

A Survey on Multiset Dimension and Its Variations

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Abstract The classical notion of metric dimension has led to a wide range of extensions, such as the local, strong, fractional, and k-metric dimensions. This naturally raises the question of whether analogous variants can be formulated and studied within the multiset framework. While some progress has been made, particularly on the local multiset dimension, outer multiset dimension, local outer multiset dimension, and edge multiset dimension, however, the area remains far from fully explored. In this paper, we survey the existing variants and consolidate the results currently available in the literature. Furthermore, we identify several directions for future work.

Keywords Metric Dimension, Metric Dimension Variations, Multiset Dimension, Outer Multiset Dimension, Local Multiset Dimension, Local Outer Multiset Dimension, Edge Multiset Dimension

AMS 2010 subject classifications 05C12

DOI: 10.19139/soic-2310-5070-4062

1. Introduction

The concept of metric dimension in graphs has roots back to 1953 in general metric spaces, though it was formally introduced for graphs by Slater [1], Harary & Melter [2] independently in the 1970s. Earlier mentions of related ideas also appeared in works by Erdős and others [3].

In addition to its theoretical interest, the work on the metric dimension also has practical significance. For example, Chartrand et al. [4] applied the concept to a real-world problem in the representation and classification of chemical compounds.

The concept of multiset dimension was first introduced by Simanjuntak et al. in 2018 [30] as part of their work to generalization of the metric dimension, the idea is to relax the constraints of metric representations in graphs. Motivation arose from applications in networks where nodes can sense or measure distances to certain reference points but cannot distinguish which specific reference produced which measurement. Since then, the multiset dimension has gained attention as it provides a more flexible framework for node identification in graphs, particularly in contexts such as anonymous networks, robotics, and communication systems.

There are numerous extensions of the metric dimension concept, for example, the local, strong, fractional, and k-metric dimensions, yet corresponding extensions within the multiset-dimension framework remain comparatively limited. In this paper, we summarize the existing results on the multiset dimension and its known variants, and discuss a few possible future directions. Relevant publications were identified primarily through searches on Google Scholar and arXiv using combinations of keywords such as *multiset dimension*, *multiset metric*

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dimension, local multiset dimension, outer multiset dimension, and edge-multiset dimension. The reference lists and citing articles of the identified works were subsequently examined to locate additional relevant studies. We included papers that introduce, develop, or establish theoretical or computational results for multiset resolvability parameters.

2. Terms and Definition

Let G be a connected graph with the vertex set $V(G)$ and edge set $E(G)$, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between them. The *diameter* of a graph G is the longest distance between any two vertices in G . The *edge distance* $d_E(e, f)$ between two edges $e, f \in E(G)$ is the length of a shortest path between vertices e and f in the line graph $L(G)$.

A $v_0 - v_l$ *walk* of a graph G is a finite alternating sequence $v_0, e_1, v_1, e_2, \dots, e_l, v_l$ of vertices and edges in G such that $e_i = v_{i-1}v_i$ for each $i, 1 \leq i \leq l$. Such a walk may also be denoted by $v_0v_1\dots v_l$. We note that there may be repetition of vertices and edges in a walk. The *length* of a walk is the number of edges in the walk. A $v_0 - v_l$ walk is *closed* if $v_0 = v_l$. If all the vertices of a $v_0 - v_l$ walk are distinct, then the walk is called a *path*. A *cycle* is a closed path. A path and a cycle of order n are denoted by P_n and C_n , respectively.

A *vertex cover* of a graph is a set S of vertices of G such that every edge of G is incident with at least one vertex of S . The *vertex covering number* of G , denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of G .

A graph is *acyclic* if it has no cycles. A *tree* T_n is a connected acyclic graph. Any graph without cycles is called a *forest*; thus, the components of a forest are trees. A *star* S_n graph is a tree consisting of one vertex adjacent to all other vertices.

Sunflower graph is a graph obtained by taking a wheel with the central vertex c and the n -cycle $v_0, v_1, \dots, v_{n-1}$ combined with additional vertices $w_0, w_1, \dots, w_{n-1}$, where w_i is joined by edges to v_i, v_{i+1} , where $i + 1$ is taken from modulo n .

A t -fold *wheel graph* $W_{t,n}$ is a graph derived from a wheel by duplicating the hub vertex one or more times, resulting in t hub vertices, each adjacent to all rim vertices, and not adjacent to each other.

A *helm graph* H_n is constructed from a wheel W_n by adding n vertices of degree 1, one adjacent to each terminal vertex.

A *friendship graph* f_n is a graph consisting of n triangles sharing a common vertex.

The n -*sunlet graph* S_n is obtained by attaching n pendant edges to the cycle C_n , having order and size $2n$

The *kayak paddle graph*, denoted by $KP(\vartheta, \lambda, \mu)$, is obtained from two cycles of length $\vartheta \geq 3$ and $\lambda \geq 3$ by joining one vertex of one cycle with a vertex of degree one in a path of length $\mu \geq 2$, and another vertex of the other cycle with the other vertex of degree one in the path of length μ . Its vertex set is $V(KP(\vartheta, \lambda, \mu)) = \{\alpha_1, \alpha_2, \dots, \alpha_\vartheta\} \cup \{\beta_1, \beta_2, \dots, \beta_\lambda\} \cup \{\gamma_1, \dots, \gamma_{\mu-1}\}$, and its edge set is $E(KP(\vartheta, \lambda, \mu)) = \{\alpha_i\alpha_{i+1} : 1 \leq i \leq \vartheta\} \cup \{\beta_j\beta_{j+1} : 1 \leq j \leq \lambda\} \cup \{\gamma_k\gamma_{k+1} : 1 \leq k \leq \mu - 2\} \cup \{\alpha_1\gamma_1, \gamma_{\mu-1}\beta_1\}$. where $\alpha_{\vartheta+1} = \alpha_1$ and $\beta_{\lambda+1} = \beta_1$ as defined in [5].

The *dragon graph* $T_{n,m}$ is obtained by joining a vertex v_n of a cycle graph C_n to a vertex u_1 of a path graph P_m with a bridge. Its vertex set is $V(T_{n,m}) = \{v_i, u_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, and its edge set is $E(T_{n,m}) = \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_ju_{j+1} : 1 \leq j \leq m - 1\} \cup \{v_nv_n, v_nu_1\}$ as defined in [5].

For graph-theoretic terminology not defined in this paper, we follow [6].

3. Metric Dimension

Given a graph G , we say a vertex w *resolves* a pair of vertices u, v if $d(u, w) \neq d(v, w)$. For an ordered set of k vertices $W = \{w_1, w_2, \dots, w_k\} \in G$, the *representation* of distances of a vertex v with respect to W is the ordered k -tuple

$$r(v \mid W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k)).$$

A set of vertices $W \subseteq V(G)$ is a *resolving set* of G if every two vertices of G have distinct representations. A resolving set with minimum cardinality is called a *metric basis*, and the number of vertices in a metric basis is called the *metric dimension*, denoted by $\dim(G)$.

The metric dimension of graphs has been extensively studied over the past decades, resulting in a rich body of literature covering its structural properties, computational complexity, and numerous variants. A comprehensive account of these developments is beyond the scope of this paper. Interested readers are referred to existing survey articles [7, 8, 9, 10] for detailed summary of classical results, algorithmic aspects, and applications of metric dimension. These surveys provide a thorough overview of known bounds, characterizations for special graph classes, and the diverse extensions that have emerged from the original concept. While these surveys provide comprehensive coverage of metric dimension and its extensions, they do not focus specifically on the multiset dimension framework. This work complements these literature surveys by providing a systematic overview of the multiset dimension and its known variants, as well as their relationships and open problems. To provide background and allow comparison with their multiset counterparts, we briefly review several important variants of metric dimension below.

4. Metric Dimension Variations

Over the years, many variations of the classical metric dimension have been introduced. These variants either impose extra structural constraints on the resolving sets, alter the conditions of resolvability, or even aim to identify different types of elements in the graph.

4.1. The Partition Dimension

The *partition dimension* of a graph was first introduced in [11]. It is a variant of the classical metric dimension in which vertices are distinguished based on their distances to subsets of vertices rather than to individual resolving vertex. Formally, a partition $\Pi = \{P_1, P_2, \dots, P_t\}$ of the vertex set $V(G)$ is called a *resolving partition* if every vertex in the graph has a unique vector of minimum distances to the sets in Π . The smallest number of sets in such a resolving partition defines the *partition dimension* of the graph, denoted by $pd(G)$.

G	$pd(G)$
Connected graph, $n \geq 4$, not a path or complete graph	$3 \leq pd(G) \leq n - 1$ [11]
n -cycle, $n \geq 3$	3 [11]
3-cube Q_3	3 [11]
Petersen graph	4 [11]
Nontrivial connected graph H , $(H \times K_2)$	$\leq pd(H) + 1$ [11]
n -cube Q_n , $n \geq 2$	$\leq n + 1$ [11]
Complete bipartite graph $K_{r,r}$	$\leq r + 1$ [11]
Complete bipartite graph $K_{r,s}$, $r \neq s$	$\leq \max\{r, s\}$ [11]
$K_{1,n-1}$, $K_n - e$, $K_1 + (K_1 \cup K_{n-1})$, $n \geq 3$	$n - 1$ [11]

Table 1. The partition dimension of graphs.

A set of fundamental results on the partition dimension was developed in [11], including bounds for bipartite graphs and general connected graphs in terms of their order and diameter; the reader is referred to [7, 11] for details.

4.2. The Edge Metric Dimension

The *edge metric dimension* extends the idea of metric identification from vertices to edges. While the classical metric dimension focuses on uniquely identifying the vertices of a graph, the edge metric dimension focuses on distinguishing edges through their distances to some vertices.

Formally, given a connected graph G , the distance between a vertex $v \in V(G)$ and an edge $e = xy \in E(G)$ is defined as $d_G(e, v) = \min\{d_G(v, x), d_G(v, y)\}$. A set of vertices $S \subset V(G)$ is called an *edge resolving set* if for every pair of distinct edges $e, f \in E(G)$, their distance vectors to the resolving set is different. The smallest such set is called an *edge metric basis*, and its cardinality is the *edge metric dimension*, denoted by $\text{edim}(G)$. This parameter was first introduced in [12].

We have noticed another definition of the edge metric dimension which was given in [13] as follows:

Definition 1

Let $W_E = \{e_1, e_2, \dots, e_k\}$ be an ordered set of the edges of G (i.e. vertices of $L(G)$) and $f \in E(G) = V(L(G))$, then representation $r(f|W_E)$ of f with respect to W_E is the k -tuple $(d_E(f, e_1), d_E(f, e_2), \dots, d_E(f, e_k))$. W_E is called edge resolving set if $r(f|W_E)$ is distinct for all $f \in E(G)$. A edge resolving set of minimum cardinality is called an edge basis for G and this cardinality is the edge metric dimension of G , denoted by $\text{dim}_E(G)$.

This definition uses the edges to resolve the edges, which translate the problem into determine the metric dimension of the line graph of the original graph. In this paper, we follow the definition of $\text{edim}(G)$ as in [12].

A number of important results on the edge metric dimension were introduced in [12]. These include values for several graph families such as complete bipartite graphs, trees, grids, wheels, and fan graphs, as well as NP-completeness and NP-hardness, an $O(\log m)$ approximation bound, structural bounds, and results for certain graph families such as hypercubes. For more details, the reader is referred to [7].

The following table summarize some of the known results:

G	$\text{edim}(G)$
P_n	1 [4]
C_n	2
K_n	$n - 1$ [14]
T	$\sigma(T) - \text{ex}(T)$ [15]
$K_{s,t}$	$\begin{cases} \left\lfloor \frac{2(s+t-1)}{3} \right\rfloor, & \text{if } s \leq t \leq 2s \\ t - 1, & \text{if } t \geq 2s \end{cases}$ [16]
$W_{1,n}$	$\begin{cases} 3, & n = 3, 4 \\ 4, & n = 5 \\ n - \lceil n/3 \rceil, & n \geq 6 \end{cases}$ [17]
B_n	$2n - 1$ [17]
n -sunlet graph S_n	$= \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$ [13]
$G_{n,k}$	$= \begin{cases} 2, & \text{if } 1 \leq k \leq \lceil n/2 \rceil, \\ 3, & \text{if } \lceil n/2 \rceil \leq k \leq n. \end{cases}$ [13]
D_n the family of prism graph	3, for $n \geq 3$. [13]

Table 2. Edge metric dimension of graphs.

4.3. The Local Metric Dimension

The *local metric dimension* of a connected graph G is defined based on a *local resolving set*, which is a subset of vertices $S \subset V(G)$ such that for every pair of adjacent vertices $u, v \in V(G)$ have different representations. The smallest such set S is called a *local metric basis*, and its cardinality is known as the *local metric dimension*, denoted $\dim_\ell(G)$, $ldim(G)$ [7], $lmd(G)$ [18] or $\beta_l(G)$ [19]. In this paper we use the symbol $ldim(G)$. This concept was first introduced in [18].

The following table summarize some of the known results:

G	$ldim(G)$
Path graph P_n	1 [20]
Cycle graph C_n	$= \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$ [20]
K_n	$n - 1$ [18]
Bipartite graph	1 [18]
Complete k -partite graph, $k \geq 2$	$k - 1$ [18]
Wheel graph $W_n, n \geq 5$	$\lceil \frac{n}{4} \rceil$ [19]
Helm graph $H_n, n \geq 5$	$\lceil \frac{n}{4} \rceil$ [19]
Sunflower graph $SF_n, n \geq 7$	$\lceil \frac{n}{4} \rceil$ [19]

Table 3. The local metric dimension of graphs.

More results on the local metric dimension can be found in [18]. These include structural properties, bounds in terms of diameter, clique number, and true twin equivalence classes. Further details are available in [7].

4.4. Other Variants of Metric Dimension

In addition to the above variations of the metric dimension, several other variants have been introduced to capture different aspects of the identification of the vertices based on distances. These include the *mixed metric dimension* which is an extension of both the classical metric dimension and the edge metric dimension, designed to distinguish all elements of the graph, vertices and edges, using a single set of reference vertices, for details, see [21].

The *adjacency dimension* relies on the adjacency representation of vertices, where this adjacency relation allows the resolving set to uniquely distinguish between pairs of vertices based on their direct connections [22][7].

The *strong metric dimension* is an extension of the classical metric dimension, developed to capture the true distance relationships within a graph more effectively. A vertex w is said to strongly resolve two vertices u and v if there is a shortest path between w and u containing v or a shortest path between w and v containing u [23].

Another variation is the *strong partition dimension* [24], which is based on partitioning the vertex set of a graph instead of selecting individual vertices. The sets of the partition, under the strong resolving condition, act as elements that uniquely determine the position of each vertex.

The *k -metric dimension* generalizes the standard metric dimension by requiring a higher level of redundancy in vertex distinction; every pair of distinct vertices is distinguished by at least k vertices in the resolving set. This concept can be useful in applications demanding fault tolerance and robustness. Notably, when $k = 1$, the k -metric dimension coincides with the classical metric dimension. This concept was introduced independently in both [25] and [26].

The *fractional metric dimension* was first described in [27]. It is a modified version of the classical metric dimension that allows assigning fractional weights to vertices rather than selecting them completely (as in 0 or 1 choices).

The *fractional strong metric dimension* assigns a weight between 0 and 1 to each vertex instead of selecting vertices entirely. For every pair of distinct vertices, the total weight of vertices that strongly resolve them must be at least 1, and the goal is to minimize the sum of all vertex weights [28].

The k -metric antidimension is a graph parameter introduced to support privacy protection in network analysis, especially in social networks. Unlike the classical metric dimension, which aims to uniquely identify all nodes based on distances to a resolving set, the k -metric antidimension aims to maximize anonymity by making some nodes indistinguishable [29][7].

5. Multiset Dimension

The *multiset dimension* is a variation of the metric dimension in which vertices are identified by unordered, rather than ordered, distance information. Let G be a simple connected graph, let $W \subseteq V(G)$, and let $v \in V(G)$. The multiset representation of v with respect to W is defined by

$$r_m(v | W) = \{\{d(v, w) : w \in W\}\},$$

where repeated distances are retained with their multiplicities. The set W is called a *multiset resolving set* if $r_m(u | W) \neq r_m(v | W)$ for every pair of distinct vertices $u, v \in V(G)$. A multiset resolving set of minimum cardinality is called a *multiset basis*, and its cardinality is the *multiset dimension* of G , denoted by $md(G)$. If no multiset resolving set exists, then G is said to have infinite multiset dimension, and we write $md(G) = \infty$. This possibility distinguishes multiset dimension from classical metric dimension, which is finite for every connected graph. In particular, high levels of symmetry, including large classes of twin vertices, may prevent unordered distance information from uniquely identifying all vertices [30].

5.1. Known Results on Multiset Dimension

Several foundational results on the multiset dimension were first established, particularly those concerning existence. In [30], the followings are given.

Lemma 1

[30] No graph has multiset dimension 2.

Theorem 1

[30] Let G be a graph other than a path. Then $md(G) \geq 3$.

Lemma 2

[30] $md(G) \geq \dim(G)$.

For positive integers n and d , let $f(n, d)$ denote the least positive integer k for which

$$\frac{(k + d - 1)!}{k! (d - 1)!} + k \geq n.$$

The following theorem gives a lower bound.

Theorem 2

[30] If G is a graph of order $n \geq 3$ and diameter d , then

$$md(G) \geq f(n, d).$$

Further study from [31], Novi et al. have shown that.

Theorem 3

[31] The multiset dimension is not monotonic to the order of the graph.

Furthermore, a construction of a tree is given which lead to the following results.

Theorem 4

[31] Given any positive integer $m \geq 3$, there exists a simple graph whose multiset dimension equals m .

Following the Theorem 2, for graphs with multiset dimension 3, Novi et al. showed the following bound and also the graph reaching the bound.

Theorem 5

[31] If G is a graph on n vertices of diameter k with multiset dimension 3, then

$$n \leq \frac{k^3 + 3k^2 + 2k + 12}{6},$$

and this bound is the best possible.

Another structural bound relating the multiset dimension to the maximum degree of a connected graph is as follows:

Theorem 6

[32] Let G be a connected graphs and let d be the maximum degree of G and $md(G) = k$, we have for $k \geq 3$ and $d < 3k$.

Several conditions for a graph to have infinite multiset dimension were also identified. Specifically,

Theorem 7

[30] If G is a graph of diameter at most 2 other than a path, then

$$md(G) = \infty.$$

This means that cycles with at most 5 vertices, complete graphs, stars, the Petersen graph, and strongly regular graphs have multiset dimension infinity.

Lemma 3

[30] If G contains a vertex which is adjacent to (at least) three pendant vertices, then

$$md(G) = \infty.$$

However, these conditions are neither sufficient nor necessary for a graph to have infinite multiset dimension. For example in Figure 1, there exists a graph that does not satisfy these conditions yet still has $md(G) = \infty$.

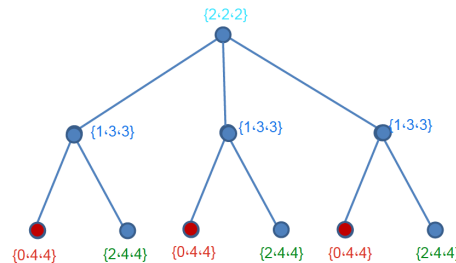


Figure 1. A tree that satisfies neither of the following: diameter at most 2, nor having a vertex adjacent to at least three pendant vertices, yet still has infinite multiset dimension.

The following conjecture was proposed.

Conjecture 1

[30] If G is a graph on n vertices having finite multiset dimension, then $md(G) \leq n - 1$.

Conjecture 1 was proved for trees in [33].

Theorem 8

[33] Let T be a tree of order n and diameter at least 2. If $md(T) < \infty$, then $md(T) \leq n - 2$.

However, recently, Allikvere [63] disproved the conjecture. An exhaustive search of all connected graphs of orders 2 through 11 identified exactly eight graphs, all of order 11, for which $md(G) = n$. Hence, the trivial bound $md(G) \leq n$ is sharp, and order 11 is the smallest order at which equality occurs. Allikvere also proved that $md(P_n \boxtimes P_n) = 4$ for every $n \geq 5$ and that $md(P_3 \boxtimes P_n) = n$ for every $n \geq 6$, revealing a substantial difference between these two classes of graphs.

Furthermore, the following was conjectured.

Conjecture 2

[33] Let T be a tree. If $md(T) < \infty$, then $md(T) \leq n - \text{diam}(T) + 1$ and the bound is sharp.

There are various of results on trees. In [33], the following was proved.

Theorem 9

[33] Let G be a lobster. If P is the minimum 2-center path of G , then the following are equivalent.

1. G has finite multiset dimension.
2. The only component of $G - E(P)$ with infinite multiset dimension is an S_4 .
3. If H is a component of $G - E(P)$ then $[H]$ has at most 4 components which are either a P_2 , a P_3 , or an S_4 , with at most two P_2 s and two S_4 s.

Theorem 10

[33] Let G be a caterpillar with P its minimum 1-center-path. $md(G)$ is finite if and only if every vertex in P has at most 2 neighbors in $G - P$.

In the paper [5], the authors defined the kayak paddle graphs, dragon graphs, and the comb products of two path graphs, and established that all of these graphs have multiset dimension 3. Marcelo et al. [34] have determined the multiset dimension of prisms. Moreover, Marcelo et al. [35] proved that all cylindrical graphs $P_m \square C_n$, with $m, n \geq 3$, have finite multiset dimension. For exact values, see Table 4.

The concepts of ID-colorings and ID-graphs were introduced and studied in [36].

Definition 2

[57] Let G be a connected graph of diameter d and a set of vertices $S \subset V(G)$. For every vertex $x \in V(G)$, the *code* of x with respect to S is the d -vector $\vec{d}(x|S) = (a_1, a_2, \dots, a_d)$, where a_i , with $i \in \{1, \dots, d\}$ represents the number of vertices in S at distance i from x . If all the codes of vertices of G are pairwise different, then S is called an *identification coloring* or *ID-coloring*. Moreover, a graph G that has an ID-coloring is called an *ID-graph*. For any ID-graph G , the cardinality of a smallest ID-coloring is the *ID-number* of G , denoted by $ID(G)$.

This concept is equivalent to m -resolving sets, where the cardinality of a smallest m -resolving set is the multiset dimension.

Theorem 11

[57] Let G be a graph of diameter d . Then $S \subset V(G)$ is an ID-coloring for G if and only if S is a multiset resolving set for G .

Later, Kono and Zhang studied identification colorings for trees in [37], for caterpillars in [38], and for grids and prisms in [39] and for spectra of grids in [40]. Lollipop graphs have been studied in [41]. The paper [42] presents some relations between ID-graphs and their ID-indices, give a lower bound on the ID-index of a graph and determine the ID-indices of paths, grids, cycles, prisms, complete graphs, some complete multipartite graphs, and some caterpillars.

We have reviewed the literature on ID-coloring, including [36, 37, 38, 39, 40, 42]. All results are already covered in the multiset dimension setting, except for lollipop graphs [41]. The following results establish the multiset metric dimension or derive upper bounds for the corresponding lollipop graphs.

Theorem 12

[41] The lollipop graph $T_{3,n+1}$ ($n \geq 2$) has an identification coloring number r for an ID-coloring if and only if $3 \leq r \leq n + 1$.

Theorem 13

[41] The lollipop graph $T_{4,n+1}$ ($n \geq 1$) has an identification coloring number r for an ID-coloring:

- when $n = 1$ or $n = 2$, if and only if $r = 3$;
- when $n \geq 3$, if and only if $3 \leq r \leq n + 2$.

Theorem 14

[41] The lollipop graph $T_{5,n+1}$ ($n \geq 1$) has an identification coloring number r for an ID-coloring:

- when $n = 1$, if and only if $r = 3$ or $r = 4$;
- when $n \geq 2$, if and only if $3 \leq r \leq n + 4$.

Known results on multiset dimension is summarized in the Table 4.

G	$md(G)$
Path graph P_n	1 [30]
Cycle graph C_n	3, $n \geq 6$ [30]
Cycle with chord graph C_n^2	3, $n \geq 7$ [67]
Complete graph K_n	∞ [30]
Star graph S_n	∞ [30]
Petersen graph	∞ [30]
Complete k -ary tree ($k = 1$ or 2)	finite [30]
Complete binary tree of height h	$2^h - 1$ [30]
$P_m \square P_n$	3, $m \geq 3, n \geq 2$ [30]
$P_n \boxtimes P_n$	$3 \leq md(P_n \boxtimes P_n) \leq 4, n \geq 7$ [57]
Kayak paddle graph $KP(\vartheta, \lambda, \mu), \vartheta, \lambda, \mu \geq 4$	3 [5] [67]
Dragon graph $T_{n,m}, n \geq 4, m \geq 3$	3 [5]
The comb product of P_n and $P_m, n, m \geq 4$	3 [5]
Tree T of order n and diameter $\geq 2, md(T) < \infty$	$\leq n - 2$ [33]
Prism graph $K_2 \square C_{13}$	≥ 4 [34]
Prism graph $C_n \square K_2, n \geq 6$	$= \begin{cases} 3, & n \geq 11, n \neq 13, \\ 4, & n = 8, 9, 10, 13, \\ 5, & n = 6, 7. \end{cases}$ [34]
Cylindrical graph $P_m \square C_3, n = 3, m \geq 3$	3 [34]
Cylindrical graph $P_m \square C_n, m \geq 6, n \geq 3$	≤ 4 [35]
Cylindrical graph $P_m \square C_n, m \geq 2, n \geq 8m + 1$	3 [35]
Circulant digraph $C_n(1)$	1 [55]
Circulant digraph $C_n(1, t), t \geq 2, n \geq 2t^2$	t [55]

Table 4. Multiset dimension of some graphs

G	$md(G)$
Circulant digraph $C_n(1, t)$, $t \geq 2$, $n \geq t(t-1) + 2$	$\leq t$ [55]
Circulant digraph $C_{2t}(1, t)$, $t \geq 4$	2 [55]
Circulant digraph $C_n(1, 2, \dots, t)$, $t \geq 3$, $n \geq 2t$	$\leq t$ [55]
Cayley digraph of $\mathbb{Z}_n \oplus \mathbb{Z}_n$ with generating set $\{(1, 0), (0, 1)\}$, $n \geq 6$	n [55]
Cayley digraph of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ with generating set $\{(1, 0), (0, 1)\}$, $n \geq 3$, $m \geq n + 1$	n [55]
V-graph $V(R)$, R a commutative ring with identity	1 iff $R \cong \mathbb{Z}_3$ [59]
V-graph $V(R)$ with a ring R	∞ iff $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R \cong \mathbb{F}_4$ [59]
Compressed zero-divisor graph $\Gamma_E(R)$ of a finite commutative ring R	0 iff $\Gamma(R) \cong K_n$ [60]
Compressed zero-divisor graph $\Gamma_E(R)$ of a finite commutative ring R	1 iff $\Gamma(R) \cong K_{m,n}$, m or $n \geq 2$ [60]
$\Gamma_E(\mathbb{Z}_n)$, $n = 2p$, $p > 2$, p prime	1 [60]
$\Gamma_E(\mathbb{Z}_n)$, $n = p^2$, p prime	0 [60]
$\Gamma_E(R)$, a realizable graph containing three vertices	1 [60]
$\Gamma_E(R)$, a realizable graph containing four vertices	1 or ∞ [60]
$\Gamma_E(R)$, a realizable graph containing five vertices	∞ [60]
$\Gamma_E(R)$, finite commutative ring R , $\text{gr}(\Gamma_E(R)) = \infty$, R reduced ring	1 [60]
$\Gamma_E(R)$, finite commutative ring R , $\text{gr}(\Gamma_E(R)) = \infty$, $R \cong \mathbb{Z}_4, \mathbb{Z}_9, \mathbb{Z}_2[x]/(x^2)$	0 [60]

Table 4. Multiset dimension of some graphs (continued)

While exact values of the multiset dimension have been obtained for several specific graph classes, Eide and Prałat [56] studied its typical behaviour in the binomial random graph $\mathcal{G}(n, p)$. Their results reveal a transition governed by the average degree $d = (n-1)p$. In the sparse regime, where d grows as a sufficiently small power of n , the multiset dimension is finite with high probability and grows polynomially with the order of the graph. The available upper and lower bounds have different exponents, however, so the precise asymptotic order remains undetermined. As the average degree increases, distinguishing vertices by unordered distance profiles becomes progressively more difficult. In the intermediate regime, the known lower bound shows that any multiset resolving set, if one exists, must already be polynomially large. More strikingly, when the average degree grows faster than $n^{1/2}$, the multiset dimension is infinite with high probability. Thus, sufficiently dense random graphs typically admit no multiset resolving set at all. This phenomenon reflects the concentration of distances in dense random graphs: many vertices have the same multiplicities of distances to a proposed landmark set, even though their ordered distance vectors may differ. The random-graph results therefore illustrate that discarding the correspondence between landmarks and distances can cause not only an increase in the required number of landmarks, but also the complete loss of resolvability.

6. Multiset Dimension Variations

In comparison to the large number of variations to the metric dimension, there is very limited work on the multiset dimension, mainly through four variants, namely the local, outer, local(outer), and edge versions. We have conducted a systemic survey of the known results.

6.1. The Outer Multiset Dimension

The outer multiset resolving set was first introduced by Gil-Pons *et al.* [44] in 2019. Instead of requiring that all vertices be distinguished, this modified version requires only vertices *outside* the set S to have distinct multiset representations. Formally, a set $S \subset V(G)$ is an outer multiset resolving set if for any two distinct vertices $u, v \in V(G) \setminus S$, their multisets of distances to S differ [45]. This approach guarantees the existence of such sets for every graph. This makes it a more practical and computable parameter, especially for graphs where the classical multiset dimension is infinite. However, further studies on the outer multiset dimension remain limited.

Definition 3 (Outer Multiset Dimension)

[45] [46] If for every pair of distinct vertices $u, v \in V(G) \setminus W$, $m(u | W) \neq m(v | W)$, then W is called an *outer*

multiset resolving set for G . This set with the smallest cardinality is an *outer multiset basis*, and its cardinality is the *outer multiset dimension* of G , denoted by $\dim_{ms}(G)$.

It is obvious that a multiset basis of a graph is also its outer multiset basis. Thus, $\dim(G) \leq \dim_{ms}(G) \leq md(G)$ [46]. Furthermore, we have.

Theorem 15

[44][45][46] If the order of G is n , then $\dim_{ms}(G) \leq n - 1$. Moreover, $\dim_{ms}(G) = n - 1$ if and only if G is a regular graph with diameter at most 2.

The quantity $f(n, d)$ defined in [4] is the smallest positive integer k such that $k + d^k \geq n$. Similarly, $f'(n, d)$ is defined as the smallest positive integer k' such that $k' + \binom{r+d-1}{d-1} \geq n$. We have the following.

Theorem 16

[44] For every graph $G = (V, E)$ of order n and diameter d such that $\dim(G) < f'(n, d)$, $\dim_{ms}(G) > \dim(G)$.

Two different vertices u, v are called *true twins* if $N[u] = N[v]$. Likewise, u, v are called *false twins* if $N(u) = N(v)$. In general, u, v are called *twins* if they are either true twins or false twins. The property of being twins induces an equivalence relation on the vertex set of a graph [44].

Corollary 1

[44] Let G be a non-trivial graph and let $\mathcal{T} = \{[u_1], [u_2], \dots, [u_t]\}$ be the set of equivalence classes induced in $V(G)$ by the twin equivalence relation. Then, $\dim_{ms}(G) \geq \sum_{i=1}^t (|[u_i]| - 1)$.

Similarly to the multiset dimension problem, the problem of determining the outer multiset dimension is NP-complete. then the attention is to graphs with small values of $\dim_{ms}(G)$. We have the following.

Lemma 4

[45] If G is a graph with $\dim_{ms}(G) = 2$ and $S = \{u, v\}$ is an outer multiset basis, then $d_G(u, v) \leq 2$.

Theorem 17

[45] Deciding whether a graph G of order n satisfies $\dim_{ms}(G) = 2$ can be done in $O(n^3)$ time.

For a graph G , $X \subseteq V(G)$, and $k \geq 0$, define $L_k(X) = \{u \in V(G) : \min_{x \in X} d_G(u, x) = k\}$. Note that $L_0(X) = X$, and that the sets $L_k(X)$, $k \geq 0$, partition $V(G)$ [45].

Lemma 5

[45] Let G be a graph with $\dim_{ms}(G) = 2$ and let S be an outer multiset basis of G . Then for every $k \geq 1$ we have $|L_k(S)| \leq 3$. Moreover, if the vertices of S are adjacent, then $|L_k(S)| \leq 2$ and $|L_{k+1}(S)| \leq |L_k(S)|$.

Furthermore, in [45], the family $\mathcal{F}$ of graphs G is constructed in the following way. The vertex set of $G \in \mathcal{F}$ is $V(G) = \{u_0, \dots, u_r\} \cup \{v_0, \dots, v_s\}$ for some $r \geq 0$ and $s \geq 1$, and the edges of G are given as follows:

- $u_0v_0, u_0v_1 \in E(G)$.
- For every $i \in [r]$ and every $j \in [s]$, $u_{i-1}u_i \in E(G)$ and $v_{j-1}v_j \in E(G)$.
- For every $i \in [\min\{r, s\}]$, the edge $u_i v_i$ might exist or not in G .
- For every $j \in [\min\{r, s-1\}]$, the edge $u_i v_{i+1}$ might exist or not in G .

Then the following structural results are shown.

Theorem 18

[45] A graph G has an outer multiset basis formed by two adjacent vertices if and only if $G \in \mathcal{F}$.

Recall that the *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G) \times V(H)$ and edges $(g, h)(g', h')$, where either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$. If $g \in V(G)$, then the set of vertices $\{(g, h) : h \in V(H)\}$ induces a subgraph of $G \circ H$ isomorphic to H which is called an H -layer and denoted by ${}^g H$. Also, a graph G is *multiset distance irregular* if for every two vertices $u, v \in V(G)$, the multisets $m_G(u \mid V(G))$ and $m_G(v \mid V(G))$ are different [45].

Theorem 19

[45] If G is a graph with $n(G) \geq 2$ and $H \in \{K_k, \overline{K_k}\}$, $k \geq 2$, then $\dim_{ms}(G \circ H) \geq n(G)(k-1)$ where $n(G)$ is the order of the graph. Moreover, equality holds if and only if G is multiset distance irregular.

The following Table 5 summarizes the results on some fundamental family of graphs.

G	$\dim_{ms}(G)$
Path graph P_n	1 [44]
Cycle graph C_4	3 [44]
Cycle graph C_5	4 [44]
Cycle graph C_n	3, $n \geq 6$ [44]
Complete graph K_n , $n \geq 2$	$n-1$ [20]
Complete k -partite $(K_{r_1, r_2, \dots, r_k}, r_1 = r_2 = \dots = r_k \geq 2)$	$n-1$ [44]
$(P_s \square P_t)$, $s \geq t \geq 2$	3[44][45]
V-graph $V(R)$, R a commutative ring with identity	1 iff $R \cong \mathbb{Z}_3$ [59]
V-graph $V(R)$, R a commutative ring with identity	0 iff $R \cong \mathbb{Z}_2$ [59]
V-graph $V(R)$, R a commutative ring with identity	3 iff $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R \cong \mathbb{F}_4$ [59]

Table 5. The outer multiset dimension of some graphs

6.2. Local Multiset Dimension

In [20], the authors introduced the concept of *local multiset dimension* $\mu_\ell(G)$ (also written as $lmd(G)$), which focuses on distinguishing only adjacent vertices using multiset representations. They studied this parameter on several fundamental families of graphs, including paths, cycles, complete graphs, and trees. Their results provide exact values and sharp bounds that establish a baseline understanding of $lmd(G)$. The definition of the local multiset dimension is given below.

Definition 4 (Local Multiset Dimension)

[20]

Let G be a connected graph with vertex set $V(G)$. Suppose $W = \{s_1, s_2, \dots, s_k\}$ of vertex set G , the multiset representation of a vertex v of G with respect to W is $r_m(v|W) = \{d(v, s_1), d(v, s_2), \dots, d(v, s_k)\}$ where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The resolving set W is a *local resolving set* of G if $r_m(v|W) \neq r_m(u|W)$ for every pair u, v of adjacent vertices of G . The minimum local resolving set W is a *local multiset basis* of G . If G has a local multiset basis, then its cardinality is called *local multiset dimension*, denoted by $lmd(G)$.

Every local multiset basis of a graph induces its local basis, and every multiset basis of a graph induces its local multiset basis. This leads to the relationship $ldim(G) \leq lmd(G) \leq md(G)$ [20].

In a later paper [32], the authors discussed the relationship between the multiset dimension and local multiset dimension of graphs. The key insights is that the multiset dimension and local multiset dimension are not monotonic with respect to the number of vertices or the number of edges of a graph, also the difference between the two parameters can be arbitrarily large i.e., $|(md(G) - lmd(G))| = \infty$. The investigation of the local multiset dimension was also extended to graphs obtained via a standard graph operation, in particular the Cartesian product and several bounds was established.

Lemma 6

[32] Given that two connected graphs G_1 and G_2 , $lmd(G_1 \square G_2) \geq \min\{lmd(G_1), lmd(G_2)\}$.

More specific, the Cartesian product of a graph G and a tree graph T , where $lmd(G) = 1$, yields the following results:

Theorem 20

[32] Given that a connected graph G and a path P_n , $lmd(G \square P_n) = lmd(G)$.

Theorem 21

[32] Given that a connected graph G and a tree T , $lmd(G \square T) \leq lmd(G)$.

Corollary 2

[32] Given that a connected graph G and a tree T . For $lmd(G) = 1$, $lmd(G \square T) = 1$.

Theorem 22

[32] For $lmd(G) = 1$ and m is even, $lmd(G \square C_m) = 1$.

Lemma 7

[32] For ($lmd(G) = 1$ and m is odd or $lmd(G) \neq 1$ and $m \geq 3$), $lmd(G \square C_m) \geq 2$.

And in the case where $lmd(G) \neq 1$, we have the following.

Lemma 8

[32] For $lmd(G) \neq 1$ and any tree T , $lmd(G \square T) \geq 2$.

In a recent study [47], the comb product of two connected graphs, demoted by $\triangleright$, was introduced. In the following, the main results on the local multiset dimension of comb product graphs with respect to the main graph is presented.

Lemma 9

[47] Let G and H be a nontrivial connected graph. $lmd(G) = lmd(H) = 1$ if and only if $lmd(G \triangleright H) = 1$.

Theorem 23

[47] Let T be a tree with order $n \geq 2$, and let T_1, T_2 be subtrees of T with $V(T_1) \cup V(T_2) = V(T)$ and $V(T_1) \cap V(T_2) = \emptyset$, then

$$lmd(T \triangleright C_3) \geq \begin{cases} n, & \text{for } T_1 \neq T_2, \\ n+1, & \text{for } T_1 = T_2. \end{cases}$$

The m -shadow of a connected graph G , denoted by $D_m(G)$, is constructed by taking m copies of G , say $G_1, G_2, \dots, G_m$, then joining each vertex u in G_i to the neighbors of the corresponding vertex v in G_j , where $1 \leq i$ and $j \leq m$ [48]. Let (G_i) be a finite collection of graphs and each G_i has a fixed vertex v called a terminal. The amalgamation $Amal(G_i, v, m)$ is formed by taking all the G_i s and joining them on the terminal.

Lemma 10

[49] Let m and n be two integers with $m \geq 2$ and $n \geq 3$. Let G be a connected graph of order n and v be a terminal vertex of G , then $lmd(Amal(G, v, m)) \leq m \cdot lmd(G)$.

The Table 6 summarize all the known results on some families of graphs.

The following open problems are proposed in the literature.

Open Problem 1

[49] Characterise the local multiset dimension of bicyclic graphs in which the two cycles have at least one vertex in common.

Open Problem 2

[66]

Determine the local multiset dimension of some families of graph namely related wheel related, generalized Petersen graphs, related cycle and others.

Subsequently, wheel related [46] and cycle related [51, 46] cases have been studied, while generalized Petersen graphs remain open.

Open Problem 3

[66]

Determine the local multiset dimension of some graph operation results.

The local multiset dimension of the comb and corona products of tree graphs has been studied in [61] and [62], respectively.

G	$lmd(G)$
Path graph P_n	$1, n \geq 3$ [20]
Path graph with homogeneous pendant edges $P_n \odot mK_1$	$1, n \geq 3$ [50]
$P_n \triangleright C_m, n \geq 2$ and $m \geq 3$	$\begin{cases} 1, & \text{for } m \text{ is even,} \\ n, & \text{for } m \text{ is odd and } m \neq 3 \text{ or for } m = 3 \text{ and } n \text{ is odd, [51]} \\ n+1, & \text{for } m = 3 \text{ and } n \text{ is even.} \end{cases}$
Tree T of order n	1 [49][52]
Complete k -ary tree of height h	1 [20]
$T_1 \triangleright T_2$	1 [51]
$T \triangleright C_m, m \geq 4$	1 for m even [51]
Caterpillar graph $C_{n,m}$	$1, n \geq 3$ and $m \geq 1$ [20]
Caterpillar with adding one edge $C'_{n,m}$	$\infty, n \geq 3$ [53]
Caterpillar graph with homogeneous pendant edges $C_{n,m} \odot mK_1$	$1, n \geq 3$ and $m \geq 2$ [50]
Bipartite graph	1 [49][52]
k -partite graph $K_{n_1, n_2, \dots, n_k}$ with $1 \leq l \leq k$ and $n_l \geq k-1$	$\frac{k(k-1)}{2}$ [20]
Star graph S_n	$1, n \geq 3$ [20]
Star graph with homogeneous pendant edges $S_n \odot mK_1$	$1, n \geq 3$ [50]
Cycle graph C_n with $n \geq 3$	$\begin{cases} 1, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \geq 7 \text{ is odd, [20][46]} \\ \infty, & \text{if } n = 3, 5. \end{cases}$
Cycle graph with two pendant $C_n^2, n \geq 4, 4 \leq i \leq n$	$\begin{cases} 1, & \text{if } m \equiv 0 \pmod{2} \\ 2, & \text{if } m \equiv 1 \pmod{2}. \end{cases}$ [53]
Cycle graph with homogeneous pendant edges $C_n \odot mK_1, n \geq 4$	$\begin{cases} 1, & \text{if } n \text{ is even, [50]} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$
$C_n \triangleright T, n \geq 3$	$\begin{cases} 1, & n \text{ even, [61]} \\ 2, & n \text{ odd.} \end{cases}$
$C_n \triangleright C_m, n, m \geq 3$	$\begin{cases} 1, & \text{both } n \text{ and } m \text{ are even,} \\ 2, & \text{one of } n \text{ and } m \text{ is even and the other is odd, [61]} \\ n, & \text{both } n \text{ and } m \text{ are odd.} \end{cases}$
$T \triangleright C_3, n \geq 2$	$\begin{cases} n, & \text{for } T_1 \neq T_2, [47] \\ n+1, & \text{for } T_1 = T_2. \end{cases}$
Complete graph K_2	1 [20]
Complete graph K_n	$\infty, n \geq 3$ [20]
Complete graph with homogeneous pendant edges $K_n \odot mK_1$	$\infty, n \geq 3$ [50]
Wheel graph $W_n, n \geq 3$	$\begin{cases} 3, & n = 4, 6, \\ \lfloor n/4 \rfloor, & n \geq 8 \text{ and even, [46]} \\ \infty, & \text{otherwise.} \end{cases}$
Pan graph $P'_n, n \geq 4, 4 \leq i \leq n$	$\begin{cases} 1, & \text{if } n \equiv 0 \pmod{2} \\ 2, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$ [53]
Sunlet graph $Sl_n, n \geq 4, 4 \leq i \leq n$	$\begin{cases} 1, & \text{if } m \equiv 0 \pmod{2} \\ 2, & \text{if } m \equiv 1 \pmod{2}. \end{cases}$ [53]
Non-cycle unicyclic graph	$\begin{cases} 1, & \text{if } G \text{ contains an even cycle, [46]} \\ 2, & \text{if } G \text{ contains an odd cycle.} \end{cases}$
Unicyclic graph of order at least 3 and contains a circle of order p	$\begin{cases} 1, & p \text{ even, [51]} \\ 2, & p \text{ odd.} \end{cases}$
Bicyclic graph of order at least 3 and contains cycles of order p_1 and p_2	$\begin{cases} 1, & \text{if } p_1 \text{ and } p_2 \text{ are even, [51]} \\ 2, & \text{otherwise.} \end{cases}$
m -shadowing path graph $D_m(P_n)$	$1, n \geq 3$ [48]
m -shadowing cycle graph $D_m(C_n)$	$\begin{cases} 1, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd, } n \geq 3 \end{cases}$ [48]
m -shadowing star graph $D_m(S_n)$	$1, n \geq 3$ [48]
m -shadowing gear graph $D_m(G_n)$	$1, n \geq 4$ [48]
m -shadowing complete graph $D_m(K_n)$	$\infty, n \geq 3$ [48]
Amalgamation of m paths $Amal(P_n, v, m)$	1 [49]
Amalgamation of m complete graphs $Amal(K_n, v, m)$	∞ [49]
Amalgamation of m wheel graph $Amal(W_n, v, m)$ ($m \geq 2, n \geq 3, v$ center)	$m \cdot lmd(W_n)$ [49]
Amalgamation of m wheel graph $Amal(W_n, v, m)$ (v not center)	$\begin{cases} m \cdot (\frac{n}{4} - 1), & n \equiv 0 \pmod{4}, \\ m \cdot \lfloor n/4 \rfloor, & n \equiv 1, 2, 3 \pmod{4}. \end{cases}$ [49]
Amalgamation of m fan graph $Amal(F_n, v, m)$ ($m \geq 2, n \geq 6, v$ center)	$m \cdot lmd(F_n)$ [49]
Amalgamation of m fan graph $Amal(F_n, v, m)$ ($m \geq 2, n \geq 3, d(v) = 2$)	$m \lfloor n/4 \rfloor$ [49]
Amalgamation of m fan graph $Amal(F_n, v, m)$ ($m \geq 2, n \geq 3, d(v) = 3$)	$m \left(\left\lfloor \frac{p}{4} \right\rfloor + \left\lfloor \frac{n-p}{4} \right\rfloor \right)$ or $m \left(\left\lfloor \frac{p}{4} \right\rfloor + \left\lceil \frac{n-p}{4} \right\rceil \right)$ [49]

Table 6. Local multiset dimension of some graphs

6.3. Local Outer Multiset Dimension

The local version of the outer multiset dimension was introduced in [46].

Definition 5

[46] If for every pair of distinct vertices $u, v \in V(G) \setminus W$, $m(u | W) \neq m(v | W)$, then W is called a *local outer multiset resolving set* for G . This set with the smallest cardinality is a *local outer multiset basis*, and its cardinality is the *local outer multiset dimension* of G , denoted by $ldim_{ms}(G)$.

Since every outer multiset basis of a graph induces its local outer multiset basis, then $ldim_{ms}(G) \leq \dim_{ms}(G)$. Additionally, since every local multiset basis of a graph induces its local outer multiset basis, and every local outer multiset basis of a graph induces its local basis, then $ldim(G) \leq ldim_{ms}(G) \leq lmd(G)$ [46].

The paper [46] established fundamental results on the local outer multiset dimension, including key properties, parameter relations, structural bounds, and results for several graph families such as amalgamations of complete graphs, non-cycle unicyclic graphs, cycles, and wheel graphs.

G	$ldim_{ms}(G)$
Amalgamation of complete graphs $Amal(K_{n_i}, m)$	$\begin{cases} 1, & m_3 = 0, m_{\geq 4} = 0, \\ 2, & m_3 = 1, m_{\geq 4} = 0, \\ \sum_{n_i \geq 3} (n_i - 2), & \text{otherwise.} \end{cases}$ [46]
Bipartite graph	1 [46]
Non-cycle unicyclic graph	$\begin{cases} 1, & \text{if } G \text{ contains an even cycle,} \\ 2, & \text{if } G \text{ contains an odd cycle.} \end{cases}$ [46]
Cycle graph $C_n, n \geq 3$	$\begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$ [46]
Complete graph $K_n, n \geq 2$	$n - 1$ [20]
Wheel graph $W_n, n \geq 3$	$\begin{cases} 3, & n = 3, 4, 6, \\ \lceil n/4 \rceil, & n \geq 8 \text{ even or } n \equiv 1 \pmod{4}, \\ \lceil n/4 \rceil + 1, & \text{otherwise.} \end{cases}$ [46]

Table 7. Local outer multiset dimension of some graphs

The following open problems have been proposed in the literature.

Open Problem 4

[46]

For $1 \leq i \leq m$ and $m_i \geq 1$, find a “good” upper bound for $lmd(G \odot K_{m_i})$ and $ldim_{ms}(G \odot K_{m_i})$.

Open Problem 5

[46]

For any integer $c \geq 3$, is there a graph G with chromatic number $\chi(G) = c$ such that $lmd(G) = g(d, c)$? $ldim_{ms}(G) = g(d, c)$?

Open Problem 6

[46]

Let G and H be two graphs.

1. Determine $lmd(G + H)$ as a function of $lmd(G)$ and $lmd(H)$.
2. Determine $ldim_{ms}(G + H)$ as a function of $ldim_{ms}(G)$ and $ldim_{ms}(H)$.

6.4. Edge Multiset Dimension

The concept of the edge multiset dimension of graphs was introduced in [5].

Definition 6**[5]**

Let G be a graph and let $B = \{b_1, b_2, \dots, b_j\}$ be a set of vertices. For an edge $e = xy$, the *edge-multiset representation* of e with respect to B is the multiset

$$r_{em}(e \mid B) = \{d_G(e, b_1), d_G(e, b_2), \dots, d_G(e, b_j)\}.$$

If for any two distinct edges e and f of G we have $r_{em}(e \mid B) \neq r_{em}(f \mid B)$, then B is called an *edge-multiset resolving set* for G . If G admits an edge-multiset resolving set, then a smallest such set is called an *edge-multiset basis* of G . The number of elements in an edge-multiset basis is called the edge-multiset dimension of G , denoted by $md_e(G)$. If G does not contain any edge-multiset resolving set, we define $md_e(G) = \infty$.

In [5], it was shown that the edge multiset dimension of a connected graph G is at least the edge metric dimension of graph G , that is, $\text{edim}(G) \leq md_e(G)$. Moreover, no connected graph has edge multiset dimension equal to 2, i.e., $md_e(G) \neq 2$. In particular, for any connected graph other than a path, $md_e(G) \geq 3$. Several exact values of this parameter for basic graph classes were also determined, see Table 8.

G	$md_e(G)$
Path graph P_n	1 [5]
Complete graph	∞ [5]
Star graph	∞ [5]
Friendship graph	∞ [5]
Wheel graph	∞ [5]
The Peterson graph	∞ [5]
Fan graph	∞ [5]
Complete bipartite graph	∞ [5]
Cycle graph with at most 6 vertices	∞ [5]
Cycle graph $C_n, n \geq 7$	3 [5]
Kayak paddle graph $KP(\vartheta, \lambda, \mu)$, with $\vartheta, \lambda, \mu \geq 4$	3 [5]
Dragon graph $T_{n,m}$ with $n \geq 4$ and $m \geq 3$	3 [5]
The comb product of P_n and P_m with $n, m \geq 4$	3 [5]
Caterpillar graph $CT_n, n \geq 5$	3 [5]
Lobster graph $L_n, n \geq 5$	3 [5]
Caterpillar graph CT_n with $n \geq 5$, $\deg(x_i) \leq 4$ where $3 \leq i \leq n-2$ and $D = \{i \mid 3 \leq i \leq n-2\}$	$3 + D $ [5]
Lobster graph L_n with $n > 5$, $\deg(x_i) = 4$, where $i \in D = \{i \mid 3 \leq i \leq n-2\}$	$3 + D $ [5]

Table 8. Edge-multiset dimension of some graphs

Several structural conditions have been identified that ensure a graph has infinite edge multiset dimension:

Lemma 11

[5] Let G be a graph with a set of vertices and edges, V and E , respectively, where $|V| \geq 2$. If the distance between a vertex and an edge is at most 2, then G does not contain an edge-multiset resolving set.

Theorem 24

[5] If any graph G has a vertex adjacent to at least three pendant vertices, then $md_e(G) = \infty$.

The following families of graphs have infinite edge multiset dimensions. Complete graph, star graph, friendship graph, wheel graph, the Peterson graph, fan graph, complete bipartite graph and cycle graph with at most 6 vertices **[5]**.

Some bounds on the edge multiset dimension of trees:

Lemma 12

[5] Let T_n be a tree with order n and diameter 3. If $md_e(T_n) \neq \infty$, then $md_e(T_n) \leq n-2$.

Theorem 25

[5] Let T_n be a tree with order n and T_n as not a path. If $md_e(T_n) \neq \infty$, then $3 \leq md_e(T_n) \leq n-2$.

7. Existence, Optimization, and Computational Perspectives

A distinctive feature of multiset resolvability is that the corresponding dimension may be infinite. This differs fundamentally from the classical metric dimension, which is finite for every connected graph. Consequently, the classical metric-dimension problem is purely an optimization problem, whereas a multiset-resolvability problem contains two computational stages:

1. determine whether the graph admits a multiset resolving set; and
2. if such a set exists, determine its minimum cardinality.

An algorithm for multiset dimension must therefore be able either to return an optimal resolving set or to certify that no such set exists.

This distinction applies to most of multiset variants. However, outer multiset parameters avoid the feasibility issue because an outer multiset resolving set always exists. For outer parameters, the problem is therefore entirely on optimization, although finding a minimum solution may still be computationally difficult.

The computational complexity of classical metric dimension is well established. Khuller et al. [58] proved that computing the metric dimension of an arbitrary graph is NP-hard, and the associated decision problem is NP-complete. For multiset dimension, the decision problem of deciding if a graph G contains a multiset resolving set of size k is NP-complete [57].

Given a proposed landmark set S , its validity can be verified in polynomial time. However, the bounded optimization problem may remain difficult even on a graph class for which finiteness has a simple structural characterization. Conversely, an algorithm that finds small resolving sets does not necessarily provide a useful certificate that the dimension is infinite when no such set exists.

A direct exact algorithm may enumerate all $2^{|V(G)|}$ possible landmark sets. These rapidly growing search spaces motivate the development of branch-and-bound algorithms, integer-programming and SAT formulations, and symmetry-aware search methods. In particular, automorphism orbits could be used to avoid examining landmark sets or partitions that are equivalent under graph symmetry.

Parameterized complexity provides another natural direction. Relevant parameters include the required solution size, diameter, maximum degree, treewidth, cyclomatic number, and number of twin classes. It would be useful to determine whether multiset dimension and its variants are fixed-parameter tractable with respect to combinations of these parameters, or whether they remain parameterized-intractable even on structurally restricted graph classes.

8. New Directions in Multiset-Based Metric Dimensions

8.1. A Unified Framework for Multiset Resolvability

The preceding discussion shows that the existing multiset parameters are not isolated definitions. They can be described within a unified framework based on four components:

1. the objects that are to be identified;
2. the observers used to produce distance information;
3. the form in which the observations are retained; and
4. the pairs of objects that are required to be distinguished.

This framework clarifies the relationships among the existing parameters and provides a systematic way to formulate new variants.

While several multiset versions of metric dimension variants and their relationships have been established, multiset analogues of the partition, k -metric, strong, mixed, and fractional variants remain unexplored.

Some of these variants may have additional extensions, whose corresponding multiset versions remain unexplored. The following hierarchy illustrates some examples:

- | Partition Dimension
 - |– Strong Partition Dimension
 - |– Local Partition Dimension
- | Fractional Metric Dimension
 - |– Fractional Strong Metric Dimension
 - |– Fractional k -Metric Dimension

Following this framework, it is possible to introduce several multiset variants, for example the *local edge multiset dimension*. However, in the absence of clear real-world applications or substantial theoretical contributions, the introduction and study of new parameters should be approached with care.

The distinction between ordered and multiset representations is currently all-or-nothing. In practice, landmarks may be distinguishable by type, region, or sensor class, but not individually. Determine the minimum number of landmarks required to tolerate missing, corrupted, or imprecise distance measurements is another possible direction, for example, one might define a k -multiset dimension of G , where the vertex representation difference, i.e. the set difference, exceeds k . Another multiset variant we would like to introduce is in the following section.

8.2. Multiset Partition Dimension

Based on the partition dimension of a graph [11], we introduce the following parameter.

Let $G = (V, E)$ be a connected graph and let $\Pi = \{P_1, P_2, \dots, P_k\}$ be an ordered partition of $V(G)$. For a vertex $v \in V(G)$, the *multiset representation* of v with respect to Π is defined as

$$r_m(v \mid \Pi) = \{d(v, P_1), d(v, P_2), \dots, d(v, P_k)\},$$

where $d(v, P_i) = \min\{d(v, x) : x \in P_i\}$. The values are taken as a multiset, allowing repetition but ignoring order.

The partition Π is called a *multiset resolving partition* if for every pair of distinct vertices $u, v \in V(G)$,

$$r_m(u \mid \Pi) \neq r_m(v \mid \Pi).$$

The *multiset partition dimension* of G , denoted by $\text{mpd}(G)$, is the minimum cardinality of such a partition.

Apparently, this parameter behaves quite differently. In [54], a graph with $md = 3$ with the largest possible number of vertices have been given, however, the mpd here is much larger.

Vertex	Multiset partition representation
1	$\{0, 2, 3, 3, 3, 3\}$
2	$\{0, 1, 2, 2, 2, 2\}$
3	$\{0, 1, 2, 2, 3, 3\}$
4	$\{0, 1, 1, 1, 3, 3\}$
5	$\{0, 1, 1, 2, 2, 2\}$
6	$\{0, 1, 1, 1, 1, 2\}$
7	$\{0, 1, 2, 2, 2, 3\}$
8	$\{0, 1, 1, 2, 2, 3\}$
9	$\{0, 1, 1, 3, 3, 3\}$
10	$\{0, 2, 2, 2, 3, 3\}$
11	$\{0, 1, 1, 1, 2, 2\}$
12	$\{0, 1, 1, 2, 3, 3\}$

Table 9. Multiset partition representations of the vertices with respect to the six-part partition shown in Figure 2.

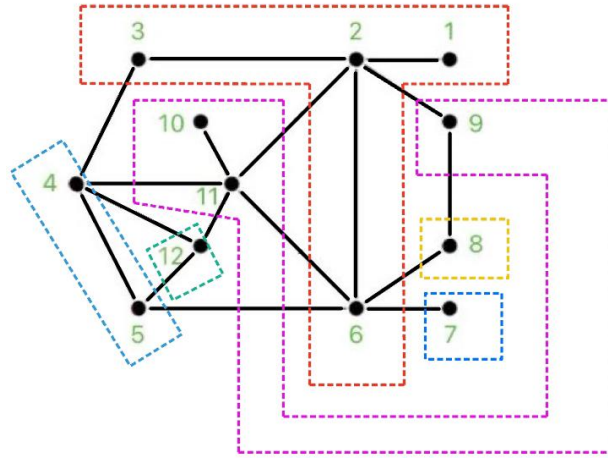


Figure 2. The graph has $\text{mpd} \leq 6$, with partitions illustrated in the figure.

Since all twelve vertices have distinct multiset partition representations with respect to the six-part partition shown in Figure 2, the partition is a multiset resolving partition. Hence, $\text{mpd}(G) \leq 6$.

It is natural to investigate how the multiset partition dimension can be bounded in terms of the multiset dimension and other metric type parameters. In particular, establishing tight upper and lower bounds is essential for understanding its behavior and for identifying extremal graph classes. It is also interesting to examine whether the multiset partition dimension is a characteristic parameter, in the sense that it captures intrinsic structural properties of graphs or distinguishes between non-isomorphic graphs in a meaningful way.

8.3. Metric Dimension Versus Partition Dimension

Metric dimension and partition dimension differ in the form of their landmarks. In metric dimension, each landmark is an individual vertex, whereas in partition dimension, each landmark is a nonempty subset of vertices.

In the classical ordered setting, metric dimension and partition dimension satisfy

$$pd(G) \leq \dim(G) + 1$$

for every nontrivial connected graph G . Indeed, if $W = \{w_1, \dots, w_k\}$ is a metric basis, then

$$\Pi_W = (\{w_1\}, \dots, \{w_k\}, V(G) \setminus W)$$

is a resolving partition. This bound is sharp for several fundamental families. For example,

$$\dim(P_n) = 1, \quad pd(P_n) = 2,$$

$$\dim(C_n) = 2, \quad pd(C_n) = 3,$$

and

$$\dim(K_n) = n - 1, \quad pd(K_n) = n.$$

However, partition dimension may also be substantially smaller than metric dimension. For example,

$$\dim(K_{r,r}) = 2r - 2,$$

whereas

$$pd(K_{r,r}) \leq r + 1.$$

Consequently, the difference $\dim(G) - pd(G)$ can grow with the order of the graph. Partition classes can therefore be more efficient than individual landmarks because one class may encode the positions of many vertices through its distance-minimum function.

The relationship changes substantially when order is discarded. The multiset partition representation is

$$r_{mp}(v \mid \Pi) = \{\{d(v, P_1), d(v, P_2), \dots, d(v, P_t)\}\}.$$

Multiset partition dimension therefore combines two information-reducing operations:

1. distances to individual vertices in each part are replaced by their minimum; and
2. the association between a distance and the part producing it is removed.

Every resolving multiset partition is also a classical resolving partition. Indeed, if two ordered partition representations were equal, then their corresponding multisets would also be equal. It follows that

$$pd(G) \leq mpd(G)$$

whenever $mpd(G)$ is finite. Together with the corresponding relation for metric dimension, we obtain

$$pd(G) \leq \min\{\dim(G) + 1, mpd(G)\}.$$

The classical construction from a resolving set does not automatically extend to the multiset setting. Appending the distance to $V(G) \setminus W$ may cause two previously distinct distance multisets to become equal. Thus, a multiset resolving set does not necessarily generate a resolving multiset partition.

The difference between $md(G)$ and $mpd(G)$ can already be observed in basic graph classes. For $n \geq 5$,

$$md(P_n) = 1, \quad mpd(P_n) = 4,$$

while for sufficiently large cycles,

$$md(C_n) = 3, \quad mpd(C_n) = 4.$$

We would like to propose a few open problems here.

Open Problem 7

Characterize the graphs satisfying

$$mpd(G) = pd(G).$$

More generally,

Open Problem 8

Determine the maximum finite value of

$$mpd(G) - pd(G)$$

among connected graphs of order n .

Open Problem 9

For a fixed integer k , determine whether the finite values of $mpd(G)$ are bounded over all graphs satisfying

$$md(G) = k.$$

Open Problem 10

Determine whether

$$mpd(G) < \infty \implies md(G) < \infty.$$

This would help to clarify all possible finiteness relationships between the two parameters.

Open Problem 11

Characterize the graphs for which a multiset basis $W = \{w_1, \dots, w_k\}$ can be extended to a resolving multiset partition of the form

$$\{\{w_1\}, \dots, \{w_k\}, V(G) \setminus W\}.$$

Identify structural conditions under which an inequality analogous to

$$\text{mpd}(G) \leq \text{md}(G) + 1$$

holds.

9. Discussion and Future Work

9.1. Information Loss from Ordered to Multiset Representations

The fundamental difference between classical metric dimension and multiset dimension is not simply a change in notation. It represents a change in the information available for identifying vertices. In the multiset setting, the correspondence between distances and landmarks is removed. Thus, two ordered representations are identified

The reduction in identification capacity can be seen through a simple counting argument. With k landmarks and diameter D , there are at most $(D+1)^k$ ordered distance vectors. By contrast, the number of multisets of cardinality k with elements from $\{0, 1, \dots, D\}$ is at most

$$\binom{D+k}{k}.$$

Although not all of these representations need be realisable in a particular graph, the comparison illustrates the substantial reduction caused by removing landmark identities.

Metric dimension versus multiset dimension. Every multiset resolving set is also a classical resolving set, since two different multisets necessarily correspond to different ordered vectors. Therefore, whenever $\text{md}(G)$ is finite,

$$\dim(G) \leq \text{md}(G).$$

The reverse does not hold. Every connected graph has finite metric dimension, whereas many connected graphs have infinite multiset dimension. For example,

$$\dim(K_n) = n - 1, \quad \text{md}(K_n) = \infty \quad (n \geq 3).$$

Similarly,

$$\dim(C_n) = 2$$

for every $n \geq 3$, whereas

$$\text{md}(C_n) = \begin{cases} \infty, & 3 \leq n \leq 5, \\ 3, & n \geq 6. \end{cases}$$

For grid graphs,

$$\dim(P_r \square P_s) = 2, \quad \text{md}(P_r \square P_s) = 3$$

for $r, s \geq 2$ and $(r, s) \neq (2, 2)$. In contrast, paths experience no increase:

$$\dim(P_n) = \text{md}(P_n) = 1.$$

These examples exhibit three different outcomes of removing order information: no change, a finite increase, and a complete loss of finite resolvability.

Another important distinction concerns monotonicity. If W is a classical resolving set, then every superset of W is also resolving, because adding a landmark appends another coordinate without destroying any existing difference. This property does not generally hold for multiset resolving sets. Adding a new landmark inserts a potentially different distance into each representation, and two previously different multisets may become equal. This lack of monotonicity makes both structural arguments and algorithmic searches more difficult.

Local metric dimension versus local multiset dimension. The same information loss occurs in the local setting, although only adjacent vertices are required to be distinguished. Since every local multiset resolving set is also a local metric resolving set,

$$ldim(G) \leq lmd(G) \leq md(G).$$

The map induced by a local multiset resolving set may be viewed as a proper vertex colouring. This interpretation connects local multiset dimension with both distance-based identification and graph colouring.

Some graph families show that the local condition can reduce, but does not eliminate, the effect of losing order. For every connected bipartite graph,

$$ldim(G) = lmd(G) = 1.$$

In particular, even cycles experience no information-loss. For odd cycles,

$$ldim(C_n) = 2,$$

whereas

$$lmd(C_n) = \begin{cases} \infty, & n = 3, 5, \\ 3, & n \geq 7 \text{ and } n \text{ is odd.} \end{cases}$$

For complete graphs, all distinct vertices are adjacent, and therefore the local problem coincides with the global problem:

$$ldim(K_n) = n - 1, \quad lmd(K_n) = \infty \quad (n \geq 3).$$

Thus, restricting the pairs that must be distinguished does not necessarily restore finite resolvability.

Outer and local outer multiset dimensions. In the ordered setting, requiring only vertices outside the resolving set to be distinguished does not produce a genuinely different parameter. The vertices belonging to W are automatically distinguished by the positions of the zero entries in their ordered representations. Moreover, a vertex in W is automatically distinguished from a vertex outside W . Therefore, the ordered outer metric dimension coincides with the ordinary metric dimension.

This equivalence fails after order information is removed. Although a vertex in W has a zero in its multiset representation, the multiset does not identify which landmark produced that zero. Consequently, two vertices of W may have the same multiset representation. By excluding these vertices from the identification requirement, outer multiset dimension becomes an relaxation of multiset dimension. The known relationships are

$$\dim(G) \leq \dim_{ms}(G) \leq md(G),$$

whenever $md(G)$ is finite. Unlike ordinary multiset dimension, outer multiset dimension is always finite and satisfies

$$\dim_{ms}(G) \leq n - 1.$$

It therefore provides a useful alternative for graphs whose ordinary multiset dimension is infinite.

The same distinction occurs between local metric dimension and local outer multiset dimension. In the ordered setting, the local outer condition again reduces to local metric dimension. In the multiset setting, however, it is a separate parameter satisfying

$$ldim(G) \leq ldim_{ms}(G) \leq lmd(G)$$

and

$$ldim_{ms}(G) \leq \dim_{ms}(G).$$

For an odd cycle C_n , the local metric and local outer multiset dimensions are both equal to 2, whereas the local multiset dimension is 3 for odd $n \geq 7$ and is infinite for $n = 3, 5$. Excluding the landmarks themselves from the required comparisons can therefore compensate for part of the information lost by removing order.

Edge metric dimension versus edge multiset dimension. For edge identification, as in the vertex case, every edge-multiset resolving set is an edge resolving set. Hence,

$$\text{edim}(G) \leq md_e(G)$$

whenever $md_e(G)$ is finite. Edge metric dimension is finite for every connected graph, but edge multiset dimension may be infinite. For example,

$$\text{edim}(C_n) = 2,$$

while

$$md_e(C_n) = \begin{cases} \infty, & 3 \leq n \leq 6, \\ 3, & n \geq 7. \end{cases}$$

Paths again incur no penalty:

$$\text{edim}(P_n) = md_e(P_n) = 1.$$

The edge version therefore displays the same three regimes as the vertex version: equality, a finite increase, and infinite multiset dimension.

The principal relationships discussed above are summarized in Table 10.

To clarify the relationships among these different dimension concepts, we provide a brief overview in the following framework:

Category	Main Idea	Classical Version	Multiset Analogue	Relationship
Basic Concept	Classical version uses ordered distance tuples, while the multiset version uses distance multiplicities.	Metric Dimension $\dim(G)$	Multiset Dimension $md(G)$	$\dim(G) \leq md(G)$ [30]
Edge Variant	Distinguish every pair of edges using vertex-to-edge distances.	Edge Metric Dimension $\text{edim}(G)$	Edge-Multiset Dimension $md_e(G)$	$\text{edim}(G) \leq md_e(G)$ [5]
Local Variant	Only adjacent pairs of vertices are required to be resolved.	Local Metric Dimension $ldim(G)$	Local Multiset Dimension $lmd(G)$	$ldim(G) \leq lmd(G) \leq md(G)$ [20] $ldim(G) \leq \dim(G)$ [46]
Outer Variant	Distinguishing vertices outside the resolving set.	Outer Dimension Equivalent to $\dim(G)$	Outer Multiset Dimension $\dim_{ms}(G)$	$\dim(G) \leq \dim_{ms}(G) \leq md(G)$ [46]
Local Outer Variant	Distinguishing vertices outside the resolving set.	Local Outer Dimension Equivalent to $ldim(G)$	Local Outer Multiset Dimension $ldim_{ms}(G)$	$ldim_{ms}(G) \leq \dim_{ms}(G)$ [46] $ldim(G) \leq ldim_{ms}(G) \leq lmd(G) \leq md(G)$ [46]

Table 10. Known Multiset Versions of Metric Variations

In general, the transition from ordered distance representations to multiset representations results in the loss of information about the identities of the landmarks.

These comparisons suggest that the effect of removing order should itself be studied as a graph invariant. Whenever both parameters are finite, one may define the corresponding *order-loss gap* Δ

$$\begin{aligned} \Delta_V(G) &= md(G) - \dim(G), \\ \Delta_L(G) &= lmd(G) - ldim(G), \\ \Delta_O(G) &= \dim_{ms}(G) - \dim(G), \\ \Delta_{LO}(G) &= ldim_{ms}(G) - ldim(G), \\ \Delta_E(G) &= md_e(G) - \text{edim}(G), \\ \Delta_P(G) &= mpd(G) - pd(G). \end{aligned}$$

A systematic research programme could seek to characterize graphs for which these penalties are zero, bounded, arbitrarily large, or undefined because the corresponding multiset parameter is infinite. This would shift the emphasis from computing individual parameter values to understanding which structural properties make landmark or partition identities essential.

Open Problem 12

Study the bounds of $\Delta_V(G)$, $\Delta_L(G)$, $\Delta_O(G)$, $\Delta_{LO}(G)$, $\Delta_E(G)$ and $\Delta_P(G)$.

Although several results are known for particular graph products and operations, most concern specific choices of the constituent graphs. A general theory explaining how multiset resolvability behaves under graph operations is still lacking. In particular, an operation may increase the dimension, preserve it, or destroy multiset resolvability altogether by introducing new symmetries or repeated distance distributions. This motivates the following problem.

Open Problem 13

For each of the multiset resolvability parameters

$$md(G), \quad lmd(G), \quad \dim_{ms}(G), \quad ldim_{ms}(G), \quad md_e(G), \quad \text{and} \quad mpd(G),$$

study its behaviour under standard graph operations, including Cartesian, strong, direct, lexicographic, comb, and corona products, as well as joins, amalgamations, and edge subdivisions. In particular:

1. characterize when these operations preserve finite multiset resolvability;
2. determine when an operation forces the corresponding parameter to become infinite; and
3. establish sharp upper and lower bounds for the resulting parameter in terms of the parameters and structural invariants of the constituent graphs.

9.2. Breaking Symmetry: Pointwise and Setwise Stabilizers

The loss of landmark identities also changes the type of graph symmetry that must be broken. Metric dimension has a well-established connection with automorphism groups and permutation-group bases [55]. The multiset setting strengthens this connection because the landmarks themselves are no longer individually labelled.

Let

$$\Gamma = \text{Aut}(G)$$

be the automorphism group of G . For a set $W \subseteq V(G)$, define its pointwise stabilizer by

$$\Gamma_{(W)} = \{\varphi \in \Gamma : \varphi(w) = w \text{ for every } w \in W\},$$

and its setwise stabilizer by

$$\Gamma_{\{W\}} = \{\varphi \in \Gamma : \varphi(W) = W\}.$$

Clearly,

$$\Gamma_{(W)} \leq \Gamma_{\{W\}},$$

but the two stabilizers need not be equal. An automorphism in $\Gamma_{\{W\}}$ may permute the vertices of W without fixing them individually.

Suppose that $W = \{w_1, \dots, w_k\}$ is a classical resolving set and that $\varphi \in \Gamma_{(W)}$. Since automorphisms preserve distances,

$$d(\varphi(v), w_i) = d(\varphi(v), \varphi(w_i)) = d(v, w_i)$$

for every $v \in V(G)$ and every $w_i \in W$. It follows that

$$r(\varphi(v) \mid W) = r(v \mid W).$$

Since W is resolving, $\varphi(v) = v$ for every $v \in V(G)$, and hence φ is the identity automorphism. We therefore have the following necessary condition.

Proposition 1

If W is a resolving set of G , then

$$\Gamma_{(W)} = \{\text{id}\}.$$

A set with trivial pointwise stabilizer is commonly called a *determining set* of the graph, or a *base* for the action of $\text{Aut}(G)$ on $V(G)$. If $b(G)$ denotes the minimum cardinality of such a set, then

$$b(G) \leq \dim(G).$$

The converse does not hold in general. A determining set removes all automorphisms, but two vertices that are not related by an automorphism may still have the same distance representation. Thus, symmetry breaking is necessary, but not sufficient, for metric resolvability.

The situation is stronger for multiset dimension. Let W be a multiset resolving set and let $\varphi \in \Gamma_{\{W\}}$. For every $v \in V(G)$ and every nonnegative integer i ,

$$\begin{aligned} |\Gamma_i(\varphi(v)) \cap W| &= |\varphi(\Gamma_i(v)) \cap \varphi(W)| \\ &= |\Gamma_i(v) \cap W|. \end{aligned}$$

Therefore,

$$h_W(\varphi(v)) = h_W(v),$$

or equivalently,

$$r_m(\varphi(v) \mid W) = r_m(v \mid W).$$

Since W is a multiset resolving set, it follows that $\varphi(v) = v$ for every vertex v . This gives the following stronger condition.

Proposition 2

If W is a multiset resolving set of G , then

$$\Gamma_{\{W\}} = \{\text{id}\}.$$

Hence, a multiset resolving set must break every automorphism that preserves the landmark set as an unlabelled set, including automorphisms that merely permute its elements. This is precisely the additional symmetry-breaking requirement created by ignoring order information.

Let $\rho(G)$ denote the minimum cardinality of a subset $W \subseteq V(G)$ whose setwise stabilizer is trivial, and let $\rho(G) = \infty$ if no such subset exists. Propositions 1 and 2 imply that, whenever $md(G)$ is finite,

$$b(G) \leq \rho(G) \leq md(G)$$

and

$$\dim(G) \leq md(G).$$

Consequently,

$$\max\{\dim(G), \rho(G)\} \leq md(G).$$

In particular, finite multiset dimension implies that the distinguishing number of G is at most 2.

The condition $\rho(G) < \infty$ is not sufficient for finite multiset dimension. Triviality of the setwise stabilizer only rules out equal representations caused by automorphisms. Two vertices belonging to different automorphism orbits may still have identical distance histograms. Thus, two distinct obstructions must be considered:

1. a *group-theoretic obstruction*, caused by automorphisms that preserve the landmark set; and
2. a *distance-distribution obstruction*, caused by vertices with identical distance histograms even though no automorphism maps one to the other.

This distinction is important because it shows that multiset dimension is not determined solely by the size of the automorphism group.

The symmetry requirement becomes weaker when only selected pairs of vertices must be distinguished. Let W be a local multiset resolving set and let $\varphi \in \Gamma_{\{W\}}$. As before,

$$h_W(\varphi(v)) = h_W(v).$$

If v and $\varphi(v)$ were adjacent and distinct, this would contradict the local resolving condition. Therefore,

$$v\varphi(v) \notin E(G)$$

whenever $\varphi(v) \neq v$. More generally, every orbit of $\Gamma_{\{W\}}$ on $V(G)$ must be an independent set. Thus, a local multiset resolving set need not eliminate the entire setwise stabilizer; it only needs to ensure that no surviving orbit contains an adjacent pair.

Similarly, if W is an outer multiset resolving set, then every $\varphi \in \Gamma_{\{W\}}$ must fix each vertex of $V(G) \setminus W$. However, φ may still permute vertices within W . For a local outer multiset resolving set, the orbits of $\Gamma_{\{W\}}$ contained in $V(G) \setminus W$ must be independent sets, while nontrivial action inside W is permitted. These observations explain why the outer and local variants may remain finite even when the ordinary multiset dimension is infinite.

If W is an edge-multiset resolving set and $\varphi \in \Gamma_{\{W\}}$, then

$$r_{em}(\varphi(e) \mid W) = r_{em}(e \mid W)$$

for every edge $e \in E(G)$. Consequently, every edge is fixed by φ . Whenever the action of $\text{Aut}(G)$ on $E(G)$ is faithful, this implies that φ is the identity. Thus, edge multiset dimension also imposes a setwise symmetry-breaking requirement, but the resulting action is considered on the edge set rather than directly on the vertex set.

For a local edge-multiset resolving set, a nontrivial orbit of edges may remain, but it cannot contain two incident edges. Equivalently, every orbit of the surviving setwise stabilizer must be an independent set in the line graph $L(G)$. This gives a natural group-theoretic interpretation of the proposed local edge-multiset dimension.

The symmetry conditions for the different parameters are summarised in Table 11.

Parameter	Necessary symmetry condition
Metric dimension	The pointwise stabilizer of W is trivial.
Multiset dimension	The setwise stabilizer of W is trivial.
Local multiset dimension	Every orbit of the setwise stabilizer of W is an independent set.
Outer multiset dimension	The setwise stabilizer of W fixes $V(G) \setminus W$ pointwise.
Local outer multiset dimension	Every orbit outside W is an independent set.
Edge multiset dimension	The setwise stabilizer of W fixes every edge.
Local edge-multiset dimension	Every edge orbit is an independent set in $L(G)$.

Table 11. Symmetry-breaking requirements for multiset parameters.

These observations suggest several directions for future work. First,

Open Problem 14

Determine which graphs with distinguishing number at most 2 have finite multiset dimension

This would separate the group-theoretic obstruction from the additional obstruction created by identical distance histograms. Second,

Open Problem 15

Study the relationships among

$$b(G), \quad \rho(G), \quad \dim(G), \quad \text{and} \quad md(G)$$

Finally, highly symmetric graph classes, including vertex-transitive graphs, Cayley graphs, distance-regular graphs, and graph products, provide natural settings in which to determine when symmetry is the principal obstruction to multiset resolvability and when additional distance-distribution obstructions occur.

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