density of $D(\mathcal{A})$, this equality extends to all $x \in \mathcal{X}$. ■

Example 2.6. On the space $X \oplus X$ we define the semigroup

$$S_t = T_t \oplus I, \qquad \text{and the operator } B(x, y) = (x, 0).$$

Then B is a non-zero operator with closed range $M = X \oplus \{0\}$, and $BS_t = S_t B$ for all $t \geq 0$. If x_0 is a hypercyclic vector for $\mathcal{T}$, we have that $(x_0, 0)$ is an M-hypercyclic vector for $\mathcal{S} = (S_t)_{t\geq 0}$, since $\mathrm{orb}(\mathcal{S}, (x_0, 0)) = \{(T_t x_0, 0) : t \geq 0\}$ is dense in M. However, $\mathcal{S}$ is not hypercyclic on $X \oplus X$.

The following result is a semigroup counterpart of the fact that any two hypercyclic operators can be related by an invertible operator carrying one dense orbit onto the other. It shows that two hypercyclic C_0-semigroups, which may have completely different dynamics, can always be made to share the same restricted dynamical behavior: one is M-hypercyclic on a closed subspace M, and the other is $T(M)$-hypercyclic on its image under an invertible operator T. The proof combines three ingredients: the result of Conejero, Müller and Peris [7], which provides a common time $t > 0$ for which $\mathcal{T}_t$ and $\mathcal{S}_t$ are both hypercyclic; Grivaux's theorem [10], which produces an invertible operator mapping one dense linearly independent orbit onto another; and the theorem of Bamerni, Kadets and Kilicman [2], which yields a closed subspace M on which the orbit of a hypercyclic operator is dense.

Theorem 2.7
Let $\mathcal{T}_1 = (T_t)_{t\geq 0}$ and $\mathcal{T}_2 = (S_t)_{t\geq 0}$ be two hypercyclic C_0-semigroups on a Banach space $\mathcal{X}$. Then, there exist a non-trivial closed subspace $M \subset \mathcal{X}$ and an invertible operator $T \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{T}_1$ is M-hypercyclic and $\mathcal{T}_2$ is $T(M)$-hypercyclic.

Proof:
By hypothesis, we have $\mathcal{T}_1 = (T_t)$ and $\mathcal{T}_2 = (S_t)$ as two hypercyclic C_0-semigroups, so according to Müller [7], there exists $t > 0$ such that T_t and S_t are two hypercyclic operators. Let a and b be two hypercyclic vectors of T_t and S_t, respectively.
Let $H_0 = orb(T_t, a)$ and $H_1 = orb(S_t, b)$. Since H_0 and H_1 are two dense and linearly independent sets, and according to Lemma 2-1 in Grivaux's work [10], we can find an invertible operator $T : \mathcal{X} \to \mathcal{X}$ such that: $T(H_0) = H_1$.
Now, according to Theorem 2-1 of [2], there exists a non-trivial closed subspace M such that $H_0 \cap M$ is dense in M, which means that T_t is M-hypercyclic, so $\mathcal{T}_1$ is M-hypercyclic. It remains to be shown that $\mathcal{T}_2$ is $T(M)$-hypercyclic. To do this, it suffices to show that $H_1 \cap T(M)$ is dense in $T(M)$.
We have:

$$T(M) = T(\overline{H_0 \cap M}) = \overline{T(H_0 \cap M)}$$

Therefore, $\overline{T(H_0 \cap M)} \cap T(M) = \overline{T(H_0 \cap M)} = T(M)$.
Now, $H_0 \cap M \subset H_0$, so $T(H_0 \cap M) \subset T(H_0) = H_1$, so we have:
$T(M) = T(\overline{H_0 \cap M}) \cap T(M) \subset \overline{H_1 \cap T(M)} \subset T(M)$.
Hence $\overline{H_1 \cap T(M)} = T(M)$, which proves that $H_1 \cap T(M)$ is dense in $T(M)$, so b is a $T(M)$-hypercyclic vector of S_t, so S_t is $T(M)$-hypercyclic, in particular $\mathcal{T}_2$ is $T(M)$-hypercyclic. ■

Remark 2.8. The adjoint semigroup $\mathcal{T}^* = (T_t^*)_{t\geq 0}$ on $\mathcal{X}^*$ is not always strongly continuous. However, it forms a C_0-semigroup on the sun-dual space $\mathcal{X}^{\odot}$. If $\mathcal{X}$ is reflexive, $\mathcal{T}^*$ is a C_0-semigroup on the entire dual space $\mathcal{X}^*$ with generator $\mathcal{A}^*$.

Proposition 2.9
Let $\mathcal{T}$ be an M-hypercyclic C_0-semigroup. If there exists a functional $a^* \in \mathcal{X}^*$ such that the adjoint orbit $orb(\mathcal{T}^*, a^*)$ is bounded, then a^* belongs to the annihilator $M^{\perp}$.

Proof: Let a be an M-hypercyclic vector for $\mathcal{T}$ and assume $\|T_t^* a^*\| \leq K$ for all $t \geq 0$. For any $x \in M$ and $\epsilon > 0$, we can find a sequence t_n such that $T_{t_n} a \to \frac{x}{\epsilon}$. It follows that:

$$|\langle x, a^*\rangle| \leq |\langle x - \epsilon T_{t_n} a, a^*\rangle| + \epsilon |\langle T_{t_n} a, a^*\rangle| \leq \|a^*\| \|x - \epsilon T_{t_n} a\| + \epsilon K \|a\|$$

Taking the limit as $n \to \infty$ and then as $\epsilon \to 0$, we conclude that $\langle x, a^*\rangle = 0$ for all $x \in M$, meaning $a^* \in M^{\perp}$. ■

Corollary 2.10
Let $\mathcal{T}$ be an M-hypercyclic C_0-semigroup with generator $\mathcal{A}$. Any eigenvector of the adjoint operator $\mathcal{A}^*$ is necessarily contained in $M^{\perp}$. Consequently, $\ker(\mathcal{A}^* - \lambda I) \subset M^{\perp}$ for all $\lambda \in \mathbb{C}$.

Corollary 2.11
Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on $\mathcal{X}$ with $K := \sup_{t\geq 0} \|T_t\| < \infty$. Then $\mathcal{T}$ is not M-hypercyclic for any non-zero closed subspace $M \subset \mathcal{X}$.

Proof: Let $a^* \in \mathcal{X}^*$. For every $t \geq 0$,

$$\|T_t^* a^*\| = \|a^* \circ T_t\| \leq K \|a^*\|,$$

so every adjoint orbit is bounded. If $\mathcal{T}$ were M-hypercyclic, Proposition 2.9 would give $a^* \in M^{\perp}$ for every $a^* \in \mathcal{X}^*$, that is, $M^{\perp} = \mathcal{X}^*$. Suppose $M \neq \{0\}$ and pick $x_0 \in M \setminus \{0\}$. By the Hahn–Banach theorem there exists $a^* \in \mathcal{X}^*$ with $\langle x_0, a^*\rangle = \|x_0\| \neq 0$, so $a^* \notin M^{\perp}$, a contradiction. Hence $M = \{0\}$. ■

Proposition 2.12
Consider a C_0-semigroup $\mathcal{T}$ on $\mathcal{X}$. If the linear span of the kernels $\bigcup_{\lambda\in\Lambda} \ker(T_t^* - \lambda I)$ is dense in $\mathcal{X}^*$ for some non-empty $\Lambda \subset \mathbb{C}$, then $\mathcal{T}$ cannot be M-hypercyclic for any non-trivial closed subspace $M \subset \mathcal{X}$.

Proof: Assume $\mathcal{T}$ is M-hypercyclic for some non-trivial closed M. By Proposition 2.5 in [17], $\ker(T_t^* - \lambda I) \subset M^\perp$ for all $\lambda \in \Lambda$. The density assumption implies that the entire dual space $\mathcal{X}^*$ is contained in $M^\perp$. This forces $M^\perp = \mathcal{X}^*$, which implies $M = \{0\}$, a contradiction. ■

Example 2.13. On the space $X \oplus X$ we define the semigroup: $S_t = T_t \oplus I$, with $(T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup

The adjoint results are also consistent with this example. Here $S_t^* = T_t^* \oplus I$ and the adjoint generator is $A^* \oplus 0$. Since $\mathcal{T}$ is hypercyclic, $T_t^* a^*$ is bounded only for $a^* = 0$; hence the functionals with bounded adjoint orbit are exactly $\{0\} \oplus X^* = \mathcal{M}^\perp$, as predicted. Moreover, $\ker(A^* \oplus 0 - \lambda I) = \{0\}$ for $\lambda \neq 0$ and $\ker(A^* \oplus 0) = \{0\} \oplus X^* = \mathcal{M}^\perp$, so the inclusion in corollary **??** holds and is sharp at $\lambda = 0$.

3. Conclusion

This paper studied M-hypercyclicity for C_0 -semigroups on separable, infinite-dimensional complex Banach spaces. We showed that M-hypercyclic semigroups can be obtained from hypercyclic ones through intertwining operators (Proposition 2.3 and corollary 2.5), and that any two hypercyclic C_0 - semigroups can be related, via an invertible operator, by M- and $T(M)$-hypercyclicity (Theorem 2.7). On the spectral side, a functional with bounded adjoint orbit must lie in $M^\perp$ (Proposition 2.9), which yields $ker(A^* - \lambda I) \subset M^\perp$ (Corollary 2.10) ; conversely, if the eigenspaces of T_t^* span a dense subspace of X^* , then T is not M-hypercyclic for any non-trivial closed M (Proposition 2.12). The examples built on the translation semigroup and on the unitary group illustrate both the constructions and the obstruction. Natural questions for further work include a characterization of M-hypercyclicity in terms of M-transitivity, and the relation between M and the spectrum of the generator. A further direction concerns the scalar field. Throughout this paper, X is a complex Banach space, a choice inherited from the spectral framework of Desch–Schappacher–Webb and from the formulation of the spectral conditions for all $\lambda \in \mathcal{C}$.

However, most of our arguments (Propositions 2.3 and 2.9, Corollaries 2.5 and 2.11, and Theorem 2.7) rely only on linearity, continuity and density. We therefore leave the real case as an open question:

Question. Do the results of Section 2 remain valid, with the same proofs, for real, separable, infinite-dimensional Banach spaces, with λ ranging over $\mathcal{R}$?

Acknowledgements

The authors extend their heartfelt gratitude to the referees for their invaluable comments and insightful suggestions.

REFERENCES

1. S.I. Ansari, *Existence of hypercyclic operators on topological vector spaces.* J. Funct. Anal. 148, 384-390, 1997.
2. N.Bamerni, V. Kadetsand, A. Kilicman, *Hypercyclic operators are subspace-hypercyclic*, J. Math. Anal. Appl., 435 , pp. 1812–1815, 2016.
3. J. Banasiak, M. Moszynski, *A generalization of Desch-Schappacher- Webb criteria for chaos.* Discr. Conti. Dyn. Syst. 12, 959-972,2005.
4. F. Bayart and E. Matheron, *Dynamics of linear operators.* Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, MR2533318 (2010m:47001),2009.
5. T. Bermudez, *On the existence of chaotic and hypercyclic semigroups on Banach spaces.* Amer. Math. Soc V.131 No. 8, 2435-2441,2003.
6. G D Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières* , C.R. Acad. Sci Paris 189; 473-475, 1929.
7. J. A. Conejero, V. Müller, and A. Peris, *Hypercyclic behavior of operators in a hypercyclic C_0-semigroup*, J. Funct. Anal, 244, 342–348, 2007.
8. W. Desch, W. Schappacher, G.F. Webb, *Hypercyclic and chaotic semigroups of linear operators.* Ergodic Theory Dynamical Systems, 17,1997.
9. K.J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations.* Grad. Texts in Math. vol. **194**, Springer-Verlag, New York, 2000.
10. Grivaux, S. *Construction of Operators with Prescribed behavior*, Arch. Math. (Basel) 81, 291–299, 2003

11. K.-G. Grosse-Erdmann, A.Peris Manguillot, *Linear chaos.* Springer 2011.
12. T.Kalmes, *On chaotic C_0-semigroups and infinity regular hypercyclic vectors.* Proc. Amer. Math. Soc.134 2997-3002, 2006.
13. C. Kitai, *Invariant closed sets for linear operators* Dissertation, University of Toronto, 1982.
14. A. A. M. Malkawi, A. M. Rabaiah, Fractional neutrosophic dynamics: entropy, recurrence, and applications in chaotic systems and signal analysis, Stat., Optim. Inf. Comput., 16 (2026), 4147–4174, doi:10.19139/soic-2310-5070-4417
15. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Appl. Math. Sci. vol. 44, Springer-Verlag, New York,(1992).
16. Rolewicz *On orbits of elements*, Studia Math 32, 17-22, 1969.
17. A.Tajmouati, A.El Bakkali, A.Toukmati, *On M-hypercyclic semigroup.* Int. J. Math. Anal Vol 9 No 9, 417-428. 2015 https://doi.org/10.12988/ijma.2015.412397
18. A.Tajmouati, A. El Bakkali, A.Toukmati, *On some properties of M-hypercyclic C_0-semigroup.* Italian Journal of pure and applied Mathematics. N 35, 351-360, 2015.
19. A.Toukmati, *M-hypercyclicity of C_0-semigroup and Svep of its generator.* Concr. Oper. 8, 187-191, 2021.