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Estimation the Reliability Function and Hazard Rate Function of Pham Distribution under Balanced Joint Progressive Type-II Censored Data

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Abstract This paper investigates the estimation of the reliability and hazard functions of the Pham distribution under a balanced joint progressive type-II censoring scheme. Two estimation approaches are considered: maximum likelihood estimation (MLE) and maximum product of spacings estimation (MPSE). In addition to point estimation, interval estimation is carried out by constructing approximate confidence intervals using MLE and MPSE methods. Moreover, bootstrap-based confidence intervals are obtained using three techniques: the percentile bootstrap, the Student's-t bootstrap and the normal distribution bootstrap. The problem of selecting an optimal censoring scheme is also addressed. A comprehensive Monte Carlo simulation study is conducted to evaluate the finite-sample performance of the proposed estimators in terms of both point and interval estimation. Finally, the practical applicability of the proposed methods is illustrated through the analysis of a real-life data set.

Keywords Maximum likelihood estimation, Maximum product of spacings estimation, Balanced joint progressive Type-II censoring scheme, Bootstrap-based confidence, Pham distribution, Optimal censoring scheme

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1. Introduction

In practical experiments, complete lifetime data are often difficult to obtain because of time and cost constraints. Consequently, censoring is commonly employed to terminate experiments before all test units have failed. Among the most widely used censoring schemes are Type-I, Type-II, and progressive Type-II censoring, which have traditionally been developed for experiments involving a single population. Accordingly, considerable research has been devoted to statistical inference for model parameters and reliability measures under various censoring schemes and lifetime distributions. Some contributions in this direction include the studies of [1], [2], [3], [4], [5], [6], and [7]. However, many real-world studies require the comparison or joint analysis of lifetime data from multiple populations, such as products manufactured under different production processes or patients subjected to different treatment regimens. In such settings, censoring schemes based on a single population are often inadequate. Although progressive Type-II censoring permits the removal of units during the course of an experiment, it may still be inefficient or costly when a sufficiently large number of observed failures is required. Furthermore, experiments conducted on a single population cannot capture dependence or interaction effects between multiple populations.

To overcome these limitations, [8] proposed the joint progressive Type-II censoring (JPTIIC) scheme, which allows simultaneous life-testing of samples drawn from multiple populations, with the flexibility to remove surviving units from any population at each observed failure time. For further developments and detailed discussions of the JPTIIC scheme, see [9], [10], [11], and the references therein. Motivated by the need for a fair and symmetric allocation of removals across populations, a balanced version of the JPTIIC scheme was subsequently introduced by [12].

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The B-JPCTII scheme can be briefly described as follows. Consider a life-testing experiment involving two populations, namely Population–1 (Pop–1) and Population–2 (Pop–2). Let n_1 and n_2 denote the sample sizes from Pop–1 and Pop–2, respectively, and let k be the predetermined total number of failures to be observed, where $k < \min\{n_1, n_2\}$. Let the preassigned removal scheme be $(R_1, \ldots, R_k)$, which must satisfy

$$\sum_{i=1}^{k-1}(R_i + 1) < \min\{n_1, n_2\}, \qquad R_i \geq 0, \quad i = 1, 2, \ldots, k.$$

Suppose that the first failure occurs at time T_1 from Pop–1. Then R_1 units are randomly removed from the remaining units of Pop–1, while $R_1 + 1$ units are randomly removed from Pop–2. At the second failure time T_2, if the failed unit belongs to Pop–2, then $R_2 + 1$ units are randomly removed from Pop–1 and R_2 units from Pop–2. In general, at the i-th failure time T_i, for $i = 1, \ldots, k-1$, R_i units are removed from the population in which the failure occurs and $R_i + 1$ units are removed from the other population. When the k-th failure occurs at time T_k, all remaining units from both populations are removed and the experiment is terminated.

Under the B-JPCTII scheme, an additional sequence of indicator variables $(M_1, \ldots, M_k)$ is observed together with $(T_1, \ldots, T_k)$, where $M_i = 1$ if the i-th failure is from Pop–1 and $M_i = 0$ otherwise. The total numbers of observed failures from Pop–1 and Pop–2 are given by

$$K_1 = \sum_{i=1}^{k} M_i, \qquad K_2 = \sum_{i=1}^{k}(1 - M_i) = k - K_1,$$

respectively. Thus, the observed data under the B-JPCTII scheme consist of $((T_1, M_1), \ldots, (T_k, M_k))$; see [13] for further details. Figure 1 illustrates the implementation of the B-JPCTII scheme.

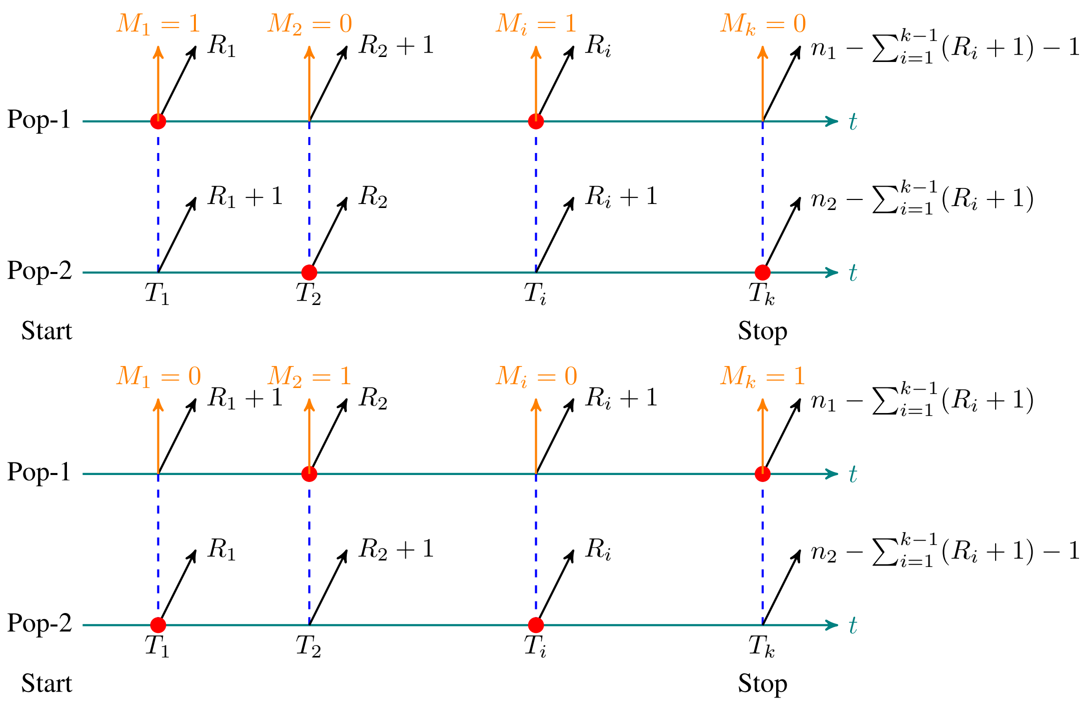


Figure 1. Graphical illustration of the B-JPCTII censoring scheme.

The B-JPCTII scheme has attracted considerable attention in comparative lifetime testing and reliability studies. Based on this scheme, [12] introduced the B-JPCTII scheme as a more analytically tractable alternative to the joint progressive Type-II censoring scheme. Assuming exponential lifetimes with different scale parameters, they developed maximum likelihood estimation, derived the exact distributions of the estimators, and constructed exact and bootstrap confidence intervals. [14] developed classical and Bayesian inference for two Lindley populations under the B-JPCTII censoring scheme. The study derived maximum likelihood estimators, asymptotic confidence intervals, Bayesian estimators under informative priors, and discussed optimal censoring criteria. [13] considered two Gompertz populations under the B-JPCTII censoring scheme with a common shape parameter and different scale parameters. They established the existence of the maximum likelihood estimators and proposed EM and stochastic EM algorithms, bootstrap and asymptotic confidence intervals, Bayesian estimation under squared-error and LINEX loss functions, and order-restricted inference. [15] investigated statistical inference for two Lomax populations under the B-JPCTII scheme. The study developed maximum likelihood estimation, constructed asymptotic confidence intervals using the observed Fisher information matrix, and obtained Bayesian estimators under informative priors. The performance of the proposed estimators was assessed through simulation studies, and a real data example was presented to illustrate the methodology. Recently, [16] studied inference for the Chen distribution under a balanced joint progressive Type-II censoring scheme. Maximum likelihood and Bayesian estimation methods, together with asymptotic, bootstrap, and HPD interval estimation procedures, were developed and evaluated through simulation and real data analysis. More recently [17] investigated competing risks under balanced joint progressive censoring using the Burr-XII distribution. Maximum likelihood and Bayesian estimation methods, along with several confidence interval procedures, were proposed and assessed using simulation studies and real data.

The Pham distribution was first introduced by [18], who applied this model to evaluate the reliability of several helicopter components. Subsequently, the same distribution was employed to model mortality rates using United States data from the year 2005; see [19]. The distribution is also referred to as the log-log distribution in the literature.

The cumulative distribution function (CDF) and the corresponding probability density function (PDF) of the two-parameter Pham distribution are given, respectively, by

$$F(x;\theta,\beta) = 1 - \exp\Big(1 - \beta^{x^{\theta}}\Big), \qquad x > 0,\ \theta > 0,\ \beta > 1, \tag{1}$$

and

$$f(x;\theta,\beta) = \theta \log(\beta)\, x^{\theta-1}\, \beta^{x^{\theta}} \exp\Big(1 - \beta^{x^{\theta}}\Big), \qquad x > 0,\ \theta > 0,\ \beta > 1. \tag{2}$$

Accordingly, the SRF and the HRF are expressed as

$$S(x) \equiv S(x;\theta,\beta) = \exp\Big(1 - \beta^{x^{\theta}}\Big), \qquad x > 0, \tag{3}$$

and

$$H(x) \equiv H(x;\theta,\beta) = \theta \log(\beta)\, x^{\theta-1}\, \beta^{x^{\theta}}, \qquad x > 0, \tag{4}$$

respectively. Moreover, the p-th quantile function of the Pham distribution is given by

$$Q(p) \equiv Q(p;\theta,\beta) = \left[\frac{\log\big(1 - \log(1-p)\big)}{\log(\beta)}\right]^{1/\theta}, \qquad 0 < p < 1. \tag{5}$$

Furthermore, [18] showed that the hazard rate function is decreasing for $x \leq x_0$ and increasing for $x \geq x_0$, where

$$x_0 = \left[\frac{1-\theta}{\theta \log(\beta)}\right]^{1/\theta}.$$

The Pham distribution has several attractive properties that make it suitable for reliability analysis. Its probability density function can be either decreasing or unimodal, while its hazard rate belongs to the class of Vtub-shaped hazard functions. This class encompasses decreasing, increasing, and bathtub-type failure-rate behaviors, thereby providing greater flexibility than models restricted to monotone hazards. Such flexibility enables the distribution to describe systems that experience an initial reduction in failure risk followed by a gradual increase due to aging or wear. Furthermore, the practical usefulness of the Pham distribution has been demonstrated through the reliability analysis of helicopter components based on data from the Maintenance Malfunction Information Reporting (MMIR) system. Motivated by these theoretical and practical advantages, together with the absence of inference procedures under the B-JPCTII censoring scheme, this paper develops both classical and Bayesian estimation methods for the Pham distribution. The PDF, CDF, and HRF for selected parameter values are illustrated in Figure 2.

Several statistical properties of the Pham distribution have been investigated in the literature, including the shapes of the hazard rate function, the mean residual lifetime, and parameter estimation. In particular, [20] derived Bayesian estimators of the model parameters using Markov chain Monte Carlo (MCMC) techniques. Additional theoretical developments and inferential results for the Pham distribution can be found in [19] and, more recently, in [21].

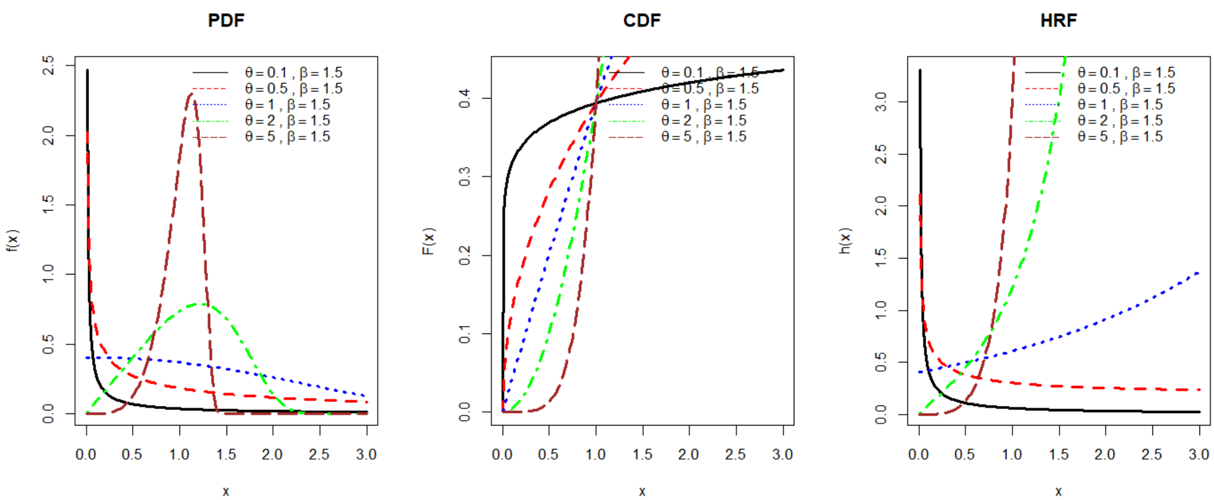


Figure 2. The PDF, CDF and HRF of Pham distribution for various parameters

As an alternative to maximum likelihood estimation method, [22] introduced the maximum product of spacings estimation method. This method was further discussed by [23]. In recent times, maximum product of spacings has been employed in different context in statistical inferences including classical and Bayesian. One may refer to the works of [24], [25], [26], [27], [28] and the cited therein. [29] pointed out that maximum product of spacings follows invariance property and is more effective than maximum likelihood estimation for skewed distributions as well as small sample cases for heavy tailed distributions. The maximum product of spacings methodology constitutes a statistical estimation paradigm employed to derive parameter estimates for probability distributions, particularly within frameworks involving censored or incomplete data sets. In contrast to the maximization of the likelihood function (as observed in Maximum Likelihood Estimation), which is contingent upon the probability density function, the maximum product of spacings framework seeks to maximize the product of the spacings delineated by consecutive values of the cumulative distribution function $F(t)$ at the observed failure times. These spacings are articulated as $F(t_i) - F(t_{i-1})$, where $t_0 = 0$ and t_{k+1} signifies a sufficiently elevated threshold. In the context of B-JPCTII, the product of spacings (PS) function is modified to accommodate both the spacings between the observed failures and the survival probabilities pertaining to the units excised at each stage of failure. The resultant estimates are procured through the numerical maximization of the log-spacing function, given that closed-form solutions are predominantly absent. The maximum product of spacings methodology is recognized for its robustness, particularly when the likelihood function exhibits irregular characteristics or when sample sizes are limited, thereby rendering it a dependable alternative to conventional maximum likelihood estimation in intricate censoring scenarios.

Although several studies have developed likelihood- and Bayesian-based inference procedures under the B-JPCTII censoring scheme, the simultaneous estimation of the HRF and SRF for two Pham populations using both the maximum likelihood estimator (MLE) and the maximum product of spacings estimator (MPSE) has not yet been investigated. This gap is particularly important because the Pham distribution can accommodate skewed lifetime data and Vtub-shaped hazard rates, while the B-JPCTII scheme produces a complex incomplete-data structure that may affect the performance of different estimation methods.

Motivated by these considerations, this study develops classical inference procedures for the HRF and SRF based on both MLE and MPSE. Approximate confidence intervals (ACIs) are derived for the two estimators, and their finite-sample performance is systematically compared. Furthermore, parametric bootstrap confidence intervals are constructed to improve interval estimation, and several optimality criteria are investigated to identify the most efficient _censoring scheme. The proposed methods are evaluated through extensive Monte Carlo simulations and illustrated using two real data sets. The findings provide practical guidance for selecting accurate and robust estimation procedures for reliability and hazard rate functions under B-JPCTII schemes.

The rest of the paper is structured as follows. The MLE and MPSE approaches, as well as the creation of ACIs under the B-JPCTII scheme, are presented in Sections 2 and 3, respectively. Confidence intervals are created in Section 4 utilizing bootstrap techniques, such as the Student's-t (boot-T), percentile bootstrap (boot-P) and normal bootstrap (boot-N) approaches. In Section 5, comprehensive simulation studies are used to assess the finite-sample performance of the suggested approaches. Section 6 discusses optimal B-JPCTII schemes. A real-world data collection is used to demonstrate the suggested approaches in Section 7. Lastly, Section 8 concludes the study by summarizing the main findings, discussing their practical implications, and suggesting directions for future research.

2. Maximum Likelihood Estimation

Let $X_1, \ldots, X_{n_1}$ be a random sample from Pop-1 following the Pham with parameters (θ_1, β_1), and let $Y_1, \ldots, Y_{n_2}$ be a random sample from Pop-2 following the same distribution with parameters (θ_2, β_2). Different censoring schemes are assumed for the two samples. Accordingly, let $R_{1i} \geq 0$ and $R_{2i} \geq 0$ denote the numbers of units removed from Pop-1 and Pop-2, respectively, at the failure time T_i, for $i = 1, 2, \ldots, k$.

The B-JPCTII scheme ensures that the total number of units removed from each population is fixed in advance. To identify the population from which each failure originates, define the indicator variable

$$M_i = I(T_i \in \text{Pop-1}), \qquad i = 1, 2, \ldots, k,$$

where $I(\cdot)$ denotes the indicator function. Consequently, the observed data can be expressed as

$$\text{data} = \big\{(T_1, M_1), (T_2, M_2), \ldots, (T_k, M_k)\big\}.$$

Hereafter, let $\Theta = (\theta_1, \theta_2, \beta_1, \beta_2)$. Using (1) and (2), the likelihood (LK) function based on the observed data is given by

$$\begin{aligned} L(\Theta) &= \prod_{i=1}^{k} [f(t_i; \theta_1, \beta_1)]^{m_i} [f(t_i; \theta_2, \beta_2)]^{1-m_i} [1 - F(t_i; \theta_1, \beta_1)]^{R_{1i}-m_i+1} [1 - F(t_i; \theta_2, \beta_2)]^{R_{2i}+m_i} \\ &= \prod_{i=1}^{k} \left[\theta_1 \ln(\beta_1) t_i^{\theta_1 - 1} \beta_1^{t_i^{\theta_1}} e^{1-\beta_1^{t_i^{\theta_1}}}\right]^{m_i} \left[\theta_2 \ln(\beta_2) t_i^{\theta_2 - 1} \beta_2^{t_i^{\theta_2}} e^{1-\beta_2^{t_i^{\theta_2}}}\right]^{1-m_i} \\ &\qquad \times \exp\left\{(R_{1i} - m_i + 1)\big(1 - \beta_1^{t_i^{\theta_1}}\big) + (R_{2i} + m_i)\big(1 - \beta_2^{t_i^{\theta_2}}\big)\right\}, \end{aligned} \tag{6}$$

where t_i and m_i denote the observed values of T_i and M_i, respectively. The corresponding log-likelihood function is

$$\ell(\Theta) = \sum_{i=1}^{k} \Bigg\{ m_i \Big[\ln\theta_1 + \ln(\ln\beta_1) + (\theta_1 - 1)\ln t_i + t_i^{\theta_1}\ln\beta_1 + 1 - \beta_1^{t_i^{\theta_1}} \Big] + (1-m_i)\Big[\ln\theta_2 + \ln(\ln\beta_2) + (\theta_2 - 1)\ln t_i + t_i^{\theta_2}\ln\beta_2 + 1 - \beta_2^{t_i^{\theta_2}} \Big] + (R_{1i} - m_i + 1)\big(1 - \beta_1^{t_i^{\theta_1}}\big) + (R_{2i} + m_i)\big(1 - \beta_2^{t_i^{\theta_2}}\big) \Bigg\}. \tag{7}$$

The likelihood equations are obtained by differentiating (7) with respect to the unknown parameters and equating the resulting expressions to zero. Specifically,

$$\frac{\partial \ell}{\partial \theta_1} = \sum_{i=1}^{k} \left\{ m_i \left(\frac{1}{\theta_1} + \ln t_i + t_i^{\theta_1}\ln t_i \ln\beta_1 \right) - (R_{1i}+1)\beta_1^{t_i^{\theta_1}} \ln\beta_1\, t_i^{\theta_1}\ln t_i \right\} = 0, \tag{8}$$

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{k} \left\{ m_i \left(\frac{1}{\beta_1 \ln\beta_1} + \frac{t_i^{\theta_1}}{\beta_1} \right) - (R_{1i}+1) t_i^{\theta_1} \beta_1^{t_i^{\theta_1}-1} \right\} = 0, \tag{9}$$

$$\frac{\partial \ell}{\partial \theta_2} = \sum_{i=1}^{k} \left\{ (1-m_i) \left(\frac{1}{\theta_2} + \ln t_i + t_i^{\theta_2}\ln t_i \ln\beta_2 \right) - (R_{2i}+1)\beta_2^{t_i^{\theta_2}} \ln\beta_2\, t_i^{\theta_2}\ln t_i \right\} = 0, \tag{10}$$

$$\frac{\partial \ell}{\partial \beta_2} = \sum_{i=1}^{k} \left\{ (1-m_i) \left(\frac{1}{\beta_2 \ln\beta_2} + \frac{t_i^{\theta_2}}{\beta_2} \right) - (R_{2i}+1) t_i^{\theta_2} \beta_2^{t_i^{\theta_2}-1} \right\} = 0. \tag{11}$$

Now, we study the existence and uniqueness of the MLEs of the parameters $\theta_1, \theta_2, \beta_1$ and β_2 of the Pham distribution under the B-JPCTII.

Theorem 2.1
Let

$$\xi_i(\theta_i) = \frac{\partial \ell}{\partial \theta_i}, \qquad \xi_i(\beta_i) = \frac{\partial \ell}{\partial \beta_i}, \qquad i = 1, 2.$$

The functions $\xi_i(\theta_i)$ and $\xi_i(\beta_i)$ attain unique solutions in the intervals $\theta_i \in (0, \infty)$ and $\beta_i \in (1, \infty)$. Let $\hat{\theta}_i$ and $\hat{\beta}_i$ be the solutions of $\xi_i(\theta_i) = 0$ and $\xi_i(\beta_i) = 0$. Then the MLEs exist and are unique provided that $K_1 > 0$ and $K_2 > 0$.

The proof of Theorem 2.1 is deferred to Appendix (B).

The MLEs of θ_1, θ_2, β_1, and β_2 are obtained by solving the system of likelihood equations in (8)-(11) simultaneously. These estimators are denoted by $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\beta}_1$, and $\hat{\beta}_2$, respectively. Due to the nonlinear nature of the estimating equations, closed-form solutions are not available. Consequently, the MLEs are obtained numerically using the `optim` function in R with the `"L-BFGS-B"` optimization algorithm. This algorithm allows explicit box constraints to be imposed on the parameter space. In particular, the lower bound for β is specified to ensure that the constraint $\beta > 1$ is satisfied throughout the optimization process.

Finally, the HRF and SRF are estimated using the invariance property of the MLEs. Specifically, the MLEs of the HRF and SRF for Pop-i are given by

$$\hat{H}_i(x) = H(x; \hat{\theta}_i, \hat{\beta}_i), \qquad \hat{S}_i(x) = S(x; \hat{\theta}_i, \hat{\beta}_i), \qquad i = 1, 2.$$

2.1. Approximate Confidence Interval Based on MLEs

In this subsection, we consider interval estimation method by constructing approximated confidence intervals (ACIs) for the model parameters utilizing the Fisher information matrix of the model parameters. In addition, the ACIs for the HRF and SRF are also obtained. Since the closed forms as well as the exact distribution of the MLEs for the parameter Θ cannot be obtained, then the ACIs are constructed here based on observed Fisher information matrix.

The second-order partial derivatives of the log-likelihood function(7) with respect to Θ are provided in Appendix.

By the asymptotic normality of the maximum likelihood estimators, we have

$$\hat{\Theta}\sim N\left(\Theta, \mathbf{I}^{-1}(\hat{\Theta})\right),$$

where $\mathbf{I}^{-1}(\hat{\Theta})$ denotes the inverse of the observed Fisher information matrix evaluated at the MLEs and is given by

$$\begin{aligned}\mathbf{I}^{-1}(\hat{\Theta}) &= \begin{bmatrix} -\frac{\partial^2\ell}{\partial\theta_1^2} & -\frac{\partial^2\ell}{\partial\theta_1\partial\beta_1} & -\frac{\partial^2\ell}{\partial\theta_1\partial\theta_2} & -\frac{\partial^2\ell}{\partial\theta_1\partial\beta_2} \\ -\frac{\partial^2\ell}{\partial\beta_1\partial\theta_1} & -\frac{\partial^2\ell}{\partial\beta_1^2} & -\frac{\partial^2\ell}{\partial\beta_1\partial\theta_2} & -\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2} \\ -\frac{\partial^2\ell}{\partial\theta_2\partial\theta_1} & -\frac{\partial^2\ell}{\partial\theta_2\partial\beta_1} & -\frac{\partial^2\ell}{\partial\theta_2^2} & -\frac{\partial^2\ell}{\partial\theta_2\partial\beta_2} \\ -\frac{\partial^2\ell}{\partial\beta_2\partial\theta_1} & -\frac{\partial^2\ell}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2\ell}{\partial\beta_2\partial\theta_2} & -\frac{\partial^2\ell}{\partial\beta_2^2} \end{bmatrix}^{-1}_{\Theta=\hat{\Theta}} \\ &= \begin{bmatrix} \widehat{\mathrm{Var}}(\hat{\theta}_1) & \widehat{\mathrm{Cov}}(\hat{\theta}_1,\hat{\beta}_1) & \widehat{\mathrm{Cov}}(\hat{\theta}_1,\hat{\theta}_2) & \widehat{\mathrm{Cov}}(\hat{\theta}_1,\hat{\beta}_2) \\ \widehat{\mathrm{Cov}}(\hat{\beta}_1,\hat{\theta}_1) & \widehat{\mathrm{Var}}(\hat{\beta}_1) & \widehat{\mathrm{Cov}}(\hat{\beta}_1,\hat{\theta}_2) & \widehat{\mathrm{Cov}}(\hat{\beta}_1,\hat{\beta}_2) \\ \widehat{\mathrm{Cov}}(\hat{\theta}_2,\hat{\theta}_1) & \widehat{\mathrm{Cov}}(\hat{\theta}_2,\hat{\beta}_1) & \widehat{\mathrm{Var}}(\hat{\theta}_2) & \widehat{\mathrm{Cov}}(\hat{\theta}_2,\hat{\beta}_2) \\ \widehat{\mathrm{Cov}}(\hat{\beta}_2,\hat{\theta}_1) & \widehat{\mathrm{Cov}}(\hat{\beta}_2,\hat{\beta}_1) & \widehat{\mathrm{Cov}}(\hat{\beta}_2,\hat{\theta}_2) & \widehat{\mathrm{Var}}(\hat{\beta}_2) \end{bmatrix}_{\Theta=\hat{\Theta}}. \end{aligned} \tag{12}$$

Then the $100(1-\alpha)\%$ confidence intervals of $\theta_1, \theta_2, \beta_1$ and β_2 are

$$\hat{\theta}_i \mp z_{\alpha/2}\sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_i)}, \quad \hat{\beta}_i \mp z_{\alpha/2}\sqrt{\widehat{\mathrm{Var}}(\hat{\beta}_i)}, \; i=1,2.$$

where $z_{\alpha/2}$ is the upper $\alpha/2$-th percentile of $N(0,1)$.

Furthermore, the corresponding empirical coverage probabilities (CP) of Θ are

$$\mathrm{CP}_{\theta_i} = \mathrm{P}\big[\hat{\theta}_i \in \mathrm{ACI}_{\theta_i}\big], \quad \mathrm{CP}_{\beta_i} = \mathrm{P}\big[\hat{\beta}_i \in \mathrm{ACI}_{\beta_i}\big], \; i=1,2.$$

Next, the approximate variance of $\hat{H}_i(x)$ and $\hat{S}_i(x), i=1,2$ can be derived using the delta method. Define the gradient vector of H and S with respect to the model parameters as:

$$\nabla H(x;\theta,\beta) = \left[\frac{\partial H(x;\theta,\beta)}{\partial\theta}, \frac{\partial H(x;\theta,\beta)}{\partial\beta}\right], \quad \nabla S(x;\theta,\beta) = \left[\frac{\partial S(x;\theta,\beta)}{\partial\theta}, \frac{\partial S(x;\theta,\beta)}{\partial\beta}\right].$$

The required partial derivatives are:

$$\begin{aligned} \frac{\partial H(x;\theta_i,\beta_i)}{\partial\theta_i} &= H(x;\theta_i,\beta_i)\left[\frac{1}{\theta_i} + \ln(x)\left(1 + x^{\theta_i}\ln\beta_i\right)\right], \\ \frac{\partial H(x;\theta_i,\beta_i)}{\partial\beta_i} &= H(x;\theta_i,\beta_i)\left[\frac{1 + x^{\theta_i}\ln\beta_i}{\beta\ln(\beta_i)}\right], \\ \frac{\partial S(x;\theta_i,\beta_i)}{\partial\theta_i} &= -e^{\,1-\beta_i^{x^{\theta_i}}}\,\beta_i^{x^{\theta_i}}\ln(\beta_i)\,x^{\theta_i}\ln(x), \\ \frac{\partial S(x;\theta_i,\beta_i)}{\partial\beta_i} &= -e^{\,1-\beta_i^{x^{\theta_i}}}\,x^{\theta_i}\beta_i^{x^{\theta_i}-1}. \end{aligned}$$

Then, the approximate variances of $\hat{H}_i(x)$ and $\hat{S}_i(x)$ are given by:

$$\widehat{\mathrm{Var}}[\hat{H}_i(x)] \approx \left[\nabla H(x;\theta_i,\beta_i)\, V_i\, [\nabla H(x;\theta_i,\beta_i)]^\top\right]_{\Theta=\hat{\Theta}},$$

$$\widehat{\mathrm{Var}}[\hat{S}_i(x)] \approx \left[\nabla S(x;\theta_i,\beta_i)\, V_i\, [\nabla S(x;\theta_i,\beta_i)]^\top\right]_{\Theta=\hat{\Theta}},$$

where V_i is the 2×2 variance-covariance matrix corresponding to $(\hat{\theta}_i, \hat{\beta}_i)$, extracted from the full estimated covariance matrix $\mathbf{I}^{-1}(\hat{\Theta})$ given in (12).

Consequently, an approximate $100(1-\alpha)\%$ confidence interval for $\hat{H}_i(x)$ and $\hat{S}_i(x)$ are:

$$\hat{H}_i(x) \mp z_{\alpha/2}\sqrt{\widehat{\mathrm{Var}}[\hat{H}_i(x)]}, \quad \hat{S}_i(x) \mp z_{\alpha/2}\sqrt{\widehat{\mathrm{Var}}[\hat{S}_i(x)]}, i=1,2.$$

3. Maximum Product of Spacings Estimation

In this section, we estimate the model parameters for both populations using the maximum product of spacings (MPS) method. The maximum product of spacings estimation method, introduced by [22], has emerged as an effective alternative to the traditional maximum likelihood estimation. Owing to its strong theoretical properties and practical robustness, it has received considerable attention in the statistical literature. As noted by [29], the maximum product of spacings estimate is invariant under one-to-one transformations and often outperforms MLE for small samples and skewed or heavy-tailed distributions. Moreover, [30] and [31] established that, under regularity conditions, maximum product of spacings estimators are consistent, asymptotically normal, and asymptotically efficient. These attractive properties have led to its widespread application in process capability analysis and the development of classical and Bayesian inferential procedures. Representative contributions include [24], [26], [32], [27], and [28].

Under the B-JPCTII scheme, the product-of-spacings (PS) function is defined as

$$\begin{aligned} P(\Theta) &= \prod_{i=1}^{k+1}\left[F(t_i;\theta_1,\beta_1) - F(t_{i-1};\theta_1,\beta_1)\right]^{m_i}\left[F(t_i;\theta_2,\beta_2) - F(t_{i-1};\theta_2,\beta_2)\right]^{1-m_i} \\ &\quad\times\prod_{i=1}^{k}\left[1-F(t_i;\theta_1,\beta_1)\right]^{R_{1i}-m_i+1}\left[1-F(t_i;\theta_2,\beta_2)\right]^{R_{2i}+m_i} \\ &= \prod_{i=1}^{k+1}\left[e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\right]^{m_i}\left[e^{1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{1-\beta_2^{t_i^{\theta_2}}}\right]^{1-m_i} \\ &\quad\times\prod_{i=1}^{k}\left[e^{1-\beta_1^{t_i^{\theta_1}}}\right]^{R_{1i}-m_i+1}\left[e^{1-\beta_2^{t_i^{\theta_2}}}\right]^{R_{2i}+m_i}. \end{aligned} \tag{13}$$

The corresponding log-PS function is

$$\begin{aligned} p(\Theta) &= \sum_{i=1}^{k+1} m_i \ln\left(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\right) + \sum_{i=1}^{k+1}(1-m_i)\ln\left(e^{1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{1-\beta_2^{t_i^{\theta_2}}}\right) \\ &\quad+\sum_{i=1}^{k}(R_{1i}-m_i+1)\left(1-\beta_1^{t_i^{\theta_1}}\right) + \sum_{i=1}^{k}(R_{2i}+m_i)\left(1-\beta_2^{t_i^{\theta_2}}\right). \end{aligned} \tag{14}$$

Here, $t_0 = 0$ and $t_{k+1} = \infty$, and $F(\cdot)$ denotes the CDF given in (1). Moreover, m_{k+1} is randomly chosen from the set $\{0, 1\}$.

The MPSEs $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\beta}_1$, and $\tilde{\beta}_2$ of $\theta_1, \theta_2, \beta_1$, and β_2 are obtained by maximizing (14) with respect to $\theta_1, \theta_2, \beta_1$, and β_2. Equivalently, they can be computed by solving the following system of nonlinear equations:

$$\begin{aligned}
\frac{\partial p}{\partial \theta_1} &= \sum_{i=1}^{k+1} m_i \left(\frac{-e^{\,1-\beta_1^{t_{i-1}^{\theta_1}}}\, \beta_1^{t_{i-1}^{\theta_1}} \ln \beta_1\, t_{i-1}^{\theta_1} \ln t_{i-1} + e^{\,1-\beta_1^{t_i^{\theta_1}}}\, \beta_1^{t_i^{\theta_1}} \ln \beta_1\, t_i^{\theta_1} \ln t_i}{e^{\,1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{\,1-\beta_1^{t_i^{\theta_1}}}} \right) \\
&\quad - \sum_{i=1}^{k} (R_{1i} - m_i + 1)\, \beta_1^{t_i^{\theta_1}} \ln \beta_1\, t_i^{\theta_1} \ln t_i, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p}{\partial \beta_1} &= \sum_{i=1}^{k+1} m_i \left(\frac{-e^{\,1-\beta_1^{t_{i-1}^{\theta_1}}}\, t_{i-1}^{\theta_1} \beta_1^{t_{i-1}^{\theta_1}-1} + e^{\,1-\beta_1^{t_i^{\theta_1}}}\, t_i^{\theta_1}\, \beta_1^{t_i^{\theta_1}-1}}{e^{\,1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{\,1-\beta_1^{t_i^{\theta_1}}}} \right) \\
&\quad - \sum_{i=1}^{k} (R_{1i} - m_i + 1)\, t_i^{\theta_1}\, \beta_1^{t_i^{\theta_1}-1}, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p}{\partial \theta_2} &= \sum_{i=1}^{k+1} (1 - m_i) \left(\frac{-e^{\,1-\beta_2^{t_{i-1}^{\theta_2}}}\, \beta_2^{t_{i-1}^{\theta_2}} \ln \beta_2\, t_{i-1}^{\theta_2} \ln t_{i-1} + e^{\,1-\beta_2^{t_i^{\theta_2}}}\, \beta_2^{t_i^{\theta_2}} \ln \beta_2\, t_i^{\theta_2} \ln t_i}{e^{\,1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{\,1-\beta_2^{t_i^{\theta_2}}}} \right) \\
&\quad - \sum_{i=1}^{k} (R_{2i} + m_i)\, \beta_2^{t_i^{\theta_2}} \ln \beta_2\, t_i^{\theta_2} \ln t_i, \qquad (17)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p}{\partial \beta_2} &= \sum_{i=1}^{k+1} (1 - m_i) \left(\frac{-e^{\,1-\beta_2^{t_{i-1}^{\theta_2}}}\, t_{i-1}^{\theta_2}\, \beta_2^{t_{i-1}^{\theta_2}-1} + e^{\,1-\beta_2^{t_i^{\theta_2}}}\, t_i^{\theta_2}\, \beta_2^{t_i^{\theta_2}-1}}{e^{\,1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{\,1-\beta_2^{t_i^{\theta_2}}}} \right) \\
&\quad - \sum_{i=1}^{k} (R_{2i} + m_i)\, t_i^{\theta_2}\, \beta_2^{t_i^{\theta_2}-1}. \qquad (18)
\end{aligned}$$

As in the maximum likelihood approach, closed-form solutions for the MPSEs are not available. Therefore, the MPSEs are computed numerically using the `optim` function in R with the "`L-BFGS-B`" optimization algorithm, which ensures that the constraint $\beta > 1$ is satisfied throughout the optimization process.

Consequently, the HRF and SRF for population i can be estimated via the invariance property of the MPSEs as

$$\tilde{H}_i(x) = H(x; \tilde{\theta}_i, \tilde{\beta}_i), \qquad \tilde{S}_i(x) = S(x; \tilde{\theta}_i, \tilde{\beta}_i), \qquad i = 1, 2.$$

3.1. Approximate Confidence Intervals Based on MPSEs

In this subsection, we construct approximate confidence intervals for Θ based on the MPSEs. Due to the complexity of the nonlinear system of equations (15) -(18), the exact sampling distribution of the MPSEs cannot be determined analytically, making exact confidence intervals unattainable. However, it is well established that, under suitable regularity conditions, the MLEs and MPSEs are asymptotically equivalent; see [29], [22] and[23]. Using the asymptotic normality of the MPS estimator (see, [29]), we have

$$\tilde{\Theta} \sim N\big(\Theta, \mathbf{J}^{-1}(\Theta)\big),$$

where $\mathbf{J}(\Theta)$ denotes the Fisher information matrix. An estimate of the asymptotic variance–covariance matrix is obtained by evaluating the observed information at $\tilde{\Theta}$, namely,

$$\mathbf{J}^{-1}(\tilde{\Theta}) = \begin{bmatrix} -\frac{\partial^2 p}{\partial \theta_1^2} & -\frac{\partial^2 p}{\partial \theta_1 \partial \beta_1} & -\frac{\partial^2 p}{\partial \theta_1 \partial \theta_2} & -\frac{\partial^2 p}{\partial \theta_1 \partial \beta_2} \\ -\frac{\partial^2 p}{\partial \beta_1 \partial \theta_1} & -\frac{\partial^2 p}{\partial \beta_1^2} & -\frac{\partial^2 p}{\partial \beta_1 \partial \theta_2} & -\frac{\partial^2 p}{\partial \beta_1 \partial \beta_2} \\ -\frac{\partial^2 p}{\partial \theta_2 \partial \theta_1} & -\frac{\partial^2 p}{\partial \theta_2 \partial \beta_1} & -\frac{\partial^2 p}{\partial \theta_2^2} & -\frac{\partial^2 p}{\partial \theta_2 \partial \beta_2} \\ -\frac{\partial^2 p}{\partial \beta_2 \partial \theta_1} & -\frac{\partial^2 p}{\partial \beta_2 \partial \beta_1} & -\frac{\partial^2 p}{\partial \beta_2 \partial \theta_2} & -\frac{\partial^2 p}{\partial \beta_2^2} \end{bmatrix}^{-1}_{\Theta=\tilde{\Theta}}$$

$$= \begin{bmatrix} \widetilde{\mathrm{Var}}(\tilde{\theta}_1) & \widetilde{\mathrm{Cov}}(\tilde{\theta}_1, \tilde{\beta}_1) & \widetilde{\mathrm{Cov}}(\tilde{\theta}_1, \tilde{\theta}_2) & \widetilde{\mathrm{Cov}}(\tilde{\theta}_1, \tilde{\beta}_2) \\ \widetilde{\mathrm{Cov}}(\tilde{\beta}_1, \tilde{\theta}_1) & \widetilde{\mathrm{Var}}(\tilde{\beta}_1) & \widetilde{\mathrm{Cov}}(\tilde{\beta}_1, \tilde{\theta}_2) & \widetilde{\mathrm{Cov}}(\tilde{\beta}_1, \tilde{\beta}_2) \\ \widetilde{\mathrm{Cov}}(\tilde{\theta}_2, \tilde{\theta}_1) & \widetilde{\mathrm{Cov}}(\tilde{\theta}_2, \tilde{\beta}_1) & \widetilde{\mathrm{Var}}(\tilde{\theta}_2) & \widetilde{\mathrm{Cov}}(\tilde{\theta}_2, \tilde{\beta}_2) \\ \widetilde{\mathrm{Cov}}(\tilde{\beta}_2, \tilde{\theta}_1) & \widetilde{\mathrm{Cov}}(\tilde{\beta}_2, \tilde{\beta}_1) & \widetilde{\mathrm{Cov}}(\tilde{\beta}_2, \tilde{\theta}_2) & \widetilde{\mathrm{Var}}(\tilde{\beta}_2) \end{bmatrix}_{\Theta=\tilde{\Theta}}. \tag{19}$$

The partial derivative terms of the Fisher information matrix are provided in Appendix. Then, the $100(1-\alpha)\%$ approximate confidence intervals for θ_i and β_i $(i = 1, 2)$ are given by

$$\mathrm{ACI}_{\theta_i} = \tilde{\theta}_i \mp z_{\alpha/2}\sqrt{\widetilde{\mathrm{Var}}(\tilde{\theta}_i)}, \qquad \mathrm{ACI}_{\beta_i} = \tilde{\beta}_i \mp z_{\alpha/2}\sqrt{\widetilde{\mathrm{Var}}(\tilde{\beta}_i)},$$

and approximate confidence intervals for the HRF and SRF can be expressed as

$$\tilde{H}_i(x) \mp z_{\alpha/2}\sqrt{\widetilde{\mathrm{Var}}[\tilde{H}_i(x)]}, \qquad \tilde{S}_i(x) \mp z_{\alpha/2}\sqrt{\widetilde{\mathrm{Var}}[\tilde{S}_i(x)]}, \qquad i = 1, 2.$$

The approximate variances of $\tilde{H}_i(x)$ and $\tilde{S}_i(x)$ can be obtained using the delta method as

$$\widetilde{\mathrm{Var}}[\tilde{H}_i(x)] \approx \left[\nabla H(x; \theta_i, \beta_i)\, U_i\, \nabla H(x; \theta_i, \beta_i)^\top\right]_{\Theta=\tilde{\Theta}},$$

$$\widetilde{\mathrm{Var}}[\tilde{S}_i(x)] \approx \left[\nabla S(x; \theta_i, \beta_i)\, U_i\, \nabla S(x; \theta_i, \beta_i)^\top\right]_{\Theta=\tilde{\Theta}},$$

where U_i is the 2×2 variance–covariance matrix of $(\tilde{\theta}_i, \tilde{\beta}_i)$ extracted from the full estimated matrix $\mathbf{J}^{-1}(\tilde{\Theta})$ in (19).

4. Bootstrap Confidence Intervals

Bootstrap resampling is a common method for assessing the accuracy of estimators, especially for small samples where standard inference methods may be unreliable. It repeatedly samples with replacement from the observed data to estimate bias, standard errors, and confidence intervals without strong distributional assumptions. In this section, we construct three bootstrap-based confidence intervals: the percentile bootstrap (boot-P) proposed by [33], the bootstrap-t method (boot-T) proposed by [34] and the bootstrap normal approximation (boot-N) described in [35, 36].

The bootstrap percentile (boot-P) interval procedure is a simple and efficient method for constructing confidence intervals using sample percentiles. However, it does not account for estimator bias and may perform poorly for small or skewed samples. Algorithm 1 outlines the boot-P procedure. The Student's-t interval procedure (boot-T) is well suited for small samples and asymmetric sampling distributions. It is generally more accurate than the percentile bootstrap but requires additional computation. Algorithm 2 summarizes the boot-T procedure. The bootstrap normal approximation (boot-N) interval is based on the asymptotic normality of the bootstrap estimator and uses the bootstrap estimate of the standard error to construct confidence intervals. It is computationally

Algorithm 1 Bootstrap percentile (boot-P) procedure

Step 1: Based on the original sample $\mathbf{T} = (T_1, T_2, \ldots, T_k)$, compute the MLEs $\widehat{\boldsymbol{\Theta}} = (\widehat{\theta}_1, \widehat{\beta}_1, \widehat{\theta}_2, \widehat{\beta}_2)^\top$ or the MPSEs $\widetilde{\boldsymbol{\Theta}} = (\widetilde{\theta}_1, \widetilde{\beta}_1, \widetilde{\theta}_2, \widetilde{\beta}_2)^\top$ by solving Eqs. (8)–(11) or Eqs. (15)–(18), respectively. For any scalar quantity of interest, $\psi = g(\boldsymbol{\Theta})$, compute its estimate as $\widehat{\psi} = g(\widehat{\boldsymbol{\Theta}})$ or $\widetilde{\psi} = g(\widetilde{\boldsymbol{\Theta}})$.

Here, ψ may represents a model parameter, the HRF, or the SRF.

Step 2: Using the parameter estimates obtained in Step 1, generate a bootstrap B-JPCTII sample, denoted by $\mathbf{T}^*$.

Step 3: From the bootstrap sample $\mathbf{T}^*$, compute the bootstrap parameter estimates

$\widehat{\boldsymbol{\Theta}}^*$ or $\widetilde{\boldsymbol{\Theta}}^*$ by solving Eqs. (8)–(11) or Eqs. (15)–(18), respectively. Then compute the corresponding bootstrap estimate of the quantity of interest: $\widehat{\psi}^* = g(\widehat{\boldsymbol{\Theta}}^*)$ or $\widetilde{\psi}^* = g(\widetilde{\boldsymbol{\Theta}}^*)$.

Step 4: Repeat Steps 2 and 3 a total of N times to obtain $\widehat{\psi}^{*1}, \widehat{\psi}^{*2}, \ldots, \widehat{\psi}^{*N}$, or, under the MPSE method, $\widetilde{\psi}^{*1}, \widetilde{\psi}^{*2}, \ldots, \widetilde{\psi}^{*N}$.

Step 5: Arrange the bootstrap estimates in ascending order: $\widehat{\psi}^{*(1)} \leq \widehat{\psi}^{*(2)} \leq \cdots \leq \widehat{\psi}^{*(N)}$, or $\widetilde{\psi}^{*(1)} \leq \widetilde{\psi}^{*(2)} \leq \cdots \leq \widetilde{\psi}^{*(N)}$.

Step 6: The approximate boot-P $100(1-\alpha)\%$ confidence interval for ψ, based on the MLE, is

$$\left(\widehat{\psi}^{*(\lfloor N\alpha/2 \rfloor)},\ \widehat{\psi}^{*(\lfloor N(1-\alpha/2) \rfloor)}\right).$$

Similarly, the corresponding boot-P confidence interval based on the MPSE is

$$\left(\widetilde{\psi}^{*(\lfloor N\alpha/2 \rfloor)},\ \widetilde{\psi}^{*(\lfloor N(1-\alpha/2) \rfloor)}\right).$$

straightforward and performs well when the sampling distribution is approximately symmetric. The boot-n procedure is described in Algorithm 3.

Here, the index $j = 1, 2$ and the symbol $\lfloor\,\rfloor$ is used here to denote the floor function, rounding down to the greatest integer less than or equal to the given value.

5. Simulation Study

In this section, simulation experiment is carried out to demonstrate the theoretical findings and assess the precision and effectiveness of several estimators for the reliability function and the hazard function under Pham from B-JPCTII. An intensive simulation study is conducted in order to accomplish this goal. The study's main objective is to evaluate the accuracy and effectiveness of various estimate methods when used with censored data that comes from Pham with different scale and shape parameters. The Algorithm 4 provides a detailed process for data generation under the B-JPCTII scheme.

Two different parameter sets are considered: $(\theta_1, \beta_1, \theta_2, \beta_2) = (0.6, 1.1, 0.4, 1.3)$ and $(1.1, 1.5, 1.4, 1.3)$. These parameter values are selected to represent decreasing and bathtub-shaped hazard functions, respectively. For the first parameter set, the true values of the HRFs and SRFs at $x = 0.5$ are

$$H_1(0.5) = 0.080, \quad H_2(0.5) = 0.194, \quad S_1(0.5) = 0.937, \quad S_2(0.5) = 0.803.$$

For the second parameter set, the corresponding true values are

$$H_1(0.5) = 0.503, \quad H_2(0.5) = 0.307, \quad S_1(0.5) = 0.812, \quad S_2(0.5) = 0.901.$$

Algorithm 2 Studentized bootstrap (boot-T) procedure

Step 1: Follow Steps 1–3 of the Bootstrap percentile (Boot-P) procedure in Algorithm 1 to obtain the bootstrap estimates $\widehat{\mathbf{\Theta}}^*$ (or $\widetilde{\mathbf{\Theta}}^*$). For the scalar quantity of interest $\psi = g(\mathbf{\Theta})$, compute the bootstrap estimate $\widehat{\psi}^* = g(\widehat{\mathbf{\Theta}}^*)$ (or $\widetilde{\psi}^* = g(\widetilde{\mathbf{\Theta}}^*)$).

Step 2: For each bootstrap sample, compute the studentized statistic

$$T_\psi^* = \frac{\widehat{\psi}^* - \widehat{\psi}}{\widehat{\mathrm{SE}}(\widehat{\psi}^*)},$$

or, under the MPSE method,

$$T_\psi^* = \frac{\widetilde{\psi}^* - \widetilde{\psi}}{\widehat{\mathrm{SE}}(\widetilde{\psi}^*)},$$

where $\widehat{\mathrm{SE}}(\widehat{\psi}^*)$ and $\widehat{\mathrm{SE}}(\widetilde{\psi}^*)$ are obtained from the inverse observed Fisher information matrix in Eq. (12) or Eq. (19), respectively, using the multivariate delta method whenever ψ is a function of the model parameters.

Step 3: Repeat Steps 1 and 2 a total of N times to obtain $T_\psi^{*(1)}, T_\psi^{*(2)}, \ldots, T_\psi^{*(N)}$.

Step 4: Arrange the studentized statistics in ascending order as

$$T_\psi^{*(1)} \leq T_\psi^{*(2)} \leq \cdots \leq T_\psi^{*(N)}.$$

Step 5: The approximate Boot-T $100(1-\alpha)\%$ confidence interval for ψ based on the MLE is

$$\left(\widehat{\psi} - \widehat{\mathrm{SE}}(\widehat{\psi})\, T_\psi^{*(\lfloor N(1-\alpha/2)\rfloor)},\ \widehat{\psi} - \widehat{\mathrm{SE}}(\widehat{\psi})\, T_\psi^{*(\lfloor N(\alpha/2)\rfloor)}\right).$$

Similarly, the corresponding Boot-T confidence interval based on the MPSE is

$$\left(\widetilde{\psi} - \widehat{\mathrm{SE}}(\widetilde{\psi})\, T_\psi^{*(\lfloor N(1-\alpha/2)\rfloor)},\ \widetilde{\psi} - \widehat{\mathrm{SE}}(\widetilde{\psi})\, T_\psi^{*(\lfloor N(\alpha/2)\rfloor)}\right).$$

For each parameter set, several censoring schemes are considered. These schemes are presented in Tables 1-2 for the two parameter sets, respectively. Here, the notation $\mathbf{R} = (0_{(19)}, 10)$ indicates that $R_1 = \cdots = R_{19} = 0$ and $R_{20} = 10$, while CS denotes the censoring scheme.

The simulation experiment is conducted over 500 iterations. In each iteration, the MLEs and MPSEs of the unknown parameters, as well as the HRFs and SRFs, are computed along with their corresponding confidence intervals. Furthermore, bootstrap-based confidence intervals, namely boot-p, boot-t, and boot-n, are constructed using 2000 bootstrap replications. Numerical optimization is performed using the `optim` function in R version 4.2.2 with the `"L-BFGS-B"` algorithm, initial values $(0.1, 1.1, 0.1, 1.1)$, and a convergence tolerance of 10^{-5}. For each estimation method, the average of estimates (Estim), the average the mean squared error (MSE), the average width of ACI (AL), and the average coverage probabilities (CP) of the confidence intervals are computed over the successful replications. The simulation results are presented in 3 and 4.

The results in Tables 3 and 4, corresponding to the decreasing and bathtub-shaped hazard functions, respectively, show that the point estimates obtained by both the MLE and MPSE are generally close to the corresponding true values. Overall, the two methods exhibit comparable performance, although the MLE tends to perform better for H_2, whereas the MPSE generally provides more accurate estimates for S_1 and S_2 and, under several censoring schemes, for H_1. The MSE results further indicate comparable performance of the two methods for the hazard

Algorithm 3 Normal bootstrap (boot-N) confidence interval procedure

Step 1. Based on the original sample $\mathbf{T} = (T_1, T_2, \ldots, T_k)$, compute the MLEs $\widehat{\theta}_j$ and $\widehat{\beta}_j$ (or the MPSEs $\widetilde{\theta}_j$ and $\widetilde{\beta}_j$), $j = 1, 2$, by solving Eqs. (8)–(11) (or Eqs. (15)–(18)).

Step 2. Using the estimates obtained in Step 1, generate a bootstrap B-JPCTII sample and compute the corresponding bootstrap estimates. Repeat this procedure N times to obtain $\widehat{\psi}^{*(1)}, \widehat{\psi}^{*(2)}, \ldots, \widehat{\psi}^{*(N)}$, where $\widehat{\psi}$ denotes any parameter or reliability measure of interest.

Step 3. Compute the mean of the bootstrap estimates:

$$\overline{\widehat{\psi}^{*}} = \frac{1}{N} \sum_{i=1}^{N} \widehat{\psi}^{*(i)}.$$

Step 4. Compute the bootstrap estimate of the standard error:

$$\widehat{\mathrm{SE}}_{\mathrm{boot}} = \left[\frac{1}{N-1} \sum_{i=1}^{N} \left(\widehat{\psi}^{*(i)} - \overline{\widehat{\psi}^{*}} \right)^{2} \right]^{1/2}.$$

Step 5. Estimate the bootstrap bias by

$$\widehat{\mathrm{Bias}}_{\mathrm{boot}} = \overline{\widehat{\psi}^{*}} - \widehat{\psi},$$

and compute the bias-corrected center

$$\widehat{\psi}_{\mathrm{bc}} = \widehat{\psi} - \widehat{\mathrm{Bias}}_{\mathrm{boot}} = 2\widehat{\psi} - \overline{\widehat{\psi}^{*}}.$$

Step 6. Let

$$z_{1-\alpha/2} = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right),$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. Then the approximate normal-bootstrap $100(1-\alpha)\%$ confidence interval for ψ is

$$\left(\widehat{\psi}_{\mathrm{bc}} - z_{1-\alpha/2}\widehat{\mathrm{SE}}_{\mathrm{boot}},\ \widehat{\psi}_{\mathrm{bc}} + z_{1-\alpha/2}\widehat{\mathrm{SE}}_{\mathrm{boot}} \right).$$

Equivalently, the interval can be written as

$$\left(2\widehat{\psi} - \overline{\widehat{\psi}^{*}} - z_{1-\alpha/2}\widehat{\mathrm{SE}}_{\mathrm{boot}},\ 2\widehat{\psi} - \overline{\widehat{\psi}^{*}} + z_{1-\alpha/2}\widehat{\mathrm{SE}}_{\mathrm{boot}} \right).$$

function, while the MLE generally yields smaller MSEs for the survival function. Moreover, smaller MSEs are generally observed for the decreasing hazard case than for the bathtub-shaped hazard case.

For the interval estimates, the AL and CP vary across the considered hazard shapes and censoring schemes. For the decreasing hazard case in Table 3, Boot-T generally produces wider intervals, whereas ACI, Boot-P, and Boot-N provide shorter intervals with competitive coverage probabilities. For the bathtub-shaped hazard case in Table 4, the relative performance of the interval methods is more variable, particularly for smaller sample sizes. Overall, ACI provides a favorable balance between AL and CP for the decreasing hazard case, whereas Boot-P generally performs well for the bathtub-shaped hazard case.

Algorithm 4 Generating samples under B-JPCTII

Require: Sizes of samples (n_1, n_2), number of failure times, (k), censoring vectors $\mathbf{R}_1 = (R_{1,1}, \ldots, R_{1,k-1})$, $\mathbf{R}_2 = (R_{2,1}, \ldots, R_{2,k-1})$, and true parameters $(\theta_1, \beta_1, \theta_2, \beta_2)$
Ensure: Ordered failure times $\mathbf{T} = (T_1, \ldots, T_k)$ and indicators $\mathbf{M} = (M_1, \ldots, M_k)$

```
Generate independent samples X_1, ..., X_{n_1} ~ f(·; θ_1, β_1) and Y_1, ..., Y_{n_2} ~ f(·; θ_2, β_2).
Merge the two samples with their group indicators and sort the pooled observations by increasing failure times to obtain (m, t)
for i = 1 to k − 1 do
  Set M_i ← m_1 and T_i ← t_1 {record the smallest observed time and its label}
  if M_i = 1 then
    Randomly remove R_{1,i} additional units from sample 1 and R_{2,i} + 1 from sample 2, then delete the smallest unit
  else
    Randomly remove R_{1,i} + 1 additional units from sample 1 and R_{2,i} from sample 2, then delete the smallest unit
  end if
end for
Set M_k ← m_1, and T_k ← t_1
Return (T, M)
```

Table 1. The notation of censoring schemes with initial parameters $(\theta_1 = 0.6, \beta_1 = 1.1, \theta_2 = 0.4, \beta_2 = 1.3)$.

Notation	$(\mathbf{n_1}, \mathbf{n_2}, \mathbf{k})$	**Censoring scheme**	
		$\mathbf{R_1}$	$\mathbf{R_2}$
CS$_1$	$(30, 35, 20)$	$(0_{(19)}, 10)$	$(0_{(19)}, 15)$
CS$_2$	$(30, 35, 20)$	$(0_{(14)}, 2_{(5)}, 0)$	$(0_{(14)}, 2_{(5)}, 5)$
CS$_3$	$(30, 35, 20)$	$(10, 0_{(18)}, 0)$	$(10, 0_{(18)}, 5)$
CS$_4$	$(30, 35, 25)$	$(0_{(24)}, 5)$	$(0_{(24)}, 10)$
CS$_5$	$(30, 35, 25)$	$(0_{(19)}, 1_{(5)}, 0)$	$(0_{(19)}, 1_{(5)}, 5)$
CS$_6$	$(30, 35, 25)$	$(5, 0_{(23)}, 0)$	$(5, 0_{(23)}, 5)$
CS$_7$	$(40, 30, 20)$	$(0_{(19)}, 20)$	$(0_{(19)}, 10)$
CS$_8$	$(40, 30, 20)$	$(2_{(5)}, 0_{(14)}, 10)$	$(2_{(5)}, 0_{(14)}, 0)$
CS$_9$	$(40, 30, 20)$	$(10, 0_{(18)}, 10)$	$(10, 0_{(19)})$
CS$_{10}$	$(40, 30, 25)$	$(0_{(24)}, 15)$	$(0_{(24)}, 5)$
CS$_{11}$	$(40, 30, 25)$	$(1_{(5)}, 0_{(19)}, 10)$	$(1_{(5)}, 0_{(19)}, 0)$
CS$_{12}$	$(40, 30, 25)$	$(5, 0_{(23)}, 10)$	$(5, 0_{(24)})$

Table 2. The notation of censoring schemes with initial parameters $(\theta_1 = 1.1, \beta_1 = 1.5, \theta_2 = 1.4, \beta_2 = 1.3)$.

Notation	$(\mathbf{n_1}, \mathbf{n_2}, \mathbf{k})$	**Censoring scheme**	
		$\mathbf{R_1}$	$\mathbf{R_2}$
CS$_1$	$(20, 20, 10)$	$(0_{(9)}, 10)$	$(0_{(9)}, 10)$
CS$_2$	$(20, 20, 15)$	$(0_{(14)}, 5)$	$(0_{(14)}, 5)$
CS$_3$	$(20, 20, 15)$	$(0_{(9)}, 1_{(5)}, 0)$	$(0_{(9)}, 1_{(5)}, 0)$
CS$_4$	$(20, 20, 15)$	$(5, 0_{(14)})$	$(5, 0_{(14)})$
CS$_5$	$(50, 50, 20)$	$(0_{(19)}, 30)$	$(0_{(19)}, 30)$
CS$_6$	$(50, 50, 20)$	$(2_{(5)}, 0_{(14)}, 20)$	$(2_{(5)}, 0_{(14)}, 20)$
CS$_7$	$(50, 50, 20)$	$(10, 0_{(18)}, 20)$	$(10, 0_{(18)}, 20)$
CS$_8$	$(50, 50, 40)$	$(0_{(39)}, 10)$	$(0_{(39)}, 10)$
CS$_9$	$(50, 50, 40)$	$(1_{(5)}, 0_{(34)}, 5)$	$(1_{(5)}, 0_{(34)}, 5)$
CS$_{10}$	$(50, 50, 40)$	$(5, 0_{(38)}, 5)$	$(5, 0_{(38)}, 5)$

Table 3. Simulation results under the balanced joint progressive censoring scheme for $(\theta_1, \beta_1, \theta_2, \beta_2) = (0.6, 1.1, 0.4, 1.3)$.

(n_1, n_2, k)	Scheme	Function	MLE											MPSE										
			Estim	MSE	ACI		boot-P		boot-T		boot-N		Estim	MSE	ACI		boot-P		boot-T		boot-N	
					AL	CP	AL	CP	AL	CP	AL	CP			AL	CP	AL	CP	AL	CP	AL	CP
$(30, 35, 20)$	CS_1	H_1	0.076	0.003	0.156	0.930	0.153	0.926	0.174	0.870	0.146	0.836	0.072	0.003	0.151	0.900	0.146	0.894	0.164	0.812	0.142	0.828
		H_2	0.237	0.007	0.241	0.904	0.287	0.836	0.271	0.978	0.282	0.954	0.237	0.006	0.239	0.916	0.256	0.896	0.249	0.966	0.253	0.940
		S_1	0.957	0.001	0.101	0.890	0.109	0.980	0.173	0.894	0.096	0.834	0.956	0.002	0.101	0.864	0.121	0.950	0.168	0.866	0.101	0.830
		S_2	0.844	0.004	0.196	0.950	0.214	0.890	0.283	0.958	0.219	0.930	0.825	0.004	0.206	0.972	0.227	0.914	0.269	0.946	0.229	0.908
$(30, 35, 20)$	CS_2	H_1	0.078	0.003	0.152	0.936	0.154	0.916	0.178	0.870	0.148	0.860	0.076	0.003	0.149	0.892	0.156	0.900	0.175	0.830	0.151	0.830
		H_2	0.231	0.005	0.235	0.880	0.278	0.940	0.271	0.990	0.274	0.914	0.229	0.004	0.231	0.894	0.243	0.986	0.246	0.980	0.239	0.912
		S_1	0.954	0.001	0.103	0.906	0.114	0.982	0.175	0.902	0.101	0.860	0.952	0.002	0.102	0.850	0.128	0.952	0.177	0.870	0.107	0.848
		S_2	0.849	0.004	0.191	0.926	0.212	0.966	0.287	0.986	0.217	0.968	0.831	0.004	0.202	0.968	0.223	0.902	0.268	0.970	0.225	0.986
$(30, 35, 20)$	CS_3	H_1	0.085	0.003	0.178	0.978	0.159	0.952	0.199	0.906	0.157	0.884	0.085	0.003	0.177	0.966	0.169	0.920	0.196	0.858	0.165	0.884
		H_2	0.232	0.005	0.243	0.912	0.249	0.874	0.277	0.962	0.249	0.928	0.237	0.004	0.238	0.906	0.239	0.906	0.256	0.952	0.237	0.920
		S_1	0.948	0.002	0.131	0.866	0.138	0.906	0.185	0.928	0.116	0.920	0.945	0.002	0.132	0.926	0.158	0.846	0.187	0.890	0.124	0.904
		S_2	0.844	0.004	0.214	0.976	0.227	0.912	0.296	0.980	0.228	0.894	0.823	0.005	0.224	0.978	0.245	0.894	0.288	0.960	0.243	0.918
$(30, 35, 25)$	CS_4	H_1	0.080	0.002	0.149	0.984	0.129	0.940	0.171	0.934	0.129	0.896	0.078	0.002	0.145	0.964	0.133	0.938	0.170	0.912	0.133	0.880
		H_2	0.232	0.004	0.218	0.906	0.216	0.856	0.239	0.978	0.214	0.944	0.238	0.003	0.216	0.934	0.213	0.918	0.227	0.970	0.210	0.938
		S_1	0.954	0.001	0.106	0.962	0.111	0.906	0.161	0.856	0.099	0.924	0.952	0.001	0.107	0.938	0.129	0.898	0.160	0.846	0.107	0.934
		S_2	0.845	0.003	0.194	0.884	0.203	0.908	0.264	0.988	0.207	0.902	0.827	0.004	0.204	0.900	0.222	0.910	0.261	0.980	0.224	0.934
$(30, 35, 25)$	CS_5	H_1	0.084	0.002	0.151	0.968	0.134	0.934	0.176	0.814	0.134	0.902	0.083	0.002	0.149	0.948	0.141	0.928	0.180	0.896	0.141	0.892
		H_2	0.231	0.004	0.216	0.906	0.216	0.956	0.238	0.976	0.213	0.944	0.235	0.003	0.213	0.934	0.209	0.922	0.222	0.972	0.205	0.948
		S_1	0.950	0.001	0.110	0.866	0.117	0.908	0.173	0.956	0.105	0.936	0.948	0.002	0.110	0.828	0.135	0.876	0.182	0.934	0.113	0.914
		S_2	0.846	0.003	0.193	0.896	0.204	0.920	0.263	0.982	0.207	0.906	0.827	0.004	0.204	0.908	0.221	0.914	0.255	0.982	0.221	0.934
$(30, 35, 25)$	CS_6	H_1	0.082	0.002	0.158	0.966	0.139	0.928	0.184	0.908	0.138	0.886	0.080	0.002	0.155	0.940	0.144	0.910	0.184	0.882	0.143	0.872
		H_2	0.240	0.004	0.227	0.930	0.222	0.904	0.245	0.944	0.220	0.928	0.248	0.003	0.223	0.944	0.216	0.932	0.228	0.940	0.213	0.924
		S_1	0.952	0.001	0.111	0.958	0.120	0.902	0.172	0.944	0.105	0.918	0.952	0.002	0.109	0.924	0.135	0.968	0.176	0.918	0.110	0.908
		S_2	0.836	0.003	0.208	0.914	0.213	0.934	0.258	0.972	0.214	0.930	0.814	0.004	0.219	0.914	0.231	0.890	0.254	0.958	0.230	0.934
$(40, 30, 20)$	CS_7	H_1	0.080	0.002	0.132	0.940	0.137	0.898	0.157	0.940	0.135	0.900	0.076	0.002	0.127	0.900	0.128	0.862	0.149	0.882	0.129	0.852
		H_2	0.233	0.007	0.264	0.896	0.303	0.936	0.305	0.984	0.297	0.924	0.236	0.006	0.264	0.902	0.283	0.906	0.288	0.974	0.279	0.934
		S_1	0.957	0.001	0.089	0.896	0.096	0.864	0.156	0.848	0.090	0.872	0.956	0.001	0.089	0.860	0.106	0.840	0.154	0.816	0.095	0.860
		S_2	0.844	0.005	0.209	0.848	0.228	0.880	0.318	0.970	0.233	0.884	0.823	0.006	0.221	0.852	0.249	0.886	0.310	0.948	0.249	0.904
$(40, 30, 20)$	CS_8	H_1	0.088	0.002	0.144	0.894	0.133	0.942	0.173	0.870	0.134	0.934	0.085	0.002	0.137	0.862	0.129	0.946	0.167	0.834	0.131	0.904
		H_2	0.230	0.007	0.285	0.876	0.292	0.948	0.334	0.962	0.290	0.908	0.241	0.007	0.289	0.906	0.313	0.906	0.327	0.956	0.311	0.926
		S_1	0.952	0.001	0.100	0.870	0.107	0.890	0.161	0.894	0.099	0.840	0.951	0.001	0.101	0.846	0.118	0.876	0.164	0.868	0.104	0.842
		S_2	0.841	0.005	0.227	0.874	0.246	0.898	0.333	0.980	0.248	0.892	0.818	0.006	0.238	0.868	0.278	0.878	0.332	0.970	0.272	0.918
$(40, 30, 20)$	CS_9	H_1	0.087	0.002	0.143	0.966	0.130	0.914	0.172	0.934	0.130	0.892	0.085	0.002	0.138	0.952	0.130	0.914	0.167	0.900	0.130	0.870
		H_2	0.223	0.006	0.283	0.906	0.283	0.870	0.338	0.984	0.280	0.924	0.235	0.006	0.287	0.912	0.305	0.922	0.330	0.980	0.302	0.928
		S_1	0.953	0.001	0.101	0.842	0.107	0.866	0.162	0.872	0.098	0.912	0.950	0.001	0.103	0.838	0.121	0.856	0.166	0.850	0.106	0.922
		S_2	0.846	0.004	0.236	0.870	0.252	0.926	0.356	0.990	0.253	0.888	0.821	0.006	0.247	0.888	0.286	0.916	0.354	0.980	0.275	0.916
$(40, 30, 25)$	CS_{10}	H_1	0.086	0.001	0.131	0.884	0.118	0.828	0.152	0.866	0.118	0.914	0.084	0.002	0.127	0.844	0.120	0.888	0.152	0.834	0.120	0.902
		H_2	0.225	0.005	0.239	0.904	0.232	0.860	0.269	0.974	0.228	0.922	0.233	0.004	0.240	0.924	0.238	0.906	0.261	0.972	0.235	0.936
		S_1	0.953	0.001	0.096	0.856	0.099	0.888	0.148	0.898	0.094	0.924	0.950	0.001	0.098	0.830	0.113	0.870	0.154	0.868	0.102	0.924
		S_2	0.848	0.004	0.208	0.894	0.217	0.928	0.299	0.988	0.222	0.912	0.827	0.004	0.220	0.900	0.244	0.920	0.300	0.980	0.246	0.934
$(40, 30, 25)$	CS_{11}	H_1	0.088	0.001	0.139	0.902	0.124	0.854	0.167	0.888	0.125	0.836	0.086	0.001	0.134	0.884	0.125	0.842	0.165	0.864	0.126	0.824
		H_2	0.231	0.005	0.256	0.900	0.246	0.852	0.287	0.958	0.242	0.904	0.243	0.005	0.260	0.924	0.264	0.930	0.285	0.952	0.261	0.950
		S_1	0.954	0.001	0.096	0.884	0.099	0.900	0.150	0.908	0.094	0.958	0.952	0.001	0.098	0.858	0.112	0.896	0.155	0.894	0.101	0.966
		S_2	0.838	0.004	0.220	0.884	0.228	0.922	0.292	0.976	0.229	0.898	0.817	0.005	0.231	0.876	0.256	0.862	0.297	0.974	0.251	0.932
$(40, 30, 25)$	CS_{12}	H_1	0.089	0.002	0.139	0.880	0.125	0.810	0.168	0.864	0.125	0.904	0.088	0.002	0.135	0.870	0.127	0.812	0.166	0.848	0.127	0.886
		H_2	0.226	0.005	0.255	0.914	0.244	0.952	0.285	0.966	0.240	0.922	0.239	0.005	0.258	0.924	0.264	0.924	0.284	0.966	0.260	0.940
		S_1	0.952	0.001	0.098	0.870	0.101	0.872	0.153	0.892	0.096	0.922	0.949	0.001	0.101	0.850	0.115	0.868	0.161	0.890	0.105	0.922
		S_2	0.842	0.004	0.224	0.908	0.230	0.930	0.299	0.986	0.232	0.906	0.820	0.005	0.234	0.896	0.260	0.894	0.305	0.988	0.254	0.924

6. Optimal Censoring Scheme

The design of an optimal censoring scheme is a fundamental component of life-testing experiments. Under the B-JPCTII framework, the removal pattern affects both the statistical efficiency of parameter estimates and practical constraints such as test duration and cost. Following the principles of optimal experimental design (see [37]),

Table 4. Simulation results under the balanced joint progressive censoring schemes for $(\theta_1, \beta_1, \theta_2, \beta_2) = (1.1, 1.5, 1.4, 1.3)$.

(n_1, n_2, k)	Scheme	Function	MLE												MPSE											
			Estim	MSE	ACI		boot-P		boot-T		boot-N		Estim	MSE	ACI		boot-P		boot-T		boot-N	
					AL	CP	AL	CP	AL	CP	AL	CP			AL	CP	AL	CP	AL	CP	AL	CP
(20, 20, 10)	CS_1	H_1	0.459	0.032	0.726	0.956	0.794	0.962	0.730	0.914	0.779	0.934	0.455	0.026	0.718	0.956	0.733	0.968	0.731	0.920	0.730	0.942
		H_2	0.240	0.020	0.515	0.930	0.587	0.946	0.585	0.878	0.513	0.868	0.214	0.019	0.475	0.898	0.516	0.934	0.602	0.840	0.471	0.830
		S_1	0.880	0.002	0.226	0.974	0.203	0.988	0.223	0.928	0.198	0.920	0.852	0.004	0.254	0.982	0.243	0.970	0.268	0.898	0.228	0.940
		S_2	0.950	0.001	0.127	0.914	0.133	0.970	0.218	0.904	0.112	0.840	0.949	0.002	0.126	0.878	0.151	0.964	0.274	0.890	0.123	0.836
(20, 20, 15)	CS_2	H_1	0.480	0.023	0.662	0.988	0.679	0.970	0.622	0.924	0.681	0.952	0.509	0.026	0.657	0.988	0.681	0.962	0.636	0.916	0.673	0.950
		H_2	0.235	0.016	0.459	0.896	0.494	0.942	0.700	0.910	0.467	0.836	0.228	0.018	0.439	0.868	0.473	0.914	0.755	0.890	0.457	0.832
		S_1	0.877	0.003	0.226	0.990	0.205	0.988	0.226	0.902	0.199	0.920	0.850	0.005	0.252	0.970	0.253	0.944	0.276	0.928	0.237	0.954
		S_2	0.948	0.001	0.126	0.840	0.134	0.940	0.331	0.892	0.120	0.812	0.942	0.002	0.132	0.832	0.162	0.928	0.399	0.906	0.138	0.822
(20, 20, 15)	CS_3	H_1	0.479	0.031	0.655	0.958	0.751	0.948	0.661	0.902	0.754	0.934	0.506	0.031	0.649	0.964	0.713	0.942	0.655	0.896	0.707	0.954
		H_2	0.222	0.019	0.436	0.838	0.508	0.898	0.860	0.920	0.474	0.818	0.217	0.020	0.420	0.822	0.471	0.876	0.857	0.902	0.452	0.800
		S_1	0.876	0.003	0.223	0.962	0.216	0.976	0.243	0.900	0.210	0.904	0.845	0.006	0.252	0.938	0.263	0.912	0.292	0.926	0.245	0.944
		S_2	0.951	0.001	0.118	0.800	0.133	0.898	0.407	0.912	0.116	0.784	0.946	0.002	0.124	0.784	0.157	0.890	0.443	0.908	0.130	0.796
(20, 20, 15)	CS_4	H_1	0.467	0.046	0.682	0.926	0.785	0.934	0.734	0.864	0.782	0.900	0.504	0.047	0.677	0.926	0.762	0.932	0.730	0.868	0.747	0.930
		H_2	0.226	0.026	0.452	0.808	0.533	0.860	1.256	0.922	0.498	0.802	0.226	0.028	0.437	0.808	0.508	0.878	1.112	0.884	0.477	0.780
		S_1	0.877	0.004	0.229	0.908	0.240	0.962	0.281	0.868	0.228	0.880	0.847	0.008	0.258	0.910	0.289	0.892	0.325	0.908	0.258	0.926
		S_2	0.947	0.002	0.127	0.786	0.153	0.892	0.583	0.934	0.129	0.790	0.941	0.003	0.134	0.794	0.179	0.896	0.553	0.928	0.144	0.818
(50, 50, 20)	CS_5	H_1	0.449	0.016	0.496	0.954	0.536	0.964	0.496	0.934	0.534	0.952	0.440	0.014	0.493	0.960	0.515	0.972	0.494	0.954	0.513	0.948
		H_2	0.241	0.011	0.373	0.930	0.411	0.926	0.409	0.920	0.402	0.906	0.229	0.011	0.363	0.904	0.385	0.916	0.413	0.906	0.382	0.896
		S_1	0.889	0.001	0.146	0.968	0.135	0.972	0.140	0.898	0.135	0.902	0.875	0.002	0.155	0.964	0.153	0.948	0.154	0.920	0.152	0.930
		S_2	0.951	0.001	0.094	0.902	0.092	0.932	0.125	0.902	0.089	0.854	0.951	0.001	0.093	0.884	0.101	0.926	0.139	0.884	0.097	0.854
(50, 50, 20)	CS_6	H_1	0.454	0.018	0.492	0.956	0.530	0.938	0.487	0.918	0.527	0.940	0.454	0.016	0.492	0.952	0.519	0.964	0.485	0.918	0.517	0.932
		H_2	0.226	0.010	0.364	0.898	0.386	0.914	0.407	0.900	0.380	0.874	0.217	0.010	0.355	0.882	0.371	0.898	0.420	0.912	0.368	0.868
		S_1	0.889	0.001	0.151	0.954	0.140	0.970	0.144	0.898	0.140	0.898	0.875	0.002	0.159	0.954	0.160	0.934	0.159	0.904	0.158	0.932
		S_2	0.954	0.001	0.093	0.884	0.092	0.924	0.134	0.896	0.088	0.840	0.953	0.001	0.093	0.856	0.104	0.924	0.154	0.894	0.098	0.842
(50, 50, 20)	CS_7	H_1	0.456	0.018	0.499	0.952	0.539	0.942	0.494	0.926	0.537	0.934	0.457	0.017	0.499	0.952	0.527	0.968	0.490	0.932	0.524	0.942
		H_2	0.231	0.009	0.376	0.920	0.398	0.940	0.418	0.924	0.390	0.876	0.221	0.010	0.364	0.892	0.377	0.904	0.425	0.900	0.373	0.854
		S_1	0.886	0.001	0.158	0.956	0.145	0.968	0.151	0.898	0.145	0.898	0.871	0.002	0.166	0.940	0.167	0.934	0.167	0.896	0.164	0.926
		S_2	0.953	0.001	0.097	0.890	0.096	0.934	0.142	0.888	0.092	0.820	0.951	0.001	0.096	0.854	0.107	0.920	0.158	0.860	0.100	0.824
(50, 50, 40)	CS_8	H_1	0.435	0.012	0.386	0.924	0.414	0.940	0.411	0.900	0.418	0.920	0.455	0.013	0.385	0.930	0.415	0.930	0.410	0.912	0.415	0.934
		H_2	0.225	0.008	0.299	0.890	0.320	0.920	0.418	0.954	0.323	0.904	0.226	0.009	0.296	0.876	0.322	0.904	0.433	0.938	0.325	0.898
		S_1	0.888	0.001	0.142	0.916	0.143	0.952	0.160	0.900	0.149	0.912	0.874	0.002	0.150	0.930	0.162	0.920	0.173	0.916	0.164	0.938
		S_2	0.950	0.001	0.091	0.874	0.096	0.916	0.160	0.966	0.094	0.880	0.947	0.001	0.093	0.840	0.108	0.906	0.175	0.956	0.104	0.890
(50, 50, 40)	CS_9	H_1	0.425	0.013	0.393	0.912	0.416	0.934	0.425	0.894	0.420	0.908	0.446	0.013	0.392	0.902	0.421	0.926	0.423	0.884	0.422	0.908
		H_2	0.218	0.008	0.304	0.892	0.321	0.918	0.449	0.956	0.324	0.902	0.222	0.009	0.302	0.870	0.327	0.908	0.458	0.950	0.331	0.902
		S_1	0.891	0.002	0.142	0.900	0.146	0.946	0.165	0.892	0.151	0.892	0.878	0.002	0.150	0.894	0.164	0.926	0.175	0.904	0.166	0.918
		S_2	0.952	0.001	0.089	0.868	0.096	0.916	0.165	0.964	0.094	0.880	0.949	0.001	0.092	0.862	0.109	0.908	0.175	0.972	0.103	0.904
(50, 50, 40)	CS_{10}	H_1	0.431	0.013	0.398	0.928	0.422	0.948	0.431	0.914	0.427	0.922	0.453	0.013	0.398	0.902	0.428	0.932	0.430	0.896	0.428	0.912
		H_2	0.222	0.008	0.307	0.876	0.325	0.906	0.457	0.976	0.328	0.882	0.224	0.010	0.303	0.862	0.331	0.896	0.461	0.956	0.333	0.882
		S_1	0.888	0.002	0.145	0.922	0.149	0.958	0.169	0.908	0.155	0.908	0.875	0.002	0.153	0.896	0.168	0.906	0.179	0.914	0.170	0.926
		S_2	0.951	0.001	0.092	0.862	0.099	0.910	0.170	0.982	0.096	0.870	0.947	0.001	0.094	0.824	0.111	0.898	0.178	0.968	0.105	0.870

and recent developments for progressive censoring plans ([38]), we summarize objective criteria and practical algorithms to determine optimum removal patterns $\mathbf{R} = (R_1, R_2)$ for B-JPCTII.

Here, we employ the following classical optimality criteria for determination of an optimum censoring scheme in the case of balanced joint progressive censoring:

Criterion D: According to this criterion, we minimize the determinant of the inverse of the observed Fisher's information matrix $I^{-1}(\hat{\Theta})$ of the parameters with respect to $\mathbf{R}$.

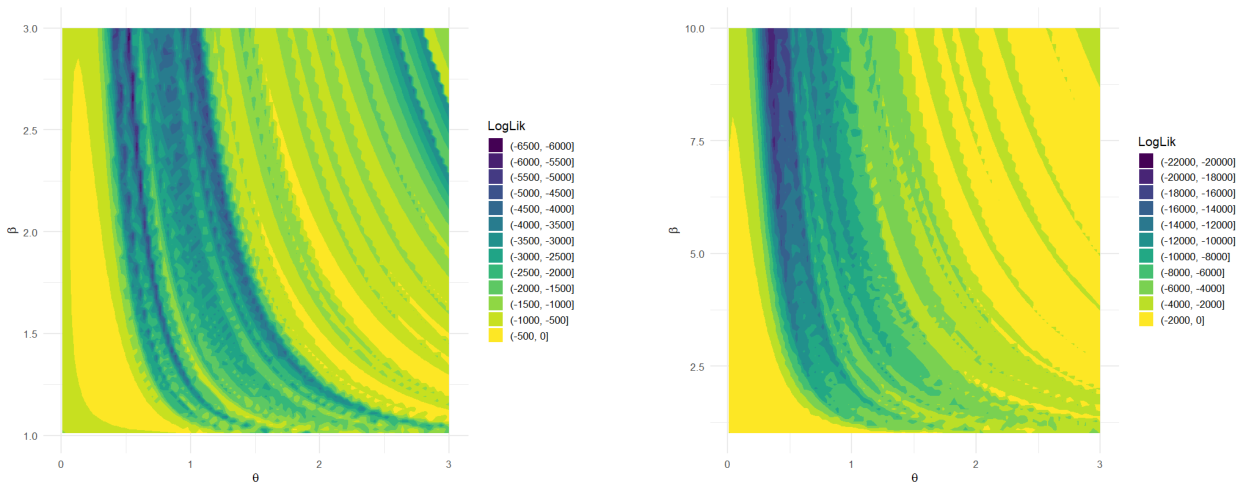
Criterion A: According to this criterion, we minimize the trace the inverse of the observed Fisher's information matrix $I^{-1}(\hat{\Theta})$ of the parameters with respect to $\mathbf{R}$.

Criterion E: This criterion is based on maximizing the smallest eigenvalue of the observed Fisher's information matrix $I(\hat{\Theta})$ of the parameters with respect to $\mathbf{R}$.

Now, we exemplify the above proposed criteria of the optimum censoring plan in the following section.

7. Real-Life Application

In this section, we illustrate the proposed methodology using two real data sets. The first consists of 104 software failure times from the CSR3.DAT data set (Table 5), while the second consists of 207 software failure times from the SYS3.DAT data set (Table 6). Both data sets were originally reported by [39]. For computational convenience, all observations were rescaled by dividing them by 10; this transformation does not affect the statistical inference. Figures 3a and 3b display the Contour plots of the log-likelihood functions for the CSR3.DAT and SYS3.DAT data sets, respectively. These plots illustrate the shape of the likelihood surface and facilitate the selection of suitable initial values for the numerical optimization algorithm.



(a) CSR3.DAT (b) SYS3.DAT

Figure 3. Contour plots of the log-likelihood function for the CSR3.DAT and SYS3.DAT data sets

Table 5. CSR3.DAT data set

3.3	0.9	0.4	6.6	0.05	1.8	14.9	1.4	1.5	5	8.1	3.4	8.5	5.4
0.3	1.5	0.6	0.8	13	1.9	1.9	11.2	1.5	1.6	15.4	5	1	0.2
2.2	5.3	1.9	5.8	2	0.3	9.2	0.5	6.6	28.9	0.3	0.9	1.2	1.8
0.9	7.5	1.5	29.1	21.2	0.4	0.5	30.8	26.9	27.6	0.1	40	29.4	22.7
11.8	1.3	4.7	8.9	24.2	9.9	60.7	8.3	0.2	2.6	58.6	70.8	0.6	0.4
5.5	40.9	3.6	1.5	57.3	58.3	6	1.9	2	7.9	2.4	54	5.2	159.6
31.4	0.1	76.3	1	2	14.4	2.8	5.6	47.6	6.5	9.8	88.4	21.2	28.7
5.3	0.3	83.1	4.3	5.5	10.9								

At the first stage of the analysis, the Pham distribution was fitted to the each data set and its adequacy was assessed by comparing it with several lifetime distributions. The competing models were evaluated using the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and the Kolmogorov–Smirnov (K–S) goodness-of-fit test. The AIC and BIC are computed as

$$\mathrm{AIC} = -2\ell(\hat{\theta}, \hat{\beta}) + 2d,$$
$$\mathrm{BIC} = -2\ell(\hat{\theta}, \hat{\beta}) + d\log(n),$$

Table 6. SYS3.DAT data set

3.9	1	0.4	3.6	0.4	0.5	0.4	9.1	4.9	0.1	2.5	0.1	0.4	3
4.2	0.9	4.9	4.4	3.2	0.3	7.8	0.1	3	20.5	0.5	12.9	10.3	22.4
18.6	5.3	1.4	0.9	0.2	1	0.1	3.4	17.0	12.9	0.4	0.4	3.5	0.5
0.5	2.2	3.6	3.5	12.1	2.3	3.3	4.8	3.2	2.1	0.4	2.3	0.9	1.3
16.5	1.4	2.2	4.1	1.2	13.8	9.5	4.9	6.2	0.2	3.5	8.9	9	6.9
2.2	1.5	1.9	4.2	1.4	1.1	4.1	21.0	1.6	3	3.7	6.6	0.9	1.6
1.4	2.4	1.2	15.9	8.9	11.8	2.9	2.1	1.8	0.2	11.4	3.7	4.6	1.7
0.1	15	38.2	16	6.6	20.6	0.9	2.6	6.2	23.9	1.3	0.4	8.5	8.5
24	17.8	3.4	10.2	0.9	14.6	5.9	4.8	2.5	2.5	11.1	0.5	3.1	5.1
0.6	19.3	2.7	2.5	9.6	2.6	3	3	1.7	32	7.8	3.9	1.3	1.3
1.9	12.8	3.4	8.4	4	17.7	34.9	27.4	8.2	5.8	3.1	11.4	3.9	8.8
8.4	23.2	10.8	3.8	8.6	0.7	2.2	8	23.9	0.3	3.9	6.3	15.2	6.3
8.0	24.5	19.6	4.6	15.2	10.2	0.9	22.8	22	20.8	7.8	0.3	8.3	0.6
21.2	9.1	0.3	1	17.2	2.1	17.3	37.1	4	4.8	12.6	9	14.9	3
31.7	50	67.3	43.2	6.6	16.8	6.6	6.6	12.8	4.9	33.2			

where d denotes the number of estimated parameters and $\ell(\hat{\theta}, \hat{\beta})$ is the maximized log-likelihood evaluated at the MLEs. In addition, the K–S statistic and its corresponding p-value were used to further assess the goodness-of-fit. The resulting MLEs, AIC, BIC, K–S statistics, and corresponding p-values for all fitted models are reported in Table 7.

Based on the AIC and BIC values, the Pham distribution provides the best fit among the candidate models for both data sets. Furthermore, for the CSR3.DAT data set, the K–S statistic is 0.112 with a corresponding p-value of 0.111, while for the SYS3.DAT data set, the K–S statistic is 0.085 with a corresponding p-value of 0.078. Therefore, the K–S test does not reject the Pham distribution at the 5% significance level for either data set. To further assess the model adequacy, Figure 4 presents empirical distribution function plots comparing the empirical and fitted cumulative distribution functions. The close agreement between the two curves provides additional graphical evidence supporting the suitability of the Pham distribution for both data sets. Overall, these numerical and graphical results indicate that the Pham distribution provides an adequate description of the observed data.

Table 7. MLEs, AIC, BIC, K–S statistics, and corresponding p-values for the candidate distributions fitted to the CSR3.DAT and SYS3.DAT data sets.

	CSR3.DAT					SYS3.DAT				
Distribution	MLE	AIC	BIC	KS	p-value	MLE	AIC	BIC	KS	p-value
Pham	(0.440,1.256)	**736**	**741**	0.112	0.111	(0.587,1.219)	**1288**	**1294**	0.085	0.078
generalized inverted exponential	(0.432,0.525)	760	766	0.154	0.011	(0.559,0.815)	1377	1384	0.182	<0.001
Fréchet	(0.576,1.809)	738	744	0.091	0.280	(0.668,1.839)	1340	1346	0.116	0.006
Burr XII	(1.387,0.400)	737	742	0.123	0.061	(1.627,0.386)	1335	1342	0.157	<0.001
generalized Rayleigh	(0.407,0.030)	2087	2092	0.217	<0.001	(0.423,0.010)	1525	1531	0.136	<0.001
Gompertz	(0.011,0.049)	810	815	0.323	<0.001	(0.022,0.112)	1292	1298	0.136	<0.001
Gumbel	(5.960,10.000)	874	880	0.214	<0.001	(4.319,5.400)	1402	1409	0.158	<0.001

Note: Bold values indicate a better fit.

Table 8 presents the point estimates (Estim), estimated standard errors (ESEs), and 95% asymptotic confidence intervals (95% ACIs) obtained by the maximum likelihood estimation method for the two real data sets.

Table 8. The point and interval estimates for CSR3.DAT and SYS3.DAT data sets.

	θ			β		
	Estim	ESE	95%ACI	Estim	ESE	95% ACI
CSR3.DAT	0.440	0.035	(0.372,0.508)	1.260	0.042	(1.170,1.340)
SYS3.DAT	0.587	0.032	(0.524,0.650)	1.220	0.026	(1.170,1.270)

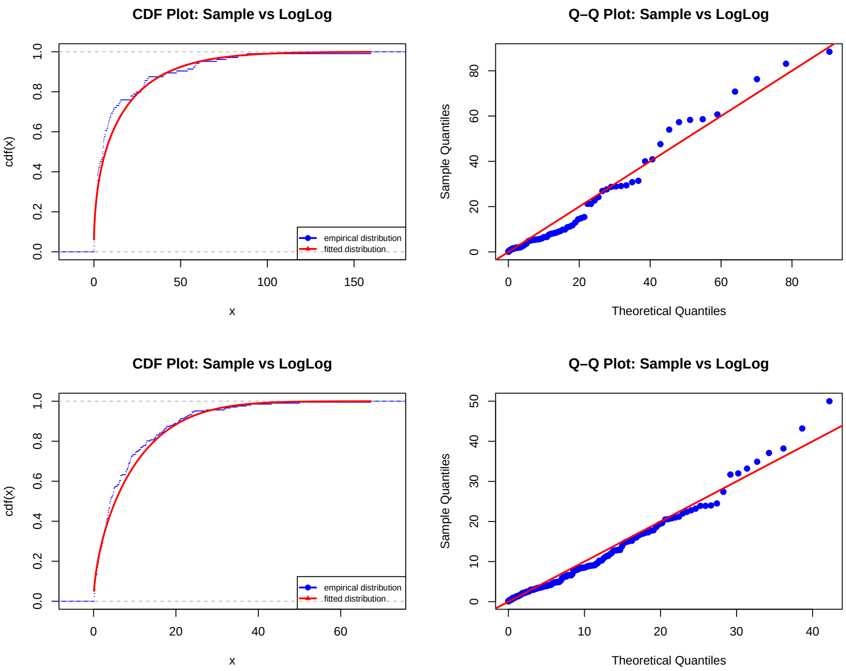


Figure 4. Left: Empirical CDF and fitted CDF of the Pham distribution based on the MLEs. Right: Q–Q plot of the fitted Pham distribution against the observed data. Upper panel for CSR3.DAT data set and lower panel for SYS3.DAT data set.

To generate B-JPCTII sample from the real data sets, we consider the candidate censoring schemes (CSs) listed in Table 9. To illustrate the proposed optimality criteria introduced in Section 6, we evaluate these schemes using both real data sets under the B-JPCTII distribution. The resulting optimality measures are summarized in Table 10. The results show that CS_6, defined in Table 9, is consistently selected as the optimal censoring scheme by all the proposed criteria. Therefore, CS_6 is selected as the censoring scheme for generating the B-JPCTII data sets, and all subsequent analyses are conducted under this scheme. From a practical perspective, unlike the terminal censoring schemes CS_1 and CS_4 or schemes with heavy early removals CS_3, CS_6 removes test units progressively throughout the experiment. This allows failure information to be collected over a wider portion of the lifetime distribution while gradually reducing the number of surviving units that must be maintained under test. Consequently, CS_6 provides an effective balance between estimation efficiency and experimental resources, reducing the cost of maintaining a large number of test units without substantially increasing the overall duration of the experiment.

Table 9. The notation of censoring schemes for the real data .

Notation	(n_1, n_2, k)	**Censoring scheme**	
		$\mathbf{R_1}$	$\mathbf{R_2}$
CS_1	$(104, 207, 50)$	$(0_{(49)}, 54)$	$(0_{(49)}, 157)$
CS_2	$(104, 207, 50)$	$(0_{(29)}, 1_{(20)}, 34)$	$(0_{(29)}, 2_{(20)}, 117)$
CS_3	$(104, 207, 50)$	$(34, 0_{(29)}, 1_{(20)})$	$(97, 0_{(29)}, 2_{(10)}, 4_{(10)})$
CS_4	$(104, 207, 70)$	$(0_{(69)}, 34)$	$(0_{(69)}, 137)$
CS_5	$(104, 207, 70)$	$(0_{(39)}, 1_{(30)}, 5)$	$(0_{(39)}, 2_{(30)}, 87)$
CS_6	$(104, 207, 70)$	$(0_{(40)}, 1_{(26)}, 2_{(4)})$	$(32, 0_{(20)}, 1_{(20)}, 2_{(10)}, 3_{(15)}, 4_{(4)})$

Table 10. Selection of optimum censoring scheme

	CS_1	CS_2	CS_3	CS_4	CS_5	CS_6
D-opt	6×10^{-8}	3×10^{-8}	4×10^{-8}	4×10^{-8}	2×10^{-8}	1×10^{-8}
A-opt	9.445×10^{-2}	7.558×10^{-2}	5.384×10^{-2}	4.839×10^{-2}	4.254×10^{-2}	3.843×10^{-2}
E-opt	17.921	22.929	32.478	34.253	36.638	48.602

The resulting B-JPCTII data sets generated under the CS_6 censoring scheme are presented in 11. Based on the generated sample, the model parameters $\theta_1, \beta_1, \theta_2$ and β_2 are estimated using the MLE and the MPSE. In addition, the HRF and SRF for both populations are evaluated at $t = 0.5$. The corresponding point and interval estimates are reported in Table 12. From Table 12, we observe that the MLE and MPSE yield very similar point estimates for the model parameters, HRF, and SRF. Overall, both methods provide stable and reliable inference under the selected CS_6 censoring scheme.

Furthermore, the estimated HRFs and SRFs for the two populations, obtained using the MLE and MPSE under the CS_6 censoring scheme, are displayed in Figure 5. These curves are evaluated over a sequence of values of t, thereby illustrating the estimated hazard and survival functions across the time domain. From Figure 5, we obtain the following comments. For the first population, the MLE yields a consistently higher hazard rate than the MPSE, resulting in a more rapid decline of the corresponding reliability function. In contrast, the MPSE indicates lower failure rates and higher survival probabilities. For the second population, both methods produce increasing hazard rates; however, the MPSE hazard increases much more rapidly for larger values of t. Consequently, the MPSE reliability is initially slightly higher than the MLE reliability, but becomes lower after the two curves intersect. Overall, Figure 5 shows that both methods capture similar qualitative features, although noticeable differences arise in their estimated hazard and reliability functions.

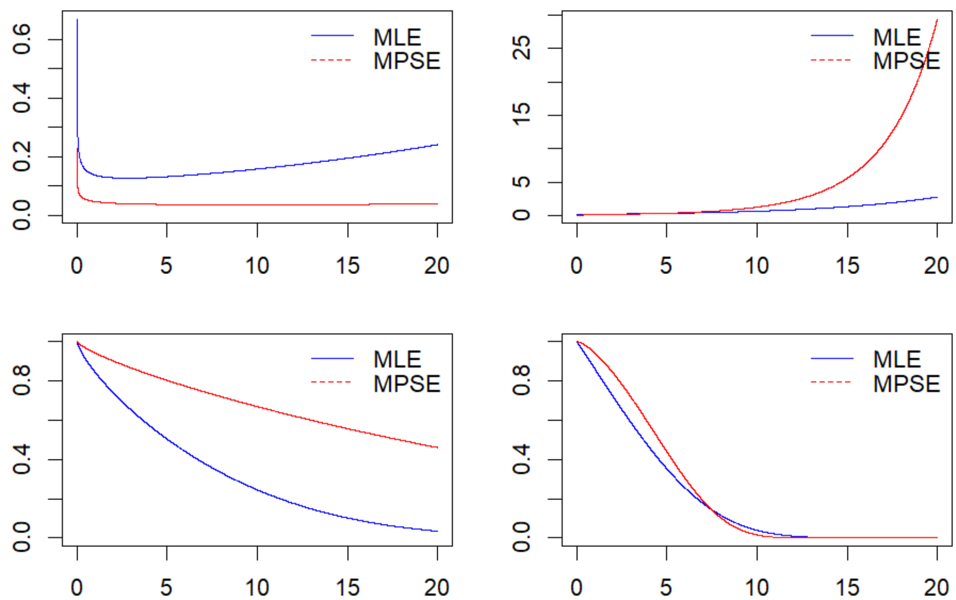


Figure 5. Estimated HRFs (upper panel) and SRFs (lower panel) based on the MLE and MPSE methods for the two real data sets under scheme C_6. Left panel: CSR3.DAT data set. Right panel: SYS3.DAT data set.

Figure 5 compares the MLE and MPSE estimates of the model parameters, HRFs, and SRFs under the six censoring schemes. Overall, the two estimation methods exhibit very similar patterns across all censoring schemes,

Table 11. Generated B-JPCTII sample from CSR3.DAT and SYS3.DAT data sets under CS_6 scheme

T	0.05, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.2, 0.2, 0.2, 0.2, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.4, 0.4,0.4, 0.4, 0.4, 0.4, 0.4, 0.4, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.6, 0.6, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 1 , 1, 1.1, 1.2, 1.3, 1.3, 1.3, 1.3, 1.4, 1.4,1.5, 1.5, 1.5, 1.7, 1.8, 1.9, 1.9, 2, 2.1, 2.2, 2.3,3,3.1, 3.4, 3.6
M	1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0

Table 12. Point and interval estimates the model parameters, HRF and SRF for B-JPCTII generated from the two data sets under the CS_6 censoring scheme.

				ACI			boot-P			boot-T			boot-N		
Parameter	Method	Estim	ESE	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length
θ_1	MLE	0.626	0.151	0.330	0.922	0.592	0.353	0.957	0.604	0.434	1.014	0.580	0.357	0.969	0.612
	MPSE	0.566	0.138	0.296	0.836	0.540	0.272	0.853	0.581	0.400	1.051	0.651	0.371	0.947	0.576
β_1	MLE	1.104	0.029	1.047	1.162	0.115	1.037	1.109	0.073	1.099	1.224	0.124	1.095	1.171	0.076
	MPSE	1.115	0.031	1.054	1.176	0.122	1.049	1.124	0.075	1.106	1.221	0.115	1.099	1.176	0.077
θ_2	MLE	0.948	0.110	0.732	1.164	0.432	0.757	1.222	0.465	0.740	1.185	0.445	0.713	1.168	0.455
	MPSE	0.951	0.112	0.732	1.170	0.438	0.955	1.331	0.376	0.689	0.948	0.258	0.605	1.005	0.400
β_2	MLE	1.142	0.025	1.092	1.191	0.099	1.086	1.171	0.085	1.115	1.213	0.098	1.112	1.195	0.084
	MPSE	1.138	0.025	1.089	1.187	0.098	1.077	1.137	0.060	1.139	1.221	0.081	1.133	1.197	0.064
H_1	MLE	0.086	0.024	0.039	0.133	0.094	0.028	0.083	0.056	0.088	0.194	0.106	0.085	0.142	0.057
	MPSE	0.090	0.025	0.041	0.138	0.097	0.032	0.087	0.054	0.093	0.192	0.100	0.089	0.145	0.056
H_2	MLE	0.139	0.021	0.097	0.181	0.084	0.089	0.157	0.068	0.122	0.197	0.075	0.119	0.186	0.067
	MPSE	0.136	0.021	0.095	0.178	0.083	0.081	0.135	0.054	0.137	0.200	0.062	0.131	0.188	0.057
S_1	MLE	0.936	0.020	0.896	0.975	0.079	0.923	0.979	0.056	0.861	0.947	0.086	0.892	0.949	0.057
	MPSE	0.927	0.022	0.884	0.969	0.084	0.909	0.971	0.062	0.855	0.942	0.088	0.887	0.948	0.061
S_2	MLE	0.931	0.016	0.901	0.962	0.062	0.908	0.964	0.056	0.885	0.951	0.066	0.898	0.954	0.056
	MPSE	0.933	0.016	0.903	0.964	0.061	0.934	0.970	0.036	0.875	0.932	0.058	0.896	0.936	0.040

Note: Note: θ_1, β_1, H_1, and S_1 denote the model parameters, HRF, and SRF, respectively, for the CSR3.DAT data set, while θ_2, β_2, H_2, and S_2 denote the corresponding quantities for the SYS3.DAT data set. The HRF and SRF values are computed at $t = 0.5$.

with the corresponding curves nearly overlapping in most panels. The estimates of β_1, β_2, H_1, S_1, and S_2 are almost identical, whereas slightly larger differences are observed for θ_1 and H_2 under a few censoring schemes. Nevertheless, both methods capture the same overall trends, indicating that the resulting estimates are robust to the choice of estimation method. These findings are consistent with the numerical results reported in Table 8, further demonstrating the close agreement between the MLE and MPSE for the two real data sets.

8. Conclusion

In this study, we investigated the estimation of the HRF and SRFfor the Pham distribution under the B-JPCTII. Both MLE and MPSE procedures were developed for point and interval estimation of the HRF and SRF. In addition, bootstrap confidence interval methods were considered, and an optimal censoring scheme was identified using several optimality criteria.

The simulation results show that neither method uniformly dominates the other; rather, their relative performance depends on the reliability characteristic, hazard shape, and censoring scheme. In particular, MPSE can provide more accurate point estimation for certain reliability characteristics, whereas MLE may be preferable when smaller

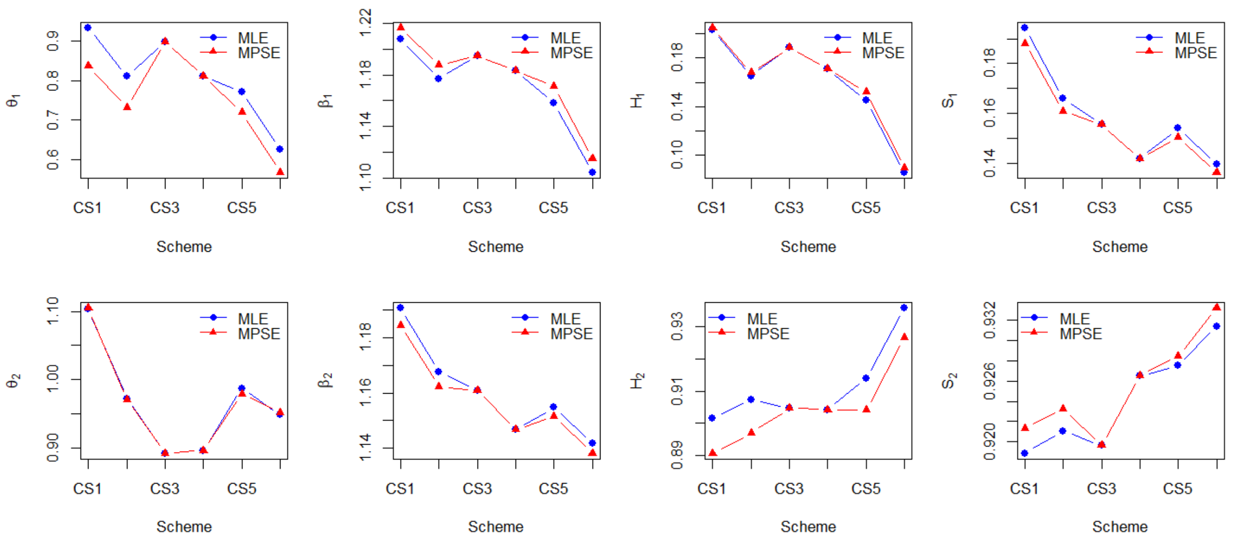


Figure 6. Point estimates of the model parameters, HRFs evaluated at $t = 0.5$, and SRFs evaluated at $t = 0.5$, obtained using the MLE and MPSE for the CSR3.DAT data set (upper panel) and the SYS3.DAT data set (lower panel) under the CS_6 censoring scheme.

MSEs and more stable numerical convergence are of primary concern. For the interval estimates, AL and CP vary across the hazard shapes and censoring schemes. ACI generally provides a favorable balance between AL and CP for the decreasing hazard case, whereas Boot-P performs well for the bathtub-shaped hazard case, particularly in terms of interval length.

The real-data applications further demonstrate the practical applicability of the proposed methodology. The Kolmogorov–Smirnov goodness-of-fit test and the empirical distribution function plots indicate that the Pham distribution provides an adequate fit to both software reliability data sets. Moreover, the proposed optimality criteria consistently identify an efficient censoring scheme, and the resulting HRF and SRF estimates obtained from the MLE and MPSE exhibit similar overall behavior, confirming the effectiveness of the proposed inferential procedures in practical applications.

From a practical perspective, the proposed methods provide a useful framework for reliability practitioners analyzing lifetime data under balanced joint progressive Type-II censoring. The simulation results suggest that the MLE is preferable when estimating the SRF, whereas the MPSE offers a competitive alternative for estimating the HRF, particularly in situations where likelihood-based estimation may be less stable. Furthermore, the proposed optimal censoring strategy can help practitioners design life-testing experiments that balance estimation accuracy with experimental cost.

Several extensions of the present work deserve further investigation. One possible direction is to extend the proposed inferential procedures to other censoring mechanisms, such as Type-I, hybrid, adaptive progressive, or generalized progressive censoring schemes, and compare their statistical efficiencies. Another important direction is to investigate the proposed estimation methods for other flexible lifetime distributions and related reliability measures, such as stress–strength reliability, mean residual life, and system reliability, under balanced joint progressive censoring. These extensions would broaden the applicability of the proposed methodology in reliability engineering and survival analysis.

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Conflict of interest:

There is no conflict of interest between all of these authors.

Appendix (A)

The second-order partial derivatives of the log-likelihood function(7)with respect to $\theta_1, \theta_2, \beta_1, \beta_2$ result in

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \theta_1^2} &= \sum_{i=1}^{k} \left\{ m_i \left(-\frac{1}{\theta_1^2} + t_i^{\theta_1} (\ln t_i)^2 \ln \beta_1 \right) - (R_{1i}+1)\, \beta_1^{t_i^{\theta_1}}\, t_i^{\theta_1} \ln \beta_1 (\ln t_i)^2\, (t_i^{\theta_1} \ln \beta_1 + 1) \right\} \\
\frac{\partial^2 \ell}{\partial \beta_1^2} &= \sum_{i=1}^{k} \left\{ -m_i \left[\frac{\ln \beta_1 + 1}{\beta_1^2 (\ln \beta_1)^2} + \frac{t_i^{\theta_1}}{\beta_1^2} \right] - (R_{1i}+1)\, t_i^{\theta_1} (t_i^{\theta_1} - 1)\, \beta_1^{t_i^{\theta_1} - 2} \right\} \\
\frac{\partial^2 \ell}{\partial \theta_2^2} &= \sum_{i=1}^{k} \left\{ (1-m_i) \left(-\frac{1}{\theta_2^2} + t_i^{\theta_2} (\ln t_i)^2 \ln \beta_2 \right) - (R_{2i}+1)\, \beta_2^{t_i^{\theta_2}}\, t_i^{\theta_2} \ln \beta_2 (\ln t_i)^2\, (t_i^{\theta_2} \ln \beta_2 + 1) \right\} \\
\frac{\partial^2 \ell}{\partial \beta_2^2} &= \sum_{i=1}^{k} \left\{ -(1-m_i) \left[\frac{\ln \beta_2 + 1}{\beta_2^2 (\ln \beta_2)^2} + \frac{t_i^{\theta_2}}{\beta_2^2} \right] - (R_{2i}+1)\, t_i^{\theta_2} (t_i^{\theta_2} - 1)\, \beta_2^{t_i^{\theta_2} - 2} \right\} \\
\frac{\partial^2 \ell}{\partial \theta_1\, \partial \beta_1} &= \frac{\partial^2 \ell}{\partial \beta_1\, \partial \theta_1} = \sum_{i=1}^{k} \left[m_i \frac{t_i^{\theta_1} \ln t_i}{\beta_1} - (R_{1i}+1)\, t_i^{\theta_1} \ln t_i\, \beta_1^{t_i^{\theta_1} - 1} \big(t_i^{\theta_1} \ln \beta_1 + 1 \big) \right] \\
\frac{\partial^2 \ell}{\partial \theta_2\, \partial \beta_2} &= \frac{\partial^2 \ell}{\partial \beta_2\, \partial \theta_2} = \sum_{i=1}^{k} \left[(1-m_i) \frac{t_i^{\theta_2} \ln t_i}{\beta_2} - (R_{2i}+1) t_i^{\theta_2} \ln t_i\, \beta_2^{t_i^{\theta_2} - 1} \big(t_i^{\theta_2} \ln \beta_2 + 1 \big) \right]
\end{aligned}
$$

and

$$
\frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_1} = \frac{\partial^2 \ell}{\partial \theta_1 \partial \beta_2} = \frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \ell}{\partial \beta_2 \partial \beta_1} = \frac{\partial^2 \ell}{\partial \beta_2 \partial \theta_1} = \frac{\partial^2 \ell}{\partial \theta_2 \partial \beta_1} = \frac{\partial^2 \ell}{\partial \beta_1 \partial \theta_2} = 0.
$$

The second derivatives of the log-PSF given by (14) are presented as follows:

$$\frac{\partial^2 p}{\partial \theta_1^2} = \sum_{i=1}^{k+1} m_i \Bigg\{ \frac{e^{1-\beta_1^{t_{i-1}^{\theta_1}}} \Big[\big(\beta_1^{t_{i-1}^{\theta_1}} \ln\beta_1 \, t_{i-1}^{\theta_1} \ln t_{i-1}\big)^2 - \beta_1^{t_{i-1}^{\theta_1}} \, t_{i-1}^{\theta_1} \, \ln\beta_1 \, (\ln t_{i-1})^2 \big(1 + t_{i-1}^{\theta_1} \ln\beta_1\big)\Big]}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2}$$
$$- \frac{e^{1-\beta_1^{t_i^{\theta_1}}} \Big[\big(\beta_1^{t_i^{\theta_1}} \ln\beta_1 \, t_i^{\theta_1} \ln t_i\big)^2 - \beta_1^{t_i^{\theta_1}} \, t_i^{\theta_1} \, \ln\beta_1 \, (\ln t_i)^2 \big(1 + t_i^{\theta_1} \ln\beta_1\big)\Big]}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2}$$
$$- \frac{\Big(- e^{1-\beta_1^{t_{i-1}^{\theta_1}}} \, \beta_1^{t_{i-1}^{\theta_1}} \ln\beta_1 \, t_{i-1}^{\theta_1} \ln t_{i-1} + e^{1-\beta_1^{t_i^{\theta_1}}} \, \beta_1^{t_i^{\theta_1}} \ln\beta_1 \, t_i^{\theta_1} \ln t_i \Big)^2}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2} \Bigg\}$$
$$- \sum_{i=1}^{k} (R_{1i} - m_i + 1) \, \beta_1^{t_i^{\theta_1}} \, t_i^{\theta_1} \, \ln\beta_1 \, (\ln t_i)^2 \big(1 + t_i^{\theta_1} \ln\beta_1\big)$$

$$\frac{\partial^2 p}{\partial \theta_2^2} = \sum_{i=1}^{k+1} (1-m_i) \Bigg\{ \frac{e^{1-\beta_2^{t_{i-1}^{\theta_2}}} \Big[\big(\beta_2^{t_{i-1}^{\theta_2}} \ln\beta_2 \, t_{i-1}^{\theta_2} \ln t_{i-1}\big)^2 - \beta_2^{t_{i-1}^{\theta_2}} \, t_{i-1}^{\theta_2} \, (\ln\beta_2) \, (\ln t_{i-1})^2 \big(1 + t_{i-1}^{\theta_2} \ln\beta_2\big)\Big]}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2}$$
$$- \frac{e^{1-\beta_2^{t_i^{\theta_2}}} \Big[\big(\beta_2^{t_i^{\theta_2}} \ln\beta_2 \, t_i^{\theta_2} \ln t_i\big)^2 - \beta_2^{t_i^{\theta_2}} \, t_i^{\theta_2} \, (\ln\beta_2) \, (\ln t_i)^2 \big(1 + t_i^{\theta_2} \ln\beta_2\big)\Big]}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2}$$
$$- \frac{\Big(- e^{1-\beta_2^{t_{i-1}^{\theta_2}}} \, \beta_2^{t_{i-1}^{\theta_2}} \ln\beta_2 \, t_{i-1}^{\theta_2} \ln t_{i-1} + e^{1-\beta_2^{t_i^{\theta_2}}} \, \beta_2^{t_i^{\theta_2}} \ln\beta_2 \, t_i^{\theta_2} \ln t_i \Big)^2}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}} - e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2} \Bigg\}$$
$$- \sum_{i=1}^{k} (R_{2i} - m_i + 1) \, \beta_2^{t_i^{\theta_2}} \, t_i^{\theta_2} \, (\ln\beta_2) \, (\ln t_i)^2 \big(1 + t_i^{\theta_2} \ln\beta_2\big)$$

$$\frac{\partial^2 p}{\partial \beta_1^2} = \sum_{i=1}^{k+1} m_i \Bigg\{ \frac{e^{1-\beta_1^{t_{i-1}^{\theta_1}}} \Big[\big(t_{i-1}^{\theta_1} \beta_1^{t_{i-1}^{\theta_1}-1}\big)^2 \big(1 - \beta_1^{t_{i-1}^{\theta_1}}\big) - t_{i-1}^{\theta_1} \beta_1^{t_{i-1}^{\theta_1}-2} \big(t_{i-1}^{\theta_1} - 1\big)\Big]}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2}$$
$$- \frac{e^{1-\beta_1^{t_i^{\theta_1}}} \Big[\big(t_i^{\theta_1} \beta_1^{t_i^{\theta_1}-1}\big)^2 \big(1 - \beta_1^{t_i^{\theta_1}}\big) - t_i^{\theta_1} \beta_1^{t_i^{\theta_1}-2} \big(t_i^{\theta_1} - 1\big)\Big]}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2}$$
$$- \frac{\Big(- e^{1-\beta_1^{t_{i-1}^{\theta_1}}} \, t_{i-1}^{\theta_1} \beta_1^{t_{i-1}^{\theta_1}-1} + e^{1-\beta_1^{t_i^{\theta_1}}} \, t_i^{\theta_1} \beta_1^{t_i^{\theta_1}-1} \Big)^2}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}} - e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2} \Bigg\} - \sum_{i=1}^{k} (R_{1i} - m_i + 1) \, t_i^{\theta_1} \big(t_i^{\theta_1} - 1\big) \beta_1^{t_i^{\theta_1}-2}$$

$$\frac{\partial^2 p}{\partial \beta_2^2} = \sum_{i=1}^{k+1}(1-m_i)\Bigg\{\frac{e^{1-\beta_2^{t_{i-1}^{\theta_2}}}\Big[\big(t_{i-1}^{\theta_2}\beta_2^{t_{i-1}^{\theta_2}-1}\big)^2\big(1-\beta_2^{t_{i-1}^{\theta_2}}\big)-t_{i-1}^{\theta_2}\beta_2^{t_{i-1}^{\theta_2}-2}\big(t_{i-1}^{\theta_2}-1\big)\Big]}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}}-e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2}$$
$$-\frac{e^{1-\beta_2^{t_i^{\theta_2}}}\Big[\big(t_i^{\theta_2}\beta_2^{t_i^{\theta_2}-1}\big)^2\big(1-\beta_2^{t_i^{\theta_2}}\big)-t_i^{\theta_2}\beta_2^{t_i^{\theta_2}-2}\big(t_i^{\theta_2}-1\big)\Big]}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}}-e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2}$$
$$-\frac{\Big(-e^{1-\beta_2^{t_{i-1}^{\theta_2}}}\,t_{i-1}^{\theta_2}\beta_2^{t_{i-1}^{\theta_2}-1}+e^{1-\beta_2^{t_i^{\theta_2}}}\,t_i^{\theta_2}\beta_2^{t_i^{\theta_2}-1}\Big)^2}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}}-e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2}\Bigg\}-\sum_{i=1}^{k}(R_{1i}+m_i)\,t_i^{\theta_2}\big(t_i^{\theta_2}-1\big)\beta_2^{t_i^{\theta_2}-2}$$

$$\frac{\partial^2 p}{\partial\theta_1\partial\beta_1}=\frac{\partial^2 p}{\partial\beta_1\partial\theta_1}=\sum_{i=1}^{k+1}\frac{m_i\big[\mathcal{N}_1\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}}-e^{1-\beta_1^{t_i^{\theta_1}}}\big)-\big(-e^{1-\beta_1^{t_{i-1}^{\theta_1}}}t_{i-1}^{\theta_1}\beta_1^{t_{i-1}^{\theta_1}-1}+e^{1-\beta_1^{t_i^{\theta_1}}}\,t_i^{\theta_1}\beta_1^{t_i^{\theta_1}-1}\big)\mathcal{D}_1\big]}{\big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}}-e^{1-\beta_1^{t_i^{\theta_1}}}\big)^2}$$
$$-\sum_{i=1}^{k}(R_{1i}-m_i+1)\,\beta_1^{t_i^{\theta_1}-1}\,t_i^{\theta_1}\ln t_i\big(1+t_i^{\theta_1}\ln\beta_1\big),$$

where

$$\mathcal{D}_1=\frac{d}{d\theta_1}\Big(e^{1-\beta_1^{t_{i-1}^{\theta_1}}}-e^{1-\beta_1^{t_i^{\theta_1}}}\Big)=-\,e^{1-\beta_1^{t_{i-1}^{\theta_1}}}\,\beta_1^{t_{i-1}^{\theta_1}}\,\ln\beta_1\,t_{i-1}^{\theta_1}\ln t_{i-1}\;+\;e^{1-\beta_1^{t_i^{\theta_1}}}\,\beta_1^{t_i^{\theta_1}}\,\ln\beta_1\,t_i^{\theta_1}\ln t_i,$$

and

$$\mathcal{N}_1=\frac{d}{d\theta_1}\Big(-e^{1-\beta_1^{t_{i-1}^{\theta_1}}}\,t_{i-1}^{\theta_1}\beta_1^{t_{i-1}^{\theta_1}-1}+e^{1-\beta_1^{t_i^{\theta_1}}}\,t_i^{\theta_1}\beta_1^{t_i^{\theta_1}-1}\Big)$$
$$=-\Bigg\{-e^{1-\beta_1^{t_{i-1}^{\theta_1}}}\beta_1^{t_{i-1}^{\theta_1}}\ln\beta_1\,t_{i-1}^{\theta_1}\ln t_{i-1}t_{i-1}^{\theta_1}\beta_1^{t_{i-1}^{\theta_1}-1}+e^{1-\beta_1^{t_{i-1}^{\theta_1}}}\,\beta_1^{t_{i-1}^{\theta_1}-1}\,t_{i-1}^{\theta_1}\ln t_{i-1}\big(1+t_{i-1}^{\theta_1}\ln\beta_1\big)\Bigg\}$$
$$+\Bigg\{-e^{1-\beta_1^{t_i^{\theta_1}}}\,\beta_1^{t_i^{\theta_1}}\ln\beta_1\,t_i^{\theta_1}\ln t_i t_i^{\theta_1}\beta_1^{t_i^{\theta_1}-1}+e^{1-\beta_1^{t_i^{\theta_1}}}\,\beta_1^{t_i^{\theta_1}-1}\,t_i^{\theta_1}\ln t_i\big(1+t_i^{\theta_1}\ln\beta_1\big)\Bigg\}$$

$$\frac{\partial^2 p}{\partial\theta_2\partial\beta_2}=\frac{\partial^2 p}{\partial\beta_2\partial\theta_2}=\sum_{i=1}^{k+1}(1-m_i)\frac{\mathcal{N}_2\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}}-e^{1-\beta_2^{t_i^{\theta_2}}}\big)-\big(-e^{1-\beta_2^{t_{i-1}^{\theta_2}}}t_{i-1}^{\theta_2}\beta_2^{t_{i-1}^{\theta_2}-1}+e^{1-\beta_2^{t_i^{\theta_2}}}\,t_i^{\theta_2}\beta_2^{t_i^{\theta_2}-1}\big)\mathcal{D}_2}{\big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}}-e^{1-\beta_2^{t_i^{\theta_2}}}\big)^2}$$
$$-\sum_{i=1}^{k}(R_{2i}+m_i)\,\beta_2^{t_i^{\theta_2}-1}\,t_i^{\theta_2}\ln t_i\big(1+t_i^{\theta_2}\ln\beta_2\big),$$

where

$$\mathcal{D}_2=\frac{d}{d\theta_2}\Big(e^{1-\beta_2^{t_{i-1}^{\theta_2}}}-e^{1-\beta_2^{t_i^{\theta_2}}}\Big)=-\,e^{1-\beta_2^{t_{i-1}^{\theta_2}}}\,\beta_2^{t_{i-1}^{\theta_2}}\,\ln\beta_2\,t_{i-1}^{\theta_2}\ln t_{i-1}\;+\;e^{1-\beta_2^{t_i^{\theta_2}}}\,\beta_2^{t_i^{\theta_2}}\,\ln\beta_2\,t_i^{\theta_2}\ln t_i,$$

and

$$\mathcal{N}_2 = \frac{d}{d\theta_2}\Big(- e^{\,1-\beta_2^{t_{i-1}^{\theta_2}}}\, t_{i-1}^{\theta_2}\beta_2^{t_{i-1}^{\theta_2}-1} + e^{\,1-\beta_2^{t_i^{\theta_2}}}\, t_i^{\theta_2}\beta_2^{t_i^{\theta_2}-1}\Big)$$
$$= -\Big\{ - e^{1-\beta_2^{t_{i-1}^{\theta_2}}}\beta_2^{t_{i-1}^{\theta_2}}\ln\beta_2\, t_{i-1}^{\theta_2}\ln t_{i-1} t_{i-1}^{\theta_2}\beta_2^{t_{i-1}^{\theta_2}-1} + e^{\,1-\beta_2^{t_{i-1}^{\theta_2}}}\,\beta_2^{t_{i-1}^{\theta_2}-1}\, t_{i-1}^{\theta_2}\ln t_{i-1}\big(1 + t_{i-1}^{\theta_2}\ln\beta_2\big)\Big\}$$
$$+ \Big\{ - e^{1-\beta_2^{t_i^{\theta_2}}}\,\beta_2^{t_i^{\theta_2}}\ln\beta_2\, t_i^{\theta_2}\ln t_i t_i^{\theta_2}\beta_2^{t_i^{\theta_2}-1} + e^{\,1-\beta_2^{t_i^{\theta_2}}}\,\beta_2^{t_i^{\theta_2}-1}\, t_i^{\theta_2}\ln t_i\big(1 + t_i^{\theta_2}\ln\beta_2\big)\Big\}$$

and

$$\frac{\partial^2 p}{\partial\theta_1\partial\theta_2} = \frac{\partial^2 p}{\partial\theta_2\partial\theta_1} = \frac{\partial^2 p}{\partial\theta_1\partial\beta_2} = \frac{\partial^2 p}{\partial\beta_1\partial\beta_2} = \frac{\partial^2 p}{\partial\beta_2\partial\beta_1} = \frac{\partial^2 p}{\partial\beta_2\partial\theta_1} = \frac{\partial^2 p}{\partial\theta_2\partial\beta_1} = \frac{\partial^2 p}{\partial\beta_1\partial\theta_2} = 0.$$

The analytical first- and second-order derivatives were verified numerically using finite-difference approximations implemented in the `numDeriv` package in R. The maximum absolute differences between the analytical and numerical derivatives were of the order of 10^{-11} for the log-likelihood function and 10^{-10} for the log-product-of-spacings function.

Appendix (B):Proof of Theorem 2.1

Proof
First consider the expressions $\xi_i(\theta_i), i = 1, 2$, where the score function are given in (8) and (10). Since,

$$\beta_j^{t_i^{\theta_j}} \longrightarrow \beta_j \quad \text{as } \theta_j \to 0^+, \text{ for every fixed } t_i > 0,$$

then

$$\xi_1(\theta_1) = \frac{K_1}{\theta_1} + O(1), \qquad \theta_1 \to 0^+,$$

and

$$\xi_2(\theta_2) = \frac{K_2}{\theta_2} + O(1), \qquad \theta_2 \to 0^+,$$

where

$$K_1 = \sum_{i=1}^{k} m_i, \qquad K_2 = \sum_{i=1}^{k}(1 - m_i).$$

Therefore, provided that $K_1 > 0$ and $K_2 > 0$,

$$\lim_{\theta_1\to 0^+} \xi_1(\theta_1) = +\infty, \qquad \lim_{\theta_2\to 0^+} \xi_2(\theta_2) = +\infty.$$

We next consider the limits as $\theta_j \to \infty$. Note that

$$t_i^{\theta_j} \longrightarrow \begin{cases} 0, & 0 < t_i < 1, \\ 1, & t_i = 1, \\ \infty, & t_i > 1. \end{cases}$$

If all relevant observations satisfy $0 < t_i < 1$, then

$$\beta_j^{t_i^{\theta_j}} \longrightarrow 1,$$

and

$$t_i^{\theta_j}\log t_i \longrightarrow 0.$$

It follows that

$$\lim_{\theta_1\to\infty}\xi_1(\theta_1)=\sum_{i=1}^{k} m_i\log t_i<0,$$

and

$$\lim_{\theta_2\to\infty}\xi_2(\theta_2)=\sum_{i=1}^{k}(1-m_i)\log t_i<0.$$

On the other hand, if at least one relevant observation satisfies $t_i>1$, then

$$\beta_j^{t_i^{\theta_j}} \longrightarrow \infty.$$

In this case, the negative term

$$-(R_{ji}+1)\beta_j^{t_i^{\theta_j}}\log\beta_j\, t_i^{\theta_j}\log t_i$$

dominates the remaining terms. Hence,

$$\lim_{\theta_j\to\infty}\xi_j(\theta_j)=-\infty,\qquad j=1,2.$$

Therefore, in either case, the score functions $\xi_j(\theta_j)$ are positive near zero and negative for sufficiently large θ_j.

Next, consider the expressions $\xi_i(\beta_i)$ where the score function are given in (9) and (11). Since

$$\log\beta_j\sim\beta_j-1,\qquad \beta_j\to 1^+,$$

we have

$$\frac{1}{\beta_j\log\beta_j}\sim\frac{1}{\beta_j-1}.$$

All remaining terms have finite limits as $\beta_j\to 1^+$. Consequently,

$$\xi_1(\beta_1)=\frac{K_1}{\beta_1-1}+O(1),\qquad \beta_1\to 1^+,$$

and

$$\xi_2(\beta_2)=\frac{K_2}{\beta_2-1}+O(1),\qquad \beta_2\to 1^+.$$

Thus,

$$\lim_{\beta_1\to 1^+}\xi_1(\beta_1)=+\infty,\qquad \lim_{\beta_2\to 1^+}\xi_2(\beta_2)=+\infty,$$

provided that $K_1>0$ and $K_2>0$.

To examine the behavior as $\beta_j\to\infty$, set

$$a_{ji}=t_i^{\theta_j}>0.$$

The individual terms in the score have the form

$$\frac{m_i}{\beta_j\log\beta_j},\qquad \frac{m_i a_{ji}}{\beta_j},\qquad -(R_{ji}+1)a_{ji}\beta_j^{a_{ji}-1}.$$

If $a_{ji}>1$, equivalently $t_i>1$, then

$$-a_{ji}\beta_j^{a_{ji}-1}\longrightarrow-\infty.$$

Hence, if at least one observation satisfies $t_i > 1$,

$$\lim_{\beta_j \to \infty} \xi_{\beta_j}(\beta_j) = -\infty.$$

If all observations satisfy $0 < t_i < 1$, then $0 < a_{ji} < 1$, and

$$a_{ji}\beta_j^{a_{ji}-1} = \frac{a_{ji}}{\beta_j^{1-a_{ji}}} \longrightarrow 0.$$

However, this negative term decreases more slowly than a_{ji}/β_j. Therefore,

$$\xi_j(\beta_j) \longrightarrow 0^-, \qquad \beta_j \to \infty.$$

If some $t_i = 1$ and none exceeds 1, the corresponding negative term converges to a finite negative constant. Thus, in all cases, $\xi_{\beta_j}(\beta_j) < 0$ for sufficiently large β_j.

By continuity of the score functions, the intermediate value theorem therefore guarantees the existence of at least one positive solution to each score equation

$$\xi_j(\theta_j) = 0, \qquad \xi_j(\beta_j) = 0, \qquad j = 1, 2.$$

For the two model parameters, direct differentiation gives

$$\frac{\partial^2 \ell}{\partial \theta_j^2} < 0, \quad \frac{\partial^2 \ell}{\partial \beta_j^2} < 0 \qquad j = 1, 2,$$

so that each score function is strictly decreasing, and the corresponding conditional solution is unique.

□

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