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Variable-Order Fractional Differential Equations: Existence, Stability, and Application to 3D Noise Evolution

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Abstract This paper studies a boundary value (BV) problem characterized by variable order(VO) Caputo fractional derivatives to study the existence, uniqueness, and stability of solutions under well-defined boundary conditions. Fixed point theory is used, where the Banach contraction principle ensures the uniqueness of solutions, while the Krasnoselskii theorem confirms their existence. Furthermore, the notion of Ulam-Hyers stability is used to investigate the response of the solutions to small perturbations. Numerical examples are presented to illustrate the theoretical results and to validate the approach under practical conditions. Additionally, an application concerning the evolution of features in three - dimensional noise fields is included. The highlights its engineering relevance, particularly in image processing and signal analysis, where modeling noise behavior and memory effects is important for tasks such as denoising and feature extraction.

Keywords Fractional boundary value problems; Variable-orer fractional calculus; Krasnoselskii fixed point theorem; Existence and uniqueness.

AMS 2010 subject classifications 26A30, 34A12, 34K37, 65L03

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1. Introduction

Over the past few decades, fractional derivative and integral operators defined for non-integer order has gained attention among mathematicians, scientists, and engineers. Their this is mainly because they can accurately describe systems exhibiting hereditary properties and memory effects and these are not adequately captured by classical integer-order models. Fractional calculus extends the conventional notions of differentiation and integration to arbitrary orders, thereby providing a flexible and powerful structure for modeling complex dynamical systems [14]. The origins of fractional calculus date back to a question posed by Leibniz in 1695, and since then, the field has developed into well -established area of mathematical research [12]. Comprehensive treatments of the subject are also available in [9].

VO fractional calculus was introduced by Samko and Ross in 1993, where the order of differentiation or integration to varies with respect to time or space [17]. This extension offers more flexibility in modeling processes whose memory characteristics change over time. Compared to fixed- order models, VO formulations are more suitable

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for representing time- dependent phenomena. These approaches have been applied in diffusion processes and heterogeneous media [11]. They are also used in areas like signal processing and machine learning [15]. These developments show the importance of VO fractional models in both theoretical studies and real- world applications.

1.1 A brief review of existing methods

Many researchers have investigated BV problems involving fractional differential equations (FDE) and established conditions for the existence of solutions. Earlier studies mainly focused on fixed - order fractional operators, which provide a basic setup for modeling systems with memory effects.

The extension to VO fractional calculus, introduced by Samko and Ross [17], allows the modeling of systems with varying memory behavior. Based on this, Parvin et al. [13] analyzed initial value problems involving Vo fractional derivatives and obtained results on the existence and behavior of solutions.

Abazid et al. [1] proposed a framework for analyzing VO FDE under broad conditions, showing that these models can be applied to time dependent systems.

In recent years has been progress in the the study of VO FDE. Yao et al. [20] examined stability properties, while Dai et al. [8] studied Ulam-Hyers stability for fractional systems. The existence and uniqueness of solutions for delayed VO problems were established by Telli et al.[18]. Bv problems for VO systems were further analyzed by Benkerrouche et al. [5]. Recent nonlinear fractional BV problems are discussed in [21], while new analytical and numerical approaches are presented in [3]. Additional computational and hybrid modeling techniques is also explored in [7].

Despite these developments, most studies focus either on purely VO fractional operators or on specific problem setting such as initial value problems or delayed systems. In particular, the analysis of Bv problems involving both variable order and constant order fractional derivatives is still limited. Also the combined investigation of existence, uniqueness, and stability properties within a unified approach has not been fully addressed in the literature.

1.2 Contributions

Based on the observations, this work addresses these gaps by developing a comprehensive system for analyzed fractional BV problems involving both variable order and constant order Caputp derivatives. The main contributions are as follows:

- Establishment of existence and uniqueness results using fixed point techniques.
- Analysis of Ulam-Hyers stability to assess robustness under perturbations.
- Extension of existing results by incorporating both variable order and constant order fractional derivatives in a unified approach.
- Demonstration of applicability through a model describing the evolution of features in three-dimensional noise fields.

Consider the following fractional BV problem:

$$\begin{cases} {}^{C}D^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = f\big(\mathrm{t}, \mathrm{h}(\mathrm{t}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{t})\big), & \text{for } \mathrm{t} \in [\rho_1, \rho_2], \\ \mathrm{u}\mathrm{h}(\rho_1) + \mathrm{u}^*\mathrm{h}(\rho_2) = \mathrm{u}_0 \end{cases} \tag{1}$$

Here, ${}^{C}D^{v(\mathrm{t})}$ represents the Caputo VO fractional derivative $v(\mathrm{t})$, while ${}^{C}D^{\mu}$ denotes the Caputo fractional derivative of constant order $\mu \in (0, 1)$. The function $f : [\rho_1, \rho_2] \times R \times R \to R$ is assumed to be smooth function. The parameters $\mathrm{u}, \mathrm{u}^* \in R$ are real constants satisfying $\mathrm{u} + \mathrm{u}^* \neq 0$ and $\mathrm{u}^* > 0$, whereas $\mathrm{u}_0 \in R$ denotes a specified constant.

Let $\mathrm{t} \in J = [\rho_1, \rho_2]$, where $0 \leq \rho_1 < \rho_2$. The VO function $v : J \to (0, 1]$ is assumed to continuous. We consider the unknown function h in the Banach space $C(J, \mathbb{R})$ equipped with the norm

$$||\mathrm{h}||_\infty = \sup_{\mathrm{t} \in J} |\mathrm{h}(\mathrm{t})|.$$

Moreover, we assume that $v(\mathrm{t}) \in C^1(J)$. Define $v_m = \min\{v(\mathrm{t}) : \mathrm{t} \in J\}$, $\quad v_M = \max\{v(\mathrm{t}) : \mathrm{t} \in J.\}$ and let $B = \{\mathrm{h} \in R : \|\mathrm{h} - \mathrm{h}_0\| \leq r\}$ be a closed ball in R.

The function $f(\mathrm{t}, \mathrm{h})$ and the order $v(\mathrm{t})$ satisfy certain regularity and continuity conditions necessary for ensuring the well-posedness of the subsequent fractional BV problem considered in the following assumption

- (H1) The order function $v(\mathrm{t})$ satisfies $0 < v_m \leq v(\mathrm{t}) \leq v_M < 0.5$, and $v(\mathrm{t})$ is continuously differentiable on J, i.e., $v(\mathrm{t}) \in C^1(J.)$
- (H2) The order function $v(\mathrm{t})$ satisfies $0.5 < v_m \leq v(\mathrm{t}) \leq v_M \leq 1.0$, and $v(\mathrm{t})$ is continuously differentiable on J.
- (H3) There exist positive constants L_f, L_g such that $|f(\mathrm{t}, \mathrm{h}_1, \mathrm{h}_1') - f(\mathrm{t}, \mathrm{h}_2, \mathrm{h}_2')| \leq L_f|\mathrm{h}_1 - \mathrm{h}_2| + L_g|\mathrm{h}_1' - \mathrm{h}_2'|$, for almost every $\mathrm{t} \in J$.
- (H4) There is a constant $C_\mu > 0$ and $K > 0$ so that for all $\mathrm{h}_1, \mathrm{h}_2 \in B$ and almost every $\mathrm{t} \in J$ and the functions satisfy $\|\mathrm{h}'\|_\infty \leq K\,\|\mathrm{h}\|_\infty, \forall\, \mathrm{h} \in B$,

$$\left|{}^C D^\mu \mathrm{h}_1(\mathrm{t}) - {}^C D^\mu \mathrm{h}_2(\mathrm{t})\right| \leq C_\mu\, |\mathrm{h}_1(\mathrm{t}) - \mathrm{h}_2(\mathrm{t})|,$$

where $C_\mu = K\,\frac{(\rho_2-\rho_1)^{1-\mu}}{\Gamma(2-\mu)}$. This ensures that the Caputo VO fractional derivative operator ${}^C D^{v(\mathrm{t})}$ is bounded and continuous on B.

For clarity, the contribution of each assumption toward the existence, uniqueness, and stability of solutions is summarized in Table.

Assumption	Mathematical Condition	Role in Analysis
(H1)	$v(\mathrm{t}) \in C^1(J),\ 0 < v(\mathrm{t}) < 0.5$	Ensures smoothness of the VO function and proper definition of the fractional operator in the low-order regime; supports the existence of solutions under appropriate conditions.
(H2)	$v(\mathrm{t}) \in C^1(J),\ 0.5 \leq v(\mathrm{t}) \leq 1$	Provides sufficient regularity required for the application of the Banach contraction principle, leading to the existence and uniqueness of solutions.
(H3)	$\|f(\mathrm{t}, h_1, h_1') - f(\mathrm{t}, h_2, h_2')\| \leq L_f\|h_1 - h_2\| + L_g\|h_1' - h_2'\|$	Ensures Lipschitz continuity of the nonlinear term, which is essential for establishing uniqueness of solutions and contributes to stability analysis.
(H4)	$\left\|{}^C D^\mu h_1(\mathrm{t}) - {}^C D^\mu h_2(\mathrm{t})\right\| \leq C_\mu\|h_1(\mathrm{t}) - h_2(\mathrm{t})\|$	Guarantees boundedness and continuity of the fractional operator, ensuring compactness of the associated operator and playing a key role in proving existence and Ulam-Hyers stability.

Table 1. Comparison of theoretical conditions under assumptions (H1)-(H4) and their roles in ensuring existence, uniqueness, and stability of solutions.

1.3 Organization

The present work focuses on examining the existence, uniqueness, and stability of solutions to the fractional BV problem. This paper is organized as follows: Section 2 establish essential preliminary concepts and theoretical results. Section 3 discusses the conditions ensuring existence, stability, and the singular behavior of solutions, along with their stability properties. Section 4 presents a numerical illustration supported by a three-dimensional representation of the proposed fractional BV problem. Finally, Section 5 summarizes the principal conclusions of this study.

2. Preliminaries

In this section, we describe the key preliminary concepts and definitions connected with the VO fractional derivative.

Definition 2.1 ([17, 19])**.** *The Riemann-Liouville VO* $v(\mathrm{t})$ *fractional integral for a function* h *is defined as*

$$I^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}\mathrm{h}(\mathrm{r})\,d\mathrm{r}, \quad \mathrm{t} > \rho_1.$$

Definition 2.2 ([17, 19])**.** *The Caputo VO fractional derivative* $v(\mathrm{t}) \in (0,1]$ *for a function* h *is defined by*

$${}^{C}D^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{-v(\mathrm{r})}}{\Gamma(1-v(\mathrm{r}))}\mathrm{h}'(\mathrm{r})\,d\mathrm{r}.$$

If $v(\mathrm{t}) = 1$*, then* ${}^{C}D_{\mathrm{t}}^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = \mathrm{h}'(\mathrm{t})$.

Lemma 2.1. *Suppose* $f : [0,\rho] \to R$ *is a smooth function and* $0 < v(\mathrm{t}) < 1$*. Then, a function* h *satisfies the following equation*

$$\mathrm{h}(\mathrm{t}) = \frac{1}{\Gamma(v(\mathrm{t}))}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\,d\mathrm{r} - \frac{1}{\mathrm{u}+\mathrm{u}^*}\left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))}\int_{\rho_1}^{\rho_2} (\rho_2 - s)^{v(\mathrm{r})-1} f(\mathrm{r})\,d\mathrm{r} - \mathrm{u}_0\right],$$

iff h *is a solution of the VO fractional BV problem*

$${}^{C}D^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = f(\mathrm{t}), \quad \mathrm{t} \in [\rho_1, \rho_2],$$

$$\mathrm{u}\,\mathrm{h}(\rho_1) + \mathrm{u}^*\,\mathrm{h}(\rho_2) = \mathrm{u}_0.$$

Proof. Assume that h satisfies the variable-order fractional BV problem

$${}^{C}D^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = f(\mathrm{t}), \quad \mathrm{t} \in [\rho_1, \rho_2],$$

with the boundary condition $\mathrm{u}\,\mathrm{h}(\rho_1) + \mathrm{u}^*\,\mathrm{h}(\rho_2) = \mathrm{u}_0$. Applying the variable-order Riemann-Liouville fractional integral $I^{v(\mathrm{t})}$ to both sides, we obtain

$$\mathrm{h}(\mathrm{t}) = \mathrm{h}(\rho_1) + \frac{1}{\Gamma(v(\mathrm{t}))}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\,d\mathrm{r}.$$

Let $\mathrm{h}(\rho_1) = C$, where C is a constant to be determined. Then,

$$\mathrm{h}(\mathrm{t}) = C + \frac{1}{\Gamma(v(\mathrm{t}))}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\,d\mathrm{r}.$$

Evaluating at $\mathrm{t} = \rho_2$, we get

$$\mathrm{h}(\rho_2) = C + \frac{1}{\Gamma(v(\rho_2))}\int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\,d\mathrm{r}.$$

Substituting $\mathrm{h}(\rho_1) = C$ and $\mathrm{h}(\rho_2)$ into the boundary condition, we obtain

$$\mathrm{u}C + \mathrm{u}^* \left[C + \frac{1}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\, dr \right] = \mathrm{u}_0.$$

$$(\mathrm{u} + \mathrm{u}^*)C + \frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\, dr = \mathrm{u}_0.$$

Solving for C, we get $C = \frac{1}{\mathrm{u}+\mathrm{u}^*} \left[\mathrm{u}_0 - \frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\, dr \right]$. Substituting this value of C back into $\mathrm{h}(\mathrm{t})$, we obtain the desired integral representation.

$$\mathrm{h}(\mathrm{t}) = \frac{1}{\mathrm{u} + \mathrm{u}^*} \left[\mathrm{u}_0 - \frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\, dr \right] + \frac{1}{\Gamma(v(\mathrm{t}))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v(\mathrm{r})-1} f(\mathrm{r})\, dr.$$

Conversely, it is straightforward to verify that any function h defined by the given integral expression satisfies the FDE and the boundary condition. Hence, the equivalence holds.

Theorem 2.1. [9] Consider a Banach space X, $B_X \subset X$ closed, and let $\psi : B_X \to B_X$ be a strict contraction, i.e., $|\psi\hat{\mathrm{h}} - \psi\mathrm{h}| \leq L|\hat{\mathrm{h}} - \mathrm{h}|$ for some $L \in (0, 1)$ and for all $\hat{\mathrm{h}}, \mathrm{h} \in B_X$. Then ψ has a unique fixed point h^*.

Theorem 2.2 (Krasnosel'skii Theorem [18])**.** *Let X be a Banach space and let $B_X \subset X$ be a convex, closed, nonempty and bounded subset. Suppose $A, D : B_X \to X$ satisfy*

1. *A is a contraction on B_X.*
2. *D is continuous and compact on B_X.*
3. *$A(B_X) + D(B_X) \subset B_X$.*

Then the operator $\psi = A + D$ admits at least one fixed point in B_X.

Theorem 2.3 (Ulam-Hyers Stability [8])**.** *Consider the operator equation $\psi(\mathrm{h})(\mathrm{t}) = 0$ is said to be stable with respect to a control function $\varepsilon \geq 0$ if there occurs a constant $C_{v(\mathrm{t})} > 0$ so that for any function $\mathrm{h}^*(\mathrm{t}) \in X$ satisfying $\|\psi\mathrm{h}^*(\mathrm{t})\| \leq \varepsilon, \quad \forall\, \mathrm{t} \in [0, T]$, there exists an exact solution $\mathrm{h}(\mathrm{t}) \in X$ of $\psi(\mathrm{h})(\mathrm{t}) = 0$ such that*

$$\|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})\| \leq C_{v(\mathrm{t})} \cdot \varepsilon, \quad \forall\, \mathrm{t} \in [0, T].$$

3. Uniqueness and Existence with Stability Analysis

In this section, we derive the conditions that ensure existence, uniqueness, and stability of solutions for the VO fractional derivative in (3). Three theorems are presented, with the first providing a fundamental analytical tool that underpins the core results.

Theorem 3.1. *Consider the assumptions* (H2)-(H4) *satisfied. Then, the BVP (3) has exactly one solution whenever* $\Lambda_v < 1$.

$$\Lambda_v = \begin{cases} L\left[\frac{\mathrm{s}^{1-v_m} \cdot (\mathrm{t}-\rho_1)^{v_m}}{v_m} + \aleph\right], & \textit{if}\ \mathrm{s} > 1, \\ L\left[\frac{(\mathrm{t}-\rho_1)^{v_m}}{v_m} + \aleph\right], & \textit{if}\ \mathrm{s} \leq 1. \end{cases}$$

Proof : Consider the operator $\psi : B_X \to B_X$ defined by

$$\begin{aligned} \psi(\mathrm{h}(\mathrm{t})) = {} & \frac{1}{\Gamma(v(\mathrm{t}))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v(\mathrm{r})-1} f\left(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\right) dr \\ & - \frac{1}{\mathrm{u} + \mathrm{u}^*} \left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v(\mathrm{r})-1} f\left(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\right) dr - \mathrm{u}_0 \right] \end{aligned} \tag{2}$$

Let $\alpha_0 = \sup_{\mathrm{t}\in J} \|f(\mathrm{t},0,0)\|$, $\quad (LR_X+\alpha_0)\left[\frac{\mathrm{s}^{1-v_m}(\mathrm{t}-\rho_1)^{v_m}}{v_m}\right]+\zeta < R_X$. To show that $\psi(B_X)\subset B_X$. If $\rho_2 - \rho_1 < \mathrm{s}$ and $\mathrm{s} > 1$, then

$$
\begin{aligned}
&\|\psi(\mathrm{h}(\mathrm{t}))\| \\
&\qquad \le \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}\|f(\mathrm{r},\mathrm{h}(\mathrm{r}),{}^{C}D^{\mu}\mathrm{h}(\mathrm{r}))-f(\mathrm{r},0,0)\|\,d\mathrm{r} \\
&\qquad + \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}\|f(\mathrm{r},0,0)\|\,d\mathrm{r} \\
&\qquad + \frac{1}{\mathrm{u}+\mathrm{u}^*}\left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))}\int_{\rho_1}^{\rho_2}(\rho_2-\mathrm{r})^{v(\mathrm{r})-1}\|f(\mathrm{r},\mathrm{h}(\mathrm{r}),{}^{C}D^{\mu}\mathrm{h}(\mathrm{r}))-f(\mathrm{r},0,0)\|\,d\mathrm{r}-\mathrm{u}_0\right] \\
&\qquad \le \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}\big(L_f\|\mathrm{h}(\mathrm{r})\|+L_gC_\mu\|\mathrm{h}(\mathrm{r}))\|\big)\,d\mathrm{r} \\
&\qquad + \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}\|f(\mathrm{r},0,0)\|\,d\mathrm{r} \\
&\qquad + \frac{1}{\mathrm{u}+\mathrm{u}^*}\left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))}\int_{\rho_1}^{\rho_2}(\rho_2-\mathrm{r})^{v(\mathrm{r})-1}\big(L_f\|\mathrm{h}(\mathrm{r})\|+L_gC_\mu\|\mathrm{h}(\mathrm{r}))\|\big)\,d\mathrm{r}+\gamma\right] \\
&\qquad \le \frac{LR_X}{\Gamma(v(t))}\int_{\rho_1}^{\mathrm{t}}(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}\,d\mathrm{r}+\frac{\alpha_0}{\Gamma(v(t))}\int_{\rho_1}^{\mathrm{t}}(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}\,d\mathrm{r} \\
&\qquad + \frac{1}{\mathrm{u}+\mathrm{u}^*}\left[\frac{LR_X\mathrm{u}^*}{\Gamma(v(\rho_2))}\int_{\rho_1}^{\rho_2}(\rho_2-\mathrm{r})^{v(\mathrm{r})-1}\,d\mathrm{r}+\mathrm{u}_0\right]
\end{aligned}
$$

where $L = L_f + L_gC_\mu$. Then,

$$
\begin{aligned}
\|\psi(\mathrm{h}(\mathrm{t}))\| \quad &\le (LR_X+\alpha_0)\left[\frac{\mathrm{s}^{1-v_m}(\mathrm{t}-\rho_1)^{v_m}}{v_m}\right]+\frac{LR_X\mathrm{u}^*}{\mathrm{u}+\mathrm{u}^*}\frac{(\rho_2-\rho_1)^{v(\rho_2)}}{\Gamma(v((\rho_2)+1)}+\frac{\mathrm{u}_0}{\mathrm{u}+\mathrm{u}^*} \\
&\le (LR_X+\alpha_0)\left[\frac{\mathrm{s}^{1-v_m}(\mathrm{t}-\rho_1)^{v_m}}{v_m}\right]+\zeta < R_X.
\end{aligned}
$$

For any $\hat{\mathrm{h}}, \mathrm{h} \in B_X$ and each $\mathrm{t} \in [\rho_1, \rho_2]$,

$$
\begin{aligned}
&\|\psi(\mathrm{h}(\mathrm{t}))-\psi(\hat{\mathrm{h}}(\mathrm{t}))\| \\
&\qquad \le \int_{\rho_1}^{\mathrm{t}} \frac{(\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}|f(\mathrm{r},\mathrm{h}(\mathrm{r}),{}^{C}D^{\mu}\mathrm{h}(\mathrm{r}))-f(\mathrm{r},\hat{\mathrm{h}}(\mathrm{r}),{}^{C}D^{\mu}\hat{\mathrm{h}}(\mathrm{r}))|\,d\mathrm{r} \\
&\qquad + \frac{\mathrm{u}^*}{\mathrm{u}+\mathrm{u}^*}\int_{\rho_1}^{\rho_2}\frac{(\rho_2-\mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}|f\mathrm{r},\mathrm{h}(\mathrm{r}),{}^{C}D^{\mu}\mathrm{h}(\mathrm{r}))-f(\mathrm{r},\hat{\mathrm{h}}(\mathrm{r}),{}^{C}D^{\mu}\hat{\mathrm{h}}(\mathrm{r}))|\,d\mathrm{r} \\
&\qquad \le L\left[\frac{\mathrm{s}^{1-v_m}}{v_m}(\mathrm{t}-\rho_1)^{v_m}+\frac{\mathrm{u}^*}{\mathrm{u}+\mathrm{u}^*}\frac{(\rho_2-\rho_1)^{v(\rho_2)}}{\Gamma(v((\rho_2)+1)}\right]\|\mathrm{h}-\hat{\mathrm{h}}\| \\
&\qquad \le L\left[\frac{\mathrm{s}^{1-v_m}}{v_m}(\mathrm{t}-\rho_1)^{v_m}+\aleph\right]\|\mathrm{h}-\hat{\mathrm{h}}\| \\
&\qquad \le \Lambda_v\|\mathrm{h}-\hat{\mathrm{h}}\| < \|\mathrm{h}-\hat{\mathrm{h}}\|.
\end{aligned}
$$

Thus, ψ is a contraction for $\Lambda_v < 1$. Hence, by Theorem (2.1), the operator equation corresponding to the given BVP has exactly one fixed point. Consequently, the problem admits a unique solution.

Moreover, noting that $\Gamma(v)$ decreases monotonically for $v \in (0, 0.5)$, an analogous argument as presented previously shows that the theorem continues to hold under the assumption (H1).

Theorem 3.2. Assume hypotheses (H2)-(H4) satisfied. Then the fractional BV problem (3) admits at least one continuous solution $\mathrm{h} \in C([\rho_1, \rho_2])$.

proof. Consider the operator $\psi : B_X \to B_X$ defined by

$$\begin{aligned}\psi(\mathrm{h}(\mathrm{t})) &= \frac{1}{\Gamma(v(\mathrm{t}))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1} f\big(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\big)\, d\mathrm{r} \\ &\quad - \frac{1}{\mathrm{u}+\mathrm{u}^*} \left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2-\mathrm{r})^{v(\mathrm{r})-1} f\big(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\big)\, d\mathrm{r} - \mathrm{u}_0\right] \end{aligned} \tag{3}$$

To apply Theorem(2.2), we decompose $\psi = A + D$, where

$$\begin{aligned} A(\mathrm{h})(\mathrm{t}) &= \frac{1}{\Gamma(v(\mathrm{t}))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1} f\big(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\big)\, d\mathrm{r}, \\ D(\mathrm{h})(\mathrm{t}) &= -\frac{1}{\mathrm{u}+\mathrm{u}^*} \left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2-\mathrm{r})^{v(\mathrm{r})-1} f\big(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\big)\, d\mathrm{r} - \mathrm{u}_0\right] \end{aligned}$$

Step : 1 If $B_X = \{\mathrm{h} \in X : \|\mathrm{h}\|_\infty \le R_X\}$, then there appear $r > 0$ so that

$$\begin{aligned} \|A\mathrm{h} + D\mathrm{h}\| \quad &\le \frac{LR_X}{\Gamma(v(t))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}\, d\mathrm{r} + \frac{\alpha_0}{\Gamma(v(t))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1}\, d\mathrm{r} \\ &\quad + \frac{1}{\mathrm{u}+\mathrm{u}^*} \left[\frac{LR_X \mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2-\mathrm{r})^{v(\mathrm{r})-1}\, d\mathrm{r} + \mathrm{u}_0\right] \\ &\le (LR_X + \alpha_0) \left[\frac{\mathrm{s}^{1-v_m} (\mathrm{t}-\rho_1)^{v_m}}{v_m}\right] + \frac{LR_X \mathrm{u}^*}{\mathrm{u}+\mathrm{u}^*} \frac{(\rho_2-\rho_1)^{v(\rho_2)}}{\Gamma(v((\rho_2)+1)} + \frac{\mathrm{u}_0}{\mathrm{u}+\mathrm{u}^*} \\ &\le (LR_X + \alpha_0) \left[\frac{\mathrm{s}^{1-v_m} (\mathrm{t}-\rho_1)^{v_m}}{v_m}\right] + \zeta < r. \end{aligned}$$

$$A(B_X) + D(B_X) \subset B_X.$$

Step : 2 A is a contraction on B_X. For $\mathrm{h}_1, \mathrm{h}_2 \in B_X$

$$\begin{aligned} &|A(\mathrm{h}_1)(\mathrm{t}) - A(\mathrm{h}_2)(\mathrm{t})| \\ &\qquad \le \frac{1}{\Gamma(v(\mathrm{t}))} \int_{\rho_1}^{\mathrm{t}} (\mathrm{t}-\mathrm{r})^{v(\mathrm{r})-1} \big(L_f |\mathrm{h}_1 - \mathrm{h}_2| + L_g |{}^{C}D^{\mu}\mathrm{h}_1 - {}^{C}D^{\mu}\mathrm{h}_2|\big)\, d\mathrm{r} \\ &\qquad \le \frac{\mathrm{s}^{1-v}}{v_m} (\mathrm{t}-\rho_1)^{v_m} \big(L_f + L_g C_\mu\big) \|\mathrm{h}_1 - \mathrm{h}_2\|. \\ &\qquad \le L \cdot \frac{\mathrm{s}^{1-v}}{v_m} (\mathrm{t}-\rho_1)^{v_m} \|\mathrm{h}_1 - \mathrm{h}_2\|. \\ &\qquad \le L_A \|\mathrm{h}_1 - \mathrm{h}_2\|. \end{aligned}$$

Hence, A is a contraction, whenever $L_A = L \cdot \frac{\mathrm{r}^{1-v}}{v_m} (\mathrm{t}-\rho_1)^{v_m} < 1$.

Step : 3 D is completely continuous. For each $\mathrm{h} \in B_X$,

$$D(\mathrm{h})(\mathrm{t}) = c(\mathrm{h}) = -\frac{1}{\mathrm{u}+\mathrm{u}^*} \left[\frac{\mathrm{u}^*}{\Gamma(v(\rho_2))} \int_{\rho_1}^{\rho_2} (\rho_2-\mathrm{r})^{v(\mathrm{r})-1} f\big(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})\big)\, d\mathrm{r} - \mathrm{u}_0\right]$$

D is a constant-type operator independent of t. Indeed, for any $\mathrm{h} \in B_X$. Hence, $D(B_X)$ is a bounded subset of constant functions in $C([\rho_1, \rho_2])$, which is relatively compact and continuous. Therefore, D is completely

continuous.
By Theorem(2.2), since A is a contraction, B_X is continuous and compact, and $A(B_X) + D(B_X) \subset B_X$, the operator $\psi = A + D$ has a nonempty set of fixed points in B_X.

Moreover, noting that $\Gamma(v)$ decreases monotonically for $v \in (0, 0.5)$, an analogous argument as presented previously shows that the theorem continues to hold under the assumption (H1).

Theorem 3.3. *Assume that the BVP (3) satisfies the hypotheses* (H2)-(H4). *If there exists a constant*

$$C_{v(\mathrm{t})} = \frac{(\rho_2 - \rho_1)^{v_m}}{\Gamma(v_m + 1)\,(1 - \Lambda_v)} > 0,$$

then every solution of the problem(3) is stable.

Proof. Let $\mathrm{h}^*(\mathrm{t})$ be an approximate solution satisfying

$$\left|{}^{C}D^{v(\mathrm{t})}\mathrm{h}^*(\mathrm{t}) - f\big(\mathrm{t}, \mathrm{h}^*(\mathrm{t}), {}^{C}D^{\mu}\mathrm{h}^*(\mathrm{t})\big)\right| \leq \varepsilon, \qquad \mathrm{t} \in [\rho_1, \rho_2].$$

i.e.,

$${}^{C}D^{v(\mathrm{t})}\mathrm{h}^*(\mathrm{t}) = f\big(\mathrm{t}, \mathrm{h}^*(\mathrm{t}), {}^{C}D^{\mu}\mathrm{h}^*(\mathrm{t})\big) + q(\mathrm{t}), \quad \text{with } |q(\mathrm{t})| \leq \varepsilon.$$

The equivalent integral form of $\mathrm{h}^*(\mathrm{t})$ can be expressed as

$$\mathrm{h}^*(\mathrm{t}) = \frac{1}{\Gamma(v(\mathrm{t}))}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v(\mathrm{t})-1} f\big(\mathrm{r}, \mathrm{h}^*(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}^*(\mathrm{r})\big)\, d\mathrm{r} + \frac{1}{\Gamma(v(\mathrm{t}))}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v(\mathrm{t})-1} q(\mathrm{r})\, d\mathrm{r}.$$

Let $\mathrm{h}(\mathrm{t})$ be the exact solution of (3). Subtracting the two integral forms yields

$$\begin{aligned}
|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq{}& \frac{1}{\Gamma(v_m)}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v_m - 1} |f(\mathrm{r}, \mathrm{h}^*(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}^*(\mathrm{r})) - f(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r}))|\, d\mathrm{r} \\
&+ \frac{\mathrm{u}^*}{\mathrm{u} + \mathrm{u}^*}\int_{\rho_1}^{\rho_2} \frac{(\rho_2 - \mathrm{r})^{v(\mathrm{r})-1}}{\Gamma(v(\mathrm{r}))}\big|f(\mathrm{r}, \mathrm{h}(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}(\mathrm{r})) - f(\mathrm{r}, \mathrm{h}^*(\mathrm{r}), {}^{C}D^{\mu}\mathrm{h}^*(\mathrm{r}))\big|\, d\mathrm{r} \\
&+ \frac{1}{\Gamma(v_m)}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v_m - 1} |q(\mathrm{r})|\, d\mathrm{r}. \\
\leq{}& \frac{L_f + L_g C_\mu}{\Gamma(v_m)}\Big[\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v_m - 1}\, d\mathrm{r} + \frac{\mathrm{u}^*}{\mathrm{u} + \mathrm{u}^*}\int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v_m - 1}\, d\mathrm{r}\Big] \|\mathrm{h}^*(\mathrm{r}) - \mathrm{h}(\mathrm{r})\| \\
&+ \frac{\varepsilon}{\Gamma(v_m)}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v_m - 1}\, d\mathrm{r}.
\end{aligned}$$

$$\begin{aligned}
|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq{}& \frac{L}{\Gamma(v_m)}\Big[\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v_m - 1}\, d\mathrm{r} + \frac{\mathrm{u}^*}{\mathrm{u} + \mathrm{u}^*}\int_{\rho_1}^{\rho_2} (\rho_2 - \mathrm{r})^{v_m - 1}\, d\mathrm{r}\Big] \|\mathrm{h}^*(\mathrm{r}) - \mathrm{h}(\mathrm{r})\| \\
&+ \frac{\varepsilon}{\Gamma(v_m)}\int_{\rho_1}^{\mathrm{t}} (\mathrm{t} - \mathrm{r})^{v_m - 1}\, d\mathrm{r}. \\
\leq{}& L\left[\frac{\mathrm{s}^{1 - v_m}}{v_m}(\mathrm{t} - \rho_1)^{v_m} + \aleph\right] \|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})\| + + \frac{\varepsilon(\rho_2 - \rho_1)^{v_m}}{\Gamma(v_m + 1)}. \\
|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq{}& \varepsilon \cdot \frac{(\rho_2 - \rho_1)^{v_m}}{\Gamma(v_m + 1)\,(1 - \Lambda_v)}.
\end{aligned}$$

Finally,

$$C_{v(\mathrm{t})} = \frac{(\rho_2 - \rho_1)^{v_m}}{\Gamma(v_m + 1)\,(1 - \Lambda_v)},$$

we get

$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq \varepsilon \cdot C_{v(\mathrm{t})}.$$

which proves that the solution is stable by Theorem (2.3).
Moreover, noting that $\Gamma(v)$ decreases monotonically for $v \in (0, 0.5)$, an analogous argument as presented previously shows that the theorem continues to hold under the assumption (H1).

4. Examples and Application

We consider two illustrative examples to validate the theoretical findings discussed in the preceding section.

Example 1: Let us consider the VO fractional derivative equation subject to the specified boundary condition:

$$\begin{cases} {}^{C}D^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = -0.2\,\mathrm{h}(\mathrm{t}) + 0.15\,z, \quad \mathrm{t} \in [0,1], \\ \mathrm{h}(0) + \mathrm{h}(1) = 0. \end{cases} \tag{4}$$

Here, the order function is chosen as $v(\mathrm{t}) = 0.6 + 0.1\mathrm{t}$, $v_m = 0.6$, $v_M = 0.7$, the constant fractional order is $\mu = 0.4$ and $z = {}^{C}D^{0.4}\mathrm{h}(\mathrm{t})$.

The function $f(\mathrm{t}, \mathrm{h}, \mathrm{h}') = -0.2\,\mathrm{h} + 0.15\,z$ is continuous on $[0,1] \times R^2$ and satisfies the Lipschitz condition with constants $L_f = -0.20$ and $L_g = 0.15$.

$$|(-0.2\,\mathrm{h}_1 + 0.15\,\mathrm{h}_1') - (-0.2\,\mathrm{h}_2 + 0.15\,\mathrm{h}_2')| \leq -0.2|\mathrm{h}_1 - \mathrm{h}_2| + 0.15|\mathrm{h}_1' - \mathrm{h}_2'|.$$

Assuming the Caputo operator ${}^{C}D^{\mu}$ satisfies

$$\left|{}^{C}D^{\mu}\mathrm{h}_1(\mathrm{t}) - {}^{C}D^{\mu}\mathrm{h}_2(\mathrm{t})\right| \leq C_\mu\, \|\mathrm{h}_1 - \mathrm{h}_2\|,$$

$$C_\mu = K\,\frac{(\rho_2 - \rho_1)^{1-\mu}}{\Gamma(2 - \mu)} = 0.8935\,\frac{(1 - 0)^{1-0.4}}{\Gamma(2 - 0.4)} = 1$$

we obtain the overall Lipschitz constant

$$L = L_f + L_g C_\mu = 0.35.$$

We take $L_f = \rho_2 - \rho_1 = 1 \leq 1$ and evaluate the $L_f \leq 1$ and evaluate the $L_f >$ branch of Λ_v at the endpoint $\mathrm{t} = \rho_2$ branch of Λ_v at the end point $\mathrm{t} = \rho_2 = 1$:

$$\Lambda_v = L\left[\frac{(\rho_2 - \rho_1)^{v_m}}{v_m} + \aleph\right], \qquad \aleph = \frac{\mathrm{u}^*}{\mathrm{u} + \mathrm{u}^*}\,\frac{(\rho_2 - \rho_1)^{v(\rho_2)}}{\Gamma(v(\rho_2) + 1)}.$$

Compute the ingredients:

$$(\rho_2 - \rho_1)^{v_m} = 1^{0.6} \approx 1,$$

$$\frac{(\rho_2 - \rho_1)^{v_m}}{v_m} \approx \frac{1}{0.6} \approx 1.6667.$$

For $\aleph$ (with $\mathrm{u} = \mathrm{u}^* = 1$ and $v(\rho_2) = v(1) = 0.60$):

$$(\rho_2 - \rho_1)^{v(\rho_2)} = 1^{0.60} \approx 1,$$
$$\Gamma(v(\rho_2) + 1) = \Gamma(1.60) \approx 0.8935.$$

Hence

$$\aleph \approx \tfrac{1}{2} \cdot \frac{1}{0.8935} \approx 0.5596.$$

With boundary parameters $\mathrm{u} = 1$, $\mathrm{u}^* = 1$, and $\gamma = 0$, the associated contraction constant is estimated as

$$\begin{aligned}\Lambda_v &= L\left[\frac{(1-0)^{v_m}}{v_m} + \frac{\mathrm{u}^*}{\mathrm{u}+\mathrm{u}^*}\,\frac{(1)^{v_M}}{\Gamma(v_M+1)}\right]\\ &\approx 0.35\,(1.6667 + 0.5596)\\ &= 0.7792 < 1.\end{aligned}$$

Therefore $\Lambda_v < 1$, and by the Theorem (3.1) the fractional BV problem above admits a exactly one solution on $J = [0, 1]$.

We now analyze the Ulam-Hyers stability

$$\Lambda_v = 0.7792,$$
$$C_{v(\mathrm{t})} = \frac{(\rho_2 - \rho_1)^{v_m}}{\Gamma(v_m + 1)\,(1 - \Lambda_v)} = \frac{1^{0.6}}{0.8935\,(1 - 0.7792)},$$

we obtain

$$C_{v(\mathrm{t})} = \frac{1}{0.8935\,(0.2208)} \approx 5.0684.$$

Therefore,

$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \le 4.8937 \cdot \varepsilon = 4.8937 \cdot 0.2,$$
$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \le 0.9787, \qquad \mathrm{t} \in [0, 1].$$

Thus, the problem (3) has stable solution on the $[0, 1]$ for the $\varepsilon = 0.2$ by Theorem(3.3).

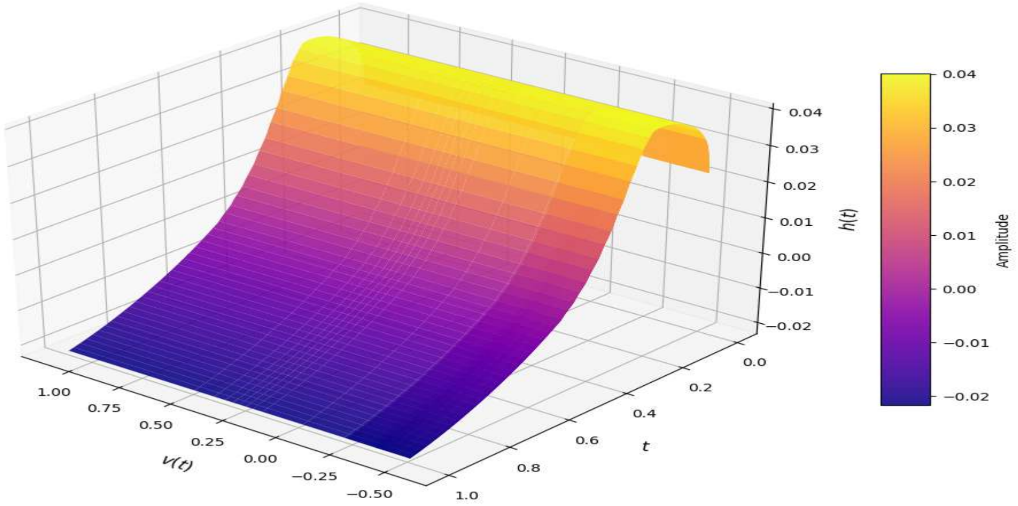

Figure 1. Three dimensional visualization of the solution $h(\mathrm{t})$ of the VO fractional BV problem (Example 1). The plot illustrates the evolution of the solution with respect to time $\mathrm{t} \in [0, 1]$ and the variable fractional order , highlighting the influence of memory effects on the system dynamics.

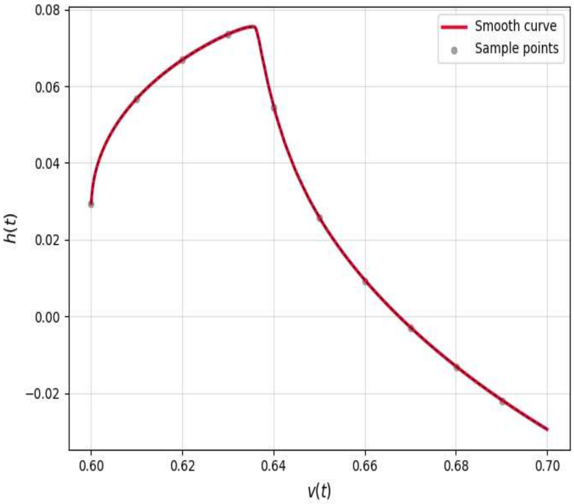

Figure 2. Solution $\mathrm{h(t)}$ for $\mu = 0.2$, plotted against time $\mathrm{t} \in [0, 1]$. The figure demonstrate the dependence of the solution behavior on the VO function $v(\mathrm{t})$.

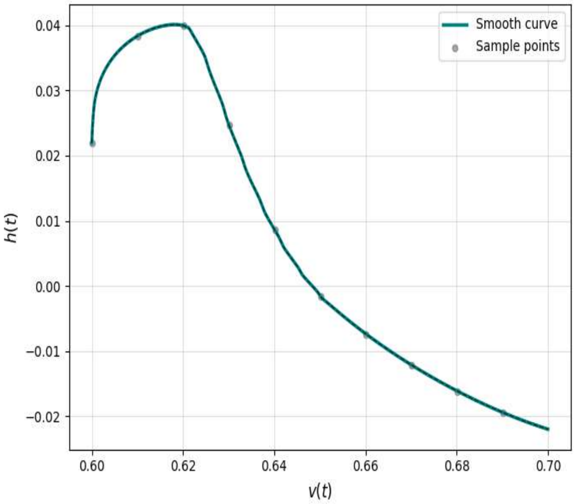

Figure 3. Solution $\mathrm{h(t)}$ for $\mu = 0.4$, illustrating the variation of the solution with respect to time t and the corresponding order function $v(\mathrm{t})$.

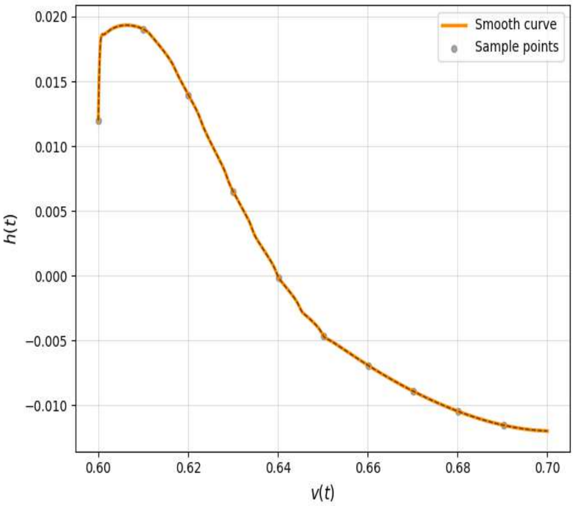

Figure 4. Solution $\mathrm{h(t)}$ for $\mu = 0.6$. The plot highlights the effect of increasing fractional order on the smoothness and evolution of the solution.

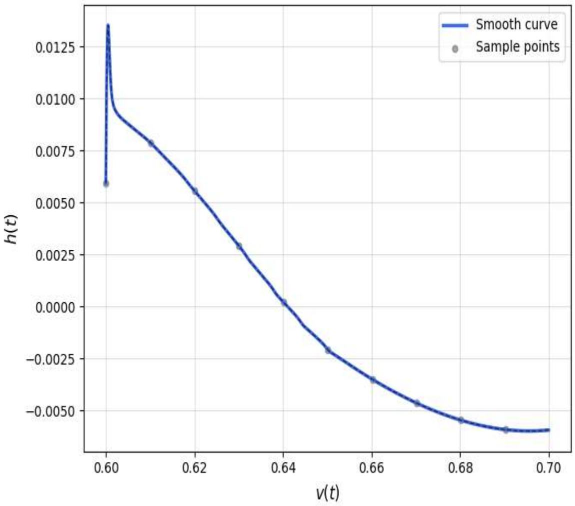

Figure 5. Solution $\mathrm{h(t)}$ for $\mu = 0.8$, showing enhanced memory influence and smoother dynamics compared to lower order cases.

Example 2: Consider the variable-order Caputo fractional BV problem on $J = [0, 2]$

$$\begin{cases} {}^{C}D^{v(\mathrm{t})}\mathrm{h(t)} = -0.03\,\mathrm{h(t)} + 0.01\,z, \quad \mathrm{t} \in [0, 2], \\ \mathrm{h}(0) + \mathrm{h}(2) = 0. \end{cases}$$

Here, the order function is chosen as $v(\mathrm{t}) = 0.12 + 0.14\mathrm{t}$, $v_m = 0.12$, $v_M = 0.40$, the constant fractional order is $\mu = 0.5$ and $z = {}^{C}D^{0.5}\mathrm{h(t)}$.

The function $f(\mathrm{t},\mathrm{h},\mathrm{h}^{'}) = -0.03\,\mathrm{h} + 0.01\,z$ is smooth on $[0,2]\times\mathrm{R}^2$ and satisfies the Lipschitz Property with constants $L_f = 0.03$ and $L_g = 0.01$.

$$|(-0.03\,\mathrm{h}_1 + 0.01\,\mathrm{h}_1^{'}) - (-0.03\,\mathrm{h}_2 + 0.01\,\mathrm{h}_2^{'})| \leq -0.03|\mathrm{h}_1 - \mathrm{h}_2| + 0.01|\mathrm{h}_1^{'} - \mathrm{h}_2^{'}|.$$

Assuming the Caputo operator ${}^{C}D^{\mu}$ satisfies

$$\left|{}^{C}D^{\mu}\mathrm{h}_1(\mathrm{t}) - {}^{C}D^{\mu}\mathrm{h}_2(\mathrm{t})\right| \leq C_\mu\,\|\mathrm{h}_1 - \mathrm{h}_2\|,$$

$$C_\mu = K\,\frac{(\rho_2-\rho_1)^{1-\mu}}{\Gamma(2-\mu)} = 0.8862\,\frac{(2-0)^{1-0.5}}{\Gamma(2-0.5)} = 1$$

we obtain the overall Lipschitz constant

$$L = L_f + L_g C_\mu = 0.04.$$

We take $L_f = \rho_2 - \rho_1 = 2 > 1$ and evaluate the $L_f > 1$ branch of Λ_v at the endpoint $\mathrm{t} = \rho_2 = 2$:

$$\Lambda_v = L\left[\frac{\mathrm{s}^{1-v_m}\,(\mathrm{t}-\rho_1)^{v_m}}{v_m} + \aleph\right], \qquad \aleph = \frac{\mathrm{u}^*}{\mathrm{u}+\mathrm{u}^*}\,\frac{(\rho_2-\rho_1)^{v(\rho_2)}}{\Gamma(v(\rho_2)+1)}.$$

Compute the ingredients:

$$L_f^{1-v_m} = 2^{\,1-0.12} = 2^{0.88} \approx 1.8404,$$
$$(\rho_2-\rho_1)^{v_m} = 2^{0.12} \approx 1.0866,$$
$$\frac{\mathrm{s}^{1-v_m}(\rho_2-\rho_1)^{v_m}}{v_m} \approx \frac{1.8404\times 1.0866}{0.12} \approx \frac{2.000}{0.12} \approx 16.6667.$$

For $\aleph$ (with $\mathrm{u} = \mathrm{u}^* = 1$ and $v(\rho_2) = v(2) = 0.40$):

$$(\rho_2-\rho_1)^{v(\rho_2)} = 2^{0.40} \approx 1.3195, \qquad \Gamma(v(\rho_2)+1) = \Gamma(1.40) \approx 0.8873,$$

hence

$$\aleph \approx \tfrac{1}{2}\cdot\frac{1.3195}{0.8873} \approx 0.7436.$$

Combining,

$$\Lambda_v \approx L\,(16.6667 + 0.7436) = 0.04\times 17.4103 \approx 0.6964 < 1.$$

Therefore $\Lambda_v < 1$, and by Theorem (3.1) the fractional BV problem above admits exactly one solution on $J = [0,2]$.

We now analyze the Ulam- Hyers stability

$$\Lambda_v = 0.6964,$$

$$C_{v(\mathrm{t})} = \frac{(\rho_2-\rho_1)^{v_m}}{\Gamma(v_m+1)\,(1-\Lambda_v)} = \frac{1^{0.12}}{0.9436\,(1-0.6964)},$$

we obtain

$$C_{v(\mathrm{t})} = \frac{1}{0.9436\,(0.3036)} \approx 3.4907.$$

Therefore,

$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq 3.4907\cdot\varepsilon = 3.4907\cdot 0.2,$$
$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq 0.6981, \qquad \mathrm{t}\in[0,1].$$

Thus, the problem (3) has stable solution on the $[0, 2]$ for the $\varepsilon = 0.2$ by Theorem (3.3).

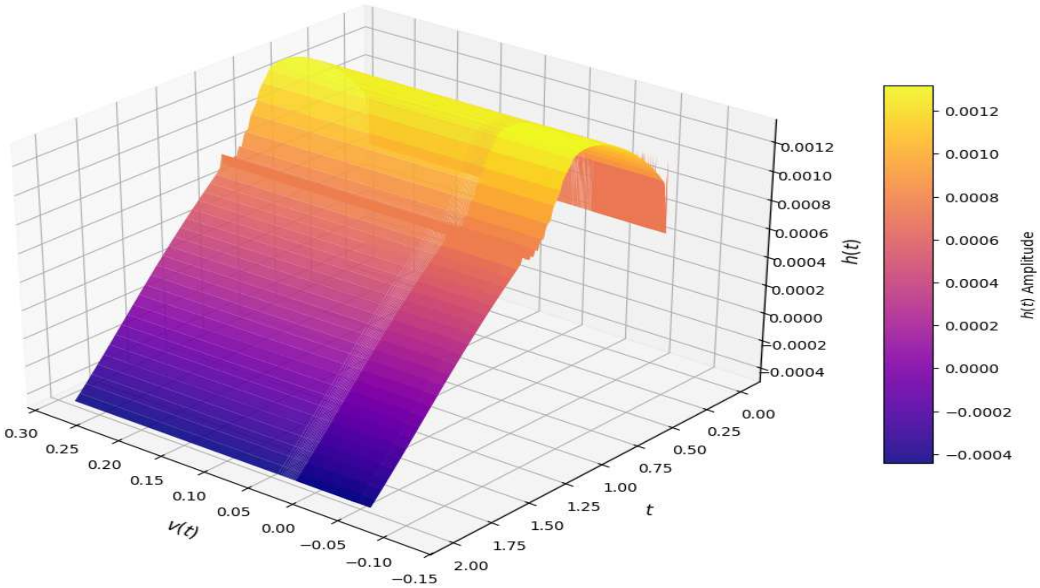

Figure 6. Three dimensional visualization of the solution $h(\mathrm{t})$ for the VO fractional BV problem (Example 2). The plot represents the solution behavior over the domain $\mathrm{t} \in [0, 2]$ under varying fractional order $v(\mathrm{t})$.

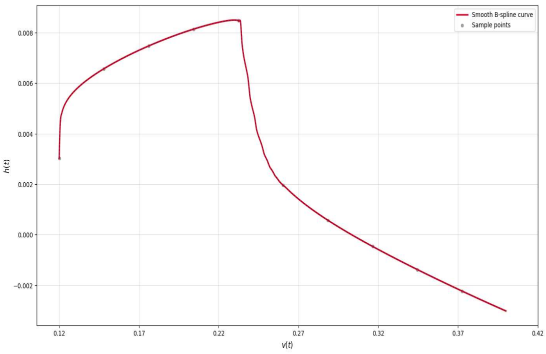

Figure 7. Solution $\mathrm{h}(\mathrm{t})$ for $\mu = 0.1$, plotted over $\mathrm{t} \in [0, 2]$, illustrating the weak memory effect in the system.

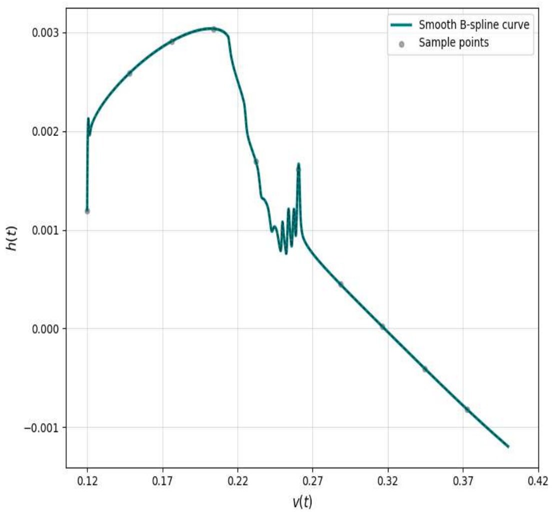

Figure 8. Solution $\mathrm{h}(\mathrm{t})$ for $\mu = 0.3$, demonstrating moderate memory influence on the solution system evolution.

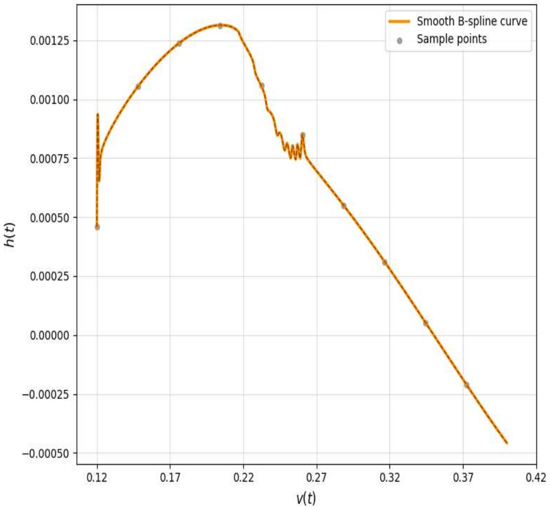

Figure 9. Solution $\mathrm{h}(\mathrm{t})$ for $\mu = 0.5$, highlighting balanced memory effects and system stability.

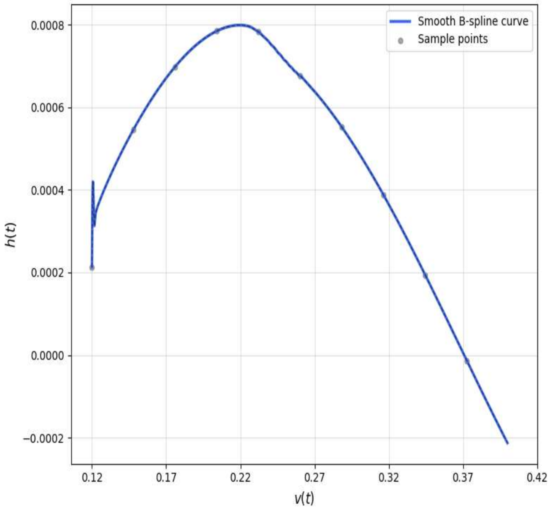

Figure 10. Solution $\mathrm{h}(\mathrm{t})$ for $\mu = 0.7$, showing stronger memory contribution and smoother solution trajectories.

Example 3: Consider the VO FDE

$$\begin{cases} {}^{C}D^{v(\mathrm{t})}\mathrm{h}(\mathrm{t}) = -0.03\,\mathrm{h}(\mathrm{t}) + 0.14\cos(z(\mathrm{t})), \quad \mathrm{t} \in [0,1], \\ \mathrm{h}(0) + \mathrm{h}(1) = 0, \end{cases} \tag{5}$$

Here, the order function is chosen as $v(\mathrm{t}) = 0.55 + 0.15\mathrm{t}^2$, $v_m = 0.55$, $v_M = 0.7$, the constant fractional order is $\mu = 0.6$ and $z = {}^{C}D^{0.6}\mathrm{h}(\mathrm{t})$.

The function $f(\mathrm{t}, \mathrm{h}, \mathrm{h}^{'}) = -0.03\,\mathrm{h} + 0.14\cos(z)$ is continuous on $[0,1] \times \mathrm{R}^2$ and satisfies the Lipschitz condition with constants $L_f = -0.030$ and $L_g = 0.14$. where $z(\mathrm{t}) = {}^{C}D^{0.6}\mathrm{h}(\mathrm{t})$. We verify the Lipschitz condition. For any $\mathrm{h}_1, \mathrm{h}_2 \in B$, let $z_1 = {}^{C}D^{0.6}\mathrm{h}_1$ and $z_2 = {}^{C}D^{0.6}\mathrm{h}_2$. Then

$$\begin{aligned} |f(\mathrm{t}, \mathrm{h}_1, z_1) - f(\mathrm{t}, \mathrm{h}_2, z_2)| &= |-0.03(\mathrm{h}_1 - \mathrm{h}_2) + 0.14(\cos(z_1) - \cos(z_2))| \\ &\le 0.03|\mathrm{h}_1 - \mathrm{h}_2| + 0.14|\cos(z_1) - \cos(z_2)| \\ &\le 0.03|\mathrm{h}_1 - \mathrm{h}_2| + 0.14|z_1 - z_2| \end{aligned}$$

Since the cosine function is Lipschitz continuous with constant 1, we have $|\cos(z_1) - \cos(z_2)| \le |z_1 - z_2|$. Thus,

$$|f(\mathrm{t}, \mathrm{h}_1, z_1) - f(\mathrm{t}, \mathrm{h}_2, z_2)| \le 0.03|\mathrm{h}_1 - \mathrm{h}_2| + 0.14|z_1 - z_2|.$$

Using the estimate

$$|z_1 - z_2| = |{}^{C}D^{0.6}\mathrm{h}_1 - {}^{C}D^{0.6}\mathrm{h}_2| \le C_\mu \|\mathrm{h}_1 - \mathrm{h}_2\|,$$

Assuming the Caputo operator ${}^{C}D^{\mu}$ satisfies

$$\left|{}^{C}D^{\mu}\mathrm{h}_1(\mathrm{t}) - {}^{C}D^{\mu}\mathrm{h}_2(\mathrm{t})\right| \le C_\mu \,\|\mathrm{h}_1 - \mathrm{h}_2\|,$$

$$C_\mu = K\,\frac{(\rho_2 - \rho_1)^{1-\mu}}{\Gamma(2-\mu)} = 0.8873\,\frac{(1-0)^{1-0.6}}{\Gamma(2-0.6)} = 1$$

we obtain the overall Lipschitz constant

$$L = L_f + L_g C_\mu = 0.03 + 0.14(1) = 0.17.$$

Hence, f satisfies the Lipschitz condition. We take $L_f = \rho_2 - \rho_1 = 1 \leq 1$ and evaluate the $L_f \leq 1$ and evaluate the $L_f >$ branch of Λ_v at the endpoint $\mathrm{t} = \rho_2$ branch of Λ_v at the end point $\mathrm{t} = \rho_2 = 1$:

$$\Lambda_v = L\left[\frac{(\rho_2 - \rho_1)^{v_m}}{v_m} + \aleph\right], \qquad \aleph = \frac{\mathrm{u}^*}{\mathrm{u} + \mathrm{u}^*} \frac{(\rho_2 - \rho_1)^{v(\rho_2)}}{\Gamma(v(\rho_2) + 1)}.$$

Compute the ingredients:

$$(\rho_2 - \rho_1)^{v_m} = 1^{0.55} \approx 1,$$
$$\frac{(\rho_2 - \rho_1)^{v_m}}{v_m} \approx \frac{1}{0.55} \approx 1.8182.$$

For $\aleph$ (with $\mathrm{u} = \mathrm{u}^* = 1$ and $v(\rho_2) = v(1) = 0.55$):

$$(\rho_2 - \rho_1)^{v(\rho_2)} = 1^{0.55} \approx 1,$$
$$\Gamma(v(\rho_2) + 1) = \Gamma(1.55) \approx 0.8889.$$

Hence

$$\aleph \approx \tfrac{1}{2} \cdot \frac{1}{0.8889} \approx 0.5625.$$

With boundary parameters $\mathrm{u} = 1$, $\mathrm{u}^* = 1$, and $\gamma = 0$, the associated contraction constant is estimated as

$$\begin{aligned}\Lambda_v &= L\left[\frac{(1-0)^{v_m}}{v_m} + \frac{\mathrm{u}^*}{\mathrm{u} + \mathrm{u}^*} \frac{(1)^{v_M}}{\Gamma(v_M + 1)}\right] \\ &\approx 0.17\,(1.8182 + 0.5625) \\ &= 0.4047 < 1.\end{aligned}$$

Therefore $\Lambda_v < 1$, and by the Theorem (3.1) the fractional BV problem above admits a exactly one solution on $J = [0, 1]$.

We now analyze the Ulam-Hyers stability

$$\Lambda_v = 0.4047,$$

$$C_{v(\mathrm{t})} = \frac{(\rho_2 - \rho_1)^{v_m}}{\Gamma(v_m + 1)\,(1 - \Lambda_v)} = \frac{1^{0.6}}{0.8889\,(1 - 0.4047)},$$

we obtain

$$C_{v(\mathrm{t})} = \frac{1}{0.8889\,(0.5953)} \approx 1.8896.$$

Therefore,

$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq 4.8937 \cdot \varepsilon = 1.8896 \cdot 0.4,$$

$$|\mathrm{h}^*(\mathrm{t}) - \mathrm{h}(\mathrm{t})| \leq 0.75584, \qquad \mathrm{t} \in [0, 1].$$

Thus, the problem (3) has stable solution on the $[0, 2]$ for the $\varepsilon = 0.4$ by Theorem (3.3).

4.1 Application: Variable-Order Fractional Modeling for 3D Noise Feature Evolution

Three- dimensional noise fields are widely used in scientific visualization and computational graphics to represent complex stochastic structures such as turbulence, environmental fields, and synthetic textures. These signals typically exhibit long-range correlations, meaning that the current spatial pattern depends on previously observed states.Variable-order fractional models are particularly suitable for describing the evolution of such 3D noise fields due to their ability to capture spatially and temporally varying memory effects.

Unlike fixed order fractional models, where the memory influence remains constant, the VO formulations allows the system to adapt dynamically to local irregularities and heterogeneous structures present in noise distributions. This adaptability is particularly important in applications like turbulence modeling, stochastic simulations, and image processing, where system properties evolve over time and space.

In contrast to fixed- order fractional models, where the memory effect remains constant throughout the evolution process, VO formulation enables adaptive memory that changes with the system dynamics. This improves modeling accuracy in heterogeneous environments like 3D noise fields. In particular, fixed order models may fail to capture local variations in correlation structure, resulting in either over smoothing or loss of important features. The proposed VO structure overcomes this limitation by dynamically adjusting the order function $v(t)$, thereby providing enhanced flexibility, improved stability, and more accurate representation of evolving noise patterns.

Consequently, the proposed system provides a more realistic and flexible representation of noise feature evolution compared to classical integer-order or fixed-order fractional approaches. Classical methods typically treat features updates as memoryless operations, which limits their ability to capture such evolving dependencies. To address this issue, the evolution of a neural feature state extracted from a 3D noise field is modeled using a variable-order fractional dynamic update. Let $h(t)$ denote a representative feature coefficient extracted from the convolutional representation of the noise field. The temporal refinement of this feature is governed by the memory-dependent update rule

$$h_{k+1} = h_k + \Delta t^{\,v(t_k)} F\big(t_k, h_k, \mathcal{M}_\mu(h)\big), \tag{6}$$

where $v(t_k) \in (0, 1]$ represents the adaptive memory order, and $\mathcal{M}_\mu(h)$ denotes the fractional memory accumulation of past states, defined by

$$\mathcal{M}_\mu(h) = \sum_{j=0}^{k} w_{k-j} h_j, \tag{7}$$

The weights w_{k-j} control the persistence of past information. The nonlinear function $F(\cdot)$ represent the neural transformation acting on the feature map and determines how the current representation interacts with memory.A global constraint is introduced to maintain stability between the initial and final representations:

$$\eta\, h(\rho_1) + \eta^* h(\rho_2) = \eta_0, \tag{8}$$

where $h(\rho_1)$ represents the initial feature extracted from the noise input and $h(\rho_2)$ represent the stabilized feature obtained after fractional evolution. This constraint ensures consistency between the input feature structure and the final encoded representation. The structure allows the system to incorporate adaptive memory during feature refinement. As a result, it produce a more coherent representation of stochastic 3D noise structure compared with classical memoryless updates. This approach improves the stability and consistency of feature evolution. It is suitable for applications such as texture synthesis, image denoising, and physically inspired simulation of complex systems.

4.2 Discussion

To demonstrate the behavior of the proposed VO fractional model, a numerical simulation is performed using a synthetic three- dimensional noise field. A representative feature coefficient is extracted from the noise field and evolved using the proposed fractional update rule. The resulting temporal dynamics are shown in the following figures.

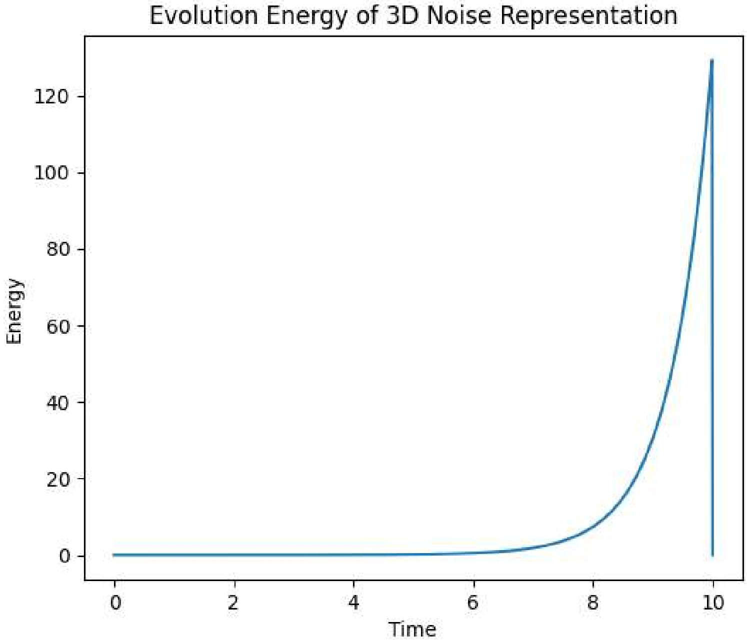

Figure 11. Energy evolution of the 3D noise representation under the VO fractional update scheme. The horizontal axis represents iteration or time steps, while the vertical axis shows the energy of the evolving feature.

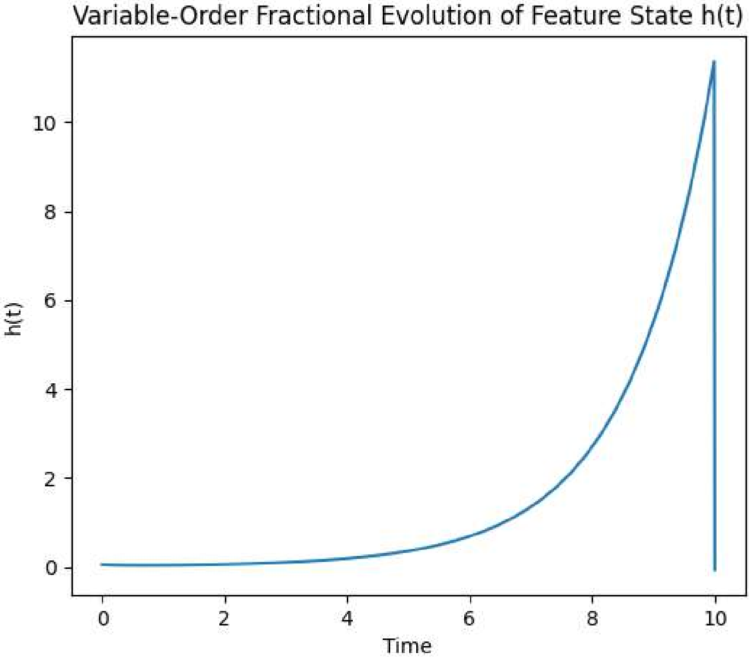

Figure 12. VO fractional evolution of the feature state $h(t)$ extracted from a 3D noise field. The figure shows how adaptive memory (via $v(\mathrm{t})$) influences features refinement and stabilization over time.

These results demonstrate that the proposed VO fractional model can effectively capture the memory dependent evolution of stochastic noise features. Compared with classical memoryless update schemes, the fractional dynamical modela provides smoother and more stable feature evolution while preserving preserving past information during the refinement process.

These observations show that the proposed VO fractional setup offers better adaptability and stability compared with classical models, and can be applied more broadly in computational physics, graphics, and data-driven modeling.

5. Conclusion

This paper studies a derivative-dependent BV problem involving Caputo VO fractional derivatives. The main results establish the existence, uniqueness, and stability of solutions under suitable assumptions. The analysis is carried out using Banach and Krasnoselskii fixed point theorems, while Ulam-Hyers stability is used to examine the response of the system to small perturbations. The proposed formulation differs from many earlier works by considering both variable order and constant order fractional derivatives within a single model. This allows the framework to capture more general memory - dependent behavior compared to models based only on fixed order or purely VO operators. The application to three dimensional noise feature evolution shows that the VO approach provides smoother and more stable dynamics. The model is able to incorporate adaptive memory effects, which are useful in applications such as image processing, stochastic simulations, and related computational problems. Nevertheless, the present study has certain limitations. The analysis is restricted to a specific class boundary conditions and assumes sufficient regularity of the nonlinear terms, which may not be valid for highly nonlinear or non smooth systems. Moreover, the work is primarily theoretical, and the computational aspects of solving VO fractional problems, particularly for large scale or real time applications, are not fully addressed. Future research may extend this structure to more general boundary conditions, strongly nonlinear systems, and nonlocal fractional models. In additional, the development of efficient numerical algorithms and adaptive computational techniques for VO fractional equations is an important direction. Exploring applications in emerging areas such as data driven modeling, machine learning, and complex dynamical systems can further improved the scope and impact of the proposed approach.

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