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Imperfect Delayed Repair Model Under Geometric-Alpha Process

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Abstract Planning for the maintenance of deteriorating systems is more difficult when repair operations cannot always be started immediately after failure due to operational and resource limitations. In this study, a stochastic maintenance model is presented for a deteriorating system whose successive operating times are modeled by a geometric–alpha process, where the deterioration of the system is modeled as a structured change in the failure behaviour in successive cycles. System failure is followed by imperfect delayed repair, meaning that repair is either immediately started or delayed, and repair action brings the system back to a state that is not as-good-as-new. It is assumed that an N-failure replacement policy is used, and the system is substituted with a new one when the N-th failure occurs. With the help of renewal reward arguments, an explicit expression of the long-term expected cost per unit time is obtained and the optimum replacement number N^* is obtained analytically. To demonstrate the use of the proposed model and to explore the impact of the model parameters on the optimal maintenance decision, a numerical study is presented. The proposed Geometric–Alpha Process is further illustrated using the Aircondit7 dataset, a benchmark reliability dataset containing operating times between failures of an aircraft air-conditioning system. For this dataset, the Geometric–Alpha Process yields a higher maximized log-likelihood and a lower Akaike Information Criterion (AIC) than the classical Geometric Process, indicating a better fit to the data. The analytical results, numerical study, and data illustration demonstrate the applicability of the proposed maintenance framework for deteriorating systems with imperfect delayed repair.

Keywords Geometric -alpha Process, Replacement Policy, Renewal Process

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1. Introduction

A very significant research area in reliability theory has been the development of effective maintenance policies for repairable systems for many decades. It was usually assumed in the early studies that, following each failure, the system is “as-good-as-new” as a result of a repair. This assumption makes mathematical analysis easier, but is not always true of real engineering systems in which components slowly degrade due to aging, repeated use, fatigue, and environmental effects. Aware of this, Barlow and Hunter [8] adapted the concepts of preventive maintenance and later, Brown and Proschan [11] and Smithson and Sarkar [26] developed the imperfect repair model in which a repair restores the system in part only, and the impact of previous deterioration remains.

It is also found that the time between failures decreases as the system ages for deteriorating systems. Lam [16, 17, 18] introduced the geometric process to describe this behaviour and it has become a popular model in replacement and maintenance modelling. The constant scaling parameter is an effective way to represent monotone deterioration, and the geometric process is an effective way to do so. But in many applications, the deterioration

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of the system is not necessarily a fixed geometric pattern. The degradation rate can be different with time due to varying operating conditions, environmental uncertainties and accumulated ageing effects. The classical geometric process, therefore, does not always have enough flexibility to describe the real deterioration behaviour of complex systems.

In recent years, several extensions to the geometric process have been suggested that address this difficulty. Babu, Govindaraju and Rizwan [6] proposed the partial product process for more modelling flexibility and Asadi and Wu [3] examined a rate-randomized geometric process for systems in uncertain environments. More recently, İnan [15] studied doubly geometric processes and their applications in reliability analysis. While these models provide valuable generalizations, there is a need for a process that can capture the geometric scaling structure and the extra ageing effect that is seen in many real systems.

In response to this need, the present study takes into account an alternative deterioration model, namely the Geometric–Alpha Process. The Geometric–Alpha Process extends the classical geometric process by incorporating an additional alpha parameter to represent different rates of system deterioration. The additional alpha parameter enables the process to capture stage-dependent degradation behaviour that may arise in practical engineering systems. Thus, the proposed process provides an alternative framework for modelling systems whose degradation behaviour evolves over time.

One of the most commonly made assumptions in the maintenance models is that repairs start as soon as the system fails. However, in real operating situations, immediate repair is not always possible. Repair personnel may be unavailable, spares may be in short supply, transport may be difficult or there may be operational scheduling constraints that can cause maintenance activities to be delayed. This means that there can be a delay between failure and repair.

In order to enhance its practical significance, the present study proposes a maintenance framework for a degrading system governed by the Geometric–Alpha Process with imperfect and delayed repair. An N-failure replacement policy is considered where the unit is renewed after the occurrence of the N-th breakdown. Closed-form results for the expected average cost rate over an extended period and the best replacement strategy are obtained. The proposed Geometric–Alpha Process is further illustrated using the Aircondit7 dataset, a benchmark reliability dataset containing operating times between failures of an aircraft air-conditioning system. For this dataset, the Geometric–Alpha Process yields a higher maximized log-likelihood and a lower Akaike Information Criterion (AIC) than the classical Geometric Process, indicating a better fit to the data. This maintenance framework provides an approach for maintenance decision making in engineering systems where deterioration, imperfect restoration, and repair delay occur simultaneously.

2. Geometric-Alpha Process

Definition 2.1. A sequence of independent non-negative random variables $\{X_i, i = 1, 2, \dots\}$ is called a geometric process if the distribution function of X_i is $F_i(x) = F(a^{i-1}x)$ for $i = 1, 2, \dots$ where a is a positive constant.

Definition 2.2. Let $\{X_i, i = 1, 2, \dots\}$ be a sequence of independent, non-negative random variables. The sequence is called a Geometric-Alpha Process (GAP) if there exists a positive parameter α such that the cumulative distribution function of X_i, denote as $F_i(x)$ is given by:

$$F_i(x) = F(\alpha^{i-1} i^{\alpha-1} x), \quad \text{for } i = 1, 2, \dots \tag{1}$$

where $F(x)$ is the baseline cumulative distribution function of X_1.

Definition 2.3. For two random variables Y and Z, Z is called stochastically less than Y (or Y is stochastically larger than Z) if for all real α,

$$P(Y > \alpha) \geq P(Z > \alpha). \tag{2}$$

This is written as $Z \leq_{st} Y$ or $Y \geq_{st} Z$.

Lemma 2.1. Given a Geometric-Alpha Process $\{X_i, i = 1, 2, 3, \dots\}$,

(i) If $\alpha > 1$, then $\{X_i, i = 1, 2, 3, \dots\}$ is stochastically decreasing.

(ii) If $0 < \alpha < 1$, then $\{X_i, i = 1, 2, 3, \dots\}$ is stochastically increasing.

Proof. Let

$$A_i = \alpha^{i-1} i^{\alpha-1}, \quad i = 1, 2, 3, \dots$$

(i) Suppose that $\alpha > 1$. Then

$$\frac{A_{i+1}}{A_i} = \frac{\alpha^i (i+1)^{\alpha-1}}{\alpha^{i-1} i^{\alpha-1}} = \alpha \left(\frac{i+1}{i}\right)^{\alpha-1} > 1,$$

since $\alpha > 1$ and $\left(\frac{i+1}{i}\right)^{\alpha-1} > 1$. Hence, $\{A_i\}$ is strictly increasing. Therefore, for every $x \geq 0$,

$$F(A_1 x) \leq F(A_2 x) \leq F(A_3 x) \leq \cdots \leq F(A_i x),$$

because F is a cumulative distribution function. Consequently,

$$P(X_1 > x) \geq P(X_2 > x) \geq P(X_3 > x) \geq \cdots \geq P(X_i > x),$$

which implies that

$$X_1 \geq_{st} X_2 \geq_{st} X_3 \geq_{st} \cdots \geq_{st} X_i.$$

Hence, $\{X_i, i = 1, 2, 3, \dots\}$ is stochastically decreasing.
(ii) Suppose that $0 < \alpha < 1$. Then

$$\frac{A_{i+1}}{A_i} = \frac{\alpha^i (i+1)^{\alpha-1}}{\alpha^{i-1} i^{\alpha-1}} = \alpha \left(\frac{i+1}{i}\right)^{\alpha-1} < 1,$$

since $0 < \alpha < 1$ and $\left(\frac{i+1}{i}\right)^{\alpha-1} < 1$. Thus, $\{A_i\}$ is strictly decreasing. It follows that, for every $x \geq 0$,

$$F(A_1 x) \geq F(A_2 x) \geq F(A_3 x) \geq \cdots \geq F(A_i x),$$

and therefore

$$P(X_1 > x) \leq P(X_2 > x) \leq P(X_3 > x) \leq \cdots \leq P(X_i > x).$$

Hence,

$$X_1 \leq_{st} X_2 \leq_{st} X_3 \leq_{st} \cdots \leq_{st} X_i,$$

so $\{X_i,\ i = 1, 2, 3, \dots\}$ is stochastically increasing. $\square$

It is clear that if $\alpha = 1$, then the Geometric-Alpha Process is a renewal process.

Lemma 2.2. Let $E(X_1) = \lambda$ and $Var(X_1) = \sigma^2$. Then for $i = 1, 2, 3, \dots$,

$$E(X_i) = \frac{\lambda}{\alpha^{i-1} i^{\alpha-1}} \quad \text{and} \quad Var(X_i) = \frac{\sigma^2}{(\alpha^{i-1} i^{\alpha-1})^2} = \frac{\sigma^2}{\alpha^{2(i-1)} i^{2(\alpha-1)}}. \tag{3}$$

3. Model Assumptions

A1. Initially, a newly commissioned repairable unit operating under imperfect delayed repair (IDR) is installed. Each time a failure occurs, the system is either restored through repair or substituted with an equivalent unit.

A2. Let X_1 denote the initial operating time before the first failure with cumulative distribution function $F(x)$ and mean $E(X_1) = \lambda$. For $i = 1, 2, 3, \dots$, the subsequent operating times X_i follow the Geometric-Alpha Process with cumulative distribution function $F_i(x) = F(\alpha^{i-1} i^{\alpha-1} x)$, where $\alpha \geq 1$ is the deterioration parameter. For $\alpha > 1$, the successive operating times are stochastically decreasing, whereas for $\alpha = 1$, the process reduces to the classical renewal process.

A3. Let Y_1 denote the initial repair time after the first failure with cumulative distribution function $G(y)$ and mean $E(Y_1) = \mu$. For $i = 1, 2, 3, \ldots$, the subsequent repair times Y_i have cumulative distribution function $G_i(y) = G(\beta^{i-1} i^{\beta-1} y)$, where $0 < \beta \leq 1$ is the repair parameter. For $0 < \beta < 1$, the successive repair times are stochastically increasing, whereas for $\beta = 1$, the repair process reduces to the classical renewal process.
A4. Upon failure, the repair action is postponed with probability p, and carried out immediately with probability $1 - p$.
A5. Let D_i denote the delayed repair (DR) time following the n-th failure. The sequence $\{D_i,\ i = 1, 2, 3, \ldots\}$ consists of independent and identically distributed random variables. Moreover, $E(D_i) = \nu \geq 0, i = 1, 2, 3, \ldots$, where $\nu = 0$ indicates the absence of repair delay.
A6. Let Z be the time required to replace the system, and assume $E(Z) = \tau$.
A7. The sequences $\{X_i\}$, $\{Y_i\}$, and $\{D_i\}$, together with the random variable Z, are mutually independent.
A8. The repair cost is incurred at rate c, and the system earns reward at rate r during operation. The replacement cost consists of two components: a fixed cost R and a variable cost proportional to the replacement time Z at rate c_p.
A9. During the waiting period prior to repair, the system neither incurs cost nor produces any reward.

Table 1. Notation

Symbol	**Description**
X_i	Operating time following the $(i-1)$-th repair
Y_i	Repair time following the i-th failure
D_i	Delay time before repair following the i-th failure
Z	Replacement time
$F(x)$	Distribution function of X_1
$G(y)$	Distribution function of Y_1
$F_i(x)$	Distribution function of X_i
$G_i(x)$	Distribution function of Y_i
λ	Mean operating time before the first failure
λ_i	Mean operating time after the $(i-1)$-th repair
μ	Mean repair time after the first failure
μ_i	Mean repair time after the i-th failure
ν	Mean repair delay time
τ	Mean replacement time
σ^2	Variance of X_1
α	Parameter for the operating-time process
β	Parameter for the repair-time process
p	Probability that repair is delayed
r	Reward rate during system operation
c	Repair cost rate
c_p	Replacement cost rate
R	Fixed replacement cost
N	Replacement number under the N-failure policy
N^*	Optimal replacement number
T	Length of one renewal cycle
$C(N)$	Long-run average cost per unit time under the N-failure policy
$B(N)$	Auxiliary function used to determine the optimal replacement number
$E(\cdot)$	Expectation operator
$Var(\cdot)$	Variance operator

4. The Replacement Policy N

The N-failure replacement strategy of order N means that the system is renewed upon the N-th failure. The goal is to determine the optimum value N^* which minimizes the long-run average cost per unit time.
The long-term average cost rate associated with the N-policy can be expressed as the ratio of the mean cost of a renewal interval to the mean duration of a renewal interval [24].

$$\begin{aligned} C(N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \\ &= \frac{cE\left(\sum\limits_{i=1}^{N-1} Y_i\right) + R + c_p E(Z) - rE\left(\sum\limits_{i=1}^{N} X_i\right)}{E\left(\sum\limits_{i=1}^{N} X_i\right) + E\left[\sum\limits_{i=1}^{N-1} (D_i + Y_i)\chi_{A_i}\right] + E\left(\sum\limits_{i=1}^{N-1} Y_i \chi_{\overline{A_i}}\right) + E(Z)} \end{aligned}$$

where $\chi_A(\cdot)$ denotes the indicator function.
Then,

$$\begin{aligned} C(N) &= \frac{cE\left(\sum\limits_{i=1}^{N-1} Y_i\right) + R + c_p E(Z) - rE\left(\sum\limits_{i=1}^{N} X_i\right)}{E\left(\sum\limits_{i=1}^{N} X_i\right) + E\left[\sum\limits_{i=1}^{N-1} (D_i + Y_i)\right] p + E\left(\sum\limits_{i=1}^{N-1} Y_i\right)(1-p) + E(Z)} \\ &= \frac{c\sum\limits_{i=1}^{N-1} \mu_i + R + c_p E(Z) - r\sum\limits_{i=1}^{N} \lambda_i}{\sum\limits_{i=1}^{N} \lambda_i + p\sum\limits_{i=1}^{N-1} E(D_i) + \sum\limits_{i=1}^{N-1} \mu_i + E(Z)} \\ &= \frac{c\sum\limits_{i=1}^{N-1} \mu_i + R + c_p\tau - r\sum\limits_{i=1}^{N} \lambda_i}{\sum\limits_{i=1}^{N} \lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1} \mu_i + \tau} \\ &= \frac{(c+r)\sum\limits_{i=1}^{N-1} \mu_i + R + (c_p + r)\tau + rp(N-1)\nu}{\sum\limits_{i=1}^{N} \lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1} \mu_i + \tau} - r \end{aligned} \tag{4}$$

$$C(N) = \frac{(c+r)\sum_{i=1}^{N-1} \frac{\mu}{\beta^{i-1} i^{\beta-1}} + R + (c_p + r)\tau + rp(N-1)\nu}{\sum_{i=1}^{N} \frac{\lambda}{\alpha^{i-1} i^{\alpha-1}} + p(N-1)\nu + \sum_{i=1}^{N-1} \frac{\mu}{\beta^{i-1} i^{\beta-1}} + \tau} - r \tag{5}$$

5. The Optimal Replacement Policy N^*

In this section, an optimal replacement policy N^* is determined for minimizing $C(N)$.
From equation (4),

$C(N+1) - C(N)$

$$= \frac{(c+r)\sum\limits_{i=1}^{N}\mu_i + R + (c_p+r)\tau + rpN\nu}{\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau} - \frac{(c+r)\sum\limits_{i=1}^{N-1}\mu_i + R + (c_p+r)\tau + rp(N-1)\nu}{\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau}$$

$$= \frac{\left(\begin{array}{l} (c+r)\sum\limits_{i=1}^{N}\mu_i\left[\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right] - (c+r)\sum\limits_{i=1}^{N-1}\mu_i\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right] \\ +\left(R + (c_p+r)\tau + rpN\nu\right)\left[\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right] \\ -\left(R + (c_p+r)\tau + rpN\nu\right)\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right] + rp\nu\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right] \end{array}\right)}{\left(\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right)\left(\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right)}$$

$$= \frac{\left(\begin{array}{l} (c+r)\left(\sum\limits_{i=1}^{N}\mu_i\left[\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right] - \sum\limits_{i=1}^{N-1}\mu_i\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right]\right) \\ +\left(R + (c_p+r)\tau + rpN\nu\right)\left(\begin{array}{r} \left[\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right] \\ -\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right] \end{array}\right) \\ +rp\nu\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right] \end{array}\right)}{\left(\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right)\left(\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right)}$$

$$= \frac{\left(\begin{array}{l} (c+r)\left(\mu_N\left[\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \tau\right] - \sum\limits_{i=1}^{N-1}\mu_i\left[\lambda_{N+1} + p\nu\right]\right) \\ -\left(R + (c_p+r)\tau + rpN\nu\right)\left[\lambda_{N+1} + p\nu + \mu_N\right] + rp\nu\left[\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right] \end{array}\right)}{\left(\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right)\left(\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right)}$$

$$= \frac{\left(\begin{array}{l} (c+r)\left(E(Y_N)\left[\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \tau\right] - \sum\limits_{i=1}^{N-1}\mu_i\left[E(X_{N+1}) + p\nu\right]\right) \\ -\left(R + (c_p+r)\tau\right)\left[E(X_{N+1}) + p\nu + E(Y_N)\right] \\ -rpN\nu\left[E(X_{N+1}) + E(Y_N)\right] + rp\nu\left[\sum\limits_{i=1}^{N+1}\lambda_i + \sum\limits_{i=1}^{N}\mu_i + \tau\right] \end{array}\right)}{\left(\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right)\left(\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right)}$$

$$= \frac{\begin{pmatrix} (c+r)\left[\mu_N\left(\sum\limits_{i=1}^{N}\lambda_i + p\,(N-1)\,\nu + \tau\right) - \sum\limits_{i=1}^{N-1}\mu_i\left(\lambda_{N+1}+p\nu\right)\right] \\ -\left(R+\left(c_p+r\right)\tau\right)\left[\lambda_{N+1}+p\nu+\mu_N\right] \\ +rp\nu\left[\sum\limits_{i=1}^{N+1}\lambda_i + \sum\limits_{i=1}^{N}\mu_i + \tau - N\left(\lambda_{N+1}+\mu_N\right)\right] \end{pmatrix}}{\left(\sum\limits_{i=1}^{N+1}\lambda_i + pN\nu + \sum\limits_{i=1}^{N}\mu_i + \tau\right)\left(\sum\limits_{i=1}^{N}\lambda_i + p\,(N-1)\,\nu + \sum\limits_{i=1}^{N-1}\mu_i + \tau\right)} \tag{6}$$

Let

$$B\,(N) \;=\; \frac{\begin{pmatrix} (c+r)\left[\mu_N\left(\sum\limits_{i=1}^{N}\lambda_i + p\,(N-1)\,\nu + \tau\right) - \sum\limits_{i=1}^{N-1}\mu_i\left(\lambda_{N+1}+p\nu\right)\right] \\ +rp\nu\left[\sum\limits_{i=1}^{N+1}\lambda_i + \sum\limits_{i=1}^{N}\mu_i + \tau - N\left(\lambda_{N+1}+\mu_N\right)\right] \end{pmatrix}}{\left(R+\left(c_p+r\right)\tau\right)\left(\lambda_{N+1}+p\nu+\mu_N\right)} \tag{7}$$

Lemma 5.1.

$$C(N+1) > C(N) \Leftrightarrow B(N) > 1$$
$$C(N+1) = C(N) \Leftrightarrow B(N) = 1$$
$$C(N+1) < C(N) \Leftrightarrow B(N) < 1$$

Proof. The denominator of $C(N+1) - C(N)$ in equation (6) is always positive. Therefore, the sign of $C(N+1) - C(N)$ is the same as the sign of its numerator. Also, $B(N)$ is defined using the numerator of $C(N+1) - C(N)$. Thus, the monotonicity of $C(N)$ is determined by the value of $B(N)$. □

from equation (7), we get,

$$B(N) = \frac{\begin{bmatrix} (c+r)\left[\dfrac{\mu}{\beta^{N-1}N^{\beta-1}}\left(\sum\limits_{i=1}^{N}\dfrac{\lambda}{\alpha^{i-1}i^{\alpha-1}} + p\nu(N-1)+\tau\right)\right. \\ \left.-\sum\limits_{i=1}^{N-1}\dfrac{\mu}{\beta^{i-1}i^{\beta-1}}\left(\dfrac{\lambda}{\alpha^{N}(N+1)^{\alpha-1}}+p\nu\right)\right] \\ +\,rp\nu\left[\sum\limits_{i=1}^{N+1}\dfrac{\lambda}{\alpha^{i-1}i^{\alpha-1}} + \sum\limits_{i=1}^{N}\dfrac{\mu}{\beta^{i-1}i^{\beta-1}} + \tau\right. \\ \left.-N\left(\dfrac{\lambda}{\alpha^{N}(N+1)^{\alpha-1}}+\dfrac{\mu}{\beta^{N-1}N^{\beta-1}}\right)\right] \end{bmatrix}}{(R+(c_p+r)\tau)\left(\dfrac{\lambda}{\alpha^{N}(N+1)^{\alpha-1}}+p\nu+\dfrac{\mu}{\beta^{N-1}N^{\beta-1}}\right)}$$
$$= \frac{1}{(R+(c_p+r)\tau)}\left[(c+r)B_1(N) + rp\nu B_2(N)\right] \tag{8}$$

where

$$B_1(N) = \frac{\mu_N\left(\sum\limits_{i=1}^{N}\lambda_i + p(N-1)\nu+\tau\right) - \sum\limits_{i=1}^{N-1}\mu_i(\lambda_{N+1}+p\nu)}{\lambda_{N-1}+p\nu+\mu_N} \tag{9}$$

and

$$B_2\left(N\right)=\frac{\sum\limits_{i=1}^{N+1}\lambda_i+\sum\limits_{i=1}^{N}\mu_i+\tau-N\left(\lambda_{N+1}+\mu_N\right)}{\lambda_{N+1}+p\nu+\mu_N} \tag{10}$$

To prove $B(N)$ is non-decreasing, we need the following results.
Lemma 5.2.

$$B_1(N+1)-B_1(N)=\frac{\left[\left(\sum_{i=1}^{N+1}\lambda_i+\sum_{i=1}^{N}\mu_i+pN\nu+\tau\right)\left[(\lambda_{N+1}\mu_{N+1}-\lambda_{N+2}\mu_N)\right]+p\nu(\mu_{N+1}-\mu_N)\right]}{(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)}$$

Proof. From equation (7)

$$B_1(N+1)-B_1(N)=\left[\begin{array}{c}\frac{\mu_{N+1}\left(\sum_{i=1}^{N+1}\lambda_i+pN\nu+\tau\right)-\sum_{i=1}^{N}\mu_i(\lambda_{N+2}+p\nu)}{\lambda_{N+2}+p\nu+\mu_{N+1}}\\ \hline \frac{\mu_N\left(\sum_{i=1}^{N}\lambda_i+p(N-1)\nu+\tau\right)-\sum_{i=1}^{N-1}\mu_i(\lambda_{N+1}+p\nu)}{\lambda_{N+1}+p\nu+\mu_N}\end{array}\right]$$

$$=\frac{\left[\begin{array}{l}(\sum_{i=1}^{N+1}\lambda_i+pN\nu+\tau)(\lambda_{N+1}\mu_{N+1}-\lambda_{N+2}\mu_N)\\ -(\sum_{i=1}^{N}\mu_i p\nu)(\lambda_{N+1}-\lambda_{N+2})\\ +p\nu\left[\begin{array}{l}(\sum_{i=1}^{N+1}\lambda_i+pN\nu+\tau)(\mu_{N+1}-\mu_N)\\ -\sum_{i=1}^{N}\mu_i(\lambda_{N+2})+(\sum_{i=1}^{N}\mu_i)(\lambda_{N+1})\end{array}\right]\\ +\mu_N\left[-\sum_{i=1}^{N}\mu_i(\lambda_{N+2}+p\nu)\right]\\ -\mu_{N+1}(\lambda_{N+1}+p\nu)\left(-\mu_N-\sum_{i=1}^{N}\mu_i\right)\end{array}\right]}{(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)}$$

$$=\frac{\left[\begin{array}{l}(\sum_{i=1}^{N+1}\lambda_i+pN\nu+\tau)(\lambda_{N+1}\mu_{N+1}-\lambda_{N+2}\mu_N)\\ +p\nu(\sum_{i=1}^{N+1}\lambda_i+pN\nu+\tau)(\mu_{N+1}-\mu_N)\\ +(\sum_{i=1}^{N}\mu_i)(\mu_{N+1}\lambda_{N+1}-\mu_N\lambda_{N+2})\\ +(\sum_{i=1}^{N}\mu_i)[p\nu(\mu_{N+1}-\mu_N)]\end{array}\right]}{(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)}$$

$$B_1(N+1)-B_1(N)=\frac{\left[\left(\sum_{i=1}^{N+1}\lambda_i+\sum_{i=1}^{N}\mu_i+pN\nu+\tau\right)\left[(\lambda_{N+1}\mu_{N+1}-\lambda_{N+2}\mu_N)\right]+p\nu(\mu_{N+1}-\mu_N)\right]}{(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)} \tag{11}$$

similarly, we have
Lemma 5.3.

$$B_2(N+1)-B_2(N)=\frac{(\sum_{i=1}^{N+1}\lambda_i+\sum_{i=1}^{N}\mu_i+pN\nu+\tau)(\lambda_{N+1}-\lambda_{N+2}+\mu_N-\mu_{N+1})}{(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)}$$

Lemma 5.4. If $\alpha>1$, then the sequence $\{\lambda_i\}$ is strictly decreasing.
Proof. For $i\geq 1$,

$$\frac{\lambda_{i+1}}{\lambda_i}=\frac{\lambda/(\alpha^i(i+1)^{\alpha-1})}{\lambda/(\alpha^{i-1}i^{\alpha-1})}=\frac{1}{\alpha}\left(\frac{i}{i+1}\right)^{\alpha-1}.$$

Since $\alpha>1$,

$$\frac{1}{\alpha}<1\quad\text{and}\quad\left(\frac{i}{i+1}\right)^{\alpha-1}<1,$$

it follows that

$$\frac{\lambda_{i+1}}{\lambda_i} < 1.$$

Hence,

$$\lambda_{i+1} < \lambda_i$$

for all $i \geq 1$. Therefore, the sequence $\{\lambda_i\}$ is strictly decreasing. □

Similarly, for $0 < \beta < 1$, the sequence $\{\mu_i\}$ is strictly increasing.

Lemma 5.5. $B(N)$ is non-decreasing in N.
Proof. Using Lemma 5.2 and Lemma 5.3 we have

$$\begin{aligned}
B(N+1)-B(N) &= \begin{bmatrix} \frac{1}{(R+(c_p+r)\tau)}[(c+r)B_1(N+1)+rp\nu B_2(N+1)] \\ -\frac{1}{(R+(c_p+r)\tau)}[(c+r)B_1(N)+rp\nu B_2(N)] \end{bmatrix} \\
&= \frac{1}{(R+(c_p+r)\tau)}\begin{bmatrix} (c+r)(B_1(N+1)-B_1(N)) \\ +rp\nu(B_2(N+1)-B_2(N)) \end{bmatrix} \\
&= \frac{\begin{bmatrix} (c+r)\left(\sum_{i=1}^{N+1}\lambda_i+\sum_{i=1}^{N}\mu_i+pN\nu+\tau\right) \\ \times[(\lambda_{N+1}\mu_{N+1}-\mu_N\lambda_{N+2})+p\nu(\mu_{N+1}-\mu_N)] \\ +rp\nu\left(\sum_{i=1}^{N+1}\lambda_i+\sum_{i=1}^{N}\mu_i+pN\nu+\tau\right) \\ \times(\lambda_{N+1}-\lambda_{N+2}+\mu_N-\mu_{N+1}) \end{bmatrix}}{(R+(c_p+r)\tau)(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)} \\
&= \frac{\begin{bmatrix} \left(\sum_{i=1}^{N+1}\lambda_i+\sum_{i=1}^{N}\mu_i+pN\nu+\tau\right) \\ \times\begin{bmatrix} c\left((\lambda_{N+1}\mu_{N+1}-\mu_N\lambda_{N+2})+p\nu(\mu_{N+1}-\mu_N)\right) \\ +r(\lambda_{N+1}\mu_{N+1}-\mu_N\lambda_{N+2})+rp\nu(\lambda_{N+1}-\lambda_{N+2}) \end{bmatrix} \end{bmatrix}}{(R+(c_p+r)\tau)(\lambda_{N+2}+p\nu+\mu_{N+1})(\lambda_{N+1}+p\nu+\mu_N)} \geq 0,
\end{aligned}$$

since λ_i is decreasing in i and μ_i is increasing in i.
This implies that $B(N)$ is non-decreasing in N. □

Theorem 5.1. The corresponding replacement number N^* can be determined as

$$N^* = \min\{N \mid B(N) \geq 1\}$$

Further, the corresponding replacement number N^* is uniquely determined $\Leftrightarrow B(N^*) > 1$.

Proof. By Lemma 5.1 and 5.5, $C(N)$ decreases for $N < N^*$ and increases for $N \geq N^*$, confirming N^* as the minimum of $C(N)$, with $B(N^*) > 1$ ensuring uniqueness.

6. Numerical Examples and Sensitivity Analysis

To validate the theoretical formulations derived in Section 5 and evaluate the performance of the proposed replacement policy, a comprehensive sensitivity analysis is performed. The baseline system parameters are set as follows: $c = 10$, $p = 0.1$, $R = 5000$, $\nu = 0.2$, $\lambda = 40$, $\alpha = 1.05$, $r = 50$, $\mu = 10$, $\beta = 0.95$, $\tau = 10$, and $c_p = 10$.

By substituting these numerical baseline values directly into the closed-form expressions for the long-run expected cost $C(N)$ (Eq. 5) and the decision function $B(N)$ (Eq. 8), the explicit cost-benefit trajectory is evaluated across various replacement thresholds N. To observe the direct impact of individual dynamics, key parameters are varied systematically around their baseline values while holding all other parameters constant. The detailed subsection-wise sensitivity evaluations are presented below.

6.1. Effect of Physical Deterioration (α)

The parameter α governs the underlying rate of physical deterioration. Table 2 reports the resulting expected costs $C(N)$ and decision boundary values $B(N)$ for $\alpha = 1.05, 1.10$, and 1.15.

Table 2. Impact of deterioration rate (α) on expected cost $C(N)$ and decision function $B(N)$

	$\alpha = 1.05$			$\alpha = 1.10$			$\alpha = 1.15$	
N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$
1	62.0000	0.1146	1	62.0000	0.1221	1	62.0000	0.1297
2	14.0483	0.1428	2	16.0043	0.1637	2	17.8685	0.1849
3	-1.7455	0.1816	3	0.9788	0.2216	3	3.5882	0.2614
4	-9.3684	0.2320	4	-6.1461	0.2962	4	-3.0556	0.3585
5	-13.6991	0.2944	5	-10.0761	0.3877	5	-6.6067	0.4743
6	-16.3738	0.3694	6	-12.3944	0.4951	6	-8.5995	0.6057
7	-18.0958	0.4571	7	-13.7842	0.6174	7	-9.6987	0.7489
8	-19.2177	0.5576	8	-14.5894	0.7528	8	-10.2409	0.8996
9	-19.9355	0.6708	9	-15.0025	0.8993	**9**	**-10.4155**	**1.0536**
10	-20.3673	0.7962	**10**	**-15.1401**	**1.0546**	10	-10.3381	1.2069
11	-20.5884	0.9336	11	-15.0776	1.2161	11	-10.0836	1.3562
12	**-20.6492**	**1.0821**	12	-14.8657	1.3815	12	-9.7028	1.4986
13	-20.5848	1.2409	13	-14.5404	1.5481	13	-9.2319	1.6324
14	-20.4207	1.4093	14	-14.1279	1.7138	14	-8.6970	1.7561
15	-20.1755	1.5860	15	-13.6480	1.8767	15	-8.1178	1.8692
16	-19.8636	1.7699	16	-13.1161	2.0349	16	-7.5092	1.9715
17	-19.4962	1.9598	17	-12.5443	2.1873	17	-6.8827	2.0633
18	-19.0822	2.1546	18	-11.9422	2.3326	18	-6.2473	2.1450
19	-18.6288	2.3529	19	-11.3179	2.4703	19	-5.6099	2.2174
20	-18.1420	2.5536	20	-10.6778	2.5998	20	-4.9761	2.2811
21	-17.6270	2.7554	21	-10.0277	2.7208	21	-4.3502	2.3371
22	-17.0879	2.9573	22	-9.3721	2.8334	22	-3.7355	2.3860
23	-16.5286	3.1583	23	-8.7149	2.9378	23	-3.1346	2.4288
24	-15.9524	3.3573	24	-8.0595	3.0340	24	-2.5495	2.4660
25	-15.3623	3.5536	25	-7.4088	3.1225	25	-1.9815	2.4983

The numerical results in Table 2 confirm that faster physical wear raises overall operating costs and accelerates replacement timing. Specifically, increasing α shifts the optimal threshold from $N^* = 12$ ($\alpha = 1.05$) to $N^* = 10$ ($\alpha = 1.10$), and down to $N^* = 9$ ($\alpha = 1.15$). Figure 1 visually traces these functional trajectories.

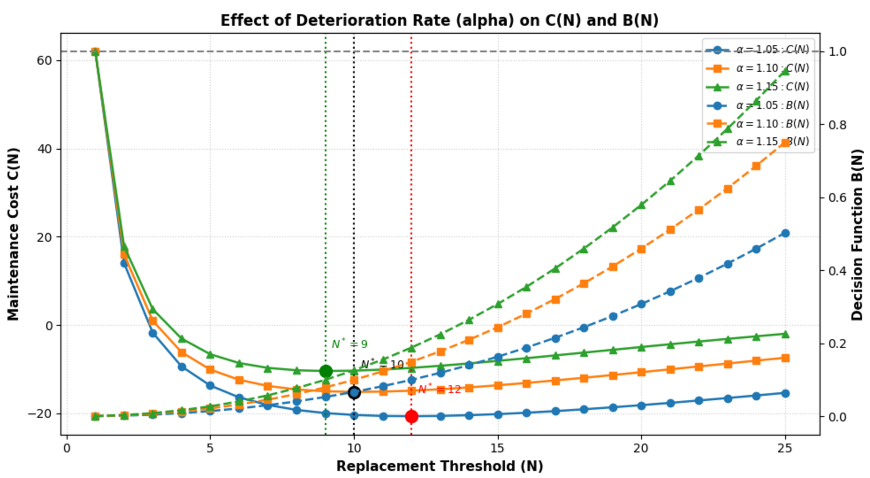

Figure 1. Trends of $C(N)$ and $B(N)$ under varying deterioration rates (α)

6.2. *Effect of Repair Delay Probability* (p)

Table 3 outlines system performance metrics under repair delay probabilities $p \in \{0.0, 0.1, 0.3\}$ to assess administrative or operational delay impacts.

Table 3. Impact of repair delay probability (p) on expected cost $C(N)$ and decision function $B(N)$

	$p = 0.0$			$p = 0.1$			$p = 0.3$	
N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$
1	62.0000	0.1145	1	62.0000	0.1146	1	62.0000	0.1149
2	14.0512	0.1427	2	14.0483	0.1428	2	14.0425	0.1431
3	-1.7460	0.1815	3	-1.7455	0.1816	3	-1.7446	0.1819
4	-9.3714	0.2318	4	-9.3684	0.2320	4	-9.3624	0.2323
5	-13.7039	0.2943	5	-13.6991	0.2944	5	-13.6896	0.2947
6	-16.3799	0.3692	6	-16.3738	0.3694	6	-16.3617	0.3697
7	-18.1028	0.4569	7	-18.0958	0.4571	7	-18.0819	0.4574
8	-19.2253	0.5574	8	-19.2177	0.5576	8	-19.2024	0.5579
9	-19.9436	0.6706	9	-19.9355	0.6708	9	-19.9193	0.6711
10	-20.3758	0.7961	10	-20.3673	0.7962	10	-20.3504	0.7965
11	-20.5971	0.9334	11	-20.5884	0.9336	11	-20.5710	0.9338
12	**-20.6580**	**1.0819**	**12**	**-20.6492**	**1.0821**	**12**	**-20.6314**	**1.0823**
13	-20.5938	1.2408	13	-20.5848	1.2409	13	-20.5669	1.2412
14	-20.4297	1.4092	14	-20.4207	1.4093	14	-20.4027	1.4095
15	-20.1844	1.5859	15	-20.1755	1.5860	15	-20.1576	1.5861
16	-19.8725	1.7698	16	-19.8636	1.7699	16	-19.8459	1.7700
17	-19.5050	1.9598	17	-19.4962	1.9598	17	-19.4787	1.9599
18	-19.0908	2.1546	18	-19.0822	2.1546	18	-19.0650	2.1547
19	-18.6372	2.3529	19	-18.6288	2.3529	19	-18.6120	2.3529
20	-18.1503	2.5536	20	-18.1420	2.5536	20	-18.1256	2.5536
21	-17.6350	2.7554	21	-17.6270	2.7554	21	-17.6110	2.7554
22	-17.0956	2.9573	22	-17.0879	2.9573	22	-17.0724	2.9573
23	-16.5361	3.1583	23	-16.5286	3.1583	23	-16.5137	3.1582
24	-15.9524	3.3573	24	-15.9524	3.3573	24	-15.9380	3.3573
25	-15.3692	3.5536	25	-15.3623	3.5536	25	-15.3485	3.5536

As shown in Table 3, variations in delay likelihood produce negligible changes in long-run expenditure, maintaining a steady optimal replacement threshold at $N^* = 12$ across all three cases ($p = 0.0, 0.1, 0.3$). Figure 2 illustrates these comparative trends.

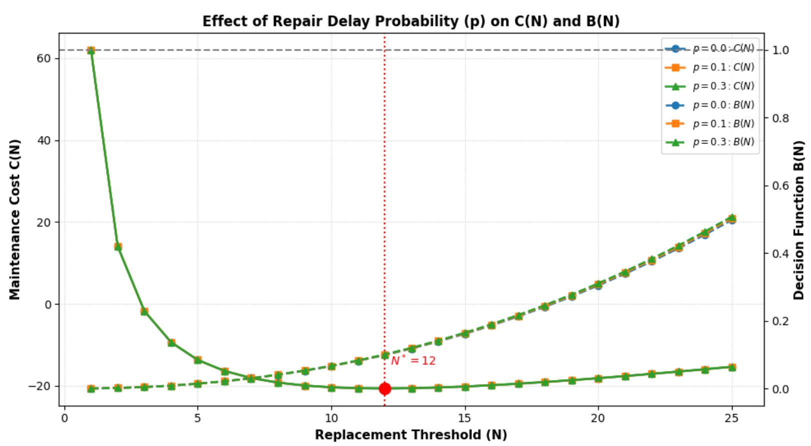

Figure 2. Trends of $C(N)$ and $B(N)$ under varying repair delay probabilities (p)

6.3. *Effect of Post-Repair Degradation* (β)

Table 4 assesses system sensitivity to post-repair degradation coefficients $\beta \in \{0.90, 0.95, 1.00\}$, capturing how imperfect repairs degrade subsequent operating life.

Table 4. Impact of post-repair degradation factor (β) on expected cost $C(N)$ and decision function $B(N)$

	$\beta = 0.90$			$\beta = 0.95$			$\beta = 1.00$	
N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$
1	62.0000	0.1146	1	62.0000	0.1146	1	62.0000	0.1146
2	14.0483	0.1600	2	14.0483	0.1428	2	14.0483	0.1269
3	-1.6626	0.2243	3	-1.7455	0.1816	3	-1.8202	0.1439
4	-9.0525	0.3098	4	-9.3684	0.2320	4	-9.6433	0.1655
5	-13.0604	0.4179	5	-13.6991	0.2944	5	-14.2396	0.1918
6	-15.3503	0.5489	6	-16.3738	0.3694	6	-17.2188	0.2227
7	-16.6398	0.7025	7	-18.0958	0.4571	7	-19.2723	0.2582
8	-17.2895	0.8777	8	-19.2177	0.5576	8	-20.7463	0.2984
9	**-17.5015**	**1.0726**	9	-19.9355	0.6708	9	-21.8334	0.3431
10	-17.3985	1.2848	10	-20.3673	0.7962	10	-22.6496	0.3925
11	-17.0604	1.5114	11	-20.5884	0.9336	11	-23.2685	0.4463
12	-16.5421	1.7492	**12**	**-20.6492**	**1.0821**	12	-23.7394	0.5047
13	-15.8833	1.9950	13	-20.5848	1.2409	13	-24.0965	0.5675
14	-15.1144	2.2456	14	-20.4207	1.4093	14	-24.3643	0.6346
15	-14.2592	2.4981	15	-20.1755	1.5860	15	-24.5607	0.7060
16	-13.3372	2.7497	16	-19.8636	1.7699	16	-24.6992	0.7816
17	-12.3650	2.9982	17	-19.4962	1.9598	17	-24.7902	0.8612
18	-11.3568	3.2416	18	-19.0822	2.1546	18	-24.8417	0.9447
19	-10.3250	3.4784	19	-18.6288	2.3529	**19**	**-24.8600**	**1.0321**
20	-9.2808	3.7075	20	-18.1420	2.5536	20	-24.8504	1.1231
21	-8.2338	3.9279	21	-17.6270	2.7554	21	-24.8170	1.2176
22	-7.1926	4.1391	22	-17.0879	2.9573	22	-24.7631	1.3154
23	-6.1650	4.3409	23	-16.5286	3.1583	23	-24.6917	1.4164
24	-5.1576	4.5332	24	-15.9524	3.3573	24	-24.6051	1.5203
25	-4.1760	4.7159	25	-15.3623	3.5536	25	-24.5053	1.6270

Table 4 and Figure 3 show that stronger post-repair degradation ($\beta = 0.90$) leads to earlier replacement ($N^* = 9$), whereas moderate ($\beta = 0.95$) and ideal ($\beta = 1.00$) restoration extend service life to $N^* = 12$ and $N^* = 19$, respectively.

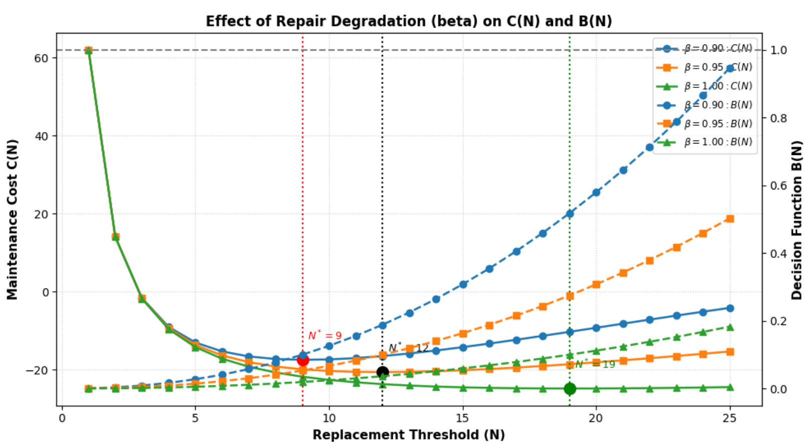

Figure 3. Trends of $C(N)$ and $B(N)$ under varying post-repair degradation factors (β)

6.4. *Effect of Repair Cost Rate* (c)

Table 5 documents expected costs $C(N)$ and decision values $B(N)$ under varying repair cost rates $c \in \{5, 10, 20\}$.

Table 5. Impact of repair cost rate (c) on expected cost $C(N)$ and decision function $B(N)$

	$c = 5$			$c = 10$			$c = 20$	
N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$
1	62.0000	0.1051	1	62.0000	0.1146	1	62.0000	0.1337
2	13.5319	0.1309	2	14.0483	0.1428	2	15.0812	0.1666
3	-2.4810	0.1665	3	-1.7455	0.1816	3	-0.2747	0.2119
4	-10.2446	0.2126	4	-9.3684	0.2320	4	-7.6159	0.2706
5	-14.6842	0.2699	5	-13.6991	0.2944	5	-11.7289	0.3434
6	-17.4519	0.3386	6	-16.3738	0.3694	6	-14.2177	0.4309
7	-19.2579	0.4190	7	-18.0958	0.4571	7	-15.7718	0.5332
8	-20.4584	0.5112	8	-19.2177	0.5576	8	-16.7363	0.6505
9	-21.2516	0.6149	9	-19.9355	0.6708	9	-17.3034	0.7825
10	-21.7566	0.7299	10	-20.3673	0.7962	10	-17.5886	0.9289
11	-22.0497	0.8558	11	-20.5884	0.9336	**11**	**-17.6658**	**1.0891**
12	-22.1816	0.9919	**12**	**-20.6492**	**1.0821**	12	-17.5843	1.2623
13	**-22.1879**	**1.1376**	13	-20.5848	1.2409	13	-17.3786	1.4477
14	-22.0942	1.2919	14	-20.4207	1.4093	14	-17.0736	1.6441
15	-21.9194	1.4538	15	-20.1755	1.5860	15	-16.6877	1.8502
16	-21.6778	1.6224	16	-19.8636	1.7699	16	-16.2352	2.0648
17	-21.3808	1.7965	17	-19.4962	1.9598	17	-15.7270	2.2864
18	-21.0372	1.9751	18	-19.0822	2.1546	18	-15.1721	2.5137
19	-20.6543	2.1568	19	-18.6288	2.3529	19	-14.5777	2.7450
20	-20.2381	2.3408	20	-18.1420	2.5536	20	-13.9499	2.9792
21	-19.7936	2.5258	21	-17.6270	2.7554	21	-13.2937	3.2146
22	-19.3250	2.7109	22	-17.0879	2.9573	22	-12.6136	3.4502
23	-18.8362	2.8951	23	-16.5286	3.1583	23	-11.9135	3.6847
24	-18.3303	3.0775	24	-15.9524	3.3573	24	-11.1968	3.9169
25	-17.8102	3.2574	25	-15.3623	3.5536	25	-10.4665	4.1460

Table 5 demonstrates that higher repair charges ($c = 20$) promote earlier replacement at $N^* = 11$ to minimize ongoing servicing fees. Conversely, standard ($c = 10$) and lower ($c = 5$) repair costs extend the optimal threshold to $N^* = 12$ and $N^* = 13$, respectively. Figure 4 depicts these financial behaviors.

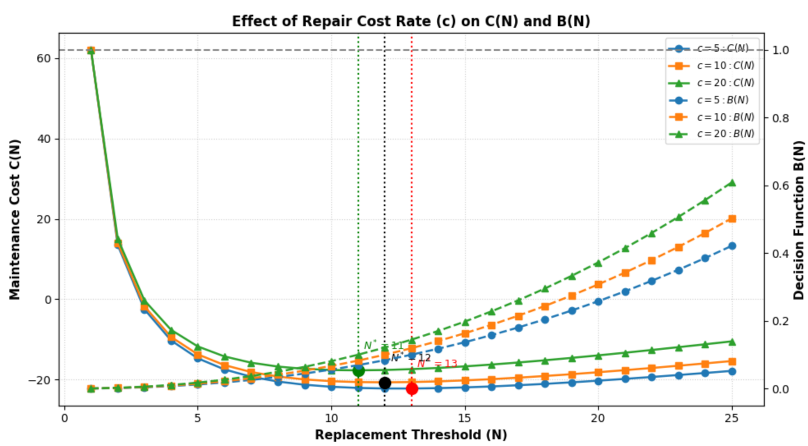

Figure 4. Trends of $C(N)$ and $B(N)$ under varying repair cost rates (c)

6.5. *Effect of Replacement Expenditure* (R)

Table 6 presents policy responses to variation in major replacement costs $R \in \{3000, 5000, 8000\}$.

Table 6. Impact of replacement cost (R) on expected cost $C(N)$ and decision function $B(N)$

	$R = 3000$			$R = 5000$			$R = 8000$	
N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$	N	$C(N)$	$B(N)$
1	22.0000	0.1783	1	62.0000	0.1146	1	122.0000	0.0746
2	-6.6091	0.2222	2	14.0483	0.1428	2	45.0344	0.0930
3	-15.8224	0.2825	3	-1.7455	0.1816	3	19.3698	0.1183
4	-20.1186	0.3608	4	-9.3684	0.2320	4	6.7569	0.1510
5	-22.4355	0.4580	5	-13.6991	0.2944	5	-0.5945	0.1917
6	-23.7564	0.5746	6	-16.3738	0.3694	6	-5.3000	0.2405
7	-24.5031	0.7111	7	-18.0958	0.4571	7	-8.4849	0.2977
8	-24.8870	0.8674	8	-19.2177	0.5576	8	-10.7138	0.3631
9	**-25.0253**	**1.0434**	9	-19.9355	0.6708	9	-12.3008	0.4368
10	-24.9887	1.2386	10	-20.3673	0.7962	10	-13.4352	0.5185
11	-24.8223	1.4522	11	-20.5884	0.9336	11	-14.2375	0.6079
12	-24.5564	1.6832	**12**	**-20.6492**	**1.0821**	12	-14.7884	0.7046
13	-24.2120	1.9304	13	-20.5848	1.2409	13	-15.1441	0.8081
14	-23.8045	2.1922	14	-20.4207	1.4093	14	-15.3449	0.9177
15	-23.3454	2.4670	15	-20.1755	1.5860	**15**	**-15.4207**	**1.0327**
16	-22.8434	2.7532	16	-19.8636	1.7699	16	-15.3939	1.1525
17	-22.3056	3.0486	17	-19.4962	1.9598	17	-15.2822	1.2762
18	-21.7375	3.3516	18	-19.0822	2.1546	18	-15.0992	1.4030
19	-21.1439	3.6601	19	-18.6288	2.3529	19	-14.8562	1.5321
20	-20.5286	3.9722	20	-18.1420	2.5536	20	-14.5622	1.6628
21	-19.8951	4.2862	21	-17.6270	2.7554	21	-14.2247	1.7942
22	-19.2464	4.6003	22	-17.0879	2.9573	22	-13.8502	1.9257
23	-18.5850	4.9129	23	-16.5286	3.1583	23	-13.4440	2.0565
24	-17.9135	5.2225	24	-15.9524	3.3573	24	-13.0108	2.1862
25	-17.2340	5.5279	25	-15.3623	3.5536	25	-12.5548	2.3140

As detailed in Table 6, lower capital outlay $(R = 3000)$ makes early replacement economically attractive at $N^* = 9$. As capital expenses increase to $R = 5000$ and $R = 8000$, management is incentivized to prolong operational life, pushing the optimal point to $N^* = 12$ and $N^* = 15$, respectively. Figure 5 illustrates these trade-offs.

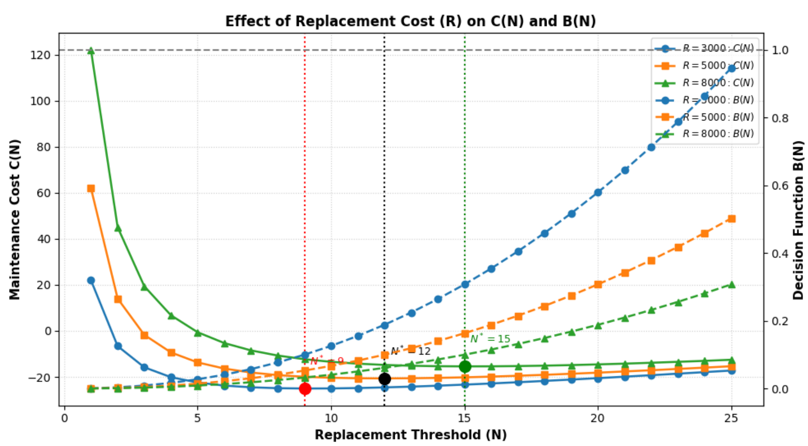

Figure 5. Trends of $C(N)$ and $B(N)$ under varying replacement costs (R)

7. Empirical Validation

We analyse the `aircondit7` data set from the `boot` package in R. The data consist of successive operating times between failures and associated repair delays of an aircraft air-conditioning system and are widely used for evaluating stochastic process models in reliability.

Parameter estimation was performed using the maximum likelihood method under a Weibull baseline distribution, incorporating imperfect repair effectiveness (α) and delayed repair time (τ). The optimization employed the L-BFGS-B algorithm with positivity constraints on all model parameters and converged successfully. The resulting maximum likelihood estimates, together with their standard errors and 95% confidence intervals, are reported in Table 7. The relatively narrow confidence intervals indicate that the parameter estimates are stable and statistically reliable for the fitted model.

The fitted GAP achieved a maximized log-likelihood of -78.9563, corresponding to an Akaike Information Criterion (AIC) of 163.9126.

Table 7. Maximum likelihood estimates of the GAP parameters with standard errors and 95% confidence intervals.

Parameter	Estimate	Standard Error	95% CI Lower	95% CI Upper
Shape	7.2244	1.2375	4.7989	9.6499
Scale	6.3079	0.3972	5.5293	7.0865
Ratio (a)	0.8721	0.0035	0.8652	0.8789

For comparison, the standard Geometric Process (GP) was fitted to the same data using the identical Weibull baseline distribution. The log-likelihood, AIC, ΔAIC, and Akaike weights for both models are presented in Table 8.

Table 8. Comparison of the GAP and the standard GP.

Model	Log-likelihood	AIC	ΔAIC	Akaike Weight
GAP	-78.9563	163.9126	0.0000	0.8826
GP	-80.9732	167.9463	4.0337	0.1174

The GAP achieved a higher maximized log-likelihood and a lower AIC than the standard GP. The AIC difference of 4.0337 favours the GAP over the standard GP. The corresponding Akaike weights indicate that the GAP receives approximately 88.3% of the model support, whereas the GP receives 11.7%. These findings indicate that, for the `aircondit7` data set, the proposed GAP with imperfect delayed repair provides a better fit than the standard GP under the same Weibull baseline distribution.

8. Conclusion

This paper presents an optimal replacement policy framework based on a Geometric α-Process (GAP) that incorporates imperfect repairs and operational delays.

The proposed framework is evaluated using the Aircondit7 dataset and a numerical sensitivity analysis. The main findings are as follows:

1. **Empirical Fit:** For the Aircondit7 aircraft air-conditioning failure data, the GAP model produces a lower AIC (163.9126) and a higher maximized log-likelihood (-78.9563) than the standard Geometric Process (GP), under the same Weibull baseline. The ΔAIC of 4.0337 and the corresponding Akaike weight of 88.3% provide support for the GAP model over the GP for this dataset.
2. **Sensitivity Analysis:** Within the parameter ranges considered in the numerical analysis, changes in physical deterioration (α) and post-repair degradation (β) are associated with changes in the optimal replacement

threshold (N^*). The analysis also examines the effects of repair and replacement costs and the repair-delay probability on the replacement policy.

3. **Economic and Operational Parameters:** For the parameter settings considered, variations in repair cost (C) and replacement cost (R) have a noticeable effect on the optimal replacement decision. The sensitivity analysis also indicates relatively limited changes in N^* for the considered variations in the repair-delay probability (p).

Overall, the proposed GAP framework provides a maintenance model that incorporates imperfect repair, degradation, and repair delays into the replacement decision. The empirical analysis using the Aircondit7 dataset supports the applicability of the model to the considered reliability data, while the sensitivity analysis illustrates the effects of selected model parameters within the ranges examined. Future work may consider multi-component systems, time-varying operating conditions, and opportunistic maintenance strategies.

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