

Representation and convergence of multivalued Aumann-Pettis integrable regular martingales

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Abstract We study multivalued Aumann–Pettis integrable regular martingales. First, we establish a representation theorem for the conditional expectation of Aumann–Pettis integrable random sets. Then, we derive a representation theorem for Aumann–Pettis integrable regular martingales. Finally, using these representation results, we establish the convergence of these martingales in the Mosco sense and in the linear topology.

Keywords Random sets, Pettis integration, Pettis conditional expectation, Set-valued martingale, Mosco and linear topologies

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1. Introduction

The concept of multivalued martingales is a generalization of real and vector-valued martingales. Here, the values of the random set are nonempty closed convex, bounded closed convex, or weakly compact convex subsets of a Banach space. Several papers on the convergence properties of multivalued martingales have been published in recent years. Among those are [14] by Hiai and Umegaki, [15] by Luu, [6] by Choukairi-Dini, [13] by Hess, and [9] by El Harami. Hiai and Umegaki [14] introduced the concept of conditional expectation of Bochner integrable (integrably bounded) random sets, which plays a major role in dealing with multivalued martingales. They also proved a generalization of Chatterji's theorem (that every uniformly integrable martingale in a Banach space with (RNP) property is regular) to the Bochner integrable random sets.

Regarding the Pettis integral framework and in comparison to the Bochner integral framework, the conditional expectation of random sets may not exist without additional assumptions (see Egghe [7]). This problem has been considered by Faik [11] and Ziat [22] when the Banach space has the Radon-Nikodym Property (RNP) and possesses a strongly separable dual. Later, El Harami and Ezzaki [10] dealt with the same problem in the setting of weakly compact and convex integrable Pettis random sets. The same problem was studied in [1] for random sets that have bounded and closed values in Banach spaces without the Radon-Nikodym Property (RNP) . In recent years, El Harami [9] investigated the conditional expectations of random sets that have convex and closed values in Banach spaces without the Radon-Nikodym Property (RNP) and Oulghazi and Ezzaki [19] studied the convergence of conditional expectations for sequences of Pettis integrable random sets and established a version of Mosco convergence in this setting. In this regard, the conditional expectation turns out to be an important element in characterizing and studying Pettis integrable random processes. It is necessary to obtain representation theorems and results concerning the convergence of multivalued Aumann–Pettis integrable martingales. The existence of martingale selections, their regularity, and their convergence are important concerns in this context.

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Our contribution in this article is to establish a representation theorem for the conditional expectation of Aumann–Pettis integrable random sets. More precisely, for an Aumann–Pettis integrable random set with bounded closed convex values, its conditional expectation can be written as the norm closure of a sequence of conditional expectations of its Pettis integrable selections. As a consequence, we prove that any Aumann–Pettis integrable regular martingale with bounded, closed, and convex values can be represented as the norm closure of a sequence of its Pettis integrable martingale selections. Additionally, some convergence properties in the Mosco sense and in the linear topology are also provided. The results obtained in this paper extend corresponding representation and convergence results established in the Bochner case; see, in particular, Lemma 1.1 and Theorem 6.1 of Hiai and Umegaki [14], Corollary 3.3 and Theorem 5.12 of Hess [13], Proposition 2.7 and Theorem 2.14 of Choukairi-Dini [6], and Theorem 3.2 of Papageorgiou [20]. In this paper, we consider Aumann–Pettis integrable random sets. Since Pettis integrability is weaker than Bochner integrability, it requires different arguments concerning Pettis-integrable selections, conditional expectations, and convergence. For example, in the Bochner case, every measurable selection of an integrably bounded random set is Bochner integrable, while this is not necessarily true in the Pettis case. Therefore, we construct selections via an explicit truncation procedure (see the proof of Theorem 3.1). In addition, Section 2.1 gives an example, based on the construction of Egghe [7], of a set-valued Aumann–Pettis integrable regular martingale that is not Bochner integrable. This example shows that the present framework is genuinely broader than the Bochner setting.

This article is structured as follows. The preliminaries and some basic assumptions are presented in Section 2. The representation theorem for the conditional expectation of Aumann–Pettis integrable random sets is established in Section 3. As a consequence, a representation theorem (Theorem 4.1) and a convergence theorem (Theorem 4.2) for multivalued Aumann–Pettis integrable regular martingales with bounded, closed, and convex values are established in Section 4. Their counterparts for such martingales with weakly compact convex values are given in Corollary 4.1 (representation) and Corollary 4.2 (convergence).

2. Notation and preliminary results

Throughout this work, we consider a complete probability space $(\Omega, \mathcal{Q}, P)$, and an increasing filtration $(\mathcal{Q}_n)_{n \geq 1}$ of sub- σ -algebras of $\mathcal{Q}$ such that $\mathcal{Q}_\infty$ is generated by the union $\bigcup_{n \geq 1} \mathcal{Q}_n$.

Let Y be a separable Banach space, Y' its topological dual, and $B_{Y'}$ the closed unit ball in Y' . We denote by $D \subset Y$ a countable dense subset of Y and by D' (resp. D'_M) a countable dense subset of Y' in strong topology (resp. with respect to the Mackey topology), we denote by $D'(B_{Y'})$ (resp. $D'_M(B_{Y'})$) a countable dense subset of $B_{Y'}$ in strong topology (resp. with respect to the Mackey topology).

Let $N_c(Y)$ be the set of all nonempty closed subsets of Y . We define the following families:

- $N_{cc}(Y)$: set of all nonempty closed convex subsets of Y .
- $N_{bcc}(Y)$: set of all nonempty bounded closed convex subsets of Y .
- $N_{wcc}(Y)$: set of all nonempty weakly compact convex subsets of Y .

For any nonempty set $Z \in 2^Y$, we define:

$$\begin{aligned} \overline{Z} &: \text{norm-closure of } Z, \\ \delta(y', Z) &= \sup_{y \in Z} \langle y', y \rangle, \quad y' \in Y' : \text{the support function,} \\ d(y, Z) &= \inf_{a \in Z} \|y - a\| : \text{the distance function,} \\ |Z| &= \sup_{y \in Z} \|y\|. \end{aligned}$$

We consider the following topologies:

- For $(Z_n)_{n \geq 1} \subset N_{bcc}(Y)$ and $Z \in N_{bcc}(Y)$. Let

$$n_t - \liminf_{n \rightarrow \infty} Z_n := \{y \in Y, \exists y_k \in Z_k, k \geq 1 : n_t - \lim_k y_k = y\}.$$

$$w_t - \limsup_{n \rightarrow \infty} Z_n := \{y \in Y, \exists y_k \in Z_{n_k}, k \geq 1 : w_t - \lim_k y_k = y\}.$$

where $(Z_{n_k})_{k \geq 1}$ is a subsequence of $(Z_n)_{n \geq 1}$. The norm topology and weak topology on Y are represented by the symbols n_t and w_t , respectively.

The sequence $(Z_n)_{n \geq 1}$ is said to converge to Z in the Mosco sense, and we write $M - \lim_n Z_n = Z$, if

$$n_t - \liminf_{n \rightarrow \infty} Z_n = Z = w_t - \limsup_{n \rightarrow \infty} Z_n.$$

- The topology induced by the convergence of support functions denoted by T_{scalar} , we say that a sequence (Z_n) is T_{scalar} convergent to some subset Z if

$$\delta(y', Z_n) \rightarrow \delta(y', Z) \quad \forall y' \in Y'.$$

- The Wijsman topology induced by the convergence of distance functions denoted by T_w , we say that a sequence (Z_n) is T_w convergent to some subset Z if

$$d(y, Z_n) \rightarrow d(y, Z) \quad \forall y \in Y.$$

- The linear topology (T_L) , proposed by Beer [2], defined as the supremum of:

1. convergence of support functions on Y' ,
2. convergence of distance functions on Y .

A sequence (Z_n) converges to Z in T_L if and only if:

$$\lim_n d(y, Z_n) = d(y, Z) \quad \forall y \in Y, \quad \text{and} \quad \lim_n \delta(y', Z_n) = \delta(y', Z) \quad \forall y' \in Y'.$$

Compared with the Mosco topology, the topology T_L is stronger. In particular, T_L -convergence implies Mosco convergence. Thus, if $Z_n \rightarrow Z$ in T_L , then

$$n_t - \liminf_{n \rightarrow \infty} Z_n = Z = w_t - \limsup_{n \rightarrow \infty} Z_n.$$

Moreover, in reflexive spaces, the Mosco topology and the linear topology coincide. For additional details on these topologies (see [3]).

A function $\zeta : \Omega \rightarrow Y$ is said to be scalarly integrable if $\langle y', \zeta \rangle$ is integrable for all $y' \in Y'$. The function ζ is called *Pettis integrable* if it is scalarly integrable and, for each $C \in \mathcal{Q}$, there exists an element $x_C \in Y$ such that

$$\langle y', x_C \rangle = \int_C \langle y', \zeta \rangle dP, \quad \forall y' \in Y',$$

in which case we write $x_C = P\text{-}\int_C \zeta dP$.

We denote $L^1(\Omega)$ (resp. L_Y^1 , P_Y^1) the space of all Bochner integrable functions defined from Ω to $\mathbb{R}$ (resp. the space of all Bochner-integrable functions from Ω to Y , and the space of all Pettis integrable functions defined from Ω to Y).

Let $\zeta \in P_Y^1$. We consider the following topologies.

1. The topology induced by the Pettis norm:

$$\|\zeta\|_P = \sup_{y' \in B_{Y'}} \|\langle y', \zeta(\cdot) \rangle\|_1.$$

2. The topology derived from the duality $(P_Y^1, L^\infty \otimes Y')$ is defined via the bilinear operator

$$(v \otimes y', \zeta) = \int_{\Omega} v(w) \langle y', \zeta(w) \rangle dP.$$

We denote this topology by w - P . It is called the weak topology on P_Y^1 (see [12]).

A random set $H : \Omega \rightarrow N_c(Y)$ is $\mathcal{Q}$ -measurable if $\{\omega \in \Omega : H(\omega) \cap O \neq \emptyset\} \in \mathcal{Q}$ for every open set $O \subset Y$. By Castaing and Valadier [5], this is equivalent to the existence of a sequence of measurable selectors (h_n) satisfying:

$$H(\omega) = \overline{\{h_n(\omega) : n \geq 1\}}, \quad \forall \omega \in \Omega.$$

Let $H : \Omega \rightarrow N_{bcc}(Y)$ be a random set.

- H is *quasi-scalarly integrable* if for all $y' \in Y'$, either $\delta(y', H(\cdot))^-$ or $\delta(y', H(\cdot))^+$ is integrable.
- H is *scalarly integrable* if $\delta(y', H(\cdot))$ is integrable for all $y' \in Y'$.

The random set H is called *Pettis integrable* if it is quasi-scalarly integrable and for each $C \in \mathcal{Q}$, there exists a set $X_C(H) \in N_c(Y)$ such that:

$$\delta(y', X_C(H)) = \int_C \delta(y', H) dP, \quad \forall y' \in Y'.$$

We write P - $\int_C H dP = X_C(H)$.

A function $h : \Omega \rightarrow Y$ is said to be a selection of H if $h(\omega) \in H(\omega)$ for all ω . Let us denote the following sets:

- The set of measurable selections of H is denoted by S_H ,
- The set of measurable and Bochner integrable selections of H is denoted by S_H^1 ,
- The set of measurable and Pettis integrable selections of H is denoted by S_H^P .

H is *Aumann-Pettis integrable* if $S_H^P \neq \emptyset$, and the definition of its Aumann-Pettis integral over C is:

$$I_C(H) = \int_C H dP = \left\{ \int_C h dP : h \in S_H^P \right\}.$$

Every scalarly and Aumann-Pettis integrable random set with values in $N_{cc}(Y)$ is Pettis integrable (see; Theorem 3.7 [8]).

Given a sub- σ -algebra $\mathcal{B} \subset \mathcal{Q}$, the conditional expectation of h with respect to $\mathcal{B}$ when it exists, is the unique $\mathcal{B}$ -measurable function ζ satisfying:

$$P\text{-}\int_B \zeta dP = P\text{-}\int_B h dP, \quad \forall B \in \mathcal{B},$$

denoted $E^{\mathcal{B}}(h)$.

If H is Aumann-Pettis integrable and $E^{\mathcal{B}}(h)$ exists for all $h \in S_H^P$, then we define (after [9]) the conditional expectation of H with respect to sub- σ -algebra $\mathcal{B}$ as the unique random set $E^{\mathcal{B}}(H)$ such that

$$S_{E^{\mathcal{B}}(H)}^P(\mathcal{B}) = \overline{\{E^{\mathcal{B}}(h) : h \in S_H^P\}}^{\|\cdot\|_P}.$$

We refer the reader to [1] and [9] for results regarding the existence of conditional expectations of Pettis-integrable random sets.

Definition 2.1. Let G be a countable subset of $B_{Y'}$. A random set $Z : \Omega \rightarrow N_{cc}(Y)$ is called G -countably supported whenever

$$Z(\omega) = \bigcap_{y' \in G} \{y \in Y, \langle y', y \rangle \leq \delta(y', Z(\omega))\}, \quad \forall \omega \in \Omega.$$

Remark 2.2. 1. Let $(x'_k)_{k \geq 1}$ be any countable subset of $B_{Y'}$ such that $x'_k = \frac{y'_k}{\|y'_k\|}$, where $(y'_k)_{k \geq 1} \subset D'_M$. Then any family taking values in $N_{wcc}(Y)$ is countably supported with respect to $(x'_k)_{k \geq 1}$ (see Lemma III.34 in [5]).

2. Assume that Y' is separable, and let $(x'_k)_{k \geq 1} \subset D'(B_{Y'})$. Then any family taking values in $N_{bcc}(Y)$ is countably supported relative to $(x'_k)_{k \geq 1}$ (see [8]).

Definition 2.3. An adapted sequence $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ of Aumann–Pettis integrable random sets taking values in $N_{bcc}(Y)$ is said to be a martingale if $E^{\mathcal{Q}_n}(Z_k)$ exists for all $k \geq n$ and

$$E^{\mathcal{Q}_n}(Z_{n+1}) = Z_n \quad \text{a.s.}$$

It is called a regular martingale if there exists an Aumann–Pettis integrable random set Z such that $E^{\mathcal{Q}_n}(Z)$ exists for each $n \geq 1$ and

$$E^{\mathcal{Q}_n} Z = Z_n \quad \text{a.s.}$$

Definition 2.4. Let $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ be an Aumann–Pettis integrable martingale taking values in $N_{bcc}(Y)$. An adapted sequence $\{e_n, \mathcal{Q}_n\}_{n \geq 1}$ of Pettis integrable functions is called a *martingale selection* of $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ if:

- $\{e_n, \mathcal{Q}_n\}_{n \geq 1}$ is a vector-valued martingale;
- for all $n \geq 1$, $e_n \in S_{Z_n}^P(\mathcal{Q}_n)$.

The set of all such selections is denoted by $PMS(Z_n)$.

2.1. Motivation: a set-valued Aumann–Pettis integrable regular martingale not covered by the Bochner framework

We note that this extension is genuinely meaningful, since there exist regular Aumann–Pettis integrable martingales that are not Bochner integrable. For example, let

$$Y = \ell^2, \quad Y' = \ell^2,$$

and let

$$(\Omega, \mathcal{Q}, P) = ([0, 1), \mathcal{B}([0, 1)), \lambda),$$

where λ denotes the Lebesgue measure. Let $(\mathcal{Q}_n)_{n \geq 1}$ be the dyadic filtration, where $\mathcal{Q}_n$ is generated by the dyadic intervals of length 2^{-n} .

Let $(b_n)_{n \geq 1}$ be the canonical orthonormal basis of ℓ^2 and define

$$x_n := \frac{b_n}{n}, \quad n \geq 1.$$

Since

$$\sum_{n=1}^{\infty} \|x_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

the series

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{b_n}{n}$$

converges unconditionally in ℓ^2 . On the other hand,

$$\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

For $n \geq 1$, set

$$I_n = \left[1 - 2^{-(n-1)}, 1 - 2^{-n} \right).$$

Then $(I_n)_{n \geq 1}$ is a measurable partition of $[0, 1)$ and

$$P(I_n) = 2^{-n}.$$

Moreover,

$$I_n \in \mathcal{Q}_n, \quad n \geq 1.$$

Define the Y -valued random variable

$$M_{\infty}(\omega) := \sum_{n=1}^{\infty} 2^n x_n \mathbf{1}_{I_n}(\omega) = \sum_{n=1}^{\infty} \frac{2^n}{n} b_n \mathbf{1}_{I_n}(\omega).$$

Since the sets $(I_n)_{n \geq 1}$ are pairwise disjoint, at most one term in this series is nonzero for each $\omega \in \Omega$. Hence M_{∞} is strongly measurable. We first show that M_{∞} is Pettis integrable. Let $y = (y_n)_{n \geq 1} \in Y' = \ell^2$. Then

$$\langle y, M_{\infty}(\omega) \rangle = \sum_{n=1}^{\infty} \frac{2^n y_n}{n} \mathbf{1}_{I_n}(\omega),$$

and therefore

$$\begin{aligned} \int_{\Omega} |\langle y, M_{\infty}(\omega) \rangle| dP(\omega) &= \sum_{n=1}^{\infty} \frac{2^n |y_n|}{n} P(I_n) \\ &= \sum_{n=1}^{\infty} \frac{|y_n|}{n} \\ &\leq \left(\sum_{n=1}^{\infty} |y_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \\ &< \infty. \end{aligned}$$

Thus $\langle y, M_{\infty} \rangle \in L^1(\Omega)$ for every $y \in Y'$.

Moreover, for every $A \in \mathcal{Q}$,

$$\int_A M_{\infty} dP = \sum_{n=1}^{\infty} 2^n x_n P(A \cap I_n) = \sum_{n=1}^{\infty} \frac{P(A \cap I_n)}{P(I_n)} \frac{b_n}{n}.$$

Since

$$0 \leq \frac{P(A \cap I_n)}{P(I_n)} \leq 1,$$

we have

$$\sum_{n=1}^{\infty} \left\| \frac{P(A \cap I_n)}{P(I_n)} \frac{b_n}{n} \right\|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Consequently, the above series converges in ℓ^2 for every $A \in \mathcal{Q}$. Hence M_∞ is Pettis integrable. However,

$$\begin{aligned} \int_{\Omega} \|M_\infty(\omega)\| dP(\omega) &= \sum_{n=1}^{\infty} \frac{2^n}{n} P(I_n) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty. \end{aligned}$$

Therefore, $M_\infty \notin L^1(\Omega; Y)$, and hence M_∞ is not Bochner integrable. For each $n \geq 1$, define

$$M_n := E^{\mathcal{Q}_n}(M_\infty).$$

Then $(M_n, \mathcal{Q}_n)_{n \geq 1}$ is a vector-valued martingale and

$$E^{\mathcal{Q}_n}(M_\infty) = M_n, \quad n \geq 1.$$

In particular, the martingale $(M_n, \mathcal{Q}_n)_{n \geq 1}$ is regular with Pettis integrable random variable M_∞ . We now construct the corresponding set-valued martingale. Define

$$Z_n(\omega) := \{M_n(\omega)\}, \quad n \geq 1,$$

and

$$Z(\omega) := \{M_\infty(\omega)\}.$$

Since singleton subsets of Y are nonempty, closed, convex and bounded, we have

$$Z_n(\omega), Z(\omega) \in N_{bcc}(Y), \quad \omega \in \Omega.$$

Moreover, Z_n is $\mathcal{Q}_n$ -measurable because M_n is $\mathcal{Q}_n$ -measurable. Thus $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ is adapted. We next verify the Aumann–Pettis integrability in the sense of the definition used in this paper. Since

$$Z(\omega) = \{M_\infty(\omega)\},$$

every measurable selection of Z coincides with M_∞ . Hence

$$S_Z^P = \{M_\infty\} \neq \emptyset,$$

because M_∞ is Pettis integrable. Therefore, Z is Aumann–Pettis integrable. Similarly,

$$S_{Z_n}^P = \{M_n\} \neq \emptyset, \quad n \geq 1,$$

so each Z_n is Aumann–Pettis integrable.

Moreover, by the definition of the conditional expectation of an Aumann–Pettis integrable random set, we have

$$S_{E^{\mathcal{Q}_n}(Z)}^P(\mathcal{Q}_n) = \overline{\{E^{\mathcal{Q}_n}(h) : h \in S_Z^P\}}^{\|\cdot\|_P}.$$

Since

$$S_Z^P = \{M_\infty\},$$

it follows that

$$S_{E^{\mathcal{Q}_n}(Z)}^P(\mathcal{Q}_n) = \{E^{\mathcal{Q}_n}(M_\infty)\} = \{M_n\}.$$

On the other hand,

$$S_{Z_n}^P(\mathcal{Q}_n) = \{M_n\}.$$

By the uniqueness in the definition of the conditional expectation, consequently we obtain

$$E^{\mathcal{Q}_n}(Z) = Z_n, \quad n \geq 1.$$

Hence, by the definition above, $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ is a set-valued Aumann–Pettis integrable regular martingale. The example also admits a Pettis integrable martingale selection. Indeed, $\{M_n, \mathcal{Q}_n\}_{n \geq 1}$ is a vector-valued martingale and

$$M_n \in S_{Z_n}^P(\mathcal{Q}_n), \quad n \geq 1.$$

Therefore,

$$\{M_n, \mathcal{Q}_n\}_{n \geq 1} \in PMS(Z_n).$$

Finally, the representing random set Z is Aumann–Pettis integrable but not Bochner integrable. Indeed,

$$\int_{\Omega} \sup_{z \in Z(\omega)} \|z\| dP(\omega) = \int_{\Omega} \|M_{\infty}(\omega)\| dP(\omega) = \infty.$$

Thus, $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ is a set-valued Aumann–Pettis integrable regular martingale whose representing random set Z is Aumann–Pettis integrable but not Bochner integrable. Hence, the class of set-valued Aumann–Pettis integrable regular martingales considered in this paper contains objects that are not covered by the corresponding Bochner integrable framework.

Furthermore, in several fields such as econometrics, finance and computer science, there are problems that can be modelled by Pettis integrable random processes but not by Bochner integrable ones; see for example, Candeloro, Sambucini and Trastulli [4]. Their paper studies an application of Pettis-integrable martingales to mathematical finance, a setting where the Bochner integral is not sufficient. Also, in the early 1980s, another important field of applications for the theory of random sets emerged, namely partial identification analysis (see Molchanov and Molinari [17], Miyashita and Ui [16]).

Before turning to the proofs, we briefly outline the strategy of the paper. In Section 3, we first prove a representation theorem for the conditional expectation of an Aumann–Pettis integrable random set with values in $N_{bcc}(Y)$ (Theorem 3.1). Using a truncation argument in the spirit of Lemma 1.1 of Hiai and Umegaki [14], we construct a sequence of Pettis integrable selections $(e^n)_{n \geq 1} \subset S_Z^P$ such that

- Z is the norm-closure of $(e^n)_{n \geq 1}$,
- $E^B(Z)$ is the norm-closure of $(E^B(e^n))_{n \geq 1}$.

Corollary 3.1 gives this result for the filtration $(\mathcal{Q}_n)_{n \geq 1}$. In Section 4, this representation is applied to show that any Aumann–Pettis integrable regular martingale $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ with values in $N_{bcc}(Y)$ can be represented as the norm-closure of a sequence of Pettis integrable martingale selections (Theorem 4.1). Likewise, any Aumann–Pettis integrable regular martingale $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ with values in $N_{wcc}(Y)$ can be represented as the norm-closure of a sequence of Pettis integrable martingale selections (Corollary 4.1). This representation by selections is the main tool for our convergence result (Theorem 4.2), in the Mosco sense. If $\sup_{n \geq 1} E^{\mathcal{Q}_n}|Z| < \infty$, then we obtain a convergence

result in the linear topology T_L . Finally, in Corollary 4.2, since $\bigcup_{n \geq 1} \overline{Z_n(\omega)}$ is convex and weakly compact, we

remove the separability assumption on Y' , and we obtain a convergence theorem for an Aumann–Pettis integrable regular martingale with values in $N_{wcc}(Y)$ in the linear topology T_L .

3. Representation theorem for the conditional expectation of Aumann–Pettis integrable random sets

In this section, we establish a representation theorem for the conditional expectation of Aumann–Pettis integrable random sets taking values in $N_{bcc}(Y)$. More precisely, we show that the conditional expectation of such a random

set can be represented as the norm-closure of a sequence of conditional expectations of its Pettis integrable selections.

We begin by establishing the representation theorem for the conditional expectation of Aumann–Pettis integrable random sets.

Theorem 3.1

Let Z be an Aumann–Pettis integrable random set with values in $N_{bcc}(Y)$. Given a sub- σ -algebra $\mathcal{B}$ of $\mathcal{Q}$ such that $E^{\mathcal{B}}(Z)$ exists. Then there exists a sequence $(e^n)_{n \geq 1} \subset S_Z^P$ such that:

$$Z = \overline{\{e^n, n \geq 1\}}^{\|\cdot\|_Y} \quad a.s.$$

$$E^{\mathcal{B}}Z = \overline{\{E^{\mathcal{B}}(e^n), n \geq 1\}}^{\|\cdot\|_Y} \quad a.s.$$

Proof

We employ methods akin to those presented in Lemma 23 of [12]. The existence of a sequence $(e^m)_{m \geq 1}$ of measurable selections dense in Z is known. Therefore, for almost every $\omega \in \Omega$, for every $e \in Z(\omega)$ and every $\varepsilon > 0$, there exists $m \geq 1$ such that

$$\|e^m(\omega) - e\| \leq \varepsilon.$$

We consider the countable family of measurable sets defined by

$$A_{m,j} = \{\omega \in \Omega : j - 1 \leq \|e^m(\omega)\| < j\} \in \mathcal{Q}.$$

Since Z is Aumann–Pettis integrable, it admits at least one Pettis integrable selection $g \in S_Z^P$. Define

$$e^{m,j} = e^m \chi_{A_{m,j}} + g \chi_{A_{m,j}^c}, \quad m, j \geq 1.$$

On $A_{m,j}$, we have $\|e^m(\omega)\| < j$, so $e^m \chi_{A_{m,j}}$ is a bounded measurable function, hence Bochner integrable and therefore Pettis integrable. Since g is Pettis integrable, then $g \chi_{A_{m,j}^c}$ is Pettis integrable. As the sum of two Pettis integrable functions, $e^{m,j}$ is Pettis integrable. Hence

$$e^{m,j} \in S_Z^P, \quad m, j \geq 1.$$

So for every $m \geq 1$ and every $\omega \in \Omega$, since $(A_{m,j})_{j \geq 1}$ is a partition of Ω , there exists a unique $j \geq 1$ such that $\omega \in A_{m,j}$, and therefore

$$e^{m,j}(\omega) = e^m(\omega).$$

It follows that, for almost every $\omega \in \Omega$, for every $e \in Z(\omega)$ and every $\varepsilon > 0$, there exists $m, j \geq 1$ such that

$$\|e^{m,j}(\omega) - e\| = \|e^m(\omega) - e\| < \varepsilon.$$

Hence, for almost every $\omega \in \Omega$

$$Z(\omega) = \overline{\{e^{m,j}(\omega), m \geq 1\}}^{\|\cdot\|_Y}.$$

Thus, we can enumerate the family $(e^{m,j})_{m,j \geq 1}$ as a sequence $(e^n)_{n \geq 1}$. Consequently,

$$Z = \overline{\{e^n : n \geq 1\}}^{\|\cdot\|_Y} \quad a.s.$$

Let us show that $E^{\mathcal{B}}(Z) = \overline{\{E^{\mathcal{B}}(e^n), n \geq 1\}}^{\|\cdot\|_Y} \quad a.s.$
Since $E^{\mathcal{B}}(Z)$ is convex and closed, it is evident that

$$\overline{\{E^{\mathcal{B}}(e^n), n \geq 1\}}^{\|\cdot\|_Y} \subset E^{\mathcal{B}}(Z). \quad (3.1)$$

To prove the converse, let $G(\omega) = \overline{\{E^{\mathcal{B}}e^n(\omega), n \geq 1\}}^{\|\cdot\|_Y}$ a.s, for every $y' \in Y'$ and every $h \in L^\infty(\mathcal{B})$, we have

$$\begin{aligned}
 \delta(h \otimes y', S_{E^{\mathcal{B}}(Z)}^P) &= \sup_{g \in S_{E^{\mathcal{B}}(Z)}^P} \langle h \otimes y', g \rangle \\
 &= \sup_{g \in S_{E^{\mathcal{B}}(Z)}^P} \int_{\Omega} h(\omega) \langle y', g(\omega) \rangle dP \\
 &= \sup_{g \in \{E^{\mathcal{B}}(e), e \in S_Z^P\}} \int_{\Omega} h(\omega) \langle y', g(\omega) \rangle dP \\
 &= \sup_{e \in S_Z^P} \int_{\Omega} h(\omega) \langle y', E^{\mathcal{B}}(e)(\omega) \rangle dP \\
 &= \sup_{e \in S_Z^P} \int_{\Omega} h(\omega) \langle y', e(\omega) \rangle dP \\
 &\leq \sup_{e \in S_Z} \int_{\Omega} h(\omega) \langle y', e(\omega) \rangle dP \quad (\text{see [12], Proposition 11}) \\
 &= \sup_{n \geq 1} \int_{\Omega} h(\omega) \langle y', e^n(\omega) \rangle dP \\
 &= \sup_{n \geq 1} \int_{\Omega} h(\omega) \langle y', E^{\mathcal{B}}(e^n)(\omega) \rangle dP \\
 &= \sup_{f \in G} \int_{\Omega} h(\omega) \langle y', f(\omega) \rangle dP \\
 &= \sup_{f \in S_G^P} \int_{\Omega} h(\omega) \langle y', f(\omega) \rangle dP \\
 &= \delta(h \otimes y', S_G^P).
 \end{aligned}$$

Since, for every $y' \in Y'$ and every $A \in P_Y^1$, we have

$$\begin{aligned}
 \delta(\chi_{\Omega} \otimes y', A) &= \sup_{g \in A} \langle \chi_{\Omega} \otimes y', g \rangle \\
 &= \sup_{g \in A} \int_{\Omega} \chi_{\Omega}(\omega) \langle y', g(\omega) \rangle dP \\
 &= \sup_{g \in A} \int_{\Omega} \langle y', g(\omega) \rangle dP \\
 &= \sup_{g \in A} \langle y', g \rangle \\
 &= \delta(y', A).
 \end{aligned}$$

Set $h = \chi_{\Omega}$, then for every $y' \in Y'$ we have

$$\delta(y', S_{E^{\mathcal{B}}(Z)}^P) \leq \delta(y', S_G^P).$$

So

$$S_{E^{\mathcal{B}}(Z)}^P \subset S_G^P.$$

Hence, by Remark 3.2 in [9], we have

$$E^{\mathcal{B}}(Z) \subset G. \quad (3.2)$$

From (3.1) and (3.2), we deduce that

$$E^{\mathcal{B}}Z(\omega) = \overline{\{E^{\mathcal{B}}e^n(\omega), n \geq 1\}}^{\|\cdot\|_Y} \text{ a.s.}$$

□

Corollary 3.1. Let Z be an Aumann–Pettis integrable random set with values in $N_{bcc}(Y)$ such that $E^{\mathcal{Q}_n}(Z)$ exists for every $n \geq 1$. Then there exists a sequence $(e^i)_{i \geq 1} \subset S_Z^P$ satisfying:

$$Z = \overline{\{e^i, i \geq 1\}}^{\|\cdot\|_Y} \text{ a.s.}$$

$$E^{\mathcal{Q}_n} Z = \overline{\{E^{\mathcal{Q}_n}(e^i), i \geq 1\}}^{\|\cdot\|_Y} \text{ a.s. } n \geq 1.$$

Proof

The result follows directly from Theorem 3.1 by taking $\mathcal{B} = \mathcal{Q}_n$ for each $n \geq 1$. □

4. Representation and convergence results for multivalued Aumann–Pettis integrable regular martingales

In this section, we establish that every Aumann–Pettis integrable regular martingale taking values in $N_{bcc}(Y)$ admits a representation as the norm-closure of a sequence of its Pettis integrable martingale selections. This representation allows us to derive new convergence results of Lévy type for such processes, both in the Mosco sense and in the linear topology. These results extend the corresponding representation and convergence results established in the Bochner setting; see, in particular, [6], [13] and [14].

We begin by establishing the following representation theorem.

Theorem 4.1

Let $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ be an Aumann–Pettis integrable regular martingale with values in $N_{bcc}(Y)$. Then, there exists a sequence $\{f_n^k, \mathcal{Q}_n, n \geq 1\}_{k \geq 1}$ in $\text{PMS}(Z_n)$ such that, for every $n \geq 1$ and almost every $\omega \in \Omega$,

$$Z_n(\omega) = \overline{\{f_n^k(\omega) : k \geq 1\}}^{\|\cdot\|_Y}.$$

Proof

By Corollary 3.1, we obtain a sequence $(e^k)_{k \geq 1} \subset S_Z^P$ such that for every $n \geq 1$

$$E^{\mathcal{Q}_n}(Z) = Z_n = \overline{\{E^{\mathcal{Q}_n}(e^k), k \geq 1\}}^{\|\cdot\|_Y}$$

Then there exists a null set N such that

$$Z_n(\omega) = \overline{\{E^{\mathcal{Q}_n}(e^k)(\omega) : k \geq 1\}}^{\|\cdot\|_Y} \quad \forall \omega \in N^c.$$

For each $k \geq 1$ and for every $n \geq 1$ define

$$f_n^k(\omega) = E^{\mathcal{Q}_n}(e^k)(\omega), \quad \forall \omega \notin N.$$

Then for each $k \geq 1$, $\{f_n^k, \mathcal{Q}_n, n \geq 1\} \in \text{PMS}(Z_n)$. Moreover, for every $n \geq 1$ and for almost every $\omega \in \Omega$

$$Z_n(\omega) = \overline{\{f_n^k(\omega) : k \geq 1\}}^{\|\cdot\|_Y}.$$

□

Corollary 4.1. Let $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ be an Aumann–Pettis integrable regular martingale with values in $N_{bcc}(Y)$. Then, there exists a sequence $\{f_n^k, \mathcal{Q}_n, n \geq 1\}_{k \geq 1}$ in $\text{PMS}(Z_n)$ such that, for every $n \geq 1$ and almost every $\omega \in \Omega$,

$$Z_n(\omega) = \overline{\{f_n^k(\omega) : k \geq 1\}}^{\|\cdot\|_Y}.$$

Proof

Since, every nonempty convex weakly compact subset of Y is convex closed and bounded. Hence, the result is an immediate consequence of Theorem 4.1. □

We are now ready to state the convergence theorem for multivalued Aumann–Pettis integrable regular martingales taking values in $N_{bcc}(Y)$.

Theorem 4.2

Suppose that Y' is separable, and let $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ be an Aumann–Pettis integrable regular martingale taking values in $N_{bcc}(Y)$. Assume that $Z_\infty = E^{\mathcal{Q}_\infty}(Z)$ exists. Let $(e^i)_{i \geq 1} \subset S_Z^P$ be the sequence given by Theorem 4.1, such that

$$e_n^i = E^{\mathcal{Q}_n}(e^i), \quad 1 \leq n \leq \infty, \quad i \geq 1, \quad \text{and} \quad Z_n(\omega) = \overline{\{e_n^i(\omega) : i \geq 1\}}^{\|\cdot\|_Y}, \quad 1 \leq n \leq \infty, \quad \text{a.s.}$$

Assume that

$$\lim_{n \rightarrow \infty} \|e_n^i(\omega) - e_\infty^i(\omega)\| = 0 \quad \text{a.e., for every } i \geq 1. \quad (\text{H})$$

Then

$$M\text{-}\lim_{n \rightarrow \infty} Z_n = Z_\infty \quad \text{a.s.}$$

If, in addition, $\sup_{n \geq 1} E^{\mathcal{Q}_n}|Z| < \infty \quad \text{a.s.}$, then

$$T_L\text{-}\lim_{n \rightarrow \infty} Z_n = Z_\infty \quad \text{a.s.}$$

Proof

From Theorem 4.1, there exists $(e^i)_{i \geq 1} \subset S_Z^P$ and there exists $\{e_n^i, \mathcal{Q}_n, n \geq 1\}_{i \geq 1} \subset PMS(Z_n)$ such that

$$Z_n(\omega) = \overline{\{e_n^i(\omega), i \geq 1\}}^{\|\cdot\|_Y} = \overline{\{E^{\mathcal{Q}_n}(e^i(\omega)), i \geq 1\}}^{\|\cdot\|_Y} \quad \text{a.s.}, \quad 1 \leq n.$$

and

$$Z_\infty(\omega) = \overline{\{E^{\mathcal{Q}_\infty}(e^i(\omega)), i \geq 1\}}^{\|\cdot\|_Y} \quad \text{a.s.}$$

Let $\Omega_m = \{m-1 \leq E^{\mathcal{Q}_1}|Z| < m\}$. It is clear that, $(\Omega_m)_{m \geq 1}$ is a partition of Ω in $\mathcal{Q}_1$. Let $(y'_k)_{k \geq 1} \subset D'$. For all $k \geq 1$ and all $m \geq 1$,

$$\begin{aligned} \sup_{n \geq 1} \int_{\Omega_m} \sup_{i \geq 1} \langle y'_k, e_n^i \rangle^+ dP &\leq \sup_{n \geq 1} \int_{\Omega_m} \delta(y'_k, Z_n)^+ dP \\ &= \sup_{n \geq 1} \int_{\Omega_m} \delta(y'_k, E^{\mathcal{Q}_n} Z)^+ dP \\ &\leq \int_{\Omega_m} \delta(y'_k, Z)^+ dP \\ &\leq \int_{\Omega_m} |Z| dP \\ &= \int_{\Omega_m} E^{\mathcal{Q}_1} |Z| dP < +\infty. \end{aligned}$$

Then by Lemma 4 of Neveu [18], for all $k \geq 1$, we have

$$\lim_{n \rightarrow \infty} \sup_{i \geq 1} \langle y'_k, e_n^i \rangle = \lim_{n \rightarrow \infty} \sup_{i \geq 1} \langle y'_k, E^{\mathcal{Q}_n} e^i \rangle = \sup_{i \geq 1} \langle y'_k, E^{\mathcal{Q}_\infty} e^i \rangle \quad \text{a.s.}$$

So, there exists a null set $N_k \in \mathcal{Q}_\infty$ such that for all $\omega \in \Omega \setminus N_k$,

$$\lim_{n \rightarrow \infty} \sup_{i \geq 1} \langle y'_k, e_n^i(\omega) \rangle = \sup_{i \geq 1} \langle y'_k, E^{\mathcal{Q}_\infty} e^i(\omega) \rangle.$$

Then for all $\omega \in \Omega \setminus N_k$,

$$\lim_{n \rightarrow \infty} \delta(y'_k, Z_n(\omega)) \leq \delta(y'_k, Z_\infty(\omega)), \quad \forall k \geq 1. \quad (4.1)$$

By the classical convergence of scalar martingales, for all $i \geq 1$ and all $k \geq 1$

$$\lim_{n \rightarrow \infty} (\langle y'_k, E^{\mathcal{Q}_n}(e^i)(\omega) - E^{\mathcal{Q}_\infty}(e^i)(\omega) \rangle) = 0, \quad a.s.$$

Since for all $i \geq 1$ and all $n \geq 1$, $E^{\mathcal{Q}_n}(e^i) \in E^{\mathcal{Q}_n}Z = Z_n$ a.s. Then for all $i \geq 1$ and all $k \geq 1$

$$\langle y'_k, E^{\mathcal{Q}_\infty}e^i(\omega) \rangle = \lim_{n \rightarrow \infty} \langle y'_k, e_n^i(\omega) \rangle \leq \lim_{n \rightarrow \infty} \delta(y'_k, Z_n(\omega)) \quad a.s.$$

Thus for all $\omega \in \Omega \setminus N_k$,

$$\delta(y'_k, Z_\infty(\omega)) = \sup_{i \geq 1} \langle y'_k, E^{\mathcal{Q}_\infty}e^i(\omega) \rangle \leq \lim_{n \rightarrow \infty} \delta(y'_k, Z_n(\omega)), \quad \forall k \geq 1. \quad (4.2)$$

From (4.1) and (4.2), there exists a null set $N \in \mathcal{Q}_\infty$ such that for all $\omega \in \Omega \setminus N$,

$$\lim_{n \rightarrow \infty} \delta(y'_k, Z_n(\omega)) = \delta(y'_k, Z_\infty(\omega)), \quad \forall k \geq 1. \quad (4.3)$$

Let $x \in w_t - \limsup_{n \rightarrow \infty} Z_n$, then, there exists a sequence $(x_j)_{j \geq 1} \subset (Z_{n_j})_{j \geq 1}$ converges weakly to x such that $(Z_{n_j})_{j \geq 1}$ is a subsequence of $(Z_n)_{n \geq 1}$, for all $k \geq 1$

$$\langle y'_k, x \rangle = \lim_{j \rightarrow \infty} \langle y'_k, x_j \rangle \leq \limsup_n \delta(y'_k, Z_n).$$

Then, for all $k \geq 1$ and for all $\omega \in \Omega \setminus N$,

$$\langle y'_k, x \rangle \leq \delta(y'_k, Z_\infty(\omega)).$$

Let $((x'_k)_{k \geq 1} \subset D'(B_{Y'}))$ be defined by

$$x'_k = \frac{y'_k}{\|y'_k\|}, \quad y'_k \neq 0.$$

Since $\delta(\cdot, Z_\infty(\omega))$ is positively homogeneous, it follows that, for all $k \geq 1$ and all $\omega \in \Omega \setminus N$,

$$\langle x'_k, x \rangle \leq \delta(x'_k, Z_\infty(\omega)).$$

Since Z_∞ takes values in $N_{bcc}(Y)$ and Y' is separable, then by Remark 2.2, it is $((x'_k)_{k \geq 1})$ -countably supported. Then $x \in Z_\infty(\omega)$. Therefore,

$$w_t - \limsup_{n \rightarrow \infty} Z_n(\omega) \subset Z_\infty(\omega) \quad a.s. \quad (4.4)$$

By condition H, for every $i \geq 1$, there exists a sequence $(e_n^i, \mathcal{Q}_n)_{n \geq 1} \in PMS(Z_n)$ and a negligible subset N_i such that, for any $\omega \in N_i^c$, $\lim_{n \rightarrow \infty} \|e_n^i(\omega) - e_\infty^i(\omega)\| = 0$ a.e.

Since $e_n^i(\omega) \in Z_n(\omega)$, for every $y \in Y$ and every $i \geq 1$ we have

$$d(y, Z_n(\omega)) \leq \|y - e_n^i(\omega)\|, \quad \omega \in N_i^c.$$

Let $N = \bigcup_{i \geq 1} N_i$. For every $y \in Y$ and every $\omega \in N^c$

$$d(y, Z_n(\omega)) \leq \|y - e_n^i(\omega)\| \leq \|y - e_\infty^i(\omega)\| + \|e_\infty^i(\omega) - e_n^i(\omega)\|.$$

Letting $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} d(y, Z_n(\omega)) \leq \|y - e_\infty^i(\omega)\|.$$

Taking the infimum over i and using the density of $\{e_\infty^i(\omega)\}_{i \geq 1}$ in $Z_\infty(\omega)$,

$$\limsup_{n \rightarrow \infty} d(y, Z_n(\omega)) \leq d(y, Z_\infty(\omega)), \quad \omega \in N^c. \quad (4.5)$$

Then, by Theorem 2.2 of Tsukada in [21]

$$Z_\infty(\omega) \subset n_t - \liminf_{n \rightarrow \infty} Z_n(\omega). \quad (4.6)$$

Combining (4.4) and (4.6), we conclude

$$M - \lim_n Z_n = Z_\infty \text{ a.s.}$$

Now, we prove that if $\sup_{n \geq 1} E^{\mathcal{Q}_n} |Z| < \infty$ a.s., then $T_L - \lim_n Z_n = Z_\infty$ a.s.

By Lemma 3.3 in [1], we have

$$|Z_n| = |E^{\mathcal{Q}_n} Z| = \sup_i \|E^{\mathcal{Q}_n}(e^i)\| \leq \sup_i E^{\mathcal{Q}_n} \|e^i\| \leq E^{\mathcal{Q}_n} |Z| \text{ a.s.}$$

Since $\sup_{n \geq 1} E^{\mathcal{Q}_n} |Z| < \infty$ a.s., then

$$\sup_{n \geq 1} |Z_n| \leq \sup_{n \geq 1} E^{\mathcal{Q}_n} |Z| < \infty \text{ a.s.}$$

So, for almost every $\omega \in \Omega$, the set $\{\delta(\cdot, Z_n(\omega)), n \geq 1\}$ is equicontinuous.

Hence, from (4.3) and the equicontinuity above, there exists a null set A such that for all $\omega \in \Omega \setminus A$,

$$\lim_{n \rightarrow \infty} \delta(y', Z_n(\omega)) = \delta(y', Z_\infty(\omega)), \quad \forall y' \in Y'. \quad (4.7)$$

We claim that $T_w - \lim_{n \rightarrow \infty} Z_n = Z_\infty$ a.s.

On the other hand, it follows from Corollary (1.5.3) of Beer [3], for every non-empty convex subset C of Y , we obtain

$$d(\cdot, C) = \sup_{y' \in B_{Y'}} [\langle y', \cdot \rangle - \delta(y', C)].$$

For any $y \in D$ and for every sequence $(y'_m)_{m \geq 1} \subset D'(B_{Y'})$, we have

$$d(y, Z_n) = \sup_{m \geq 1} [\langle y'_m, y \rangle - \delta(y'_m, Z_n)].$$

and similarly

$$d(y, Z_\infty) = \sup_{m \geq 1} (\langle y'_m, y \rangle - \delta(y'_m, Z_\infty)).$$

Since for all $n \geq 1$, $Z_n \in N_{bcc}(Y)$, then the function $y' \rightarrow \delta(y', Z_n)$ is continuous in the norm topology of Y' (see [8], Proposition 1.5), so we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} d(y, Z_n) &= \liminf_{n \rightarrow \infty} \sup_{m \geq 1} [\langle y'_m, y \rangle - \delta(y'_m, Z_n)] \\ &\geq \sup_{m \geq 1} \liminf_{n \rightarrow \infty} [\langle y'_m, y \rangle - \delta(y'_m, Z_n)] \\ &= \sup_{m \geq 1} [\langle y'_m, y \rangle - \delta(y'_m, Z_\infty)] \\ &= d(y, Z_\infty) \text{ a.s.} \end{aligned} \quad (4.8)$$

From (4.5) and (4.8),

$$\lim_{n \rightarrow \infty} d(y, Z_n) = d(y, Z_\infty) \quad \forall y \in Y. \quad (4.9)$$

Combining (4.7) and (4.9), we obtain

$$T_L - \lim_n Z_n = Z_\infty \text{ a.s.}$$

□

Remark 4.3. For an Aumann–Pettis integrable regular martingale taking values in $N_{wcc}(Y)$, we can omit the separability assumption on Y' . Then the following corollary can be drawn.

Corollary 4.2. Suppose that $\{Z_n, \mathcal{Q}_n\}_{n \geq 1}$ is an Aumann–Pettis integrable regular martingale taking values in $N_{wcc}(Y)$, and $Z_\infty := E^{\mathcal{Q}_\infty}(Z)$ exists. Let $(f^i)_{i \geq 1} \subset S_Z^P$ be the sequence given by Corollary 4.1, such that

$$f_n^i = E^{\mathcal{Q}_n}(f^i), \quad 1 \leq n \leq \infty, \quad i \geq 1, \quad \text{and} \quad Z_n(\omega) = \overline{\{f_n^i(\omega) : i \geq 1\}}^{\|\cdot\|_Y}, \quad 1 \leq n \leq \infty, \quad \text{a.s.}$$

Assume that

$$\lim_{n \rightarrow \infty} \|f_n^i(\omega) - f_\infty^i(\omega)\| = 0 \quad \text{a.e., for every } i \geq 1. \quad (\text{H})$$

Additionally, assume that for almost every $\omega \in \Omega$,

$$\overline{\bigcup_{n \geq 1} Z_n(\omega)} \quad \text{belongs to } N_{wcc}(Y).$$

Then

$$T_L - \lim_n Z_n = Z_\infty \quad \text{a.s.}$$

Proof

Let $(y'_k)_{k \geq 1} \subset D'_M$. As in the proof of Theorem 4.2 (see (4.3)), there exists a null set N such that for all $\omega \in \Omega \setminus N$

$$\lim_{n \rightarrow \infty} \delta(y'_k, Z_n(\omega)) = \delta(y'_k, Z_\infty(\omega)), \quad \forall k \geq 1.$$

Moreover, for all $\omega \in \Omega \setminus N$, the set $\overline{\bigcup_{n \geq 1} Z_n(\omega)}$ is weakly compact, convex and contains each $Z_n(\omega)$, we apply Lemma 5.1 of Hess [13] to conclude that

$$\lim_{n \rightarrow \infty} \delta(y', Z_n(\omega)) = \delta(y', Z_\infty(\omega)), \quad \forall y' \in Y', \quad \forall \omega \in \Omega \setminus N. \quad (4.10)$$

Furthermore, let $(y'_m)_{m \geq 1} \subset D'_M(B_{Y'})$, by (4.10), for all $m \geq 1$ and all $y \in Y$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} d(y, Z_n(\omega)) &= \liminf_{n \rightarrow \infty} \sup_{m \geq 1} [\langle y'_m, y \rangle - \delta(y'_m, Z_n(\omega))] \\ &\geq \sup_{m \geq 1} \liminf_{n \rightarrow \infty} [\langle y'_m, y \rangle - \delta(y'_m, Z_n(\omega))] \\ &= \sup_{m \geq 1} [\langle y'_m, y \rangle - \delta(y'_m, Z_\infty(\omega))] \\ &= d(y, Z_\infty(\omega)) \quad \text{a.s.} \end{aligned} \quad (4.11)$$

Following the argument of Theorem 4.2, we show that

$$\limsup_{n \rightarrow \infty} d(y, Z_n(\omega)) \leq d(y, Z_\infty) \quad \text{a.s.} \quad (4.12)$$

By (4.11) and (4.12), we have

$$\lim_{n \rightarrow \infty} d(y, Z_n) = d(y, Z_\infty) \quad \forall y \in Y. \quad (4.13)$$

Combining (4.10) and (4.13), we obtain

$$T_L - \lim_n Z_n = Z_\infty \quad \text{a.s.}$$

□

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