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On α–Lindelöf in bitopological spaces

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Abstract This paper studies α-Lindelöf covering properties in bitopological spaces by means of the associated α-topologies. For a bitopological space $(\mathfrak{E}, \xi_1, \xi_2)$, the central method is the identification of α-open subsets of $(\mathfrak{E}, \xi_i)$ with ordinary open subsets of $(\mathfrak{E}, \xi_i^\alpha)$. This allows α-Lindelöfness and related covering properties to be treated through classical Lindelöf theory. We establish the equivalence between α-Lindelöfness of $(\mathfrak{E}, \xi_i)$ and Lindelöfness of $(\mathfrak{E}, \xi_i^\alpha)$, formulate componentwise and pairwise-cover versions in bitopological spaces, and introduce corresponding α-Lindelöf numbers. Cardinal bounds in terms of network weight and weight are obtained, and a countable-sum formula for the α-Lindelöf number is proved. We also analyze the relation between α-Lindelöfness, α-compactness, and α-countable compactness, supported by examples separating the principal notions. Finally, preservation under strongly α-continuous maps and product results under explicit compatibility hypotheses are discussed.

Keywords α-open set; α-Lindelöf space; α-compact space; bitopological space; pairwise cover.

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1. Introduction

Covering properties occupy a central position in general topology because they control many structural phenomena related to compactness, Lindelöfness, countability conditions, continuous images, sums, products, and quotient-type constructions. In bitopological spaces, these questions become more delicate: two topologies are present on the same underlying set, and a covering property may be studied either componentwise, with respect to each topology separately, or in a genuinely pairwise sense, where covers are allowed to draw their members from both topologies. This distinction is particularly important when one passes from ordinary open sets to generalized open sets.

The notion of α-open sets was introduced by Njåstad [22]. For a topological space $(\mathfrak{E}, \xi)$, a subset $A \subseteq \mathfrak{E}$ is called α-open if

$$A \subseteq \mathrm{Int}_\xi\big(\mathrm{Cl}_\xi(\mathrm{Int}_\xi(A))\big).$$

The family of all α-open subsets of $(\mathfrak{E}, \xi)$ forms a topology, usually denoted by ξ^α, and $\xi \subseteq \xi^\alpha$. Thus the study of α-open covers of $(\mathfrak{E}, \xi)$ can be translated into the ordinary study of open covers of the associated topological space $(\mathfrak{E}, \xi^\alpha)$. This observation provides the main methodological principle of the present paper: many α-covering properties can be analyzed by transporting them to classical covering properties in the α-topology.

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The theory of α-open sets has been developed in several directions, including generalized continuity, compactness-type properties, fuzzy and soft topological structures, and bitopological variants; see, for example, [1, 2, 3, 4, 5, 10, 11, 16, 23, 24, 26, 27, 28].

On the other hand, bitopological spaces were introduced by Kelly [19], and pairwise covering properties have since become an important part of the subject. The notion of pairwise open cover, due to Fletcher et al. [15], plays a basic role in the development of pairwise compactness. In the α-setting, Kar and Bhattacharyya [18] introduced and studied bitopological α-compactness. Later, pairwise Lindelöf-type properties were investigated in bitopological spaces by Kilicman and Salleh [20]. These works motivate a systematic study of α-Lindelöfness in bitopological spaces.

The purpose of this paper is to develop a coherent framework for α-Lindelöfness in bitopological spaces $(\mathfrak{E}, \xi_1, \xi_2)$. We distinguish between two related but different viewpoints. The first is the componentwise viewpoint, where one requires $(\mathfrak{E}, \xi_i)$ to be α-Lindelöf for $i = 1, 2$. The second is the direct pairwise-cover viewpoint, where one studies covers whose members belong to $\xi_1^\alpha \cup \xi_2^\alpha$. This distinction is essential, since componentwise covering properties and mixed pairwise-cover properties need not coincide without additional hypotheses.

The main tool used throughout the paper is the α-transport principle: $(\mathfrak{E}, \xi)$ is α-Lindelöf $\iff$ $(\mathfrak{E}, \xi^\alpha)$ is Lindelöf. This principle allows us to convert questions about α-open covers into ordinary Lindelöf questions in the associated α-topology. In the bitopological setting, the same method leads naturally to the associated bitopological space $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$. Thus, instead of treating α-open covers as an external generalized notion, we study them as ordinary open covers in the transported topology.

The contributions of the paper may be summarized as follows. First, we formalize componentwise and direct pairwise versions of α-Lindelöfness and introduce the corresponding α-Lindelöf numbers. Second, we establish the basic equivalence between α-Lindelöfness of $(\mathfrak{E}, \xi_i)$ and ordinary Lindelöfness of $(\mathfrak{E}, \xi_i^\alpha)$, together with its bitopological forms. Third, we derive cardinal estimates involving the network weight and weight of ξ_i^α, and show that countable networks yield hereditary Lindelöfness of the associated α-topological spaces. Fourth, we prove a countable-sum formula for the α-Lindelöf number. Fifth, we analyze the relation between α-Lindelöfness, α-compactness, and α-countable compactness, including examples that separate these properties and show the necessity of the hypotheses. Finally, preservation under suitable maps and product-type results are discussed under explicit compatibility assumptions between product α-topologies and products of α-topologies.

Recent developments in Lindelöf-type covering properties and related variants, including productively Lindelöf spaces [13], cellular Lindelöf spaces [30], R-star-Lindelöf spaces [12], and newer Lindelöf-type constructions in generalized and bitopological settings [6, 7, 8, 9], show that refined covering properties remain an active and flexible area of research. The present work contributes to this direction by clarifying how α-Lindelöfness behaves under the natural passage from (ξ_1, ξ_2) to $(\xi_1^\alpha, \xi_2^\alpha)$.

The paper is organized as follows. Section 2 recalls the necessary definitions concerning α-open sets, α-topologies, and bitopological covering notions. Section 3 introduces componentwise and direct pairwise α-Lindelöfness, establishes the basic transport equivalences, and studies cardinal bounds, networks, countable sums, and related examples. Section 4 examines the relationship between α-Lindelöfness and α-compactness, including the role of α-countable compactness and normality-type assumptions. The final part summarizes the main conclusions and indicates possible directions for further study.

2. Preliminaries

For any $(\mathfrak{E}, \xi)$, "α–open in ξ" equals "open in ξ^α".Thus α–cover properties in $(\mathfrak{E}, \xi)$ are ordinary cover properties in $(\mathfrak{E}, \xi^\alpha)$. We routinely prove results in $(\mathfrak{E}, \xi^\alpha)$ and translate back.

Definition 2.1. [22] In the topological space (G, θ), a subset $K \subseteq G$ is called an α–set iff

$$K \subseteq \mathrm{Int}_\theta\big(\mathrm{Cl}_\theta(\mathrm{Int}_\theta(K))\big).$$

Definition 2.2. [22] A set $U \subseteq \mathfrak{E}$ is α-open in ξ if

$$U \subseteq \mathrm{Int}_\xi\big(\mathrm{Cl}_\xi(\mathrm{Int}_\xi(U))\big).$$

The family ξ^α of all α-open sets is a topology and $\xi \subseteq \xi^\alpha$.

Definition 2.3. [18] Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space and let ξ_i^α be the α–topology of ξ_i. A family $\mathcal{U}$ of subsets of $\mathfrak{E}$ is a *pairwise α-cover* of $(\mathfrak{E}, \xi_1, \xi_2)$ if $\mathfrak{E} = \bigcup \mathcal{U}, \quad \mathcal{U} \subset (\xi_1^\alpha \cup \xi_2^\alpha)$, and each side is represented nontrivially: $\mathcal{U} \cap \xi_i^\alpha \neq \varnothing \quad (i = 1, 2)$.

Definition 2.4. [18] A bitopological space $(\mathfrak{E}, \xi_1, \xi_2)$ is called *pairwise α-normal* if $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$ is pairwise normal; that is, whenever F_1 is ξ_1^α-closed and F_2 is ξ_2^α-closed with $F_1 \cap F_2 = \varnothing$, there exist $U \in \xi_2^\alpha$ and $V \in \xi_1^\alpha$ such that $F_1 \subseteq U, \quad F_2 \subseteq V, \quad U \cap V = \varnothing$.

Definition 2.5. [21] A space $(\mathfrak{E}, \xi)$ is *α–countably compact* if every countable α–open cover of $\mathfrak{E}$ has a finite subcover.

Definition 2.6. Let $\psi\colon (\mathfrak{E}, \xi) \to (\mathfrak{H}, \eta)$.

- [21] ψ is called α-continuous if $\psi^{-1}(V) \in \xi^\alpha$ for every $V \in \eta$.
- [22] ψ is *α–closed* if for every ξ^α–closed $F \subseteq \mathfrak{E}$, the image $\psi(F)$ is η^α–closed.
- [4] ψ is *α–perfect* if it is α–continuous and α–closed, and every fiber $\psi^{-1}(\mathfrak{h})$ is ξ^α–compact for all $\mathfrak{h} \in \mathfrak{H}$.

Definition 2.7. [14] Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space. A family $\mathcal{U} \subseteq \xi_1 \cup \xi_2$ with $\bigcup \mathcal{U} = \mathfrak{E}$ is a *pairwise open cover*. For a cardinal κ, $\mathfrak{E}$ is *κ–bi-Lindelöf* if every pairwise open cover of $\mathfrak{E}$ has a subcover of cardinality $\leq \kappa$. The *bi-Lindelöf number* of $(\mathfrak{E}, \xi_1, \xi_2)$ is $\mathrm{bL}(\mathfrak{E}) = \min\{\kappa : \ \mathfrak{E} \text{ is } \kappa\text{–bi-Lindelöf}\}$. When $\kappa = \aleph_0$ we simply say that $\mathfrak{E}$ is bi-Lindelöf.

3. Pairwise α-Lindelöf

In this section, we prove $(\mathfrak{E}, \xi)$ is α–Lindelöf $\Leftrightarrow$ $(\mathfrak{E}, \xi^\alpha)$ Lindelöf and the join version for pairwise–cover. We give base criteria with cardinal bounds, the exact countable–sum formula, and a product bound via the tube lemma with one α–compact factor.

Definition 3.1. Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space. For $i = 1, 2$, let ξ_i^α denote the topology consisting of all α-open subsets of $(\mathfrak{E}, \xi_i)$.

A family $\mathcal{U}$ of subsets of $\mathfrak{E}$ is called a *pairwise α-open cover* of $(\mathfrak{E}, \xi_1, \xi_2)$ if $\mathfrak{E} = \bigcup \mathcal{U}, \quad \mathcal{U} \subseteq \xi_1^\alpha \cup \xi_2^\alpha$, and $\mathcal{U} \cap \xi_i^\alpha \neq \varnothing \quad (i = 1, 2)$.

The bitopological space $(\mathfrak{E}, \xi_1, \xi_2)$ is called *pairwise α-Lindelöf* if every pairwise α-open cover of $(\mathfrak{E}, \xi_1, \xi_2)$ contains a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathfrak{E} = \bigcup \mathcal{V}$.

Definition 3.2. Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space. For $i = 1, 2$, let

$$\xi_i^\alpha = \{A \subseteq \mathfrak{E} : A \subseteq \mathrm{Int}_{\xi_i} \mathrm{Cl}_{\xi_i} \mathrm{Int}_{\xi_i}(A)\}$$

be the α-topology associated with ξ_i.

A family $\mathcal{U}_i \subseteq \xi_i^\alpha$ is called an $\alpha(\xi_i)$-open cover of $\mathfrak{E}$ if $\mathfrak{E} = \bigcup \mathcal{U}_i$.

The α-Lindelöf number of $(\mathfrak{E}, \xi_i)$ is defined by

$$L^\alpha(\mathfrak{E}, \xi_i) = \min\{\kappa \geq \omega : \text{every } \alpha(\xi_i)\text{-open cover of } \mathfrak{E} \text{ has a subcover of cardinality at most } \kappa\}.$$

The *componentwise pairwise α-Lindelöf number* of $(\mathfrak{E}, \xi_1, \xi_2)$ is

$$L^\alpha_{\mathrm{cw}}(\mathfrak{E}) = \max\{L^\alpha(\mathfrak{E}, \xi_1), L^\alpha(\mathfrak{E}, \xi_2)\}.$$

A family $\mathcal{U}$ is called a *mixed pairwise α-cover* of $(\mathfrak{E}, \xi_1, \xi_2)$ if $\mathcal{U} \subseteq \xi_1^\alpha \cup \xi_2^\alpha$ and $\mathfrak{E} = \bigcup \mathcal{U}$. The *direct pairwise α-Lindelöf number* of $(\mathfrak{E}, \xi_1, \xi_2)$ is defined by

$$\widehat{L}^{\alpha}_{\mathrm{pair}}(\mathfrak{E}) = \min\{\kappa \geq \omega : \text{every mixed pairwise } \alpha\text{-cover of } \mathfrak{E} \text{ has a subcover of cardinality at most } \kappa\}.$$

Remark 3.3. The notation used above separates three related but different invariants. For a single component topology ξ_i, $L^\alpha(\mathfrak{E}, \xi_i)$ denotes the α-Lindelöf number of the space $(\mathfrak{E}, \xi_i)$. The componentwise invariant is

$$L^{\alpha}_{\mathrm{cw}}(\mathfrak{E}) = \max\{L^\alpha(\mathfrak{E}, \xi_1), L^\alpha(\mathfrak{E}, \xi_2)\}.$$

On the other hand, $\widehat{L}^{\alpha}_{\mathrm{pair}}(\mathfrak{E})$ denotes the direct mixed-cover pairwise invariant, where covers are allowed to have members from $\xi_1^\alpha \cup \xi_2^\alpha$. The hat is used to distinguish this mixed-cover invariant from the componentwise maximum. As shown in Proposition 3.6, every componentwise α-open cover is a special case of a mixed α-cover. Hence $L^{\alpha}_{\mathrm{cw}}(\mathfrak{E}) \leq \widehat{L}^{\alpha}_{\mathrm{pair}}(\mathfrak{E})$. Example 3.7 shows that this inequality reflects a genuine distinction between the two notions.

Lemma 3.4. *Let $(\mathfrak{E}, \xi)$ be a topological space. Then $(\mathfrak{E}, \xi)$ is α-Lindelöf if and only if every family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ satisfying $\mathfrak{E} = \bigcup_{\lambda \in \Lambda} U_\lambda$ and*

$$U_\lambda \subseteq \operatorname{Int}_\xi \operatorname{Cl}_\xi \operatorname{Int}_\xi(U_\lambda) \qquad (\lambda \in \Lambda)$$

contains a countable subfamily $\{U_{\lambda_n} : n \in \mathbb{N}\}$ such that $\mathfrak{E} = \bigcup_{n \in \mathbb{N}} U_{\lambda_n}$.

Proof
By definition, a subset $U \subseteq \mathfrak{E}$ is α-open in $(\mathfrak{E}, \xi)$ precisely when $U \subseteq \operatorname{Int}_\xi \operatorname{Cl}_\xi \operatorname{Int}_\xi(U)$. Hence a family satisfying the displayed condition is exactly an α-open cover of $(\mathfrak{E}, \xi)$. Therefore the assertion that every such cover has a countable subcover is precisely the definition of α-Lindelöfness. □

Proposition 3.5. *Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space.*

1. *The space $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf if and only if, for each $i = 1, 2$, every cover $\mathcal{U}_i = \{U_\lambda : \lambda \in \Lambda\}$ of $\mathfrak{E}$ satisfying*

$$U_\lambda \subseteq \operatorname{Int}_{\xi_i} \operatorname{Cl}_{\xi_i} \operatorname{Int}_{\xi_i}(U_\lambda) \qquad (\lambda \in \Lambda)$$

   has a countable subcover.
2. *The space $(\mathfrak{E}, \xi_1, \xi_2)$ is mixed-cover α-Lindelöf if and only if every cover $\mathcal{U}$ of $\mathfrak{E}$ such that each $U \in \mathcal{U}$ satisfies $U \subseteq \operatorname{Int}_{\xi_i} \operatorname{Cl}_{\xi_i} \operatorname{Int}_{\xi_i}(U)$ for at least one $i \in \{1, 2\}$, has a countable subcover.*

Proof
For (1), apply Lemma 3.4 separately to $(\mathfrak{E}, \xi_1)$ and $(\mathfrak{E}, \xi_2)$.
For (2), a member U belongs to $\xi_1^\alpha \cup \xi_2^\alpha$ exactly when it satisfies the displayed α-openness condition with respect to at least one of the two topologies ξ_1 or ξ_2. Hence a mixed-cover α-cover is precisely a cover of $\mathfrak{E}$ by subsets that are α-open with respect to at least one component topology. The countable-subcover condition is therefore exactly mixed-cover α-Lindelöfness. □

Proposition 3.6. *Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space. Then $\widehat{L}^{\alpha}_{\mathrm{pair}}(\mathfrak{E}) \geq L^{\alpha}_{\mathrm{cw}}(\mathfrak{E})$. In particular, if $(\mathfrak{E}, \xi_1, \xi_2)$ is mixed-cover α-Lindelöf, then it is componentwise α-Lindelöf.*

Proof
For $i = 1, 2$, every $\alpha(\xi_i)$-open cover $\mathcal{U}_i \subseteq \xi_i^\alpha$ is also a mixed α-cover, since $\mathcal{U}_i \subseteq \xi_i^\alpha \subseteq \xi_1^\alpha \cup \xi_2^\alpha$. Therefore every cardinal that bounds the sizes of subcovers of all mixed α-covers also bounds the sizes of subcovers of all $\alpha(\xi_i)$-open covers. Hence

$$L^\alpha(\mathfrak{E}, \xi_i) \leq \widehat{L}^{\alpha}_{\mathrm{pair}}(\mathfrak{E}) \qquad (i = 1, 2).$$

Taking the maximum over $i = 1, 2$, we obtain

$$L^{\alpha}_{\mathrm{cw}}(\mathfrak{E}) = \max\{L^{\alpha}(\mathfrak{E}, \xi_1), L^{\alpha}(\mathfrak{E}, \xi_2)\} \le \widehat{L}^{\alpha}_{\mathrm{pair}}(\mathfrak{E}).$$

Thus mixed-cover α-Lindelöfness implies componentwise α-Lindelöfness. □

Example 3.7. Let $\mathfrak{E}$ be an uncountable set, and choose two distinct points $p, q \in \mathfrak{E}$. Let ξ_1 be the excluded-point topology at p, namely

$$\xi_1 = \{\varnothing, \mathfrak{E}\} \cup \{U \subseteq \mathfrak{E} : p \notin U\},$$

and let ξ_2 be the excluded-point topology at q, namely

$$\xi_2 = \{\varnothing, \mathfrak{E}\} \cup \{U \subseteq \mathfrak{E} : q \notin U\}.$$

Each of $(\mathfrak{E}, \xi_1)$ and $(\mathfrak{E}, \xi_2)$ is compact. Indeed, in $(\mathfrak{E}, \xi_1)$, the only open set containing p is $\mathfrak{E}$, so every ξ_1-open cover of $\mathfrak{E}$ contains $\mathfrak{E}$. Hence it has a one-element subcover. The same argument applies to $(\mathfrak{E}, \xi_2)$. Thus both component spaces are compact and therefore Lindelöf. We next observe that $\xi_1^{\alpha} = \xi_1$ and $\xi_2^{\alpha} = \xi_2$. We prove this for ξ_1; the proof for ξ_2 is identical. Every ξ_1-open set is α-open. Conversely, let $U \notin \xi_1$. Then $p \in U$ and $U \neq \mathfrak{E}$. Hence $\mathrm{Int}_{\xi_1}(U) = U \setminus \{p\}$. If $U \setminus \{p\} = \varnothing$, then $\mathrm{Int}_{\xi_1} \mathrm{Cl}_{\xi_1} \mathrm{Int}_{\xi_1}(U) = \varnothing$, so U is not α-open. If $U \setminus \{p\} \neq \varnothing$, then $\mathrm{Cl}_{\xi_1}(U \setminus \{p\}) = U$, and therefore

$$\mathrm{Int}_{\xi_1} \mathrm{Cl}_{\xi_1} \mathrm{Int}_{\xi_1}(U) = \mathrm{Int}_{\xi_1}(U) = U \setminus \{p\}.$$

Thus $U \nsubseteq \mathrm{Int}_{\xi_1} \mathrm{Cl}_{\xi_1} \mathrm{Int}_{\xi_1}(U)$, and U is not α-open. Consequently, $\xi_1^{\alpha} = \xi_1$. Similarly, $\xi_2^{\alpha} = \xi_2$. It follows that $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact and hence componentwise α-Lindelöf. However, it is not mixed-cover α-Lindelöf. Consider the singleton cover $\mathcal{S} = \big\{\{\mathfrak{e}\} : \mathfrak{e} \in \mathfrak{E}\big\}$. If $\mathfrak{e} \neq p$, then $\{\mathfrak{e}\} \in \xi_1 \subseteq \xi_1^{\alpha}$. If $\mathfrak{e} = p$, then $p \neq q$, so $\{p\} \in \xi_2 \subseteq \xi_2^{\alpha}$. Hence $\mathcal{S} \subseteq \xi_1^{\alpha} \cup \xi_2^{\alpha}$. Clearly, $\mathfrak{E} = \bigcup \mathcal{S}$, so $\mathcal{S}$ is a mixed α-cover of $(\mathfrak{E}, \xi_1, \xi_2)$. Since $\mathfrak{E}$ is uncountable, no countable subfamily of $\mathcal{S}$ covers $\mathfrak{E}$. Therefore $(\mathfrak{E}, \xi_1, \xi_2)$ is not mixed-cover α-Lindelöf.

Remark 3.8. Proposition 3.6 and Example 3.7 show that, for the mixed-cover notion used in this paper, the implication relation is one-directional:

$$\text{mixed-cover } \alpha\text{-Lindelöf} \Longrightarrow \text{componentwise } \alpha\text{-Lindelöf},$$

but the converse fails in general. Thus a space cannot be mixed-cover α-Lindelöf and fail to be componentwise α-Lindelöf under this definition. The interaction between the two topologies is therefore encoded in the possible failure of the converse.

Lemma 3.9. *Let $\mathfrak{E}$ be an infinite set.*

1. *If ξ is the cofinite topology on $\mathfrak{E}$, then $\xi^{\alpha} = \xi$.*
2. *If $\mathfrak{E}$ is uncountable and ξ is the cocountable topology on $\mathfrak{E}$, then $\xi^{\alpha} = \xi$.*

Proof
Let

$$\Phi_{\xi}(U) = \mathrm{Int}_{\xi}\big(\mathrm{Cl}_{\xi}(\mathrm{Int}_{\xi}(U))\big).$$

Suppose first that ξ is the cofinite topology on the infinite set $\mathfrak{E}$. Every ξ-open set is α-open: this is clear for $U = \varnothing$, while if $U \neq \varnothing$ is cofinite open, then

$$\mathrm{Int}_{\xi}(U) = U, \qquad \mathrm{Cl}_{\xi}(U) = \mathfrak{E},$$

and therefore $\Phi_{\xi}(U) = \mathfrak{E}$, so $U \subseteq \Phi_{\xi}(U)$.

Conversely, if $U \notin \xi$, then $\mathfrak{E} \setminus U$ is infinite. Hence U contains no nonempty cofinite open set, and so $\mathrm{Int}_{\xi}(U) = \varnothing$. Thus $\Phi_{\xi}(U) = \varnothing$, and U cannot be α-open unless $U = \varnothing$. Hence $\xi^{\alpha} = \xi$.

The proof for the cocountable topology is analogous. If $U \neq \varnothing$ is cocountable open, then U is dense, so $\Phi_{\xi}(U) = \mathfrak{E}$. If $U \notin \xi$, then $\mathfrak{E} \setminus U$ is uncountable, so U contains no nonempty cocountable open set; hence $\mathrm{Int}_{\xi}(U) = \varnothing$ and again $\Phi_{\xi}(U) = \varnothing$. Therefore $\xi^{\alpha} = \xi$. □

Example 3.10. Let $\mathfrak{E}$ be an infinite set, and let $\xi_1 = \xi_2 = \xi_{\mathrm{cof}}$, where ξ_{cof} denotes the cofinite topology on $\mathfrak{E}$. By Lemma 3.9, $\xi_i^\alpha = \xi_i \quad (i = 1, 2)$. Since the cofinite topology is compact, each $(\mathfrak{E}, \xi_i)$ is α-compact. Hence $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact, and therefore componentwise α-Lindelöf.

Example 3.11. Let $\mathfrak{E}$ be an uncountable set and let $D \subseteq \mathfrak{E}$ be countable. Let ξ_1 be the cocountable topology on $\mathfrak{E}$, and let ξ_2 be the topology generated by the cocountable subsets of $\mathfrak{E}$ and the singletons $\{d\}$, $d \in D$. Then

$$\xi_2 = \mathcal{P}(D) \cup \{U \subseteq \mathfrak{E} : \mathfrak{E} \setminus U \text{ is countable}\}.$$

We claim that $\xi_2^\alpha = \xi_2$. Since every open set is α-open, it remains to show that no non-open subset of $\mathfrak{E}$ is α-open.

Let $U \notin \xi_2$. By the definition of ξ_2, this means that U is not contained in D and that U is not cocountable. Equivalently, $U \nsubseteq D \quad \text{and} \quad \mathfrak{E} \setminus U$ is uncountable. We first compute the ξ_2-interior of U. Since

$$U \cap D \in \mathcal{P}(D) \subseteq \xi_2$$

and $U \cap D \subseteq U$, we have $U \cap D \subseteq \mathrm{Int}_{\xi_2}(U)$. Conversely, let $O \in \xi_2$ with $O \subseteq U$. Then, by the definition of ξ_2, either $O \subseteq D$, or O is cocountable. If O were cocountable, then $\mathfrak{E} \setminus U \subseteq \mathfrak{E} \setminus O$, and since $\mathfrak{E} \setminus O$ is countable, it would follow that $\mathfrak{E} \setminus U$ is countable. This contradicts the fact that U is not cocountable. Hence every ξ_2-open subset O of U must satisfy $O \subseteq D$. Therefore $O \subseteq U \cap D$. Taking the union of all ξ_2-open subsets of U, we obtain $\mathrm{Int}_{\xi_2}(U) = U \cap D$.

Now $U \cap D \subseteq D$, and hence $U \cap D \in \xi_2$. Also, $U \cap D$ is countable, so its complement $\mathfrak{E} \setminus (U \cap D)$ is cocountable and therefore ξ_2-open. Thus $U \cap D$ is ξ_2-closed. Consequently,

$$\mathrm{Cl}_{\xi_2} \mathrm{Int}_{\xi_2}(U) = \mathrm{Cl}_{\xi_2}(U \cap D) = U \cap D.$$

Since $U \cap D \in \xi_2$, we also have

$$\mathrm{Int}_{\xi_2} \mathrm{Cl}_{\xi_2} \mathrm{Int}_{\xi_2}(U) = \mathrm{Int}_{\xi_2}(U \cap D) = U \cap D.$$

But $U \nsubseteq D$, and hence $U \nsubseteq U \cap D$. Therefore

$$U \nsubseteq \mathrm{Int}_{\xi_2} \mathrm{Cl}_{\xi_2} \mathrm{Int}_{\xi_2}(U).$$

So U is not α-open. Hence no non-open subset of $\mathfrak{E}$ is α-open with respect to ξ_2, and therefore $\xi_2^\alpha = \xi_2$.

Also, by Lemma 3.9, $\xi_1^\alpha = \xi_1$. Both $(\mathfrak{E}, \xi_1)$ and $(\mathfrak{E}, \xi_2)$ are Lindelöf. Since

$$\xi_1^\alpha = \xi_1, \qquad \xi_2^\alpha = \xi_2,$$

the space is componentwise α-Lindelöf.

Moreover, since $\xi_1 \subseteq \xi_2$, every mixed α-cover $\mathcal{U} \subseteq \xi_1^\alpha \cup \xi_2^\alpha$ is a ξ_2-open cover. Since $(\mathfrak{E}, \xi_2)$ is Lindelöf, such a cover has a countable subcover. Hence the space is also mixed-cover α-Lindelöf.

Example 3.12. Let $\mathfrak{E} = \mathfrak{H} \sqcup \mathfrak{M}$, where $\mathfrak{H}$ and $\mathfrak{M}$ are uncountable disjoint sets. Define

$$\xi_1 = \tau_{\mathrm{coc}}(\mathfrak{H}) \sqcup \tau_{\mathrm{cof}}(\mathfrak{M}), \qquad \xi_2 = \tau_{\mathrm{cof}}(\mathfrak{H}) \sqcup \tau_{\mathrm{coc}}(\mathfrak{M}),$$

where τ_{coc} denotes the cocountable topology and τ_{cof} denotes the cofinite topology. Thus ξ_1 and ξ_2 are topological-sum topologies on $\mathfrak{E}$.

By Lemma 3.9, the cofinite and cocountable topologies are α-invariant. Since the operation

$$U \longmapsto \mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}(U)$$

is computed componentwise in a topological sum, we obtain

$$\xi_1^\alpha = \xi_1, \qquad \xi_2^\alpha = \xi_2.$$

We show that $(\mathfrak{E},\xi_1)$ is Lindelöf. Let $\mathcal{U}$ be a ξ_1-open cover of $\mathfrak{E}$. Choose $U_{\mathfrak{H}}\in\mathcal{U}$ with $U_{\mathfrak{H}}\cap\mathfrak{H}\neq\varnothing$, and choose $U_{\mathfrak{M}}\in\mathcal{U}$ with $U_{\mathfrak{M}}\cap\mathfrak{M}\neq\varnothing$. Then $U_{\mathfrak{H}}\cap\mathfrak{H}$ is cocountable in $\mathfrak{H}$, and $U_{\mathfrak{M}}\cap\mathfrak{M}$ is cofinite in $\mathfrak{M}$. Hence

$$\mathfrak{E}\setminus(U_{\mathfrak{H}}\cup U_{\mathfrak{M}})\subseteq(\mathfrak{H}\setminus U_{\mathfrak{H}})\cup(\mathfrak{M}\setminus U_{\mathfrak{M}})$$

is countable. Choosing one member of $\mathcal{U}$ for each remaining point gives a countable subcover. Therefore $(\mathfrak{E},\xi_1)$ is Lindelöf.

The same argument shows that $(\mathfrak{E},\xi_2)$ is Lindelöf. Since $\xi_i^\alpha=\xi_i$ for $i=1,2$, the bitopological space $(\mathfrak{E},\xi_1,\xi_2)$ is componentwise α-Lindelöf.

It is not componentwise α-compact. Indeed, $\mathfrak{H}$ is clopen in $(\mathfrak{E},\xi_1)$, and the induced topology on $\mathfrak{H}$ is the cocountable topology. Since an uncountable cocountable space is not compact, $(\mathfrak{E},\xi_1)$ is not compact. As $\xi_1^\alpha=\xi_1$, $(\mathfrak{E},\xi_1)$ is not α-compact. Hence $(\mathfrak{E},\xi_1,\xi_2)$ is componentwise α-Lindelöf but not componentwise α-compact.

Theorem 3.13. *If $(\mathfrak{E},\xi_1,\xi_2)$ is componentwise α-Lindelöf, then $(\mathfrak{E},\xi_1,\xi_2)$ is componentwise Lindelöf; that is, $(\mathfrak{E},\xi_i)$ is Lindelöf for each $i=1,2$.*

Proof
Fix $i\in\{1,2\}$. Since $(\mathfrak{E},\xi_i)$ is α-Lindelöf, $(\mathfrak{E},\xi_i^\alpha)$ is Lindelöf. Moreover, $\xi_i\subseteq\xi_i^\alpha$. Hence every ξ_i-open cover of $\mathfrak{E}$ is also a ξ_i^α-open cover. Since $(\mathfrak{E},\xi_i^\alpha)$ is Lindelöf, every ξ_i-open cover has a countable subcover. Therefore $(\mathfrak{E},\xi_i)$ is Lindelöf. Since this holds for $i=1,2$, $(\mathfrak{E},\xi_1,\xi_2)$ is componentwise Lindelöf. □

Example 3.14. Let $\mathfrak{E}$ be an uncountable set. Let $\xi_1=\tau_{\text{coc}}$ be the cocountable topology on $\mathfrak{E}$, and let $\xi_2=\tau_{\text{cof}}$ be the cofinite topology on $\mathfrak{E}$. By Lemma 3.9, $\xi_i^\alpha=\xi_i \quad (i=1,2)$.

The space $(\mathfrak{E},\xi_1)$ is Lindelöf. Indeed, if $\mathcal{U}$ is a ξ_1-open cover of $\mathfrak{E}$, choose a nonempty member $U_0\in\mathcal{U}$. Then U_0 is cocountable, so $\mathfrak{E}\setminus U_0$ is countable. For each $\mathfrak{e}\in\mathfrak{E}\setminus U_0$, choose $U_{\mathfrak{e}}\in\mathcal{U}$ such that $\mathfrak{e}\in U_{\mathfrak{e}}$. Then

$$\{U_0\}\cup\{U_{\mathfrak{e}}:\mathfrak{e}\in\mathfrak{E}\setminus U_0\}$$

is a countable subcover. Thus $(\mathfrak{E},\xi_1)$ is Lindelöf.

The space $(\mathfrak{E},\xi_2)$ is compact, since every open cover contains a nonempty cofinite member and only finitely many points remain uncovered.

Since $\xi_i^\alpha=\xi_i$, it follows that $(\mathfrak{E},\xi_1,\xi_2)$ is componentwise α-Lindelöf. Moreover, because $\xi_2\subseteq\xi_1$, every mixed pairwise α-cover $\mathcal{U}\subseteq\xi_1^\alpha\cup\xi_2^\alpha$ is in fact a ξ_1-open cover. Hence it has a countable subcover, and so $(\mathfrak{E},\xi_1,\xi_2)$ is pairwise α-Lindelöf in the mixed-cover sense as well.

However, $(\mathfrak{E},\xi_1,\xi_2)$ is not componentwise α-compact. Indeed, $(\mathfrak{E},\xi_1)$ is not compact. To see this, choose a countably infinite set

$$D=\{\mathfrak{e}_n:n\in\mathbb{N}\}\subseteq\mathfrak{E}$$

and put $C_n=\{\mathfrak{e}_k:k\geq n\}$. Then

$$\mathcal{V}=\{\mathfrak{E}\setminus C_n:n\in\mathbb{N}\}$$

is a ξ_1-open cover of $\mathfrak{E}$, since each C_n is countable and

$$\bigcup_{n\in\mathbb{N}}(\mathfrak{E}\setminus C_n)=\mathfrak{E}.$$

But no finite subfamily of $\mathcal{V}$ covers $\mathfrak{E}$. Hence $(\mathfrak{E},\xi_1)$ is not compact. Since $\xi_1^\alpha=\xi_1$, the space $(\mathfrak{E},\xi_1)$ is not α-compact. Therefore $(\mathfrak{E},\xi_1,\xi_2)$ is not componentwise α-compact.

Theorem 3.15. *For any topological space $(\mathfrak{E},\xi)$,*

$$(\mathfrak{E},\xi)\text{ is }\alpha\text{-Lindelöf}\iff(\mathfrak{E},\xi^\alpha)\text{ is Lindelöf.}$$

Consequently, for any bitopological space $(\mathfrak{E},\xi_1,\xi_2)$, the following hold:

1. $(\mathfrak{E}, \xi_1, \xi_2)$ *is componentwise* α*-Lindelöf if and only if* $(\mathfrak{E}, \xi_i^\alpha)$ *is Lindelöf for* $i = 1, 2$.
2. $(\mathfrak{E}, \xi_1, \xi_2)$ *is pairwise* α*-Lindelöf in the mixed-cover sense if and only if* $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$ *is pairwise Lindelöf in the ordinary mixed-cover sense.*

Proof
First let $(\mathfrak{E}, \xi)$ be a topological space. By definition, ξ^α is the topology whose open sets are precisely the α-open subsets of $(\mathfrak{E}, \xi)$. Hence a family $\mathcal{U}$ is an α-open cover of $(\mathfrak{E}, \xi)$ if and only if $\mathcal{U}$ is an ordinary open cover of $(\mathfrak{E}, \xi^\alpha)$. Therefore, the statement that every α-open cover of $(\mathfrak{E}, \xi)$ has a countable subcover is exactly the statement that every open cover of $(\mathfrak{E}, \xi^\alpha)$ has a countable subcover. Thus

$$(\mathfrak{E}, \xi) \text{ is } \alpha\text{-Lindelöf} \quad \Longleftrightarrow \quad (\mathfrak{E}, \xi^\alpha) \text{ is Lindelöf.}$$

Now let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space.

(1) Suppose first that $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf. Then, for each $i = 1, 2$, the space $(\mathfrak{E}, \xi_i)$ is α-Lindelöf. By the one-topology equivalence proved above, $(\mathfrak{E}, \xi_i^\alpha)$ is Lindelöf $\quad (i = 1, 2)$. Conversely, if $(\mathfrak{E}, \xi_i^\alpha)$ is Lindelöf for $i = 1, 2$, then again by the one-topology equivalence, each $(\mathfrak{E}, \xi_i)$ is α-Lindelöf. Hence $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf.

(2) Suppose now that pairwise α-Lindelöfness is understood in the mixed-cover sense. A family $\mathcal{U}$ is a pairwise α-cover of $(\mathfrak{E}, \xi_1, \xi_2)$ precisely when

$$\mathcal{U} \subseteq \xi_1^\alpha \cup \xi_2^\alpha \quad \text{and} \quad \mathfrak{E} = \bigcup \mathcal{U}.$$

But this is exactly the definition of a pairwise open cover of the bitopological space $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$. Therefore, every pairwise α-cover of $(\mathfrak{E}, \xi_1, \xi_2)$ has a countable subcover if and only if every pairwise open cover of $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$ has a countable subcover. Hence $(\mathfrak{E}, \xi_1, \xi_2)$ is pairwise α-Lindelöf in the mixed-cover sense if and only if $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$ is pairwise Lindelöf in the ordinary mixed-cover sense. $\square$

Remark 3.16. The α-transport principle gives a complete reduction for properties formulated purely in terms of α-open covers of the whole space. However, some questions require direct verification in terms of α-open sets or additional compatibility hypotheses. This occurs, for example, in subspace problems, since $(\xi|_A)^\alpha$ need not coincide with $(\xi^\alpha)|_A$, and in product problems, since $(\xi \times \eta)^\alpha \subseteq \xi^\alpha \times \eta^\alpha$ need not hold in general. Mapping results also require care, because ordinary α-continuity is weaker than continuity $\psi : (\mathfrak{E}, \xi^\alpha) \to (\mathfrak{H}, \eta^\alpha)$. Thus the transport principle is an organizing tool, while subspaces, products, mixed covers, and mappings may require direct α-open-set arguments.

Example 3.17. Let $\mathfrak{E}$ be an uncountable set, and let ξ be the cocountable topology on $\mathfrak{E}$. By Lemma 3.9, $\xi^\alpha = \xi$.

We show that $(\mathfrak{E}, \xi)$ is Lindelöf. Let $\mathcal{U}$ be an open cover of $\mathfrak{E}$. Choose a nonempty member $U_0 \in \mathcal{U}$. Since U_0 is cocountable, the set $\mathfrak{E} \setminus U_0$ is countable. For each $\mathfrak{e} \in \mathfrak{E} \setminus U_0$, choose $U_\mathfrak{e} \in \mathcal{U}$ such that $\mathfrak{e} \in U_\mathfrak{e}$. Then

$$\{U_0\} \cup \{U_\mathfrak{e} : \mathfrak{e} \in \mathfrak{E} \setminus U_0\}$$

is a countable subcover. Hence $(\mathfrak{E}, \xi)$ is Lindelöf.

Therefore, since $\xi^\alpha = \xi$, the space $(\mathfrak{E}, \xi)$ is α-Lindelöf. This example is nontrivial because the cocountable topology on an uncountable set is neither discrete nor metrizable.

Example 3.18. Let C be the classical Cantor subset of $\mathbb{R}$. Consider the bitopological space $(\mathbb{R}, \xi_1, \xi_2)$, where ξ_1 is the usual Euclidean topology on $\mathbb{R}$, and ξ_2 is the lower-limit topology on $\mathbb{R}$. For each $c \in C$, define

$$A_c = (\mathbb{R} \setminus C) \cup \{c\}.$$

We claim that $A_c \in \xi_1^\alpha$, although $A_c \notin \xi_1$. Indeed, since C is closed and has empty interior, $\mathbb{R} \setminus C$ is open and dense in $\mathbb{R}$. Hence $\mathrm{Int}_{\xi_1}(A_c) = \mathbb{R} \setminus C$, and therefore

$$\mathrm{Cl}_{\xi_1} \mathrm{Int}_{\xi_1}(A_c) = \mathrm{Cl}_{\xi_1}(\mathbb{R} \setminus C) = \mathbb{R}.$$

Consequently, $\mathrm{Int}_{\xi_1}\mathrm{Cl}_{\xi_1}\mathrm{Int}_{\xi_1}(A_c)=\mathbb{R}$, and so

$$A_c \subseteq \mathrm{Int}_{\xi_1}\mathrm{Cl}_{\xi_1}\mathrm{Int}_{\xi_1}(A_c).$$

Thus A_c is α-open with respect to ξ_1. However, A_c is not ξ_1-open, because $c \in C$ and every Euclidean neighbourhood of c meets $C \setminus \{c\}$, which is not contained in A_c. Hence $\xi_1 \subsetneq \xi_1^\alpha$. Now consider the family

$$\mathcal{U} = \{\mathbb{R} \setminus C\} \cup \{A_c : c \in C\}.$$

Since $\mathbb{R} \setminus C$ is usual open, it is also open in the lower-limit topology ξ_2, and hence $\mathbb{R} \setminus C \in \xi_2 \subseteq \xi_2^\alpha$. Moreover, $A_c \in \xi_1^\alpha$ for every $c \in C$. Therefore $\mathcal{U} \subseteq \xi_1^\alpha \cup \xi_2^\alpha$. Also, $\mathbb{R} = \bigcup\mathcal{U}$, because every point of $\mathbb{R} \setminus C$ is covered by $\mathbb{R} \setminus C$, while each point $c \in C$ is covered by A_c. Thus $\mathcal{U}$ is a mixed α-cover of $(\mathbb{R}, \xi_1, \xi_2)$. However, $\mathcal{U}$ has no countable subcover. Indeed, if $\{c_n : n \in \mathbb{N}\} \subseteq C$, then

$$(\mathbb{R} \setminus C) \cup \bigcup_{n\in\mathbb{N}} A_{c_n} = (\mathbb{R} \setminus C) \cup \{c_n : n \in \mathbb{N}\}.$$

Since C is uncountable, there exists $c^* \in C \setminus \{c_n : n \in \mathbb{N}\}$. This point is not covered by the chosen countable subfamily. Hence no countable subfamily of $\mathcal{U}$ covers $\mathbb{R}$. Therefore $(\mathbb{R}, \xi_1, \xi_2)$ is not mixed-cover α-Lindelöf. In particular, this example shows that the α-topology may be strictly finer than the original topology and that the α-Lindelöf property can fail for genuinely α-open covers that are not ordinary open covers in the original topology.

Corollary 3.19. *For any topological space $(\mathfrak{E}, \xi)$,*

$$L^\alpha(\mathfrak{E}, \xi) \leq nw(\xi^\alpha) \leq w(\xi^\alpha).$$

If ξ^α has a countable network, then $(\mathfrak{E}, \xi^\alpha)$ is hereditarily Lindelöf.

Proof
By Theorem 3.15, $L^\alpha(\mathfrak{E}, \xi) = L(\mathfrak{E}, \xi^\alpha)$, where $L(\mathfrak{E}, \xi^\alpha)$ denotes the ordinary Lindelöf number of the space $(\mathfrak{E}, \xi^\alpha)$.

For every topological space, the Lindelöf number is bounded above by the network weight. Indeed, if $\mathcal{N}$ is a network for $(\mathfrak{E}, \xi^\alpha)$ and $\mathcal{U}$ is an open cover, then for each $N \in \mathcal{N}$ such that $N \subseteq U$ for some $U \in \mathcal{U}$, choose one such $U_N \in \mathcal{U}$. The family of all selected U_N's has cardinality at most $|\mathcal{N}|$, and it covers $\mathfrak{E}$. Hence $L(\mathfrak{E}, \xi^\alpha) \leq nw(\xi^\alpha)$. Therefore $L^\alpha(\mathfrak{E}, \xi) \leq nw(\xi^\alpha)$.

Moreover, every base is a network, so $nw(\xi^\alpha) \leq w(\xi^\alpha)$. Thus

$$L^\alpha(\mathfrak{E}, \xi) \leq nw(\xi^\alpha) \leq w(\xi^\alpha).$$

Finally, if ξ^α has a countable network, then every subspace of $(\mathfrak{E}, \xi^\alpha)$ also has a countable network. Hence every subspace of $(\mathfrak{E}, \xi^\alpha)$ is Lindelöf. Therefore $(\mathfrak{E}, \xi^\alpha)$ is hereditarily Lindelöf. □

Remark 3.20. Example 3.18 also shows that $nw(\xi^\alpha)$ may be strictly larger than $nw(\xi)$. Indeed, for the usual topology τ on $\mathbb{R}$, $nw(\tau) = \omega$. However, the Cantor set C becomes a discrete subspace of $(\mathbb{R}, \tau^\alpha)$, because for each $c \in C$ the set

$$A_c = (\mathbb{R} \setminus C) \cup \{c\}$$

is α-open and satisfies $A_c \cap C = \{c\}$. Hence

$$nw(\tau^\alpha) \geq |C| > \omega = nw(\tau).$$

Thus no general estimate of $L^\alpha(\mathfrak{E}, \xi)$ purely in terms of $nw(\xi)$ is possible without an additional compatibility assumption.

Example 3.21. Let $\mathfrak{E} = \mathbb{N}$ and let $\xi = \mathcal{P}(\mathbb{N})$ be the discrete topology. Then $\xi^\alpha = \xi$. Indeed, every subset of $\mathbb{N}$ is open, hence every subset is α-open.

The family $\mathcal{B} = \{\{n\} : n \in \mathbb{N}\}$ is a countable base for ξ^α. Therefore every ξ^α-open cover of $\mathbb{N}$ has a countable subcover. Hence $(\mathbb{N}, \xi^\alpha)$ is Lindelöf. By Theorem 3.15, the space $(\mathbb{N}, \xi)$ is α-Lindelöf.

Example 3.22. Let $\mathfrak{E}$ be an uncountable set, and let $\xi = \mathcal{P}(\mathfrak{E})$ be the discrete topology on $\mathfrak{E}$. Then $\xi^\alpha = \xi$, because every subset of $\mathfrak{E}$ is open and hence α-open.

Consider the singleton cover $\mathcal{U} = \{\{\mathfrak{e}\} : \mathfrak{e} \in \mathfrak{E}\}$. This is a ξ^α-open cover of $\mathfrak{E}$. However, no countable subfamily of $\mathcal{U}$ covers $\mathfrak{E}$, since $\mathfrak{E}$ is uncountable. Therefore $(\mathfrak{E}, \xi^\alpha)$ is not Lindelöf. By Theorem 3.15, $(\mathfrak{E}, \xi)$ is not α-Lindelöf.

Lemma 3.23. *Let $\{(\mathfrak{E}_n, \xi_n)\}_{n\in\mathbb{N}}$ be a pairwise disjoint family of topological spaces, and let*

$$\mathfrak{E} = \bigsqcup_{n\in\mathbb{N}} \mathfrak{E}_n, \qquad \xi = \bigsqcup_{n\in\mathbb{N}} \xi_n$$

be the corresponding topological sum. Then

$$\xi^\alpha = \bigsqcup_{n\in\mathbb{N}} \xi_n^\alpha.$$

Equivalently, for every $U \subseteq \mathfrak{E}$,

$$U \in \xi^\alpha \iff U \cap \mathfrak{E}_n \in \xi_n^\alpha \quad \textit{for every } n \in \mathbb{N}.$$

Proof
Let $U \subseteq \mathfrak{E}$. Since $(\mathfrak{E}, \xi)$ is the topological sum of the spaces $(\mathfrak{E}_n, \xi_n)$, interiors and closures are computed componentwise. That is, for every $n \in \mathbb{N}$,

$$\mathrm{Int}_\xi(U) \cap \mathfrak{E}_n = \mathrm{Int}_{\xi_n}(U \cap \mathfrak{E}_n),$$

and

$$\mathrm{Cl}_\xi(U) \cap \mathfrak{E}_n = \mathrm{Cl}_{\xi_n}(U \cap \mathfrak{E}_n).$$

Therefore,

$$\mathrm{Int}_\xi \mathrm{Cl}_\xi \mathrm{Int}_\xi(U) \cap \mathfrak{E}_n = \mathrm{Int}_{\xi_n} \mathrm{Cl}_{\xi_n} \mathrm{Int}_{\xi_n}(U \cap \mathfrak{E}_n).$$

Hence $U \subseteq \mathrm{Int}_\xi \mathrm{Cl}_\xi \mathrm{Int}_\xi(U)$ if and only if, for every $n \in \mathbb{N}$,

$$U \cap \mathfrak{E}_n \subseteq \mathrm{Int}_{\xi_n} \mathrm{Cl}_{\xi_n} \mathrm{Int}_{\xi_n}(U \cap \mathfrak{E}_n).$$

This means precisely that U is α-open in $(\mathfrak{E}, \xi)$ if and only if $U \cap \mathfrak{E}_n$ is α-open in $(\mathfrak{E}_n, \xi_n)$ for every $n \in \mathbb{N}$.

Thus the α-topology of the topological sum is exactly the topological sum of the α-topologies:

$$\xi^\alpha = \bigsqcup_{n\in\mathbb{N}} \xi_n^\alpha.$$

□

Theorem 3.24. *Let $\{(\mathfrak{E}_n, \xi_n)\}_{n\in\mathbb{N}}$ be a pairwise disjoint family of topological spaces, and let*

$$\mathfrak{E} = \bigsqcup_{n\in\mathbb{N}} \mathfrak{E}_n, \qquad \xi = \bigsqcup_{n\in\mathbb{N}} \xi_n$$

be the corresponding topological sum. Then

$$L^\alpha(\mathfrak{E}, \xi) = \max\left\{\omega, \sup_{n\in\mathbb{N}} L^\alpha(\mathfrak{E}_n, \xi_n)\right\}.$$

Proof
By Lemma 3.23,

$$\xi^\alpha = \bigsqcup_{n\in\mathbb{N}} \xi_n^\alpha.$$

Hence an α-open cover of $(\mathfrak{E}, \xi)$ is precisely an ordinary open cover of the topological sum

$$\bigsqcup_{n\in\mathbb{N}} (\mathfrak{E}_n, \xi_n^\alpha).$$

Put

$$\kappa_n = L^\alpha(\mathfrak{E}_n, \xi_n), \qquad \kappa = \sup_{n\in\mathbb{N}} \kappa_n.$$

By the convention used in the definition of Lindelöf numbers, each κ_n is an infinite cardinal, since $\kappa_n \geq \omega$. Hence κ is also an infinite cardinal and $\omega \leq \kappa$. All cardinal estimates below are understood in the usual ZFC setting.

Upper bound. Let $\mathcal{U}$ be an α-open cover of $(\mathfrak{E}, \xi)$. Equivalently, $\mathcal{U}$ is an open cover of $(\mathfrak{E}, \xi^\alpha)$. For each $n \in \mathbb{N}$, the family

$$\mathcal{U}|_{\mathfrak{E}_n} = \{U \cap \mathfrak{E}_n : U \in \mathcal{U}\}$$

is an open cover of $(\mathfrak{E}_n, \xi_n^\alpha)$. Since

$$L(\mathfrak{E}_n, \xi_n^\alpha) = L^\alpha(\mathfrak{E}_n, \xi_n) = \kappa_n,$$

there exists a subfamily $\mathcal{V}_n \subseteq \mathcal{U}$ with $|\mathcal{V}_n| \leq \kappa_n$ such that $\mathfrak{E}_n \subseteq \bigcup \mathcal{V}_n$.
Set $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$. Then $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathfrak{E}$. Moreover,

$$|\mathcal{V}| \leq \sum_{n\in\mathbb{N}} |\mathcal{V}_n| \leq \sum_{n\in\mathbb{N}} \kappa_n.$$

Since $\kappa_n \leq \kappa$ for every $n \in \mathbb{N}$, we have

$$\sum_{n\in\mathbb{N}} \kappa_n \leq \sum_{n\in\mathbb{N}} \kappa = \omega \cdot \kappa.$$

Because κ is an infinite cardinal, the standard cardinal arithmetic identity gives $\omega \cdot \kappa = \kappa = \max\{\omega, \kappa\}$. Therefore, $|\mathcal{V}| \leq \max\{\omega, \kappa\}$. Thus

$$L^\alpha(\mathfrak{E}, \xi) \leq \max\{\omega, \kappa\} = \max\left\{\omega, \sup_{n\in\mathbb{N}} L^\alpha(\mathfrak{E}_n, \xi_n)\right\}.$$

Lower bound. For each fixed $m \in \mathbb{N}$, the set $\mathfrak{E}_m$ is clopen in the topological sum $(\mathfrak{E}, \xi^\alpha)$. We claim that

$$L^\alpha(\mathfrak{E}_m, \xi_m) \leq L^\alpha(\mathfrak{E}, \xi).$$

Indeed, let $\mathcal{W}$ be an α-open cover of $(\mathfrak{E}_m, \xi_m)$. Since $\mathfrak{E}_m$ is clopen in $(\mathfrak{E}, \xi^\alpha)$, the family $\mathcal{W} \cup \{\mathfrak{E} \setminus \mathfrak{E}_m\}$ is an open cover of $(\mathfrak{E}, \xi^\alpha)$, equivalently an α-open cover of $(\mathfrak{E}, \xi)$. Hence it has a subcover of cardinality at most $L^\alpha(\mathfrak{E}, \xi)$. Restricting this subcover to $\mathfrak{E}_m$, we obtain a subcover of $\mathcal{W}$ of cardinality at most $L^\alpha(\mathfrak{E}, \xi)$. Therefore

$$L^\alpha(\mathfrak{E}_m, \xi_m) \leq L^\alpha(\mathfrak{E}, \xi).$$

Since m was arbitrary,

$$\sup_{n\in\mathbb{N}} L^\alpha(\mathfrak{E}_n, \xi_n) \leq L^\alpha(\mathfrak{E}, \xi).$$

Moreover, by definition $L^\alpha(\mathfrak{E},\xi)$ is an infinite cardinal, so $\omega \le L^\alpha(\mathfrak{E},\xi)$. Hence

$$\max\left\{\omega,\ \sup_{n\in\mathbb{N}} L^\alpha(\mathfrak{E}_n,\xi_n)\right\} \le L^\alpha(\mathfrak{E},\xi).$$

Combining the upper and lower bounds gives

$$L^\alpha(\mathfrak{E},\xi) = \max\left\{\omega,\ \sup_{n\in\mathbb{N}} L^\alpha(\mathfrak{E}_n,\xi_n)\right\}.$$

□

Example 3.25. For each $n \in \mathbb{N}$, let $\mathfrak{E}_n$ be an infinite set, and let $\xi_n = \tau_{\mathrm{cof}}(\mathfrak{E}_n)$ be the cofinite topology on $\mathfrak{E}_n$. By Lemma 3.9, $\xi_n^\alpha = \xi_n \qquad (n \in \mathbb{N})$. Moreover, each $(\mathfrak{E}_n,\xi_n)$ is compact, and therefore $L^\alpha(\mathfrak{E}_n,\xi_n) = \aleph_0$.

Let

$$\mathfrak{E} = \bigsqcup_{n\in\mathbb{N}} \mathfrak{E}_n, \qquad \xi = \bigsqcup_{n\in\mathbb{N}} \xi_n.$$

By Theorem 3.24,

$$L^\alpha(\mathfrak{E},\xi) = \max\left\{\omega, \sup_{n\in\mathbb{N}} L^\alpha(\mathfrak{E}_n,\xi_n)\right\} = \aleph_0.$$

Hence the countable topological sum $\left(\mathfrak{E}, \bigsqcup_{n\in\mathbb{N}} \xi_n\right)$ is α-Lindelöf.

Lemma 3.26. *If $(\mathfrak{H},\eta_i)$ is α-compact, then $(\mathfrak{H},\eta_i^\alpha)$ is compact. Consequently, for every space $(\mathfrak{E},\xi_i^\alpha)$, the product $(\mathfrak{E},\xi_i^\alpha)\times(\mathfrak{H},\eta_i^\alpha)$ satisfies the usual tube lemma.*

Proof
Since $(\mathfrak{H},\eta_i)$ is α-compact, the associated space $(\mathfrak{H},\eta_i^\alpha)$ is compact.

Let $x_0 \in \mathfrak{E}$, and let $W \in \xi_i^\alpha \times \eta_i^\alpha$ satisfy $\{x_0\}\times\mathfrak{H} \subseteq W$. For each $y \in \mathfrak{H}$, since $(x_0,y)\in W$ and W is open in the product topology, there exist $U_y \in \xi_i^\alpha$ and $V_y \in \eta_i^\alpha$ such that

$$x_0 \in U_y, \qquad y \in V_y, \qquad U_y \times V_y \subseteq W.$$

The family

$$\{V_y : y \in \mathfrak{H}\}$$

is an open cover of $(\mathfrak{H},\eta_i^\alpha)$. Since $(\mathfrak{H},\eta_i^\alpha)$ is compact, there exist $y_1,\dots,y_n \in \mathfrak{H}$ such that $\mathfrak{H} = \bigcup_{k=1}^n V_{y_k}$. Set $U = \bigcap_{k=1}^n U_{y_k}$. Then $U \in \xi_i^\alpha$ and $x_0 \in U$. Moreover, if $(x,y) \in U \times \mathfrak{H}$, then $y \in V_{y_k}$ for some k. Since $x \in U \subseteq U_{y_k}$, we have

$$(x,y) \in U_{y_k} \times V_{y_k} \subseteq W.$$

Therefore $U \times \mathfrak{H} \subseteq W$. This proves the tube lemma in $(\mathfrak{E},\xi_i^\alpha)\times(\mathfrak{H},\eta_i^\alpha)$. □

Theorem 3.27. *Let $(\mathfrak{E},\xi_1,\xi_2)$ and $(\mathfrak{H},\eta_1,\eta_2)$ be bitopological spaces, and fix $i \in \{1,2\}$. Suppose that $(\mathfrak{E},\xi_i)$ is α-Lindelöf and $(\mathfrak{H},\eta_i)$ is α-compact. If*

$$(\xi_i \times \eta_i)^\alpha \subseteq \xi_i^\alpha \times \eta_i^\alpha,$$

then $(\mathfrak{E}\times\mathfrak{H}, \xi_i\times\eta_i)$ is α-Lindelöf.

Consequently, if the hypotheses hold for $i = 1,2$, then the product is componentwise α-Lindelöf.

Proof
Fix $i \in \{1,2\}$. Since $(\mathfrak{E}, \xi_i)$ is α-Lindelöf, Theorem 3.15 implies that $(\mathfrak{E}, \xi_i^\alpha)$ is Lindelöf. Similarly, since $(\mathfrak{H}, \eta_i)$ is α-compact, $(\mathfrak{H}, \eta_i^\alpha)$ is compact.

By the classical product theorem, the product of a Lindelöf space with a compact space is Lindelöf. Hence $(\mathfrak{E} \times \mathfrak{H}, \xi_i^\alpha \times \eta_i^\alpha)$ is Lindelöf.

Now assume that
$$(\xi_i \times \eta_i)^\alpha \subseteq \xi_i^\alpha \times \eta_i^\alpha.$$
Let $\mathcal{U}$ be an α-open cover of $(\mathfrak{E} \times \mathfrak{H}, \xi_i \times \eta_i)$. Then
$$\mathcal{U} \subseteq (\xi_i \times \eta_i)^\alpha.$$
By the assumed inclusion,
$$\mathcal{U} \subseteq \xi_i^\alpha \times \eta_i^\alpha.$$
Thus $\mathcal{U}$ is an ordinary open cover of the Lindelöf space
$$(\mathfrak{E} \times \mathfrak{H}, \xi_i^\alpha \times \eta_i^\alpha).$$
Therefore $\mathcal{U}$ has a countable subcover. Hence $(\mathfrak{E} \times \mathfrak{H}, \xi_i \times \eta_i)$ is α-Lindelöf.

If the same hypotheses hold for $i = 1, 2$, then the above argument applies to both component topologies. Therefore the product bitopological space is componentwise α-Lindelöf. □

Remark 3.28. The compatibility assumption
$$(\xi_i \times \eta_i)^\alpha \subseteq \xi_i^\alpha \times \eta_i^\alpha$$
in Theorem 3.27 is not automatic. It means that every α-open subset of the product topology $\xi_i \times \eta_i$ is open with respect to the product of the associated α-topologies. This is a strong condition because α-openness in a product need not be generated locally by rectangles whose sides are α-open in the factors.

The condition holds in particular when the product topology is α-invariant, that is, when
$$(\xi_i \times \eta_i)^\alpha = \xi_i \times \eta_i,$$
because then
$$(\xi_i \times \eta_i)^\alpha = \xi_i \times \eta_i \subseteq \xi_i^\alpha \times \eta_i^\alpha.$$
This includes, but is not limited to, the discrete case. For instance, it holds for products of partition topologies, as shown below. However, the condition is not a consequence of regularity, normality, or metrizability, as the following example shows.

Proposition 3.29. *Let $\mathcal{P}$ be a partition of $\mathfrak{E}$, and let $\mathcal{Q}$ be a partition of $\mathfrak{H}$. Let $\xi_{\mathcal{P}}$ and $\eta_{\mathcal{Q}}$ be the corresponding partition topologies; that is, their open sets are arbitrary unions of blocks of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then*
$$(\xi_{\mathcal{P}} \times \eta_{\mathcal{Q}})^\alpha = \xi_{\mathcal{P}}^\alpha \times \eta_{\mathcal{Q}}^\alpha.$$
In particular,
$$(\xi_{\mathcal{P}} \times \eta_{\mathcal{Q}})^\alpha \subseteq \xi_{\mathcal{P}}^\alpha \times \eta_{\mathcal{Q}}^\alpha.$$

Proof
In a partition topology, every open set is a union of blocks, and every such union is also closed, since its complement is again a union of blocks. Hence, for any subset $A \subseteq \mathfrak{E}$, $\mathrm{Cl}_{\xi_{\mathcal{P}}} \mathrm{Int}_{\xi_{\mathcal{P}}}(A) = \mathrm{Int}_{\xi_{\mathcal{P}}}(A)$. Therefore $\mathrm{Int}_{\xi_{\mathcal{P}}} \mathrm{Cl}_{\xi_{\mathcal{P}}} \mathrm{Int}_{\xi_{\mathcal{P}}}(A) = \mathrm{Int}_{\xi_{\mathcal{P}}}(A)$. Thus A is α-open if and only if $A \subseteq \mathrm{Int}_{\xi_{\mathcal{P}}}(A)$, which is equivalent to $A \in \xi_{\mathcal{P}}$. Hence $\xi_{\mathcal{P}}^\alpha = \xi_{\mathcal{P}}$. Similarly, $\eta_{\mathcal{Q}}^\alpha = \eta_{\mathcal{Q}}$. Moreover, the product topology $\xi_{\mathcal{P}} \times \eta_{\mathcal{Q}}$ is again a partition topology on $\mathfrak{E} \times \mathfrak{H}$, whose blocks are $P \times Q, \quad P \in \mathcal{P}, \quad Q \in \mathcal{Q}$. Therefore the same argument gives
$$(\xi_{\mathcal{P}} \times \eta_{\mathcal{Q}})^\alpha = \xi_{\mathcal{P}} \times \eta_{\mathcal{Q}}.$$

Combining this with $\xi_{\mathcal{P}}^{\alpha} = \xi_{\mathcal{P}}$ and $\eta_{\mathcal{Q}}^{\alpha} = \eta_{\mathcal{Q}}$, we obtain

$$(\xi_{\mathcal{P}} \times \eta_{\mathcal{Q}})^{\alpha} = \xi_{\mathcal{P}} \times \eta_{\mathcal{Q}} = \xi_{\mathcal{P}}^{\alpha} \times \eta_{\mathcal{Q}}^{\alpha}.$$

□

Example 3.30. Let

$$\mathfrak{E} = A_1 \cup A_2 \quad \text{and} \quad \mathfrak{H} = B_1 \cup B_2,$$

where A_1, A_2 are nonempty disjoint subsets of $\mathfrak{E}$, and B_1, B_2 are nonempty disjoint subsets of $\mathfrak{H}$. Assume that at least one of the blocks has more than one point, so the corresponding partition topology is not discrete. Let ξ be the partition topology on $\mathfrak{E}$ generated by $\mathcal{P} = \{A_1, A_2\}$, and let η be the partition topology on $\mathfrak{H}$ generated by $\mathcal{Q} = \{B_1, B_2\}$. Thus

$$\xi = \{\varnothing, A_1, A_2, \mathfrak{E}\}, \qquad \eta = \{\varnothing, B_1, B_2, \mathfrak{H}\}.$$

These topologies are generally non-discrete. By Proposition 3.29,

$$(\xi \times \eta)^{\alpha} = \xi^{\alpha} \times \eta^{\alpha}.$$

Therefore the product compatibility assumption used in Theorem 3.27 holds in this non-discrete setting. Moreover, if the partitions are finite, then $(\mathfrak{E}, \xi)$ and $(\mathfrak{H}, \eta)$ are compact, because every open cover must cover finitely many blocks and hence has a finite subcover. Consequently, they are α-compact and α-Lindelöf. This gives a nontrivial class of spaces to which Theorem 3.27 applies.

Example 3.31. Let τ be the usual Euclidean topology on $\mathbb{R}$, and let $\Delta = \{(x, x) : x \in \mathbb{R}\}$ be the diagonal in $\mathbb{R}^2$. Define

$$A = (\mathbb{R}^2 \setminus \Delta) \cup \{(0, 0)\}.$$

We show that

$$A \in (\tau \times \tau)^{\alpha} \quad \text{but} \quad A \notin \tau^{\alpha} \times \tau^{\alpha}.$$

First, $\mathbb{R}^2 \setminus \Delta$ is open and dense in $(\mathbb{R}^2, \tau \times \tau)$. Moreover, $\mathrm{Int}_{\tau\times\tau}(A) = \mathbb{R}^2 \setminus \Delta$. Hence

$$\mathrm{Cl}_{\tau\times\tau}\,\mathrm{Int}_{\tau\times\tau}(A) = \mathrm{Cl}_{\tau\times\tau}(\mathbb{R}^2 \setminus \Delta) = \mathbb{R}^2.$$

Therefore

$$\mathrm{Int}_{\tau\times\tau}\,\mathrm{Cl}_{\tau\times\tau}\,\mathrm{Int}_{\tau\times\tau}(A) = \mathbb{R}^2,$$

and so $A \in (\tau \times \tau)^{\alpha}$. We now prove that $A \notin \tau^{\alpha} \times \tau^{\alpha}$. Suppose, to the contrary, that A is open in $\tau^{\alpha} \times \tau^{\alpha}$. Since $(0, 0) \in A$, there exist $P, Q \in \tau^{\alpha}$ such that $0 \in P, \quad 0 \in Q, \quad P \times Q \subseteq A$. Since $P \in \tau^{\alpha}$ and $0 \in P$, we have $0 \in \mathrm{Int}_{\tau}\,\mathrm{Cl}_{\tau}\,\mathrm{Int}_{\tau}(P)$. Thus there is an open interval I about 0 such that $I \subseteq \mathrm{Cl}_{\tau}\,\mathrm{Int}_{\tau}(P)$. Similarly, there is an open interval J about 0 such that $J \subseteq \mathrm{Cl}_{\tau}\,\mathrm{Int}_{\tau}(Q)$. Let $K = I \cap J$, which is a nonempty open interval about 0. Then $\mathrm{Int}_{\tau}(P) \cap K$ and $\mathrm{Int}_{\tau}(Q) \cap K$ are open dense subsets of K. Hence their intersection is nonempty. Choose

$$t \in \mathrm{Int}_{\tau}(P) \cap \mathrm{Int}_{\tau}(Q) \cap K$$

with $t \neq 0$. Then $t \in P \cap Q$, and so $(t, t) \in P \times Q$. But $(t, t) \in \Delta$ and $t \neq 0$, hence $(t, t) \notin A$. This contradicts $P \times Q \subseteq A$. Therefore $A \notin \tau^{\alpha} \times \tau^{\alpha}$. Consequently,

$$(\tau \times \tau)^{\alpha} \nsubseteq \tau^{\alpha} \times \tau^{\alpha}.$$

Thus the compatibility condition in Theorem 3.27 fails even for the usual metrizable, regular, normal topology on $\mathbb{R}$.

Example 3.32. Let $\mathfrak{E}$ be a countable set with the discrete topology ξ_i, and let $\mathfrak{H}$ be a finite set with the discrete topology η_i. Then $\xi_i^\alpha = \xi_i \quad \text{and} \quad \eta_i^\alpha = \eta_i$.

Since $\mathfrak{E}$ is countable and discrete, $(\mathfrak{E}, \xi_i)$ is α-Lindelöf. Since $\mathfrak{H}$ is finite and discrete, $(\mathfrak{H}, \eta_i)$ is α-compact.
Moreover,

$$(\xi_i \times \eta_i)^\alpha = \xi_i^\alpha \times \eta_i^\alpha,$$

because $\xi_i \times \eta_i$ is again a discrete topology on $\mathfrak{E} \times \mathfrak{H}$.

Therefore, by Theorem 3.27, $(\mathfrak{E} \times \mathfrak{H}, \xi_i \times \eta_i)$ is α-Lindelöf.

Lemma 3.33. *For any topological space $(\mathfrak{E}, \xi)$, the following hold:*

1. *$(\mathfrak{E}, \xi)$ is α-compact if and only if $(\mathfrak{E}, \xi^\alpha)$ is compact.*
2. *$(\mathfrak{E}, \xi)$ is α-Lindelöf if and only if $(\mathfrak{E}, \xi^\alpha)$ is Lindelöf.*
3. *$(\mathfrak{E}, \xi)$ is α-countably compact if and only if $(\mathfrak{E}, \xi^\alpha)$ is countably compact.*

Proof
By definition, the open sets of ξ^α are precisely the α-open subsets of $(\mathfrak{E}, \xi)$. Therefore, a family $\mathcal{U}$ is an α-open cover of $(\mathfrak{E}, \xi)$ if and only if $\mathcal{U}$ is an ordinary open cover of $(\mathfrak{E}, \xi^\alpha)$.

(1) Hence every α-open cover of $(\mathfrak{E}, \xi)$ has a finite subcover if and only if every open cover of $(\mathfrak{E}, \xi^\alpha)$ has a finite subcover. This is exactly the equivalence between α-compactness of $(\mathfrak{E}, \xi)$ and compactness of $(\mathfrak{E}, \xi^\alpha)$.

(2) Similarly, every α-open cover of $(\mathfrak{E}, \xi)$ has a countable subcover if and only if every open cover of $(\mathfrak{E}, \xi^\alpha)$ has a countable subcover. Thus $(\mathfrak{E}, \xi)$ is α-Lindelöf if and only if $(\mathfrak{E}, \xi^\alpha)$ is Lindelöf.

(3) Finally, a countable α-open cover of $(\mathfrak{E}, \xi)$ is precisely a countable open cover of $(\mathfrak{E}, \xi^\alpha)$. Therefore, every countable α-open cover of $(\mathfrak{E}, \xi)$ has a finite subcover if and only if every countable open cover of $(\mathfrak{E}, \xi^\alpha)$ has a finite subcover. Hence $(\mathfrak{E}, \xi)$ is α-countably compact if and only if $(\mathfrak{E}, \xi^\alpha)$ is countably compact. □

Remark 3.34. Let $\psi : (\mathfrak{E}, \xi_1, \xi_2) \to (\mathfrak{H}, \eta_1, \eta_2)$ be a map between bitopological spaces. The map ψ is ordinarily called pairwise continuous if, for each $i = 1, 2$, $\psi : (\mathfrak{E}, \xi_i) \to (\mathfrak{H}, \eta_i)$ is continuous. On the other hand, the preservation theorem used in this paper requires continuity with respect to the associated α-topologies, namely $\psi : (\mathfrak{E}, \xi_i^\alpha) \to (\mathfrak{H}, \eta_i^\alpha) \quad (i = 1, 2)$. We call this property componentwise strongly α-continuity. These two notions are not comparable in general. Ordinary pairwise continuity implies only that $\psi^{-1}(V) \in \xi_i \subseteq \xi_i^\alpha \quad (V \in \eta_i)$, and hence gives ordinary α-continuity with respect to η_i-open sets. It does not generally imply continuity $(\mathfrak{E}, \xi_i^\alpha) \to (\mathfrak{H}, \eta_i^\alpha)$, because an η_i^α-open set need not be η_i-open. Conversely, strongly α-continuity gives $\psi^{-1}(W) \in \xi_i^\alpha \quad (W \in \eta_i^\alpha)$, but this need not belong to ξ_i. Hence it does not generally imply ordinary pairwise continuity. The two notions coincide under suitable α-invariance assumptions. For example, if $\xi_i^\alpha = \xi_i \quad \text{and} \quad \eta_i^\alpha = \eta_i \quad (i = 1, 2)$, then pairwise continuity and componentwise strongly α-continuity are equivalent. Thus strongly α-perfect maps should be understood as perfect maps between the associated α-topological spaces, not merely as ordinary pairwise continuous maps between the original bitopological spaces.

Theorem 3.35. *Let $\psi : (\mathfrak{E}, \xi) \to (\mathfrak{H}, \eta)$ be a surjective map such that $\psi : (\mathfrak{E}, \xi^\alpha) \longrightarrow (\mathfrak{H}, \eta^\alpha)$ is continuous. Then:*

1. *if $(\mathfrak{E}, \xi)$ is α-compact, then $(\mathfrak{H}, \eta)$ is α-compact;*
2. *if $(\mathfrak{E}, \xi)$ is α-Lindelöf, then $(\mathfrak{H}, \eta)$ is α-Lindelöf.*

In particular, the conclusions hold whenever ψ is a surjective strongly α-perfect map, that is, whenever $\psi : (\mathfrak{E}, \xi^\alpha) \to (\mathfrak{H}, \eta^\alpha)$ is a surjective perfect map in the ordinary topological sense.

Proof
Assume first that $(\mathfrak{E}, \xi)$ is α-compact. By Lemma 3.33, $(\mathfrak{E}, \xi^\alpha)$ is compact. Since $\psi : (\mathfrak{E}, \xi^\alpha) \to (\mathfrak{H}, \eta^\alpha)$ is continuous and onto, its image $(\mathfrak{H}, \eta^\alpha)$ is compact. Applying Lemma 3.33 again, we conclude that $(\mathfrak{H}, \eta)$ is α-compact.

Now assume that $(\mathfrak{E}, \xi)$ is α-Lindelöf. Again by Lemma 3.33, $(\mathfrak{E}, \xi^\alpha)$ is Lindelöf. Since continuous images of Lindelöf spaces are Lindelöf and ψ is onto, the space $(\mathfrak{H}, \eta^\alpha)$ is Lindelöf. Hence, by Lemma 3.33, $(\mathfrak{H}, \eta)$ is α-Lindelöf.

If $\psi : (\mathfrak{E}, \xi^\alpha) \to (\mathfrak{H}, \eta^\alpha)$ is a surjective perfect map in the ordinary topological sense, then in particular it is continuous and onto. Therefore the previous argument applies. □

4. Relation to α-Compactness

In this section, we compare componentwise α-compactness with componentwise α-Lindelöfness and componentwise α-countable compactness. We prove that componentwise α-compactness implies componentwise α-Lindelöfness, and that componentwise α-Lindelöfness together with componentwise α-countable compactness implies componentwise α-compactness. Examples show that the converse implications fail in general.

Remark 4.1. We use two different normality conventions in this paper. The term *pairwise α-normal* is used in the standard bitopological sense: $(\mathfrak{E}, \xi_1, \xi_2)$ is pairwise α-normal when $(\mathfrak{E}, \xi_1^\alpha, \xi_2^\alpha)$ is pairwise normal. This is the notion stated in the preliminaries. In Section 4, however, the normality assumption needed is only componentwise. Thus, whenever we say that $(\mathfrak{E}, \xi_1, \xi_2)$ is *componentwise α-normal*, we mean that the associated topological space $(\mathfrak{E}, \xi_i^\alpha)$ is normal for each $i = 1, 2$. This convention is used because the arguments in this section are applied separately to the two associated topologies ξ_1^α and ξ_2^α, and do not require the cross-separation condition involved in pairwise α-normality.

Lemma 4.2. *Let $(\mathfrak{E}, \xi)$ be a topological space. Under the convention that $(\mathfrak{E}, \xi)$ is α-regular, respectively α-normal, exactly when $(\mathfrak{E}, \xi^\alpha)$ is regular, respectively normal, we have*

$$(\mathfrak{E}, \xi) \text{ is } \alpha\text{-regular} \iff (\mathfrak{E}, \xi^\alpha) \text{ is regular,}$$

and

$$(\mathfrak{E}, \xi) \text{ is } \alpha\text{-normal} \iff (\mathfrak{E}, \xi^\alpha) \text{ is normal.}$$

Moreover, for a bitopological space $(\mathfrak{E}, \xi_1, \xi_2)$, the corresponding componentwise statements hold for $i = 1, 2$.

Proof
The assertions follow immediately from the definitions. □

Theorem 4.3. *Every componentwise α-compact bitopological space is componentwise α-Lindelöf. The converse does not hold in general.*

Proof
Let $(\mathfrak{E}, \xi_1, \xi_2)$ be componentwise α-compact. Fix $i \in \{1, 2\}$. By Lemma 3.33(1),

$$(\mathfrak{E}, \xi_i) \text{ is } \alpha\text{-compact} \iff (\mathfrak{E}, \xi_i^\alpha) \text{ is compact.}$$

Thus $(\mathfrak{E}, \xi_i^\alpha)$ is compact. Since compactness implies Lindelöfness, $(\mathfrak{E}, \xi_i^\alpha)$ is Lindelöf. By Lemma 3.33(2), $(\mathfrak{E}, \xi_i)$ is α-Lindelöf. Since this holds for $i = 1, 2$, $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf.

The converse fails; see Examples 4.4 and 4.5. □

Example 4.4. Let $\mathfrak{E} = \mathbb{N}$, and let $\xi_1 = \xi_2 = \mathcal{P}(\mathbb{N})$ be the discrete topology on $\mathbb{N}$. Since every subset of $\mathbb{N}$ is open, every subset is also α-open. Hence $\xi_i^\alpha = \xi_i \quad (i = 1, 2)$.

The space $(\mathbb{N}, \xi_i)$ is Lindelöf for $i = 1, 2$, because every open cover of the countable set $\mathbb{N}$ has a countable subcover. Therefore $(\mathbb{N}, \xi_i^\alpha)$ is Lindelöf for $i = 1, 2$, and so $(\mathbb{N}, \xi_i)$ is α-Lindelöf for $i = 1, 2$. Hence $(\mathbb{N}, \xi_1, \xi_2)$ is componentwise α-Lindelöf.

On the other hand, $(\mathbb{N}, \xi_i)$ is not compact. Indeed, the open cover $\mathcal{U} = \{\{n\} : n \in \mathbb{N}\}$ has no finite subcover. Since $\xi_i^\alpha = \xi_i$, it follows that $(\mathbb{N}, \xi_i)$ is not α-compact for $i = 1, 2$. Therefore $(\mathbb{N}, \xi_1, \xi_2)$ is not componentwise α-compact.

Thus componentwise α-Lindelöfness does not imply componentwise α-compactness.

Example 4.5. Let $\mathfrak{E}$ be an uncountable set, and let $\xi_1 = \xi_2 = \tau_{\text{coc}}$, where τ_{coc} denotes the cocountable topology on $\mathfrak{E}$. By Lemma 3.9, $\xi_i^\alpha = \xi_i \quad (i = 1, 2)$.

Each space $(\mathfrak{E}, \xi_i)$ is Lindelöf. Indeed, let $\mathcal{U}$ be an open cover of $(\mathfrak{E}, \xi_i)$. Choose a nonempty member $U_0 \in \mathcal{U}$. Then U_0 is cocountable, so $\mathfrak{E} \setminus U_0$ is countable. For each $\mathfrak{e} \in \mathfrak{E} \setminus U_0$, choose $U_\mathfrak{e} \in \mathcal{U}$ such that $\mathfrak{e} \in U_\mathfrak{e}$. Hence $\{U_0\} \cup \{U_\mathfrak{e} : \mathfrak{e} \in \mathfrak{E} \setminus U_0\}$ is a countable subcover. Thus $(\mathfrak{E}, \xi_i)$ is Lindelöf for $i = 1, 2$. Since $\xi_i^\alpha = \xi_i$, the space $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf.

However, $(\mathfrak{E}, \xi_i)$ is not compact. To see this, choose a countably infinite subset

$$D = \{\mathfrak{e}_n : n \in \mathbb{N}\} \subseteq \mathfrak{E},$$

and define $C_n = \{\mathfrak{e}_k : k \geq n\}$. Then each C_n is countable, so $U_n = \mathfrak{E} \setminus C_n$ is cocountable open. The family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover of $\mathfrak{E}$, because $\bigcap_{n \in \mathbb{N}} C_n = \varnothing$. But no finite subfamily covers $\mathfrak{E}$. Indeed, for any finite set $n_1, \ldots, n_m$,

$$U_{n_1} \cup \cdots \cup U_{n_m} = \mathfrak{E} \setminus C_N, \qquad N = \max\{n_1, \ldots, n_m\},$$

and $C_N \neq \varnothing$. Therefore $(\mathfrak{E}, \xi_i)$ is not compact.

Since $\xi_i^\alpha = \xi_i$, it follows that $(\mathfrak{E}, \xi_i)$ is not α-compact. Hence $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf but not componentwise α-compact.

Theorem 4.6. *If $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf and componentwise α-countably compact, then $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact.*

Proof
Fix $i \in \{1, 2\}$. Since $(\mathfrak{E}, \xi_i)$ is α-Lindelöf, by Lemma 3.33(2), the associated space $(\mathfrak{E}, \xi_i^\alpha)$ is Lindelöf. Similarly, since $(\mathfrak{E}, \xi_i)$ is α-countably compact, Lemma 3.33(3) implies that $(\mathfrak{E}, \xi_i^\alpha)$ is countably compact.

In ordinary topology, Lindelöfness together with countable compactness implies compactness. Indeed, let $\mathcal{U}$ be an open cover of $(\mathfrak{E}, \xi_i^\alpha)$. Since the space is Lindelöf, $\mathcal{U}$ has a countable subcover. Since the space is countably compact, this countable subcover has a finite subcover. Hence $(\mathfrak{E}, \xi_i^\alpha)$ is compact.

By Lemma 3.33(1), compactness of $(\mathfrak{E}, \xi_i^\alpha)$ is equivalent to α-compactness of $(\mathfrak{E}, \xi_i)$. Therefore $(\mathfrak{E}, \xi_i)$ is α-compact.

Since this holds for $i = 1, 2$, the bitopological space $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact. $\square$

Example 4.7. Let $\mathfrak{E}$ be an infinite set, and let $\xi_1 = \xi_2 = \tau_{\text{cof}}$, where τ_{cof} is the cofinite topology on $\mathfrak{E}$. By Lemma 3.9, $\xi_i^\alpha = \xi_i \quad (i = 1, 2)$. Since the cofinite topology is compact, each $(\mathfrak{E}, \xi_i)$ is compact. Thus each $(\mathfrak{E}, \xi_i)$ is α-compact, α-Lindelöf, and α-countably compact. Hence $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact.

Example 4.8. Let $\mathfrak{E}$ be an uncountable set, and let ξ be the cocountable topology on $\mathfrak{E}$. By Lemma 3.9, $\xi^\alpha = \xi$. Moreover, $(\mathfrak{E}, \xi)$ is Lindelöf. Indeed, every open cover contains a nonempty cocountable member, whose complement is countable; choosing one additional member for each remaining point gives a countable subcover.

We show that $(\mathfrak{E}, \xi)$ is not countably compact. Choose a countably infinite subset

$$D = \{\mathfrak{e}_n : n \in \mathbb{N}\} \subseteq \mathfrak{E},$$

and for each $n \in \mathbb{N}$, define $C_n = \{\mathfrak{e}_k : k \geq n\}$. Then each C_n is countable, so $U_n = \mathfrak{E} \setminus C_n$ is open in the cocountable topology. The family

$$\mathcal{U} = \{U_n : n \in \mathbb{N}\} = \{\mathfrak{E} \setminus C_n : n \in \mathbb{N}\}$$

is a countable open cover of $\mathfrak{E}$, because $\bigcap_{n\in\mathbb{N}} C_n = \varnothing$. However, no finite subfamily of $\mathcal{U}$ covers $\mathfrak{E}$. Indeed, if $n_1, \ldots, n_m \in \mathbb{N}$ and $N = \max\{n_1, \ldots, n_m\}$, then

$$U_{n_1} \cup \cdots \cup U_{n_m} = \mathfrak{E} \setminus C_N,$$

which leaves the nonempty set C_N uncovered. Hence $(\mathfrak{E}, \xi)$ is not countably compact.

Since $\xi^\alpha = \xi$, it follows that $(\mathfrak{E}, \xi)$ is α-Lindelöf but not α-countably compact. In particular, $(\mathfrak{E}, \xi)$ is not α-compact.

Now set $\xi_1 = \xi_2 = \xi$. Then $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf, but it is not componentwise α-countably compact and hence not componentwise α-compact. Therefore the α-countable compactness hypothesis in Theorem 4.6 is necessary.

Lemma 4.9. *Let $(\mathfrak{E}, \mathcal{T})$ be a normal topological space. If F and G are disjoint $\mathcal{T}$-closed subsets of $\mathfrak{E}$, then there exist $U, V \in \mathcal{T}$ such that $F \subseteq U, \qquad G \subseteq V$, and $\overline{U}^{\mathcal{T}} \cap \overline{V}^{\mathcal{T}} = \varnothing$.*

Proof
Since F and G are disjoint closed subsets of the normal space $(\mathfrak{E}, \mathcal{T})$, there exist disjoint open sets $O_F, O_G \in \mathcal{T}$ such that

$$F \subseteq O_F, \qquad G \subseteq O_G, \qquad O_F \cap O_G = \varnothing.$$

We use the standard shrinking property of normal spaces. Since $F \subseteq O_F$ and F is closed, there exists an open set $U \in \mathcal{T}$ such that

$$F \subseteq U \subseteq \overline{U}^{\mathcal{T}} \subseteq O_F.$$

Similarly, since $G \subseteq O_G$, there exists an open set $V \in \mathcal{T}$ such that

$$G \subseteq V \subseteq \overline{V}^{\mathcal{T}} \subseteq O_G.$$

Therefore

$$\overline{U}^{\mathcal{T}} \cap \overline{V}^{\mathcal{T}} \subseteq O_F \cap O_G = \varnothing.$$

Hence

$$\overline{U}^{\mathcal{T}} \cap \overline{V}^{\mathcal{T}} = \varnothing.$$

□

The following result uses the componentwise normality convention described in Remark 4.1; it does not require pairwise α-normality.

Theorem 4.10. *Let $(\mathfrak{E}, \xi_1, \xi_2)$ be a bitopological space. Suppose that, for each $i = 1, 2$, the associated space $(\mathfrak{E}, \xi_i^\alpha)$ is normal. Then, for each $i \in \{1, 2\}$ and for any two disjoint ξ_i^α-closed sets $F, G \subseteq \mathfrak{E}$, there exist $U, V \in \xi_i^\alpha$ such that $F \subseteq U, \qquad G \subseteq V$, and $\overline{U}^{\xi_i^\alpha} \cap \overline{V}^{\xi_i^\alpha} = \varnothing$. Moreover, if $(\mathfrak{E}, \xi_1, \xi_2)$ is also componentwise α-Lindelöf and componentwise α-countably compact, then $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact. In this case, the sets U, V above may be chosen so that $\overline{U}^{\xi_i^\alpha}$ and $\overline{V}^{\xi_i^\alpha}$ are compact in $(\mathfrak{E}, \xi_i^\alpha)$.*

Proof
Fix $i \in \{1, 2\}$. Since $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-normal, the associated topological space $(\mathfrak{E}, \xi_i^\alpha)$ is normal.

Let F and G be two disjoint ξ_i^α-closed subsets of $\mathfrak{E}$. Since $(\mathfrak{E}, \xi_i^\alpha)$ is normal, there exist disjoint ξ_i^α-open sets O_F and O_G such that

$$F \subseteq O_F, \qquad G \subseteq O_G, \qquad O_F \cap O_G = \varnothing.$$

We now apply the standard shrinking property of normal spaces. Since F is ξ_i^α-closed and $F \subseteq O_F$, there exists $U \in \xi_i^\alpha$ such that

$$F \subseteq U \subseteq \overline{U}^{\xi_i^\alpha} \subseteq O_F.$$

Similarly, since G is ξ_i^α-closed and $G \subseteq O_G$, there exists $V \in \xi_i^\alpha$ such that

$$G \subseteq V \subseteq \overline{V}^{\xi_i^\alpha} \subseteq O_G.$$

Therefore,

$$\overline{U}^{\xi_i^\alpha} \cap \overline{V}^{\xi_i^\alpha} \subseteq O_F \cap O_G = \varnothing.$$

Hence

$$\overline{U}^{\xi_i^\alpha} \cap \overline{V}^{\xi_i^\alpha} = \varnothing.$$

This proves the first assertion.

Now assume, in addition, that $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-Lindelöf and componentwise α-countably compact. Then, for each $i = 1, 2$, the associated space $(\mathfrak{E}, \xi_i^\alpha)$ is Lindelöf and countably compact. By the classical result that every Lindelöf countably compact space is compact, it follows that $(\mathfrak{E}, \xi_i^\alpha)$ is compact for each $i = 1, 2$. Hence $(\mathfrak{E}, \xi_1, \xi_2)$ is componentwise α-compact.

Finally, since each $(\mathfrak{E}, \xi_i^\alpha)$ is compact, the closed subsets $\overline{U}^{\xi_i^\alpha}$ and $\overline{V}^{\xi_i^\alpha}$ are compact in $(\mathfrak{E}, \xi_i^\alpha)$. Therefore the open sets U, V constructed above have compact closures in the corresponding associated α-topology. □

5. Conclusion

In this paper, we studied α-Lindelöf-type covering properties in bitopological spaces through the associated α-topologies. The central method was the α-transport principle, namely the correspondence between α-open covers of $(\mathfrak{E}, \xi)$ and ordinary open covers of $(\mathfrak{E}, \xi^\alpha)$. This principle allows many questions concerning α-Lindelöfness to be reformulated as classical Lindelöf-type questions in the transported topology.

We distinguished between componentwise α-Lindelöfness and the direct pairwise-cover version, emphasizing that these notions should not be identified without additional hypotheses. Several structural consequences were obtained. In particular, we established the equivalence

$$(\mathfrak{E}, \xi) \text{ is } \alpha\text{-Lindelöf} \quad \Longleftrightarrow \quad (\mathfrak{E}, \xi^\alpha) \text{ is Lindelöf},$$

derived cardinal estimates of the form

$$L^\alpha(\mathfrak{E}, \xi) \leq nw(\xi^\alpha) \leq w(\xi^\alpha),$$

and proved a countable-sum formula for the α-Lindelöf number. We also analyzed the relationship between α-Lindelöfness, α-compactness, and α-countable compactness. In particular, componentwise α-compactness implies componentwise α-Lindelöfness, while componentwise α-Lindelöfness together with componentwise α-countable compactness implies componentwise α-compactness. The examples presented throughout the paper clarify the limits of these implications and show the necessity of the main hypotheses.

The product and mapping results further demonstrate the usefulness of working with the transported bitopological structure $(\xi_1^\alpha, \xi_2^\alpha)$. However, product theorems require explicit compatibility assumptions between $(\xi_i \times \eta_i)^\alpha$ and $\xi_i^\alpha \times \eta_i^\alpha$. Similarly, preservation results for α-compactness and α-Lindelöfness are most naturally formulated for maps that are continuous with respect to the associated α-topologies. An explicit example shows that this compatibility may fail even for the usual topology on $\mathbb{R}$, and therefore the product theorem cannot be stated without an additional hypothesis.

Several directions remain open for further investigation. First, one may study conditions under which $(\xi \times \eta)^\alpha = \xi^\alpha \times \eta^\alpha$, or at least determine useful sufficient conditions for the inclusion needed in product preservation theorems. Second, the precise relationship between componentwise and direct pairwise α-Lindelöf numbers deserves a systematic treatment. Third, sharper cardinal estimates involving $L^\alpha(\mathfrak{E}, \xi)$,

$nw(\xi^\alpha)$, and $w(\xi^\alpha)$ may lead to a more refined classification of α-Lindelöf spaces. Finally, it would be natural to investigate broader preservation theorems under strongly α-closed, strongly α-perfect, or related classes of maps between the associated α-topologies.

These directions suggest that the α-topological transport method provides a coherent framework for studying Lindelöf-type covering properties in bitopological spaces and for extending classical covering theory to generalized open-set settings.

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