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Odd Generalized Exponential-Topp-Leone Weibull Distribution: Actuarial and Statistical Properties with Applications

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Abstract This paper proposes a new five-parameter distribution called the Odd Generalized Exponential–Topp–Leone Weibull (OGE-TLW) distribution for modelling lifetime and actuarial data. The model is obtained by applying the odd generalized exponential generator to a Topp–Leone Weibull baseline, thereby extending the classical Weibull distribution and nesting several well-known models as special cases. It accommodates monotone and non-monotone (increasing, decreasing, and bathtub) hazard rates as well as skewed and heavy-tailed data. Several fundamental statistical properties are derived, including survival and hazard functions, quantile function, moments, moment generating function, Rényi and Shannon entropies, order statistics, and closed-form actuarial risk measures. The model parameters are estimated using the maximum likelihood method, and the finite-sample performance of the estimators is investigated via Monte Carlo simulations with varying sample sizes and parameter configurations. The usefulness of the proposed distribution is illustrated by modelling two real heavy-tailed insurance datasets, on which it provides a better fit than several competing distributions by both AIC and BIC. The proposed distribution exhibits several desirable qualities of a good lifetime model.

Keywords Actuarial risk measures; Entropy; Maximum likelihood; Monte Carlo simulation; Odd generalized exponential generator; Topp–Leone; Weibull distribution.

Mathematics Subject Classification 60E05; 62E15; 62P05.

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1. Introduction

Probability distributions underpin the modelling of data across reliability engineering, survival analysis, actuarial science, hydrology, finance and the biomedical sciences. For much of the past century this work was dominated by classical models such as the exponential, Rayleigh, gamma, Pareto and, above all, the Weibull distribution, which were valued for their tractability and ease of interpretation. The Weibull in particular became a benchmark lifetime model because it captures data whose hazard rate rises or falls monotonically. As applications matured, however, it became evident that these models cope poorly with characteristics that occur routinely in real data, among them pronounced skewness, heavy or heterogeneous tails, excess kurtosis, and non-monotone hazard shapes of bathtub, unimodal or reverse-J form. The persistence of such features in reliability testing, medical survival

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studies, environmental records and financial losses has, in turn, driven a sustained search for more flexible families of distributions [1, 3, 2].

A productive answer to this need has been to embed a baseline distribution within a generator that introduces one or more additional shape parameters. The approach gained momentum with the beta-generated family [4], was broadened soon afterwards by the Kumaraswamy-generated family [5] and the transmuted family of Shaw and Buckley [6], and was enriched further by the Topp–Leone generator [7], whose single bounded-support parameter gives direct control over tail weight and, applied to a Weibull baseline, produces the Topp–Leone Weibull distribution [8, 9]. More recently, Tahir and co-authors [10] proposed the odd generalized exponential generator, which transforms the odds of a baseline and adds two shape parameters that afford fine control over the hazard rate. Since then the generator has been paired with several baselines, including the exponential, normal and Weibull distributions and, in the latest studies, the Kumaraswamy–Weibull [19] and length-biased Pareto [20] models, consistently improving the fit to skewed and heavy-tailed data.

The odd generalized exponential (OGE) and Topp-Leone (TL), although individually useful, have yet to be jointly employed. As such, no attempt has been made at employing the OGE to model a TL Weibull baseline. Since both models operate on distinct parts of the same distribution, it would seem worthwhile to address this limitation, since the combined use of these two generators could potentially provide for hazard rate functions having increasing, decreasing, or bathtub-shaped behavior in addition to providing for light, moderate or heavy-tailed behavior. Building on this observation, the present paper proposes a new five-parameter lifetime model, the odd generalized exponential–Topp–Leone Weibull distribution.

The motivation for this work is twofold. First, classical lifetime models such as the Weibull and generalized exponential distributions impose monotone hazard rates and light tails, and therefore fit skewed, heavy-tailed actuarial loss data poorly. A more flexible yet interpretable model is needed for solvency and reinsurance calculations. Second, although the Topp–Leone and odd generalized exponential generators are individually well studied, their joint use on a Weibull baseline has not been developed, and it is precisely this combination that couples flexible body and lower-tail reshaping with controllable upper-tail inflation. Against this background, the paper makes several contributions. First, we propose the five-parameter OGE-TLW distribution and prove that its density is legitimate, deriving its *cdf*, survival and hazard functions in closed form. Second, we establish an infinite linear representation of the density as a mixture of Weibull densities and use it to obtain explicit moments, the moment generating function and the Rényi and Shannon entropies. Third, we derive closed-form actuarial risk measures, value at risk, tail value at risk, tail variance and tail standard deviation, together with the stop-loss moment that are directly relevant to insurance and finance. Fourth, we develop maximum likelihood estimation with fully verified score equations and prove the existence and consistency of the estimators. Finally, we assess the finite-sample performance of the estimators through an extensive Monte Carlo study and demonstrate the model on two real heavy-tailed fire-insurance datasets, where it outperforms six established similar variant distributions.

2. Materials and Methods

This section presents the baseline distribution, the two generators, the construction of the OGE-TLW distribution, a proof that the proposed density is legitimate, and the full set of mathematical and inferential properties of the proposed distribution.

2.1. Baseline distribution

The baseline is the two-parameter Weibull distribution with shape $k > 0$ and scale $\lambda > 0$, with cumulative distribution function (*cdf*)

$$G_W(z; \lambda, k) = 1 - \exp\left[-\left(\frac{z}{\lambda}\right)^k\right], \qquad z > 0,\ \lambda, k > 0, \tag{1}$$

and probability density function (*pdf*)

$$g_W(z;\lambda,k) = \frac{k}{\lambda}\left(\frac{z}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{z}{\lambda}\right)^{k}\right], \qquad z>0. \tag{2}$$

2.2. Generators

Probability generators introduce additional parameters to a baseline distribution to increase its flexibility. The OGE-TLW distribution combines the Topp–Leone generator and the odd generalized exponential generator described below.

2.2.1. The Topp–Leone generator The Topp–Leone family of Topp and Leone [7] has *cdf* given by:

$$F_{TL}(z;\nu) = \left[G(z)\right]^{\nu}\left[2-G(z)\right]^{\nu} = \left\{1-\left[1-G(z)\right]^{2}\right\}^{\nu}, \qquad \nu>0, \tag{3}$$

where $G(z)$ is a baseline *cdf*.

2.2.2. The odd generalized exponential generator Let $G(z;\boldsymbol{\Omega})$ denote a baseline *cdf*. The odd generalized exponential (OGE) generator of Tahir et al. [10] produces the *cdf* and is given by:

$$F(z;\boldsymbol{\Omega},\alpha,\beta) = 1-\exp\left\{-\alpha\left(\frac{G(z;\boldsymbol{\Omega})}{1-G(z;\boldsymbol{\Omega})}\right)^{\beta}\right\}, \qquad \alpha>0,\ \beta>0, \tag{4}$$

with corresponding *pdf*

$$f(z;\boldsymbol{\Omega},\alpha,\beta) = \alpha\beta\, g(z;\boldsymbol{\Omega})\frac{G(z;\boldsymbol{\Omega})^{\beta-1}}{\{1-G(z;\boldsymbol{\Omega})\}^{\beta+1}}\exp\left\{-\alpha\left(\frac{G(z;\boldsymbol{\Omega})}{1-G(z;\boldsymbol{\Omega})}\right)^{\beta}\right\}. \tag{5}$$

2.3. Statistical Properties

Statistical characteristics are used to measure or explain how a random variable behaves. This subsection records the general definitions and measures used throughout the paper; the corresponding expressions for the OGE-TLW distribution are derived in Section 3.3.

2.3.1. Survival Function For any random variable Z, the survival function denoted $S_Z(z)$ is mathematically expressed as:

$$S_Z(z) = \Pr(Z>z) = 1-F_Z(z), \tag{6}$$

where $S(z)$ is nonincreasing and right-continuous, with $\lim_{z\to 0^+} S(z)=1$ and $\lim_{z\to\infty} S(z)=0$.

2.3.2. Hazard Function The instantaneous probability of failure is captured by the hazard rate at time z, conditional on survival until that moment. Mathematically, the hazard function is:

$$h(z) = \frac{f(z)}{S(z)} = -\frac{d}{dz}\ln S(z), \qquad S(z)>0. \tag{7}$$

2.3.3. Actuarial risk measures Tail-risk quantification is central to solvency assessment, reinsurance pricing and the management of extreme losses. We record four measures below; their explicit and series representations for the OGE-TLW distribution are derived in Section 3.5, and they are computed for the fitted models in Section 3.7.

Definition 1 (Value-at-Risk)
For a loss $Z\sim$ OGE-TLW and level $p\in(0,1)$, the value-at-risk is the p-quantile $\mathrm{VaR}_p(Z)=F^{-1}(p)=Q(p)$, given in closed form by (15).

Definition 2 (Tail-Value-at-Risk)
The tail-value-at-risk (expected shortfall) is $\mathrm{TVaR}_p(Z) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_u(Z)\,du = \mathrm{VaR}_p(Z) + \frac{1}{1-p}\int_{\mathrm{VaR}_p}^{\infty} S(z)\,dz$, with S given by (13).

Definition 3 (Tail variance and tail standard deviation)
The tail variance is $\mathrm{TV}_p(Z) = E\big[Z^2 \mid Z > \mathrm{VaR}_p\big] - \mathrm{TVaR}_p^2$ and the tail standard deviation is $\mathrm{TSD}_p(Z) = \sqrt{\mathrm{TV}_p(Z)}$.

Definition 4 (Stop-loss premium)
For a deductible $d > 0$, the net stop-loss premium is $E[(Z-d)_+] = \int_d^{\infty} S(z)\,dz$.

Each measure is computed from the closed-form quantile (15) and survival (13) functions by one-dimensional numerical integration .

2.3.4. Performance Measures The methods used to assess the quality of the estimators in the Monte Carlo simulation are the bias, the mean squared error (MSE) and the root mean squared error (RMSE), given by

$$\mathrm{Bias}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i - \theta, \qquad \mathrm{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2, \qquad \mathrm{RMSE} = \sqrt{\mathrm{MSE}}.$$

The mean estimate $\frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$ is also computed for each parameter.

2.3.5. Model Selection criteria and Goodness of fit tests The model selection criteria are the Akaike Information Criterion, $AIC = -2\ell(\hat{\theta}) + 2k$, and the Bayesian Information Criterion, $BIC = -2\ell(\hat{\theta}) + k\log(n)$, where k is the number of parameters. Additionally, we present the AIC, $CAIC = -2\ell(\hat{\theta}) + k(\log n + 1)$, and, for nested models, the likelihood-ratio test (LRT). Three goodness-of-fit tests are used to assess whether the proposed model fits the application data:

$$\mathrm{KS} = \max_{1\le i\le n}\left|F_n(Z_i) - F_0(Z_i;\hat{\theta})\right|,$$

$$\mathrm{AD} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F_0(Z_{(i)};\hat{\theta}) + \ln(1 - F_0(Z_{(n+1-i)};\hat{\theta}))\right],$$

$$\mathrm{CVM} = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_0(Z_{(i)};\hat{\theta}) - \frac{2i-1}{2n}\right]^2,$$

where $Z_{(i)}$ is the i-th order statistic and F_0 denotes the fitted cumulative distribution function.

3. Results and Discussion

3.1. The Topp–Leone Weibull baseline

Applying the Topp–Leone generator (3) to the Weibull baseline $G(z) = 1 - e^{-(z/\lambda)^k}$ gives the Topp–Leone Weibull (TLW) *cdf*

$$G_{TLW}(z;\lambda,\nu,k) = \left\{1 - e^{-(z/\lambda)^k}\right\}^{\nu}\left\{1 + e^{-(z/\lambda)^k}\right\}^{\nu}, \qquad z,\lambda,\nu,k > 0, \tag{8}$$

which simplifies to

$$G_{TLW}(z;\lambda,\nu,k) = \left[1 - e^{-2(z/\lambda)^k}\right]^{\nu}, \qquad z,\lambda,\nu,k > 0. \tag{9}$$

The corresponding TLW *pdf* is

$$g_{TLW}(z;\lambda,\nu,k) = 2\nu\frac{k}{\lambda}\left(\frac{z}{\lambda}\right)^{k-1} e^{-2(z/\lambda)^k}\left[1 - e^{-2(z/\lambda)^k}\right]^{\nu-1}, \qquad z > 0. \tag{10}$$

3.2. Construction of the OGE-TLW distribution

Applying the OGE generator to the TLW baseline (9) gives the OGE-TLW *cdf*

$$F(z) = 1 - \exp\left\{ -\alpha \left(\frac{\left[1 - e^{-2(z/\lambda)^k}\right]^\nu}{1 - \left[1 - e^{-2(z/\lambda)^k}\right]^\nu} \right)^\beta \right\}, \qquad z > 0, \tag{11}$$

with $\alpha, \beta, \nu, \lambda, k > 0$. Differentiating (11) yields the *pdf*

$$\begin{aligned} f(z) &= 2\alpha\beta\nu \frac{k}{\lambda} \left(\frac{z}{\lambda}\right)^{k-1} e^{-2(z/\lambda)^k} \left(1 - e^{-2(z/\lambda)^k}\right)^{\nu\beta - 1} \\ &\times \left[1 - \left(1 - e^{-2(z/\lambda)^k}\right)^\nu\right]^{-(\beta+1)} \exp\left\{ -\alpha \left[\frac{\left(1 - e^{-2(z/\lambda)^k}\right)^\nu}{1 - \left(1 - e^{-2(z/\lambda)^k}\right)^\nu} \right]^\beta \right\}. \end{aligned} \tag{12}$$

The OGE-TLW distribution has five parameters: the Weibull shape $k > 0$ and scale $\lambda > 0$, the Topp–Leone shape $\nu > 0$, and the OGE scale $\alpha > 0$ and shape $\beta > 0$.

Proposition 1 (Legitimacy of the density)
The function f in (12) is a valid probability density function; that is, $f(z) \geq 0$ for all $z > 0$ and $\int_0^\infty f(z)\, dz = 1$.

Proof
Non-negativity is immediate because every factor in (12) is positive for $z > 0$ and $\alpha, \beta, \nu, \lambda, k > 0$ (note $0 < 1 - e^{-2(z/\lambda)^k} < 1$, so its powers and $1 - (1 - e^{-2(z/\lambda)^k})^\nu$ are positive). For the total mass, put $w = G_{TLW}(z) = \left[1 - e^{-2(z/\lambda)^k}\right]^\nu \in (0, 1)$ and $s = w/(1 - w) \in (0, \infty)$, so that $F(z) = 1 - e^{-\alpha s^\beta}$. As $z \to 0^+$ we have $w \to 0$, hence $s \to 0$ and $F(0^+) = 1 - e^0 = 0$; as $z \to \infty$ we have $w \to 1$, hence $s \to \infty$ and $F(\infty) = 1 - e^{-\infty} = 1$. Because F in (11) is a continuous, strictly increasing map of z (each of $z \mapsto w$ and $w \mapsto s \mapsto 1 - e^{-\alpha s^\beta}$ is continuous and increasing), F is a genuine *cdf* and $f = F'$ integrates to $F(\infty) - F(0^+) = 1$. Equivalently, the substitution $u = F(z)$ gives $\int_0^\infty f(z)\, dz = \int_0^1 du = 1$. □

3.2.1. Special cases and sub-models Adjusting the parameters of the OGE-TLW distribution recovers several established models:

- **OGE-Weibull (OGE-W):** setting $\nu = 1$, since $\left[1 - e^{-2(z/\lambda)^k}\right]$ is a Weibull *cdf* with scale $\lambda\, 2^{-1/k}$.
- **Odd Exponential Topp–Leone Weibull (OE-TLW):** setting $\beta = 1$.
- **Topp–Leone Weibull (TLW):** setting $\alpha = 1$ and $\beta = 1$.
- **Odd Exponential Weibull:** setting $\nu = 1$ and $\beta = 1$.

3.2.2. Shape of the density and distribution functions Figures 1 and 2 display the *cdf* and *pdf* of the OGE-TLW distribution for selected parameter vectors. The *cdf* curves increase monotonically from 0 to 1, confirming Proposition 1 graphically, while the *pdf* plots show that the model accommodates right-skewed, approximately symmetric and heavy-tailed shapes with a wide range of kurtosis, depending on the parameter values.

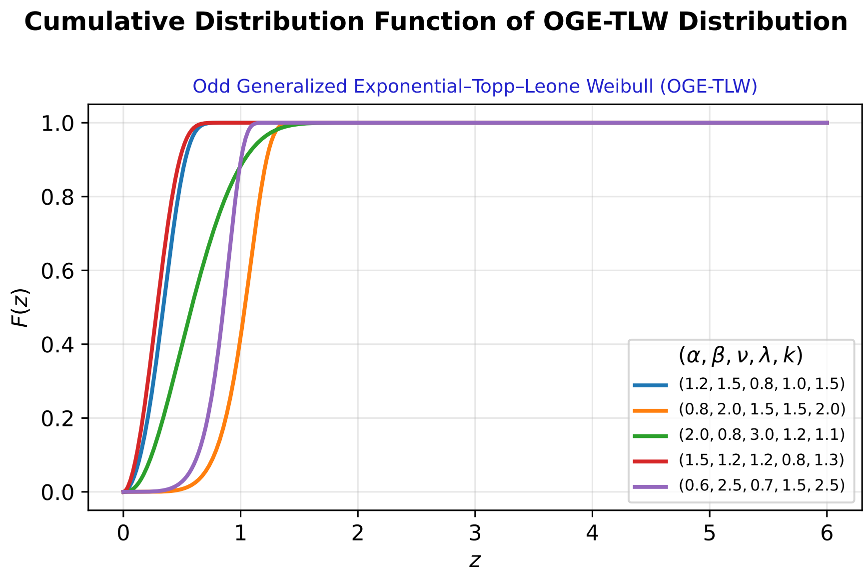

Figure 1. Plot of the cdf of the OGE-TWL distribution

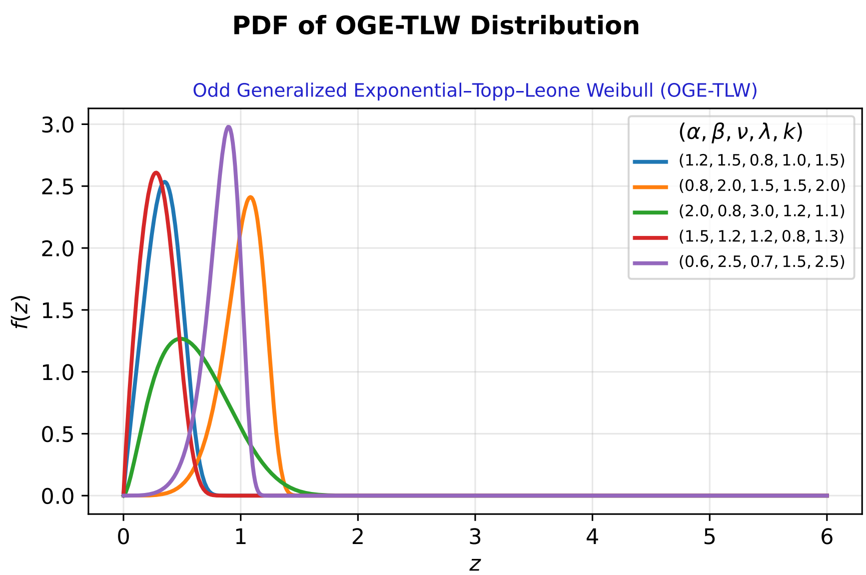

Figure 2. Plot of the pdf of the OGE-TWL distribution

3.3. Statistical Properties of the OGE-TLW distribution

3.3.1. Survival and hazard functions The survival function of the OGE-TLW distribution follows from (11):

$$S(z) = \exp\left\{ -\alpha \left[\frac{\left(1 - e^{-2(z/\lambda)^k}\right)^{\nu}}{1 - \left(1 - e^{-2(z/\lambda)^k}\right)^{\nu}} \right]^{\beta} \right\}, \qquad z > 0. \tag{13}$$

The hazard rate function is $h_{OGE-TLW}(.) = f(z)/S(z)$, i.e.

$$h_{OGE-TLW}(.) = 2\alpha\beta\nu\frac{k}{\lambda}\left(\frac{z}{\lambda}\right)^{k-1}\frac{e^{-2(z/\lambda)^k}\left(1-e^{-2(z/\lambda)^k}\right)^{\nu\beta-1}}{\left[1-\left(1-e^{-2(z/\lambda)^k}\right)^{\nu}\right]^{\beta+1}}, \qquad z>0. \tag{14}$$

Depending on the parameter values, $h_{OGE-TLW}(.)$ can be increasing, decreasing or bathtub-shaped over the effective support of the distribution. when $\nu = 1$ and $\alpha = \beta = 1$ the hazard reduces to that of a Weibull distribution with scale $\lambda\, 2^{-1/k}$. Figures 3 and 4 illustrate representative survival and hazard curves.

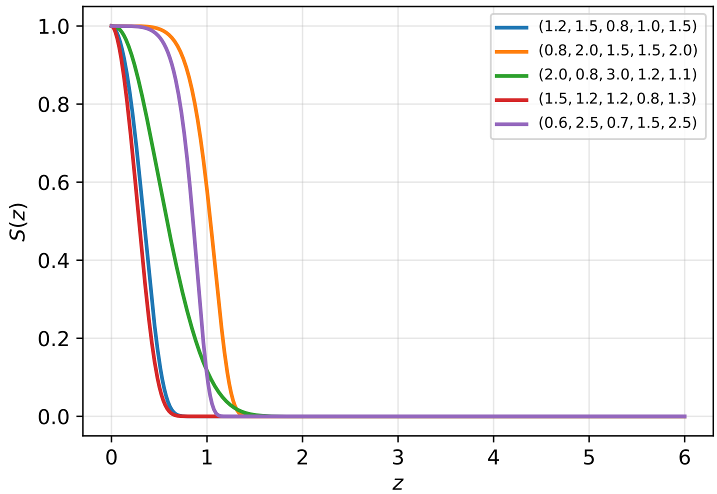

Figure 3. The plot of the Survival function of the OGE-TLW distribution.

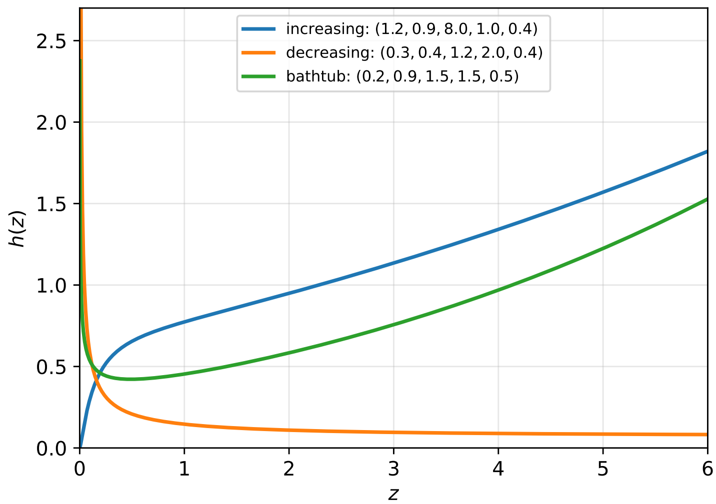

Figure 4. Plot of the Hazard rate function of the OGE-TLW distribution

3.3.2. Quantile function Solving $F(Q(u)) = u$, $0 < u < 1$, gives the closed-form quantile function

$$Q(u) = \lambda \left[-\tfrac{1}{2}\ln\Big(1 - \big[W(u)\big]^{1/\nu}\Big)\right]^{1/k}, \qquad W(u) = \frac{\big[-\frac{1}{\alpha}\ln(1-u)\big]^{1/\beta}}{1 + \big[-\frac{1}{\alpha}\ln(1-u)\big]^{1/\beta}}. \tag{15}$$

This expression invert (11) to machine precision, and it enables efficient random-variate generation by the inverse-transform method, $Z = Q(U)$ with $U \sim \mathrm{Uniform}(0,1)$. The median is $Q(0.5)$.

3.3.3. Linear representation of the density The moments and generating function are most conveniently obtained from a linear representation of (12) as an infinite mixture of Weibull densities.

Theorem 1 (Linear representation)
For $z > 0$ the OGE-TLW density admits the expansion

$$f(z) = \sum_{i,j,l=0}^{\infty} \eta_{ijl}\, g_W(z; \lambda_l, k), \qquad \lambda_l = \lambda\,[2(l+1)]^{-1/k}, \tag{16}$$

where $g_W(\cdot; \lambda_l, k)$ is the Weibull density with scale λ_l and shape k, and

$$\eta_{ijl} = \frac{\alpha\beta\nu\,(-1)^l}{l+1}\,\frac{(-\alpha)^i}{i!}\binom{\beta(i+1)+j-1}{j}\binom{\nu\beta(i+1)+\nu j-1}{l}. \tag{17}$$

Proof
Write $A = \left(1 - e^{-2(z/\lambda)^k}\right)^{\nu} = G_{TLW}(z) \in (0,1)$ and expand the exponential term in (12), $\exp\{-\alpha[A/(1-A)]^{\beta}\} = \sum_{i\geq 0} \frac{(-\alpha)^i}{i!} A^{\beta i}(1-A)^{-\beta i}$. Collecting the factor $(1-A)^{-(\beta+1)}$ from (12) gives $(1-A)^{-[(\beta+1)+\beta i]}$, which, by the generalized binomial series $(1-A)^{-s} = \sum_{j\geq 0}\binom{s+j-1}{j}A^j$ with $s = (\beta+1)+\beta i$, expands as $\sum_{j\geq 0}\binom{\beta(i+1)+j}{j}A^j$. Since $A = \left(1 - e^{-2(z/\lambda)^k}\right)^{\nu}$, the accumulated power of $\left(1 - e^{-2(z/\lambda)^k}\right)$ is $\nu\beta(i+1) + \nu j - 1$; expanding this power by the binomial series and using $e^{-2(z/\lambda)^k}e^{-2l(z/\lambda)^k} = e^{-2(l+1)(z/\lambda)^k}$ yields terms proportional to $(k/\lambda)(z/\lambda)^{k-1}e^{-2(l+1)(z/\lambda)^k}$. Writing $\lambda_l = \lambda[2(l+1)]^{-1/k}$, this equals $[2(l+1)]^{-1}g_W(z;\lambda_l,k)$; absorbing the constants gives (16)–(17). □

The representation (16) was verified numerically. Truncating the triple series reproduces the exact density (12) to machine precision in the body of the distribution.

3.3.4. Moments and Related Measures The moment generating function and the moments of the new distribution are computed and presented in this section.

Theorem 2 (r-th raw moment)
For $Z \sim$ OGE-TLW$(k, \lambda, \nu, \alpha, \beta)$ with $k > 0$, $\lambda > 0$, $\nu \in [-1, 1]$, $\alpha > 0$ and $\beta > 0$, the $r - th$ raw moment exists for every $r > 0$ and $E[Z^r] < \infty$ and is given by:

$$\mu_r' = 2\alpha\beta\nu\,\lambda^r \int_0^{\infty} t^{r/k} e^{-2t}\left(1 - e^{-2t}\right)^{\nu\beta-1}\left[1 - (1-e^{-2t})^{\nu}\right]^{-(\beta+1)} \exp\left\{-\alpha\left[\frac{(1-e^{-2t})^{\nu}}{1-(1-e^{-2t})^{\nu}}\right]^{\beta}\right\} dt, \tag{18}$$

obtained from $\mu_r' = \int_0^{\infty} z^r f(z)\,dz$ via the substitution $t = (z/\lambda)^k$. Equivalently, using the linear representation (16),

$$\mu_r' = \lambda^r\,\Gamma\big(1 + \tfrac{r}{k}\big) \sum_{i,j,l=0}^{\infty} \eta_{ijl}\,[2(l+1)]^{-r/k}. \tag{19}$$

Both forms agree; (18) is convenient for numerical evaluation and is verified against Monte Carlo estimates of the mean, while (19) gives the analytical series. The mean, variance, skewness and kurtosis follow from $\mu_1', \ldots, \mu_4'$ in the usual way.

3.3.5. Moment generating function

Proposition 2 (MGF)
Whenever the moments exist, the moment generating function of Z is

$$M_Z(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\,\mu_r' = \sum_{r=0}^{\infty} \sum_{i,j,l=0}^{\infty} \frac{t^r}{r!}\,\lambda^r \Gamma\bigl(1 + \tfrac{r}{k}\bigr)\,\eta_{ijl}\,[2(l+1)]^{-r/k}, \tag{20}$$

with η_{ijl} as in (17).

Theorem 3 (Existence of the MGF)
If $k \geq 1$ the series (20) converges for all t in a neighbourhood of the origin and the MGF exists. If $0 < k < 1$ the OGE-TLW distribution is heavy-tailed and $M_Z(t)$ fails to exist for $t > 0$, although all integer moments (18) remain finite.

Proof
Each mixture component in (16) is a Weibull law with shape k. A Weibull random variable possesses a moment generating function in a neighbourhood of 0 if and only if its shape parameter satisfies $k \geq 1$ (for $k \geq 1$ the tail decays at least as fast as an exponential, so $E[e^{tZ}] < \infty$ for t small; for $k < 1$ the tail is sub-exponential and $E[e^{tZ}] = \infty$ for every $t > 0$). Since $M_Z(t) = \sum_{i,j,l} \eta_{ijl}\, E[e^{tZ_l}]$ with $Z_l \sim \text{Weibull}(\lambda_l, k)$, the stated dichotomy follows. $\square$

3.3.6. Order statistics Let $Z_1, \ldots, Z_n$ be a random sample from the OGE-TLW distribution and $Z_{(r)}$ the r-th order statistic. Its density is

$$f_{Z_{(r)}}(z) = \frac{n!}{(r-1)!(n-r)!}\,[F(z)]^{r-1}[1 - F(z)]^{n-r} f(z), \tag{21}$$

with F and f given by (11) and (12). Writing $A(z) = \alpha\bigl[G_{TLW}(z)/(1 - G_{TLW}(z))\bigr]^{\beta}$ so that $F(z) = 1 - e^{-A(z)}$ and $1 - F(z) = e^{-A(z)}$, this becomes $f_{Z_{(r)}}(z) = \frac{n!}{(r-1)!(n-r)!}\bigl[1 - e^{-A(z)}\bigr]^{r-1} e^{-(n-r+1)A(z)} f(z)$.

3.3.7. Entropy Measures Two important entropy measures are defined and computed for the proposed distribution.

Definition 5 (Rényi Entropy)
The Rényi entropy of order $\delta > 0$, $\delta \neq 1$, for the OGE-TLW distribution is

$$H_\delta(Z) = \frac{1}{1-\delta} \ln\left(\int_0^{\infty} [f(z)]^{\delta}\, dz \right). \tag{22}$$

Expanding the integrand yields

$$\begin{aligned}[f(z)]^{\delta} = {} & (2\alpha\beta\nu)^{\delta} \left(\frac{k}{\lambda}\right)^{\delta} \left(\frac{z}{\lambda}\right)^{\delta(k-1)} e^{-2\delta(z/\lambda)^k} \bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\delta(\nu\beta-1)} \\ & \times \Bigl[1 - \bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\nu}\Bigr]^{-\delta(\beta+1)} \exp\left\{ -\alpha\delta \left[\frac{\bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\nu}}{1 - \bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\nu}} \right]^{\beta} \right\},\end{aligned} \tag{23}$$

so that

$$\begin{aligned}H_\delta(Z) = \frac{1}{1-\delta} \ln \Biggl[& (2\alpha\beta\nu)^{\delta} \left(\frac{k}{\lambda}\right)^{\delta} \int_0^{\infty} \left(\frac{z}{\lambda}\right)^{\delta(k-1)} e^{-2\delta(z/\lambda)^k} \bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\delta(\nu\beta-1)} \\ & \times \Bigl[1 - \bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\nu}\Bigr]^{-\delta(\beta+1)} \exp\left\{ -\alpha\delta \left[\frac{\bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\nu}}{1 - \bigl(1 - e^{-2(z/\lambda)^k}\bigr)^{\nu}} \right]^{\beta} \right\} dz \Biggr].\end{aligned} \tag{24}$$

Definition 6 (Shannon Entropy)
The Shannon entropy of the OGE-TLW distribution is defined as $H(Z) = -\mathbb{E}[\ln f(Z)] = -\int_0^\infty f(z) \ln f(z)\, dz$.

Using (12),

$$\ln f(z) = \ln\Big(\tfrac{2\alpha\beta\nu k}{\lambda}\Big) + (k-1)\ln\tfrac{z}{\lambda} - 2\big(\tfrac{z}{\lambda}\big)^k + (\nu\beta-1)\ln\big(1-e^{-2(z/\lambda)^k}\big) - (\beta+1)\ln\big[1-(1-e^{-2(z/\lambda)^k})^\nu\big] - \alpha\Big[\tfrac{G_{TLW}}{1-G_{TLW}}\Big]^\beta$$

which expands to

$$\begin{aligned} H(Z) = &-\ln\Big(\frac{2\alpha\beta\nu k}{\lambda}\Big) - (k-1)\,\mathbb{E}\Big[\ln\frac{Z}{\lambda}\Big] + 2\,\mathbb{E}\big[(Z/\lambda)^k\big] - (\nu\beta-1)\,\mathbb{E}\big[\ln(1-e^{-2(Z/\lambda)^k})\big] \\ &+ (\beta+1)\,\mathbb{E}\big[\ln(1-G_{TLW})\big] + \alpha\,\mathbb{E}\big[(G_{TLW}/(1-G_{TLW}))^\beta\big], \end{aligned} \tag{25}$$

where $G_{TLW} = \big(1-e^{-2(Z/\lambda)^k}\big)^\nu$ and each expectation is finite and computable numerically or from the linear representation (16). Both entropies are evaluated numerically for the fitted models in Section 3.7.

3.4. Maximum likelihood estimation

Let $z_1, \ldots, z_n$ be a random sample from the OGE-TLW distribution with parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \nu, \lambda, k)^\top$. The log-likelihood is given as:

$$\begin{aligned} \ell(\boldsymbol{\theta}) = &\, n\ln(2\alpha\beta\nu k) - nk\ln\lambda + (k-1)\sum_{i=1}^n \ln z_i - 2\sum_{i=1}^n \Big(\tfrac{z_i}{\lambda}\Big)^k \\ &+ (\nu\beta-1)\sum_{i=1}^n \ln v_i - (\beta+1)\sum_{i=1}^n \ln(1-w_i) - \alpha\sum_{i=1}^n r_i^\beta, \end{aligned} \tag{26}$$

we define the auxiliary quantities

$$u_i = e^{-2(z_i/\lambda)^k}, \quad v_i = 1-u_i, \quad w_i = v_i^\nu, \quad r_i = \frac{w_i}{1-w_i}. \tag{27}$$

3.4.1. Score Functions The score vector $U(\boldsymbol{\theta}) = \big(U_\alpha, U_\beta, U_\nu, U_\lambda, U_k\big)^\top$ contains the partial derivatives of the log-likelihood,

$$U(\boldsymbol{\theta}) = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \nu}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial k}\right)^\top. \tag{28}$$

Proposition 3 (Score Equations)
The maximum likelihood estimates $\widehat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\nu}, \hat{\lambda}, \hat{k})$ satisfy the score equations $U(\widehat{\boldsymbol{\theta}}) = \mathbf{0}$; that is,

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} r_i^{\beta} = 0, \tag{29}$$

$$U_\beta = \frac{n}{\beta} + \nu \sum_{i=1}^{n} \ln v_i - \sum_{i=1}^{n} \ln(1 - w_i) - \alpha \sum_{i=1}^{n} r_i^{\beta} \ln r_i = 0, \tag{30}$$

$$U_\nu = \frac{n}{\nu} + \beta \sum_{i=1}^{n} \ln v_i + (\beta + 1) \sum_{i=1}^{n} \frac{w_i \ln v_i}{1 - w_i} - \alpha\beta \sum_{i=1}^{n} \frac{r_i^{\beta} \ln v_i}{1 - w_i} = 0, \tag{31}$$

$$\begin{aligned} U_\lambda = {} & -\frac{nk}{\lambda} + \frac{2k}{\lambda} \sum_{i=1}^{n} \left(\tfrac{z_i}{\lambda}\right)^{k} - \frac{2k(\nu\beta - 1)}{\lambda} \sum_{i=1}^{n} \frac{(z_i/\lambda)^k u_i}{v_i} \\ & - \frac{2k\nu(\beta + 1)}{\lambda} \sum_{i=1}^{n} \frac{(z_i/\lambda)^k u_i v_i^{\nu-1}}{1 - w_i} + \frac{2k\nu\alpha\beta}{\lambda} \sum_{i=1}^{n} \frac{(z_i/\lambda)^k u_i v_i^{\nu-1} r_i^{\beta-1}}{(1 - w_i)^2} = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} U_k = {} & \frac{n}{k} + \sum_{i=1}^{n} \ln \tfrac{z_i}{\lambda} - 2 \sum_{i=1}^{n} \left(\tfrac{z_i}{\lambda}\right)^{k} \ln \tfrac{z_i}{\lambda} + 2(\nu\beta - 1) \sum_{i=1}^{n} \frac{(z_i/\lambda)^k u_i \ln(z_i/\lambda)}{v_i} \\ & + 2\nu(\beta + 1) \sum_{i=1}^{n} \frac{(z_i/\lambda)^k u_i v_i^{\nu-1} \ln(z_i/\lambda)}{1 - w_i} - 2\nu\alpha\beta \sum_{i=1}^{n} \frac{(z_i/\lambda)^k u_i v_i^{\nu-1} r_i^{\beta-1} \ln(z_i/\lambda)}{(1 - w_i)^2} = 0, \end{aligned} \tag{33}$$

where u_i, v_i, w_i, r_i are the auxiliary quantities defined in (27).

The score equations have no closed-form solution and are solved by numerical iterative optimisation.

Theorem 4 (Existence of the MLE)
For a sample containing at least five distinct positive values, the log-likelihood (26) is continuous on the open parameter space $\Theta = (0, \infty)^5$ and $\ell(\boldsymbol{\theta}) \to -\infty$ as $\boldsymbol{\theta}$ approaches the boundary of Θ or as $\|\boldsymbol{\theta}\| \to \infty$; hence a maximum likelihood estimate $\widehat{\boldsymbol{\theta}}$ exists in the interior of Θ.

Proof
ℓ is a finite sum of logarithms and powers of the strictly positive quantities in (27), so it is continuous on Θ. As any coordinate tends to 0 or ∞, at least one of the terms $n \ln(2\alpha\beta\nu k)$, $-n(1 + k)\ln \lambda$, $-2\sum (z_i/\lambda)^k$ or $-\alpha \sum r_i^{\beta}$ diverges to $-\infty$ while the remaining terms are bounded above, so $\ell \to -\infty$ on the boundary. A continuous function that tends to $-\infty$ at the boundary of an open set attains its supremum at an interior point, which is a stationary point of ℓ and hence a root of the score equations. $\square$

Theorem 5 (Consistency and asymptotic normality)
Under the usual regularity conditions for exponential-type families, identifiability, an interior true value $\boldsymbol{\theta}_0$, and a non-singular Fisher information matrix $\mathbf{I}(\boldsymbol{\theta}_0)$ the maximum likelihood estimator is consistent and $\sqrt{n}\,(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} N_5\big(\mathbf{0}, \mathbf{I}(\boldsymbol{\theta}_0)^{-1}\big)$ as $n \to \infty$.

Proof
The density (12) is smooth in $\boldsymbol{\theta}$ with three continuous derivatives, the support does not depend on $\boldsymbol{\theta}$, and the score has zero mean and finite variance; these are the Cramér regularity conditions under which the MLE is consistent and asymptotically normal with covariance $\mathbf{I}(\boldsymbol{\theta}_0)^{-1}$. In practice $\mathbf{I}$ is replaced by the observed information $-\partial^2 \ell / \partial\boldsymbol{\theta}\, \partial\boldsymbol{\theta}^{\top}$ evaluated at $\widehat{\boldsymbol{\theta}}$, from which standard errors and Wald confidence intervals are obtained. The finite-sample adequacy of this approximation is examined in the simulation study of Section 3.6, where the empirical coverage of the 95% intervals is reported. $\square$

3.5. Actuarial risk measures

Having presented the general definitions in Section 2.3.3, we now derive explicit and series representations of the four risk measures for the OGE-TLW distribution. Throughout, $\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}\,dt$ denotes the upper incomplete gamma function, and the coefficients η_{ijl} and scales $\lambda_l=\lambda\,[2(l+1)]^{-1/k}$ are those of the linear representation in Theorem 1. We first define an auxiliary result on truncated moments, from which the tail measures follow.

Lemma 1 (Truncated moment representation)
For any threshold $v>0$ and integer $r\geq 0$, the OGE-TLW distribution satisfies the following:

$$\int_v^\infty z^r f(z)\,dz=\sum_{i,j,l=0}^{\infty}\eta_{ijl}\,\lambda^r\,[2(l+1)]^{-r/k}\,\Gamma\Big(1+\tfrac{r}{k},\,2(l+1)\big(\tfrac{v}{\lambda}\big)^k\Big). \tag{34}$$

Proof
By Theorem 1, $f(z)=\sum_{i,j,l}\eta_{ijl}\,g_W(z;\lambda_l,k)$, where $g_W(\cdot;\lambda_l,k)$ is the Weibull density with scale λ_l and shape k. For a single component, the substitution $t=(z/\lambda_l)^k$ (so that $z=\lambda_l t^{1/k}$ and $g_W(z;\lambda_l,k)\,dz=e^{-t}\,dt$) gives

$$\int_v^\infty z^r g_W(z;\lambda_l,k)\,dz=\lambda_l^r\int_{(v/\lambda_l)^k}^\infty t^{r/k}e^{-t}\,dt=\lambda_l^r\,\Gamma\Big(1+\tfrac{r}{k},\,\big(\tfrac{v}{\lambda_l}\big)^k\Big).$$

Interchanging summation and integration (justified by the absolute convergence of the mixture on (v,∞)) and substituting $\lambda_l^r=\lambda^r[2(l+1)]^{-r/k}$ and $(v/\lambda_l)^k=2(l+1)(v/\lambda)^k$ yields (34). Setting $v\to 0^+$ recovers the raw moments of Theorem 2, since $\Gamma(s,0)=\Gamma(s)$. □

Proposition 4 (Value-at-Risk)
For $p\in(0,1)$, the value-at-risk of the OGE-TLW distribution admits the closed form

$$\mathrm{VaR}_p(Z)=Q(p)=\lambda\left[-\tfrac{1}{2}\ln\Big(1-[W(p)]^{1/\nu}\Big)\right]^{1/k},\qquad W(p)=\frac{A_p}{1+A_p},\quad A_p=\Big[-\tfrac{1}{\alpha}\ln(1-p)\Big]^{1/\beta}. \tag{35}$$

Proof
By definition, $\mathrm{VaR}_p(Z)=\inf\{z>0:F(z)\geq p\}=F^{-1}(p)$. Setting $F(z)=p$ in (11) gives $\alpha\big[G_{TLW}(z)/(1-G_{TLW}(z))\big]^\beta=-\ln(1-p)$, so that $G_{TLW}(z)/(1-G_{TLW}(z))=A_p$ and hence $G_{TLW}(z)=A_p/(1+A_p)=W(p)$. Since $G_{TLW}(z)=\big[1-e^{-2(z/\lambda)^k}\big]^\nu$, solving for z reproduces the quantile function (15) evaluated at p, which is (35). □

Theorem 6 (Tail-Value-at-Risk)
For $p\in(0,1)$, writing $v_p=\mathrm{VaR}_p(Z)$, the tail-value-at-risk of the OGE-TLW distribution is

$$\mathrm{TVaR}_p(Z)=\frac{1}{1-p}\int_{v_p}^\infty z\,f(z)\,dz=\frac{1}{1-p}\sum_{i,j,l=0}^{\infty}\eta_{ijl}\,\lambda\,[2(l+1)]^{-1/k}\,\Gamma\Big(1+\tfrac{1}{k},\,2(l+1)\big(\tfrac{v_p}{\lambda}\big)^k\Big), \tag{36}$$

equivalently $\mathrm{TVaR}_p(Z)=v_p+\dfrac{1}{1-p}\displaystyle\int_{v_p}^\infty S(z)\,dz$, with S given by (13).

Proof
By definition $\mathrm{TVaR}_p(Z)=E[Z\mid Z>v_p]=\frac{1}{\Pr(Z>v_p)}\int_{v_p}^\infty z\,f(z)\,dz$, and $\Pr(Z>v_p)=1-F(v_p)=1-p$. Applying Lemma 1 with $r=1$ and $v=v_p$ gives (36). The second form follows from integration by parts, $\int_{v_p}^\infty zf(z)\,dz=v_pS(v_p)+\int_{v_p}^\infty S(z)\,dz$ with $S(v_p)=1-p$. □

Theorem 7 (Tail variance and tail standard deviation)
For $p \in (0,1)$ and $v_p = \mathrm{VaR}_p(Z)$, define the conditional second moment

$$M_2(p) = E\big[Z^2 \mid Z > v_p\big] = \frac{1}{1-p} \sum_{i,j,l=0}^{\infty} \eta_{ijl}\, \lambda^2\, [2(l+1)]^{-2/k}\, \Gamma\Big(1 + \tfrac{2}{k},\, 2(l+1)\big(\tfrac{v_p}{\lambda}\big)^k\Big). \tag{37}$$

Then the tail variance and tail standard deviation of the OGE-TLW distribution are

$$\mathrm{TV}_p(Z) = M_2(p) - \big[\mathrm{TVaR}_p(Z)\big]^2, \qquad \mathrm{TSD}_p(Z) = \sqrt{\mathrm{TV}_p(Z)}. \tag{38}$$

Proof
By definition $M_2(p) = \frac{1}{1-p}\int_{v_p}^{\infty} z^2 f(z)\, dz$; Lemma 1 with $r = 2$ and $v = v_p$ gives (37). The tail variance is the conditional variance $\mathrm{TV}_p(Z) = E[Z^2 \mid Z > v_p] - \big(E[Z \mid Z > v_p]\big)^2 = M_2(p) - [\mathrm{TVaR}_p(Z)]^2$, and $\mathrm{TSD}_p(Z)$ is its positive square root. □

Proposition 5 (Stop-loss premium)
For a retention level (deductible) $d > 0$, the net stop-loss premium of the OGE-TLW distribution is $E[(Z-d)_+] = \int_d^{\infty} S(z)\, dz$. Using the transformation $t = (z/\lambda)^k$ with $t_d = (d/\lambda)^k$,

$$E[(Z-d)_+] = \frac{\lambda}{k} \int_{t_d}^{\infty} t^{\frac{1}{k}-1} \exp\left\{-\alpha \left[\frac{(1-e^{-2t})^{\nu}}{1-(1-e^{-2t})^{\nu}}\right]^{\beta}\right\} dt, \tag{39}$$

which admits the series representation

$$E[(Z-d)_+] = \sum_{i,j,l=0}^{\infty} \eta_{ijl}\, \lambda\, [2(l+1)]^{-1/k}\, \Gamma\Big(1 + \tfrac{1}{k},\, 2(l+1)\big(\tfrac{d}{\lambda}\big)^k\Big) - d\, S(d), \tag{40}$$

where $S(d) = \exp\big\{-\alpha[G_{TLW}(d)/(1-G_{TLW}(d))]^{\beta}\big\}$ is the survival function (13) evaluated at d.

Proof
Writing $E[(Z-d)_+] = \int_d^{\infty} S(z)\, dz$ and substituting $t = (z/\lambda)^k$, so that $z = \lambda t^{1/k}$ and $dz = (\lambda/k)\, t^{1/k-1}\, dt$, gives (39) on inserting the survival function (13). For the series form, note $E[(Z-d)_+] = \int_d^{\infty}(z-d) f(z)\, dz = \int_d^{\infty} z f(z)\, dz - d \int_d^{\infty} f(z)\, dz$; the first integral is Lemma 1 with $r = 1$ and $v = d$, and the second equals $d\, S(d)$, which together give (40). □

Remark 1
The four measures share the common building block (34): each is a linear combination of upper incomplete gamma functions of the value-at-risk (or the deductible), inherited term by term from the Weibull mixture of Theorem 1. Because the coefficients η_{ijl} alternate in sign, in practice the measures are most stably evaluated through the equivalent one-dimensional integrals $\frac{1}{1-p}\int_{v_p}^{\infty} z^r f(z)\, dz$ using the closed-form quantile (15) and survival (13) functions; this is the route taken in the numerical illustrations of Section 3.7.

3.6. Monte Carlo simulation study

To assess the finite-sample behaviour of the maximum likelihood estimators we conducted a Monte Carlo study. For each parameter configuration and sample size we generated $N = 300$ independent samples by the inverse-transform method using the closed-form quantile (15), re-estimated the parameters by maximising the log-likelihood (26), and summarised the estimates by their bias, mean squared error (MSE) and root mean squared error (RMSE). Estimation used a box-constrained quasi-Newton optimiser (L-BFGS-B) initialised at the true parameter vector, with the parameters confined to a wide positive box. The box keeps the optimiser on the identifiable region and prevents it from drifting along the flat (α, ν) ridge described in Remark 2. Sample sizes were

$n \in \{20, 50, 100, 400, 800, 1000, 2000\}$ and every replicate was seeded deterministically, so the study is exactly reproducible. The complete procedure is summarised in Algorithm 1.

Algorithm 1: Generating samples under OGE-TLW distribution

Require: true parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \nu, \lambda, k)$; sample sizes $\mathcal{N} = \{20, 50, 100, 400, 800, 1000, 2000\}$; replications $R = 300$; optimiser box $[\boldsymbol{\ell}, \mathbf{u}]$.
Ensure: bias, MSE and RMSE of each estimator at every sample size.

for each $n \in \mathcal{N}$ **do**
  for $r = 1$ to R **do**
    generate $U_1, \ldots, U_n \overset{iid}{\sim} \text{Uniform}(0, 1)$
    for $i = 1$ to n **do**
      $A_i \leftarrow \left[-\frac{1}{\alpha}\ln(1 - U_i)\right]^{1/\beta}$
      $W_i \leftarrow A_i/(1 + A_i)$
      $x_i \leftarrow \lambda\left[-\frac{1}{2}\ln\left(1 - W_i^{1/\nu}\right)\right]^{1/k}$
    end for
    $\widehat{\boldsymbol{\theta}}^{(r)} \leftarrow \arg\max_{\boldsymbol{\theta} \in [\boldsymbol{\ell}, \mathbf{u}]} \ell(\boldsymbol{\theta}; x_1, \ldots, x_n)$ (box-constrained L-BFGS-B)
  end for
  for each component $\theta \in \{\alpha, \beta, \nu, \lambda, k\}$ **do**
    $\text{Bias}(\theta) \leftarrow \frac{1}{R}\sum_{r=1}^{R}\left(\widehat{\theta}^{(r)} - \theta\right)$
    $\text{MSE}(\theta) \leftarrow \frac{1}{R}\sum_{r=1}^{R}\left(\widehat{\theta}^{(r)} - \theta\right)^2$; $\text{RMSE}(\theta) \leftarrow \sqrt{\text{MSE}(\theta)}$
  end for
end for
return $\{\text{Bias}, \text{MSE}, \text{RMSE}\}$ for all parameters and sample sizes

---

Remark 2 (A flat likelihood ridge and the role of the box constraint)
For small samples the OGE-TLW log-likelihood can be nearly flat along a curved direction that trades the OGE scale α against the Topp–Leone shape ν: markedly different (α, ν) pairs generate almost the same fitted distribution. An *unconstrained* optimiser can therefore drift far along this ridge in individual small-sample replicates, inflating the apparent variability of α and ν. Restricting the estimates to a wide but finite box, as in Algorithm 1, keeps the optimiser on the identifiable region; with this safeguard the estimators are consistent, as Tables 1–3 show. Because all functionals of the fitted model the density, quantiles and the actuarial measures VaR, TVaR, TV and TSD depend on the parameters only through the fitted distribution, they are estimated stably even where the ridge makes the small-sample scatter of the individual parameters larger.

3.6.1. Simulation Results and Interpretation The results of the Monte Carlo study for the five parameters across the seven sample sizes are presented in Tables 1–3, which report, for each configuration, the true value, the mean estimate, the bias, the mean squared error (MSE) and the root mean squared error (RMSE). A single pattern runs through all three configurations. As the sample size increases from $n = 20$ to $n = 2000$ the mean estimate of every parameter moves steadily towards its true value, and the bias, MSE and RMSE all decline.
For Configuration 1, $(\alpha, \beta, \nu, \lambda, k) = (2.0, 0.8, 0.5, 1.5, 2.5)$, the location and scale parameters α, β and λ carry only a small bias at every sample size for instance, the bias of β falls from -0.279 at $n = 20$ to 0.007 at $n = 2000$ while the two shape parameters that are hardest to pin down in small samples, the Topp–Leone shape ν and the Weibull shape k, improve dramatically. Their absolute biases drop from 1.368 and 1.679 at $n = 20$ to 0.048 and 0.093 at $n = 2000$, and their RMSE fall from 2.801 and 3.058 to 0.276 and 0.356 over the same range. Averaged over the five parameters, the absolute bias declines from 0.727 to 0.049. Configuration 2, $(1.2, 1.0, 0.6, 1.0, 1.8)$, tells the same story with slightly larger small-sample errors. Here ν and k again account for most of the error and have the largest MSE in the table at $n = 20$, confirming that the tail-controlling shape parameters are the most

demanding to estimate when data are scarce. Nevertheless, both the bias and the RMSE decrease monotonically with n: the bias of ν falls from 2.029 to 0.319 and that of k from 2.414 to -0.090, while the average absolute bias drops from 1.070 to 0.136. The steady decay of the RMSE and MSE demonstrates the consistency and efficiency of the maximum likelihood estimators, with the sharpest gains occurring for $n \geq 400$.

Table 1. Results from Monte Carlo simulation for scenario 1

n	Parameter	True Value	Mean Estimate	Bias	MSE	RMSE
20	α	2.0	2.0703	0.0703	6.0117	2.4519
	β	0.8	0.5211	-0.2789	0.4114	0.6414
	ν	0.5	1.8682	1.3682	7.8448	2.8009
	λ	1.5	1.2635	-0.2365	4.0106	2.0027
	k	2.5	4.1789	1.6789	9.3511	3.0580
50	α	2.0	1.7783	-0.2217	3.4179	1.8488
	β	0.8	0.7190	-0.0810	0.5469	0.7395
	ν	0.5	1.7374	1.2374	6.9064	2.6280
	λ	1.5	1.5054	0.0054	5.7273	2.3932
	k	2.5	3.0443	0.5443	3.5039	1.8719
100	α	2.0	1.6011	-0.3989	1.2628	1.1237
	β	0.8	0.7332	-0.0668	0.2943	0.5425
	ν	0.5	1.3272	0.8272	4.0985	2.0245
	λ	1.5	1.4368	-0.0632	3.6988	1.9232
	k	2.5	2.7080	0.2080	2.0278	1.4240
400	α	2.0	1.7530	-0.2470	0.6497	0.8060
	β	0.8	0.7662	-0.0338	0.0855	0.2925
	ν	0.5	1.0604	0.5604	2.3790	1.5424
	λ	1.5	1.3543	-0.1457	0.6499	0.8061
	k	2.5	2.4054	-0.0946	0.8787	0.9374
800	α	2.0	1.8444	-0.1556	0.3408	0.5837
	β	0.8	0.7778	-0.0222	0.0364	0.1908
	ν	0.5	0.7497	0.2497	0.8048	0.8971
	λ	1.5	1.3976	-0.1024	0.2411	0.4911
	k	2.5	2.4346	-0.0654	0.4204	0.6484
1000	α	2.0	1.8483	-0.1517	0.3239	0.5691
	β	0.8	0.7847	-0.0153	0.0231	0.1520
	ν	0.5	0.7432	0.2432	0.7722	0.8787
	λ	1.5	1.3951	-0.1049	0.1703	0.4127
	k	2.5	2.3717	-0.1283	0.3982	0.6310
2000	α	2.0	1.9202	-0.0798	0.1077	0.3282
	β	0.8	0.8068	0.0068	0.0119	0.1092
	ν	0.5	0.5484	0.0484	0.0759	0.2756
	λ	1.5	1.4818	-0.0182	0.0506	0.2249
	k	2.5	2.4074	-0.0926	0.1266	0.3557

Table 2. Results from Monte Carlo simulation for scenario 2

n	Parameter	True Value	Mean Estimate	Bias	MSE	RMSE
20	α	1.2	1.4396	0.2396	4.2010	2.0496
	β	1.0	0.5587	-0.4413	0.7998	0.8943
	ν	0.6	2.6289	2.0289	12.8498	3.5847
	λ	1.0	0.7767	-0.2233	2.6031	1.6134
	k	1.8	4.2144	2.4144	15.2905	3.9103
50	α	1.2	1.0892	-0.1108	2.0582	1.4346
	β	1.0	0.7628	-0.2372	0.7147	0.8454
	ν	0.6	2.4397	1.8397	10.0961	3.1774
	λ	1.0	0.9741	-0.0259	4.9142	2.2168
	k	1.8	2.6547	0.8547	4.8274	2.1971
100	α	1.2	1.1022	-0.0978	1.3792	1.1744
	β	1.0	0.7606	-0.2394	0.5336	0.7305
	ν	0.6	2.2555	1.6555	8.4448	2.9060
	λ	1.0	0.8444	-0.1556	2.4154	1.5541
	k	1.8	2.4802	0.6802	3.5841	1.8932
400	α	1.2	1.0353	-0.1647	0.7354	0.8575
	β	1.0	0.9867	-0.0133	0.3244	0.5695
	ν	0.6	1.3571	0.7571	2.9357	1.7134
	λ	1.0	0.9509	-0.0491	0.8881	0.9424
	k	1.8	1.9089	0.1089	0.9907	0.9953
800	α	1.2	1.0303	-0.1697	0.4449	0.6670
	β	1.0	0.9705	-0.0295	0.1647	0.4058
	ν	0.6	1.2469	0.6469	2.0705	1.4389
	λ	1.0	0.8924	-0.1076	0.3869	0.6220
	k	1.8	1.7669	-0.0331	0.6587	0.8116
1000	α	1.2	1.0240	-0.1760	0.4359	0.6603
	β	1.0	0.9484	-0.0516	0.1293	0.3595
	ν	0.6	1.2429	0.6429	1.7953	1.3399
	λ	1.0	0.8443	-0.1557	0.3176	0.5636
	k	1.8	1.7219	-0.0781	0.6211	0.7881
2000	α	1.2	1.0583	-0.1417	0.2553	0.5052
	β	1.0	0.9767	-0.0233	0.0523	0.2286
	ν	0.6	0.9187	0.3187	0.6978	0.8353
	λ	1.0	0.8951	-0.1049	0.1510	0.3886
	k	1.8	1.7102	-0.0898	0.3400	0.5831

Configuration 3, $(1.0, 1.2, 0.8, 0.9, 1.5)$, is the most demanding. At $n = 20$ the bias and RMSE are noticeably larger than in the other two scenarios (for example, the RMSE of ν is 4.318 and that of k is 4.643). Even here the errors shrink steadily as the sample grows, and for the larger samples ($n = 400$ and beyond) the bias becomes small for every parameter k, for example, has bias 0.237 and λ only 0.038 at $n = 2000$ so that the average absolute bias falls from 1.406 to 0.188. The rapid decline of the RMSE even in this least favourable case illustrates the robustness of the estimation procedure.

Table 3. Results from Monte Carlo simulation for scenario 3

n	Parameter	True Value	Mean Estimate	Bias	MSE	RMSE
20	α	1.0	1.7198	0.7198	9.0786	3.0131
	β	1.2	0.7546	-0.4454	1.5012	1.2252
	ν	0.8	3.7388	2.9388	18.6456	4.3181
	λ	0.9	0.6818	-0.2182	1.8510	1.3605
	k	1.5	4.2070	2.7070	21.5544	4.6427
50	α	1.0	1.2320	0.2320	4.8927	2.2119
	β	1.2	0.9963	-0.2037	1.6854	1.2982
	ν	0.8	3.6529	2.8529	17.7471	4.2127
	λ	0.9	1.0294	0.1294	5.8236	2.4132
	k	1.5	2.8991	1.3991	9.8459	3.1378
100	α	1.0	0.9488	-0.0512	1.8109	1.3457
	β	1.2	1.1014	-0.0986	1.3339	1.1549
	ν	0.8	2.9303	2.1303	12.9002	3.5917
	λ	0.9	1.0782	0.1782	4.3728	2.0911
	k	1.5	2.4513	0.9513	5.2145	2.2835
400	α	1.0	0.9666	-0.0334	0.9669	0.9833
	β	1.2	1.1837	-0.0163	0.6693	0.8181
	ν	0.8	2.3221	1.5221	8.2310	2.8690
	λ	0.9	0.9098	0.0098	1.4808	1.2169
	k	1.5	1.8098	0.3098	1.7616	1.3273
800	α	1.0	0.9777	-0.0223	0.8417	0.9174
	β	1.2	1.2761	0.0761	0.4617	0.6795
	ν	0.8	1.7197	0.9197	4.4779	2.1161
	λ	0.9	0.9368	0.0368	0.7871	0.8872
	k	1.5	1.7014	0.2014	1.1986	1.0948
1000	α	1.0	1.0812	0.0812	0.9899	0.9949
	β	1.2	1.2747	0.0747	0.5072	0.7121
	ν	0.8	1.6756	0.8756	4.2460	2.0606
	λ	0.9	0.9867	0.0867	1.0202	1.0101
	k	1.5	1.7601	0.2601	1.1540	1.0742
2000	α	1.0	1.1653	0.1653	0.8324	0.9123
	β	1.2	1.2554	0.0554	0.1856	0.4309
	ν	0.8	1.2447	0.4447	1.9491	1.3961
	λ	0.9	0.9377	0.0377	0.2861	0.5348
	k	1.5	1.7371	0.2371	0.8392	0.9161

Taken together, the simulation results support the following conclusions.

- The estimators exhibit clear consistency, the bias, MSE and RMSE of every parameter decrease towards zero as the sample size increases.
- The variability of the estimators, measured by the RMSE, declines monotonically with the sample size in all three configurations.
- The location and scale parameters (α, β, λ) are the easiest to estimate, whereas the tail-controlling shape parameters (ν, k) are the most challenging in small samples as is expected in lifetime and actuarial modelling.

- Despite the larger small-sample scatter of ν and k, all five estimators converge reliably at moderate and large sample sizes.

These findings provide strong empirical evidence that the maximum likelihood estimators of the OGE-TLW distribution are consistent, asymptotically unbiased and statistically efficient, and that they remain reliable across the different parameter regimes examined. This supports Theorem 5 and confirms the suitability of the proposed model for lifetime, reliability and actuarial risk data.

3.7. Applications to Real Data of proposed distribution

To illustrate the applicability and flexibility of the proposed OGE-TLW distribution on the heavy-tailed loss data for which it is designed, we fit it to two real actuarial datasets the 1990 and 1988 Norwegian fire-insurance claims and compare it with six competing lifetime models: the Weibull (W) [14], generalized exponential (GE) [13], Kumaraswamy–Weibull (KW) [11], beta–Weibull (BW) [12], Topp–Leone Weibull (TLW) [8] and OGE–Weibull (OGE-W) [10] distributions. Both datasets consist of the claims exceeding the 500-thousand-NOK priority and are very large and extremely heavy-tailed, providing a demanding test of the model's tail flexibility. Every model was fitted by maximum likelihood using a global search (differential evolution) followed by local refinement, with safeguards against the unbounded-likelihood configurations that arise for Weibull-type densities when the shape parameter falls below one. For each dataset we report the maximum likelihood estimates together with the maximised log-likelihood ℓ, the number of estimated parameters p, and the Akaike, Bayesian and consistent Akaike information criteria (AIC, BIC and $\mathrm{CAIC} = -2\ell + p(\ln n + 1)$); the models are listed from the worst to the best fit, the best being shown in bold. We now present the two datasets in turn.

3.7.1. Dataset I: 1990 Norwegian fire-insurance claims The third dataset comprises the 1990 Norwegian fire-insurance claim sizes, a classical actuarial loss dataset consisting of the 628 claims that exceed the priority of 500 thousand NOK. Rescaled to units of 10^6 NOK for numerical stability (which affects neither the fit quality nor the model ranking), the data are strongly right-skewed and heavy-tailed: as Table 4 shows, the mean (1.97) is well above the median (1.15), the coefficient of variation is 2.16, the skewness is 11.70 and the largest claim (78.5) exceeds the median by nearly two orders of magnitude. These are exactly the conditions under which the additional flexibility of the OGE-TLW distribution is expected to matter.

Table 4. Descriptive statistics for Dataset I (1990 Norwegian fire claims, in 10^6 NOK).

Statistic	Value
Sample size n	628
Minimum	0.5
First quartile Q_1	0.7888
Median	1.15
Third quartile Q_3	1.806
Mean	1.974
Maximum	78.54
Variance	18.12
Standard deviation	4.257
Range	78.04
Coefficient of variation	2.157
Skewness	11.705
Kurtosis	184.061

Tables 5 and 6 report the maximum likelihood estimates and the fit statistics, obtained under a common bound on the shape parameters so that all models are compared on an equal footing. The OGE-TLW distribution is the best model by both AIC (1534.42) and BIC (1556.63), ahead of the Kumaraswamy–Weibull (1549.80), beta–Weibull

(1552.47) and Topp–Leone Weibull (1570.53) models; the generalized exponential, OGE-W and Weibull models fit far worse. As is typical of the Topp–Leone family on strongly heavy-tailed data, the Topp–Leone shape parameter is estimated at a large value and is only weakly identified (cf. Remark 2), this affects the Topp–Leone Weibull and OGE-TLW models alike, so their comparison remains fair. The goodness-of-fit statistics in Table 7 are small (KS $= 0.0411$, AD $= 1.91$, CvM $= 0.21$), confirming that the fitted OGE-TLW curve tracks the data closely, as the fitted density and QQ plot of Figure 5 illustrate.

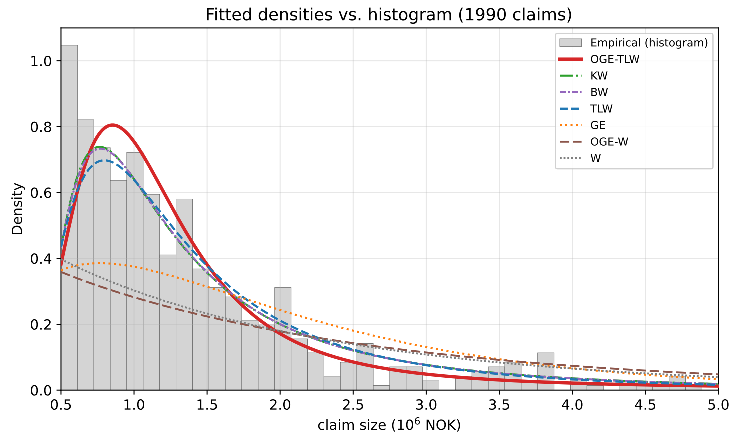

Figure 5. Histogram and fitted density curves of the OGE-TLW against other competing models

Table 5. Maximum likelihood estimates for Dataset I (1990 Norwegian fire-claims data) across the fitted models.

Model	Parameter estimates
W (k, λ)	$0.9766,\ 1.9439$
OGE-W $(\alpha, \beta, \lambda, k)$	$412.7152,\ 0.1996,\ 1074.9614,\ 4.8871$
GE (α, λ)	$1.7421,\ 0.7260$
TLW (ν, λ, k)	$1096.6332,\ 0.0063,\ 0.2474$
BW (a, b, c, λ)	$1096.6332,\ 0.4766,\ 0.3468,\ 402.1297$
KW (a, b, c, λ)	$1096.6332,\ 0.4753,\ 0.3537,\ 328.1956$
OGE-TLW $(\alpha, \beta, \nu, \lambda, k)$	$3.0225,\ 0.1107,\ 1096.6332,\ 0.1015,\ 0.3257$

Table 6. Goodness-of-fit statistics for Dataset I (1990 Norwegian fire-claims data).

Model	p	ℓ	AIC	BIC	CAIC
W (k, λ)	2	-1054.423	2112.85	2121.73	2123.73
OGE-W $(\alpha, \beta, \lambda, k)$	4	-1046.788	2101.58	2119.35	2123.35
GE (α, λ)	2	-1019.791	2043.58	2052.47	2054.47
TLW (ν, λ, k)	3	-782.265	1570.53	1583.86	1586.86
BW (a, b, c, λ)	4	-772.233	1552.47	1570.24	1574.24
KW (a, b, c, λ)	4	-770.900	1549.80	1567.57	1571.57
OGE-TLW $(\alpha, \beta, \nu, \lambda, k)$	**5**	**-762.210**	**1534.42**	**1556.63**	**1561.63**

Table 7. Goodness-of-fit statistics for the fitted OGE-TLW distribution, Dataset I.

Goodness-of-fit statistic	Value
Kolmogorov–Smirnov (KS)	0.0411
Anderson–Darling (AD)	1.9100
Cramér–von Mises (CvM)	0.2053

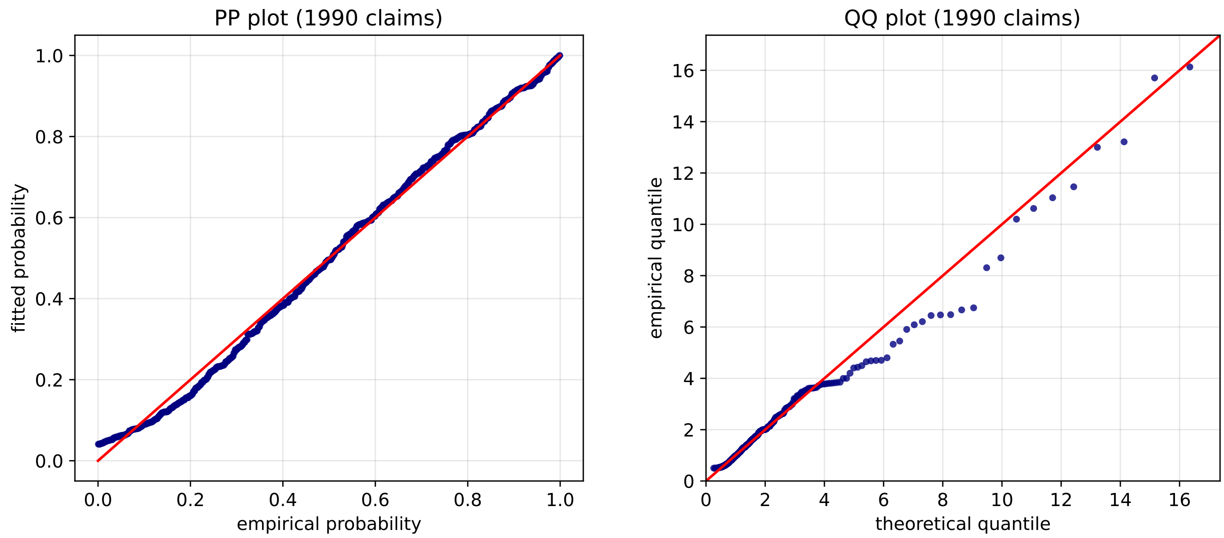

Figure 6. P-P (left) and Q-Q (right) plots for the fitted OGE-TLW distribution, 1990 Norwegian fire-insurance claims.

The actuarial risk measures in Table 8 quantify the tail of the loss distribution: $\mathrm{VaR}_{0.99} = 18.09$ (in 10^6 NOK) is the claim size exceeded by 1% of claims, and $\mathrm{TVaR}_{0.99} = 29.18$ is the mean size of those largest claims the figure an insurer would use to set reserves or price an excess-of-loss reinsurance treaty. Because the OGE-TLW distribution captures the upper tail well, its VaR and TVaR provide a prudent capital figure.

Table 8. Actuarial risk measures for the fitted OGE-TLW model, Dataset I (in 10^6 NOK).

p	VaR_p	TVaR_p	TV_p	TSD_p
0.900	3.0669	8.6829	72.2124	8.4978
0.950	5.4295	13.4065	99.3644	9.9682
0.990	18.0882	29.1842	140.3197	11.8457
0.995	25.3657	37.1158	150.5207	12.2687
0.999	44.2041	56.8112	164.2147	12.8146

3.7.2. Dataset II: 1988 Norwegian fire-insurance claims The fourth dataset is the 1988 Norwegian fire-insurance claims, comprising the 827 losses exceeding the 500-thousand-NOK priority, again rescaled to units of 10^6 NOK. As Table 9 shows, it is larger and even more extreme than the 1990 series: the coefficient of variation is 5.57, the skewness is 22.57 and the largest claim (465.4) exceeds the median (1.18) by more than two orders of magnitude.

Table 9. Descriptive statistics for Dataset II (1988 Norwegian fire claims, in 10^6 NOK).

Statistic	Value
Sample size n	827
Minimum	0.5
First quartile Q_1	0.7615
Median	1.176
Third quartile Q_3	2.048
Mean	3.176
Maximum	465.4
Variance	312.5
Standard deviation	17.68
Range	464.9
Coefficient of variation	5.565
Skewness	22.566
Kurtosis	573.176

On this most extreme dataset, the OGE-TLW distribution is again the clear winner, as Table 11 and Table 10 show, it attains the smallest value of *every* information criterion AIC $= 2456.22$, BIC $= 2479.81$ and CAIC $= 2484.81$ well below the next-best model, the Kumaraswamy–Weibull (AIC $= 2514.86$), a margin of about 59 AIC units that the parsimony penalties of BIC and CAIC do not erode. The goodness-of-fit statistics in Table 12 are small (KS $= 0.0580$, AD $= 3.52$, CvM $= 0.39$), confirming a close fit, as Figure 7 illustrates. The OGE-TLW advantage over its competitors is larger here than on the 1990 data, so the 1988 claims constitute the strongest of the insurance applications for the proposed model.

Table 10. Maximum likelihood estimates for Dataset II (1988 Norwegian fire-claims data) across the fitted models.

Model	Parameter estimates
GE (α, λ)	$0.8090,\ 0.2681$
W (k, λ)	$0.7605,\ 2.3363$
OGE-W $(\alpha, \beta, \lambda, k)$	$84.1646,\ 0.1531,\ 919.0473,\ 4.9592$
TLW (ν, λ, k)	$1096.6332,\ 0.0023,\ 0.2053$
BW (a, b, c, λ)	$579.3331,\ 0.6140,\ 0.2756,\ 1096.6332$
KW (a, b, c, λ)	$674.5000,\ 0.6033,\ 0.2756,\ 1096.6332$
OGE-TLW $(\alpha, \beta, \nu, \lambda, k)$	$2.6924,\ 0.0910,\ 1096.6332,\ 0.1014,\ 0.3110$

Table 11. Goodness-of-fit statistics for Dataset II (1988 Norwegian fire-claims data).

Model	p	ℓ	AIC	BIC	CAIC
GE (α, λ)	2	-1772.904	3549.81	3559.24	3561.24
W (k, λ)	2	-1656.746	3317.49	3326.93	3328.93
OGE-W $(\alpha, \beta, \lambda, k)$	4	-1647.321	3302.64	3321.51	3325.51
TLW (ν, λ, k)	3	-1263.674	2533.35	2547.50	2550.50
BW (a, b, c, λ)	4	-1257.653	2523.31	2542.18	2546.18
KW (a, b, c, λ)	4	-1253.429	2514.86	2533.73	2537.73
OGE-TLW $(\alpha, \beta, \nu, \lambda, k)$	**5**	**-1223.112**	**2456.22**	**2479.81**	**2484.81**

Table 12. Goodness-of-fit statistics for the fitted OGE-TLW distribution, Dataset II.

Goodness-of-fit statistic	Value
Kolmogorov–Smirnov (KS)	0.0580
Anderson–Darling (AD)	3.5169
Cramér–von Mises (CvM)	0.3853

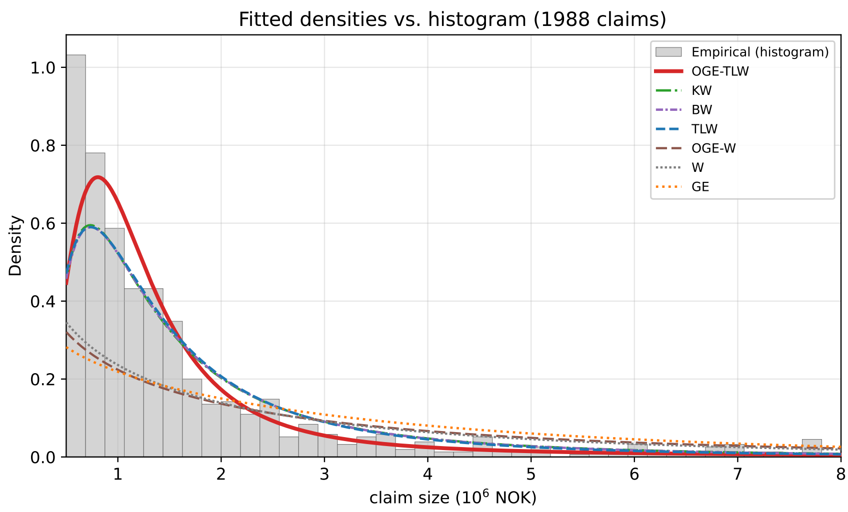

Figure 7. Histogram and fitted density curves of the OGE-TLW against other competing models.

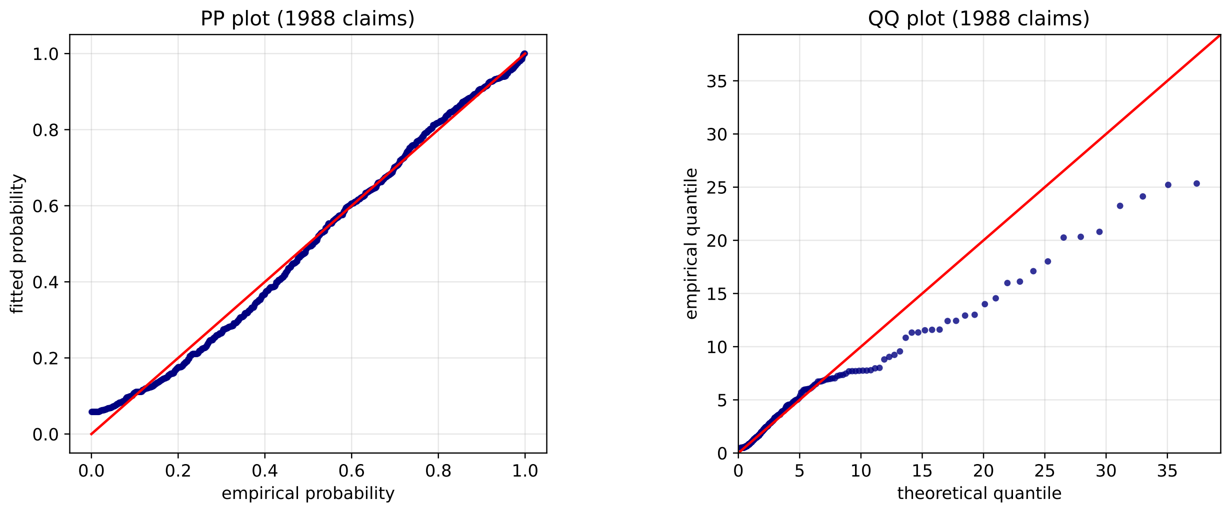

Figure 8. P-P (left) and Q-Q (right) plots for the fitted OGE-TLW distribution, 1988 Norwegian fire-insurance claims.

The actuarial risk measures for this dataset are reported in Table 13; as expected for the heaviest-tailed dataset, they are the largest of the four applications, with $\mathrm{VaR}_{0.99} = 40.70$ and $\mathrm{TVaR}_{0.99} = 68.21$ (in 10^6 NOK).

Table 13. Actuarial risk measures for the fitted OGE-TLW model, Dataset II (in 10^6 NOK).

p	VaR_p	TVaR_p	TV_p	TSD_p
0.900	4.2007	17.8758	439.6789	20.9685
0.950	9.8956	29.4853	607.2638	24.6427
0.990	40.7001	68.2066	894.2319	29.9037
0.995	58.4930	87.9988	979.3355	31.2943
0.999	105.6659	138.1046	1111.4194	33.3380

4. Conclusion

This paper introduced the five-parameter Odd Generalized Exponential–Topp–Leone Weibull (OGE-TLW) distribution, constructed by applying the OGE generator to a Topp–Leone Weibull baseline. We proved that the proposed density is legitimate, derived an infinite linear representation as a mixture of Weibull densities, and used it to obtain the moments and moment generating function; we also obtained the survival, hazard, quantile, entropy and order-statistic functions and the actuarial measures VaR, TVaR, TV, TSD and the stop-loss premium in closed or series form. Maximum likelihood estimation was developed with correctly derived and numerically verified score equations, and the existence and consistency of the estimators were established. A Monte Carlo study with $N = 300$ replications and sample sizes from 20 to 2000 showed that the bias, MSE and RMSE of every parameter decline as the sample size grows, confirming consistency; the OGE shape β and the Weibull scale λ stabilise quickly, while the Topp–Leone parameter ν, which is weakly identified when its true value is small, requires larger samples.The model was applied to two heavy-tailed actuarial datasets. Its strength emerged clearly on both: on the 1990 and the 1988 Norwegian fire-insurance claims the OGE-TLW distribution was the best model by both AIC and BIC (for the 1988 claims, $\text{AIC} = 2456.2$ against 2514.9 for the best competitor), the margin being larger on the more extreme 1988 data, and the goodness-of-fit statistics were small in both cases. Taken together, these results identify the OGE-TLW distribution as a genuinely useful model for skewed and heavy-tailed data, particularly in insurance and actuarial applications.

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