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Decision-Making Method Based on Interval-valued Fuzzy Soft Expert Set

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Abstract Soft expert sets, introduced by Alkhazaleh and Salleh in 2011, give decision makers a way to fold several experts' opinions into a single mathematical object rather than aggregating them by hand; a fuzzy version of the same idea followed shortly after, adding graded (fuzzy) membership on top of the expert structure. Separately, Yang et al. (2009) showed how interval-valued fuzzy sets – membership expressed as a range rather than one number – combine with soft sets to capture cases where a single crisp membership value is too precise for the information available. This paper brings these two lines together: we build the *interval-valued fuzzy soft expert set* (IVFSES), a structure in which every expert's opinion on every parameter is recorded as an interval rather than a point, while still keeping the multi-expert, agree/disagree machinery of soft expert sets intact. We work out complement, union, intersection, AND and OR for this structure and prove their basic algebraic properties, then build a step-by-step decision algorithm on top of it. Because opinions no longer have to be squeezed into a single number, the model can represent an expert's uncertainty about their own rating, not just disagreement between experts, and it does so without needing any extra aggregation machinery – one integrated object holds everyone's input. We illustrate the algorithm on a hospital-expansion decision problem and work through it end to end. We further test the model with a second, independent case study, benchmark it against a classical point-valued soft expert set method, and run a sensitivity analysis showing the recommended decision is stable under random perturbation of the input data.

Keywords soft set, soft expert set, fuzzy soft expert set, interval-valued fuzzy soft expert set, decision-making, sensitivity analysis

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1. Introduction

Real-world decision problems in engineering, medicine, economics, and environmental planning routinely involve information that is incomplete, vague, or contested among the people assessing it. Molodtsov [3] addressed this gap by proposing soft set theory, a parameterized family of subsets of a universe that avoids the restrictive assumptions of earlier uncertainty models. Maji, Roy and Biswas [4] worked out a set of basic operations for soft sets and showed how they could be applied in practice, and in a companion paper [6] combined soft sets with fuzzy sets to allow graded, rather than binary, membership. Building on that combination, Yang et al. [1] replaced the point-valued fuzzy grades with interval-valued ones, giving each element a membership *range* instead of a single number, and supplied a matching decision-making procedure. Feng et al. [2] pushed this interval-valued fuzzy soft model further with a level-soft-set approach to decision problems.

A separate line of work tackled the opposite limitation: rather than the shape of the membership grade, Alkhazaleh and Salleh [9, 10] focused on how to record *several experts'* judgements at once, introducing soft expert sets and their fuzzy extension so that a panel's opinions sit inside one object instead of being pre-aggregated by hand.

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Since then, soft-expert-set-based decision algorithms have been carried into a range of concrete application domains: an interval-valued neutrosophic soft expert set was used to rank suppliers in construction supply chain management [11], a fuzzy hypersoft expert set was applied to product selection [12], and a neutrosophic hypersoft expert set with fuzzy-parameterized degrees was used for heart-disease diagnosis [13]. The soft-set family itself keeps growing in this direction, too: a linguistic SuperHypersoft set, published in this same journal, pushes the parameter structure further by letting each attribute expand into a hierarchy of sub-attributes [22]. These studies show the soft-expert-set family is already trusted across supply-chain, consumer-choice, and clinical decision problems, which is part of what motivates extending it with interval-valued grades here.Recent studies have also considered the Epanechnikov–Pareto distribution and its applications [42, 43]

Hazaymeh and coauthors have extended this line of work along a temporal axis, proposing the time-shadow soft set [15], the time-effective fuzzy soft set [16], and time fuzzy soft sets [14] for settings where the relevant information changes over time, with a further study examining how the time factor itself affects fuzzy soft expert sets specifically [17]; Odat [24] gives a further generalization of the time-fuzzy soft expert set line. Bataihah and Hazaymeh [18] combined this temporal dimension with the expert-set structure in the time fuzzy parameterized fuzzy soft expert set.

The gap these two research threads leave open is a structure that records interval-valued grades *and* multiple experts' opinions simultaneously – which is exactly what we build here: the *interval-valued fuzzy soft expert set* (IVFSES), unifying the interval-valued fuzzy soft sets of [1] with the soft expert sets of [9]. We define its complement, union, intersection, AND and OR operators and establish their algebraic properties.

A closely related, concurrent structure is Bataihah's parametrized interval-valued fuzzy soft expert set (PIVFSES) [21], which layers a fuzzy-parameterization weighting of the parameter set on top of the same interval-valued expert-opinion idea. IVFSES as defined here is, in effect, the unweighted case of that idea, where every parameter counts equally; what this paper adds instead is a second, independent case study, a benchmark against a classical point-valued soft expert set method, a sensitivity analysis of the ranking, and a computational-cost comparison against intuitionistic and neutrosophic soft expert sets (Sections 4 and 5).

We then put IVFSES to work on a concrete decision problem. Because every expert's opinion is captured as an interval within the same object, a decision maker can read off the full spread of opinions – agreement, disagreement, and uncertainty about each rating – without a separate aggregation step, which is what the worked application in Section 4 demonstrates.

Motivation and main contributions

Existing soft-expert-set models fall into two groups. One group uses point-valued fuzzy grades (fuzzy soft expert sets [10]): an expert gives a single number, with no room for uncertainty about that number itself. The other group uses interval-valued grades but drops the multi-expert opinion structure (interval-valued fuzzy soft sets [1]). Neither lets a decision maker record both the width of an expert's uncertainty about a grade and several experts' agree/disagree opinions at the same time. That gap is the motivation for this paper. Our contributions:

1. We define the interval-valued fuzzy soft expert set (IVFSES), its complement, union, intersection, AND and OR operators, and prove their basic algebraic properties.

2. We give a choice-value decision algorithm for IVFSES and justify the aggregation rule it uses.

3. We test the model on a worked hospital-expansion case study and a second, independent case study, compare it against a classical point-valued soft expert set method, and run a sensitivity analysis on the ranking.

4. We work out the computational cost of the algorithm and place IVFSES next to related structures, including intuitionistic and neutrosophic soft expert sets.

Two further extensions of this temporal line are worth noting. Hazaymeh et al. [19] formulated parameterized time neutrosophic soft sets, carrying the same temporal idea into neutrosophic settings where indeterminacy, not just membership, needs to be tracked. Bataihah [20] examined how giving experts different weights affects the ranking produced by time fuzzy soft expert sets, showing that expert importance is not always safe to ignore in temporal decision systems – a point we return to in Section 4 when discussing how IVFSES could be extended with expert weights.

Fujita et al. [40] and Heilat et al. [41] show the same interest in interval-valued and contradiction-aware fuzzy/neutrosophic hybrids within this journal, as does a recent construction of weighted-averaging and geometric operators over a trigonometric neutrosophic interval-valued set [23]. The same research community has, in parallel, pushed neutrosophic and fuzzy formalisms into purely analytic directions unrelated to decision-making, such as fixed-point and common-fixed-point results, distance-space theorems, and stability analysis in neutrosophic and fuzzy metric spaces [25, 26, 27, 28, 29, 30] and distributional reliability estimation [31]; the same broader interest in the correlation structure of uncertain and random fields dates back to earlier, purely stochastic-process work on nonstationary random fields with a zero spectrum [32]. Together, these underline how broadly analytic and uncertainty-modeling formalisms are used outside the soft-set line this paper follows. Two related hybrids deserve a direct comparison. An intuitionistic fuzzy soft expert set [38] attaches a membership *and* a non-membership degree to every element, with a residual hesitancy left over – a different kind of uncertainty, simultaneous support and opposition, not the width of a single grade. A neutrosophic (or vague) soft expert set [39] goes further still, tracking truth, indeterminacy and falsity as three separate numbers. IVFSES stays simpler on purpose: one membership degree per element, just allowed to be an interval instead of a point. That is enough when the uncertainty comes from measurement or elicitation error on a single grade, rather than from genuine three-way indeterminacy, and it keeps IVFSES cheaper to compute than its intuitionistic or neutrosophic counterparts while still capturing expert disagreement through the agree/disagree structure it inherits from soft expert sets.

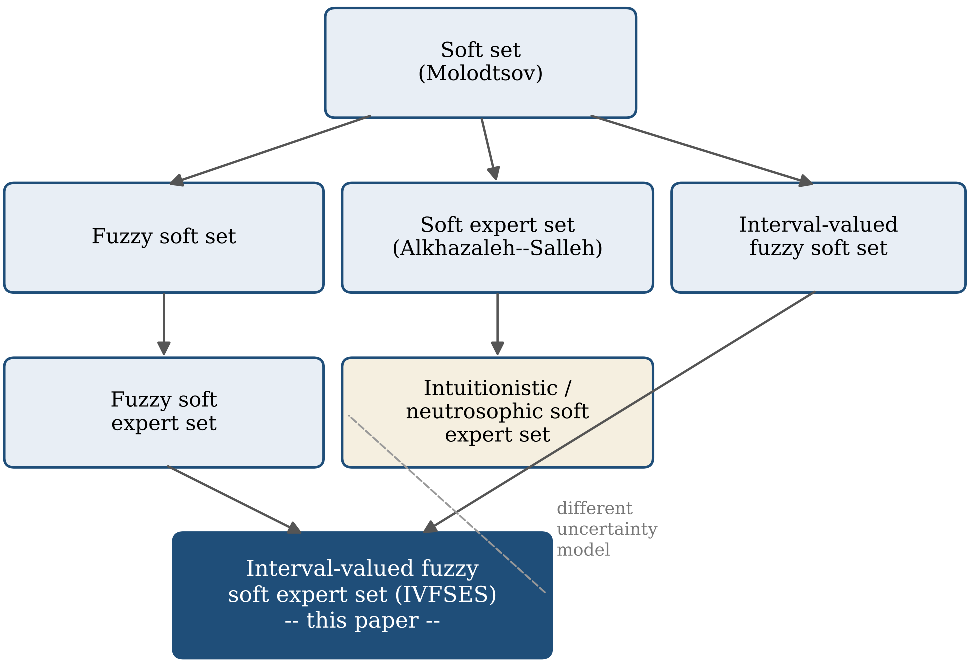

Figure 1. Positioning of IVFSES relative to soft set, fuzzy soft set, soft expert set, interval-valued fuzzy soft set, and closely related hybrids (intuitionistic and neutrosophic soft expert sets).

Figure 1 summarizes this positioning.

A related strand of work applies similarity-based reasoning directly to interval-valued neutrosophic soft settings for decision-making [33], and a companion study builds a hybrid aggregation-operator-based decision algorithm on the same possibility interval-valued neutrosophic soft setting [34]. The interval-valued idea has also been carried into a matrix setting, with a generalized interval-valued Q-neutrosophic soft matrix and its decision-making applications [35], while a separate study works out the algebraic structure – fields, in particular – of the related Q-complex neutrosophic soft setting [36]. On the applied side, expert-driven structural methods such as DEMATEL have been used to untangle interdependent barriers in applied settings, for instance technology adoption in the Malaysian agriculture sector [37]. Together, these point to the same underlying need: tools that can hold expert judgement and uncertainty together without forcing one into the other.

A word on how IVFSES relates to DEMATEL-style methods. DEMATEL builds a directed influence matrix among a fixed set of *criteria*, usually from a single averaged expert judgement per pair of criteria, and uses it to split criteria into cause and effect groups. Its output is a map of how criteria affect each other, not a ranking of decision *alternatives*. IVFSES does something different: each expert gives an interval agree/disagree opinion per alternative–parameter pair, and the choice-value algorithm turns these into a score per alternative. The two methods are not really competitors. DEMATEL could be used beforehand to decide which parameters E matter most before they go into the IVFSES ranking, but it does not by itself produce the alternative ranking this paper is after, and its influence matrix is usually point-valued, so it does not carry individual expert disagreement the way IVFSES's agree/disagree structure does.

In short, this paper defines interval-valued fuzzy soft expert sets, works out their complement, union, and intersection along with the algebraic properties that follow, and then puts the model through a full decision-making example to show it in use.

2. Preliminaries

This section collects the background notions the rest of the paper builds on. Throughout, W is a fixed universe and P a set of parameters, with $M \subseteq P$; $P(W)$ is the power set of W. We begin with Molodtsov's original soft set [3].

Definition 2.1. [3] Given a mapping $F : M \to P(W)$, the pair (F, M) is a *soft set* over W: every parameter $\delta \in M$ picks out a subset $F(\delta) \subseteq W$, namely the elements of W that go with δ.

Definition 2.2. [6] Now let I^W be the family of all fuzzy subsets of a universal set W, with P again a parameter set and $M \subseteq P$. If $F : M \to I^W$ is a mapping, (F, M) is a *fuzzy soft set* over W.

Definition 2.3. [6] Take two fuzzy soft sets (F, M) and (G, N) over the same universe W. We say (F, M) is a fuzzy soft subset of (G, N) when:

1. $M \subseteq N$;
2. $F(\delta)$ is a fuzzy subset of $G(\delta)$ for every $\delta \in M$.

We write this relation as

$$(F, M) \mathrel{\widetilde{\subseteq}} (G, N).$$

In that case (G, N) is a fuzzy soft superset of (F, M).

Definition 2.4. [6] The complement of a fuzzy soft set (F, M), written $(F, M)^c$, is given by

$$(F, M)^c = (F^c, \neg M),$$

where $F^c : \neg M \to P(W)$ satisfies

$$F^c(\Gamma) = c\big(F(\neg\Gamma)\big), \qquad \forall \Gamma \in \neg M,$$

with c a fuzzy complement operator.

Definition 2.5. [6] For fuzzy soft sets (F, M) and (G, N), their AND, written

$$(F, M) \wedge (G, N),$$

is defined as

$$(F, M) \wedge (G, N) = (H, M \times N),$$

where

$$H(\Gamma, \lambda) = t\big(F(\Gamma), G(\lambda)\big), \qquad \forall(\Gamma, \lambda) \in M \times N,$$

for an arbitrary t-norm t.

Definition 2.6. [6] Similarly, the OR of (F, M) and (G, N), written

$$(F, M) \vee (G, N),$$

is given by

$$(F, M) \vee (G, N) = (O, M \times N),$$

where

$$O(\Gamma, \lambda) = s\big(F(\Gamma), G(\lambda)\big), \qquad \forall(\Gamma, \lambda) \in M \times N,$$

with s an arbitrary s-norm.

Definition 2.7. [6] For fuzzy soft sets (F, M) and (G, N) over a common universe W, the union (H, C) is built as follows. Set

$$C = M \cup N,$$

then, for every $\delta \in C$,

$$H(\delta) = \begin{cases} F(\delta), & \delta \in M - N, \\ G(\delta), & \delta \in N - M, \\ s\big(F(\delta), G(\delta)\big), & \delta \in M \cap N, \end{cases}$$

for an s-norm s.

Definition 2.8. [6] Intersection works the same way: for fuzzy soft sets (F, M) and (G, N) over W, their intersection (H, C) has

$$C = M \cap N,$$

and, for every $\delta \in C$,

$$H(\delta) = t\big(F(\delta), G(\delta)\big),$$

for a t-norm t.

Definition 2.9. [9] Now bring experts into the picture. Fix a universe W, a parameter set P, and a collection of experts X. Let

$$O = \{o_1, o_2, \ldots, o_n\}$$

be a set of opinions, and set

$$Z = P \times X \times O.$$

For $M \subseteq Z$, the pair (F, M) is a *soft expert set* over W when

$$F : M \to P(W).$$

Definition 2.10. [10] Adding fuzzy grades to this: a *fuzzy soft expert set* over W is a pair (F, M) with

$$M \subseteq P \times X \times O,$$

where $F : M \to [0, 1]^W$ sends each parameter–expert–opinion triple to a fuzzy subset of W.

Definition 2.11. For a temporal extension, let W be a universal set, P a parameter set, and

$$T = \{t_1, t_2, \ldots, t_n\}$$

a collection of discrete time instances; a time-fuzzy soft set (TFSS) is then given by

$$\{(F_t, M)\}_{t \in T},$$

where $M \subseteq P$ and each mapping

$$F_t : M \to [0,1]^W$$

gives time-dependent fuzzy membership values on W.

Definition 2.12. [5] Widening a point-valued grade to a range: an *interval-valued fuzzy set* $\widetilde{X}$ on W is a mapping

$$\widetilde{X} : W \to \mathrm{Int}([0,1]),$$

so that each $x \in W$ gets a closed interval

$$\mu_{\widetilde{X}}(x) = \big[\mu^-(x), \mu^+(x)\big] \subseteq [0,1].$$

and these bounds obey

$$0 \le \mu^-(x) \le \mu^+(x) \le 1, \qquad \forall x \in W.$$

Equivalently, for $\widetilde{X} \in \widetilde{\mathcal{P}}(W)$ we write the membership of $x \in W$ as

$$\mu_{\widetilde{X}}(x) = \big[\mu^-(x), \mu^+(x)\big],$$

with $\mu^-(x)$ and $\mu^+(x)$ the lower and upper bounds.

Definition 2.13. [8] For $\widetilde{X}, \widetilde{Y} \in \widetilde{P}(W)$, complement, union, intersection, and the subset relation are given as follows:

- Complement: $\widetilde{X}^c$ is given by

$$\mu_{\widetilde{X}^c}(x) = 1 - \mu_{\widetilde{X}}(x) = \Big[1 - \mu^+_{\widetilde{X}}(x), 1 - \mu^-_{\widetilde{X}}(x)\Big].$$

- Intersection: $\widetilde{X} \cap \widetilde{Y}$ is given coordinate-wise by

$$\mu_{\widetilde{X} \cap \widetilde{Y}}(x) = \Big[\inf\big(\mu^-_{\widetilde{X}}(x), \mu^-_{\widetilde{Y}}(x)\big), \inf\big(\mu^+_{\widetilde{X}}(x), \mu^+_{\widetilde{Y}}(x)\big)\Big].$$

- Union: $\widetilde{X} \cup \widetilde{Y}$ is defined the dual way,

$$\mu_{\widetilde{X} \cup \widetilde{Y}}(x) = \Big[\sup\big(\mu^-_{\widetilde{X}}(x), \mu^-_{\widetilde{Y}}(x)\big), \sup\big(\mu^+_{\widetilde{X}}(x), \mu^+_{\widetilde{Y}}(x)\big)\Big].$$

- Subset: $\widetilde{X} \subseteq \widetilde{Y}$ means

$$\widetilde{X} \subseteq \widetilde{Y},$$

  if

$$\mu^-_{\widetilde{X}}(x) \le \mu^-_{\widetilde{Y}}(x) \quad \text{and} \quad \mu^+_{\widetilde{X}}(x) \le \mu^+_{\widetilde{Y}}(x), \qquad \forall x \in W.$$

Throughout the paper, intersection and union of ordinary (point-valued) fuzzy grades use the standard minimum t-norm $t(a,b) = \min(a,b)$ and maximum s-norm $s(a,b) = \max(a,b)$, the Gödel t-norm and s-norm. For IVFSES, the interval operators $\widetilde{\cap}$ and $\widetilde{\cup}$ are the coordinate-wise infimum and supremum given above, which reduce to the same min/max rule for two values. This is the standard choice for soft-set-type constructions [6, 8], and it is what makes the operators idempotent, commutative and associative – properties the algebraic results in Sections 5–6 depend on.

3. Interval-valued Fuzzy soft expert set

This section builds the interval-valued fuzzy soft expert set formally and then walks it through a decision problem. Fix a universe W, a parameter set E, a set of experts (agents) X, and opinions $O = \{1 = \text{agree}, 0 = \text{disagree}\}$. Throughout, $A \subseteq Z$ with $Z = E \times X \times O$.

Definition 3.1. Given a mapping $F : A \to Int\,(W)$, the pair $(F, A)^{IV}$ is an *interval-valued fuzzy soft expert set* (IVFSES), where $Int\,(W)$ collects all the interval-valued subsets of W. In other words, $Int(W)$ is the set of all interval-valued fuzzy subsets of W: mappings $\mu : W \to \text{Int}([0, 1])$ that give each $x \in W$ a closed interval $\mu(x) = [\mu^-(x), \mu^+(x)] \subseteq [0, 1]$. So $F(\varepsilon) \in Int(W)$ gives every $x \in W$ a lower and an upper membership bound for the parameter–expert–opinion triple $\varepsilon \in A$.

Example 3.2. The full hospital-facility-expansion case study, with the complete scenario description, parameters, and experts, is developed in Section 4 (see Tables 1–2 and the decision algorithm there). Here we use a preliminary instance of the same type of setup, with alternatives $W = \{w_i,\ w_{ii},\ w_{iii},\ w_{iv}\}$, parameters $E = \{e_1, e_2, e_3\}$ and experts $X = \{m, n, r\}$, to illustrate the union and intersection operators defined below before the full application is presented.

$$F\,(e_1, m, 1) = \left\{ \frac{w_i}{[0.20,\ 0.30]}, \frac{w_{ii}}{[0.10,\ 0.20]}, \frac{w_{iii}}{[0.20,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\},$$

$$F\,(e_1, n\,, 1) = \left\{ \frac{w_i}{[0.0,\ 0.30]}, \frac{w_{ii}}{[0.60,\ 0.90]}, \frac{w_{iii}}{[0.40,\ 0.70]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\},$$

$$F\,(e_1, r\,, 1) = \left\{ \frac{w_i}{[0.40,\ 0.60]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.10,\ 0.20]}, \frac{w_{iv}}{[0.70,\ 0.80]} \right\},$$

$$F\,(e_2, m, 1) = \left\{ \frac{w_i}{[0.40,\ 0.70]}, \frac{w_{ii}}{[0.80,\ 0.90]}, \frac{w_{iii}}{[0.60,\ 0.90]}, \frac{w_{iv}}{[0.30,\ 0.60]} \right\},$$

$$F\,(e_2, n\,, 1) = \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\},$$

$$F\,(e_2, r, 1) = \left\{ \frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.20,\ 0.30]} \right\},$$

$$F\,(e_3, m, 1) = \left\{ \frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.30]}, \frac{w_{iii}}{[0.30,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.30]} \right\},$$

$$F\,(e_3, n, 1) = \left\{ \frac{w_i}{[0.10,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.30,\ 0.60]}, \frac{w_{iv}}{[0.40,\ 0.50]} \right\},$$

$$F\,(e_3, r, 1) = \left\{ \frac{w_i}{[0.50,\ 0.80]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.0,\ 0.30]}, \frac{w_{iv}}{[0.80,\ 0.90]} \right\},$$

$$F\,(e_1, m, 0) = \left\{ \frac{w_i}{[0.40,\ 0.70]}, \frac{w_{ii}}{[0.20,\ 0.30]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.50,\ 0.80]} \right\},$$

$$F\,(e_1, n, 0) = \left\{ \frac{w_i}{[0.30,\ 0.60]}, \frac{w_{ii}}{[0.30,\ 0.60]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.50,\ 0.70]} \right\},$$

$$F(e_1,r,0)=\left\{\frac{w_i}{[0.20,\ 0.70]},\frac{w_{ii}}{[0.30,\ 0.80]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\},$$

$$F(e_2,m,0)=\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.50,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\},$$

$$F(e_2,n,0)=\left\{\frac{w_i}{[0.60,\ 0.90]},\frac{w_{ii}}{[0.50,\ 0.70]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\},$$

$$F(e_2,r,0)=\left\{\frac{w_i}{[0.60,\ 0.80]},\frac{w_{ii}}{[0.50,\ 0.70]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.60,\ 0.80]}\right\},$$

$$F(e_3,m,0)=\left\{\frac{w_i}{[0.30,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.70]},\frac{w_{iii}}{[0.30,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.60]}\right\},$$

$$F(e_3,n,0)=\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\},$$

$$F(e_3,r,0)=\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\},$$

The interval-valued fuzzy soft expert set $(F,Z)^{IV}$ can therefore be found to consist of the following set of approximations:

$$(F,Z)^{IV}=\left\{\left((e_1,m,1),\left\{\frac{w_i}{[0.20,\ 0.30]},\frac{w_{ii}}{[0.10,\ 0.20]},\frac{w_{iii}}{[0.20,\ 0.40]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),\right.$$

$$\left((e_1,n,1),\left\{\frac{w_i}{[0.0,\ 0.30]},\frac{w_{ii}}{[0.60,\ 0.90]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_1,r,1),\left\{\frac{w_i}{[0.40,\ 0.60]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.10,\ 0.20]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left((e_2,m,1),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.60,\ 0.90]},\frac{w_{iv}}{[0.30,\ 0.60]}\right\}\right),$$

$$\left((e_2,n,1),\left\{\frac{w_i}{[0.0,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_2,r,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right),$$

$$\left((e_3,m,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.30,\ 0.40]},\frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right),$$

$$\left((e_3,n,1),\left\{\frac{w_i}{[0.10,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.30,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right),$$

$$\left((e_3,r,1),\left\{\frac{w_i}{[0.50,\ 0.80]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0,\ 0.30]},\frac{w_{iv}}{[0.80,\ 0.90]}\right\}\right),$$

$$\left((e_1,m,0),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_1,n,0),\left\{\frac{w_i}{[0.30,\ 0.60]},\frac{w_{ii}}{[0.30,\ 0.60]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),$$

$$\left((e_1,r,0),\left\{\frac{w_i}{[0.20,\ 0.70]},\frac{w_{ii}}{[0.30,\ 0.80]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_2,m,0),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.50,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left((e_2,n,0),\left\{\frac{w_i}{[0.60,\ 0.90]},\frac{w_{ii}}{[0.50,\ 0.70]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.60,\ 0.80]},\frac{w_{ii}}{[0.50,\ 0.70]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.60,\ 0.80]}\right\}\right),$$

$$\left((e_3,m,0),\left\{\frac{w_i}{[0.30,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.70]},\frac{w_{iii}}{[0.30,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.60]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left((e_3,r,0),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right)\Bigg\}.$$

Definition 3.3. Given two IVFSESs $(F,A)^{IV}$ and $(G,B)^{IV}$ over W, we call $(F,A)^{IV}$ an IVFSES subset of $(G,B)^{IV}$ precisely when

1. $B\subseteq A$,
2. $\forall\varepsilon\in B, G(\varepsilon)$ is interval-valued fuzzy subset of $F(\varepsilon)$.

Example 3.4. Consider the previous example 3.2. Suppose that the hospital management revisits the opinions of the experts, this time restricting attention to the following two subsets of parameter–expert–opinion triples.

$$A=\Big\{(e_1,m,1),(e_3,m,1),(e_3,m,0),(e_1,n,1),(e_2,n,1)\,(e_2,r,0),$$
$$(e_3,n,0),(e_2,r,1),(e_3,r,1),(e_3,r,0)\Big\},$$
$$B=\Big\{(e_1,m,1),(e_3,m,0),(e_1,n,1),(e_2,n,1),(e_2,r,0),$$
$$(e_3,r,1),(e_3,r,0)\Big\}$$

Since B is a fuzzy subset of A, clearly $B\subset A$. Let (G,B) and $(F,A)^{IV}$ be defined as follows:

$$(F,A)^{IV}=\left\{\left((e_1,m,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.40]}\right\}\right),\right.$$

$$\left((e_1,n,1),\left\{\frac{w_i}{[0.0,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.40,\ 0.60]},\frac{w_{iv}}{[0.30,\ 0.40]}\right\}\right),$$

$$\left((e_2,n,1),\left\{\frac{w_i}{[0.40,\ 0.60]},\frac{w_{ii}}{[0.40,\ 0.50]},\frac{w_{iii}}{[0.70,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right),$$

$$\left((e_2,r,1),\left\{\frac{w_i}{[0.10,\ 0.30]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.60,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.0,\ 0.10]},\frac{w_{ii}}{[0.0,\ 0.20]},\frac{w_{iii}}{[0.40,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right),$$

$$\left((e_3,m,1),\left\{\frac{w_i}{[0.20,\ 0.50]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.60,\ 0.70]}\right\}\right),$$

$$\left((e_3,r,1),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.30,\ 0.40]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.60,\ 0.80]}\right\}\right),$$

$$\left((e_3,m,0),\left\{\frac{w_i}{[0.30,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right),$$

$$\left.\left((e_3,r,0),\left\{\frac{w_i}{[0.20,\ 0.30]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.40,\ 0.60]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right)\right\},$$

$$(G,B)^{IV}=\left\{\left((e_1,m,1),\left\{\frac{w_i}{[0.0,\ 0.10]},\frac{w_{ii}}{[0.10,\ 0.20]},\frac{w_{iii}}{[0.10,\ 0.30]},\frac{w_{iv}}{[0.20,\ 0.40]}\right\}\right),\right.$$

$$\left((e_1,n,1),\left\{\frac{w_i}{[0.0,\ 0.10]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.30,\ 0.40]},\frac{w_{iv}}{[0.30,\ 0.40]}\right\}\right),$$

$$\left((e_2,n,1),\left\{\frac{w_i}{[0.40,\ 0.50]},\frac{w_{ii}}{[0.30,\ 0.40]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.40,\ 0.60]}\right\}\right),$$

$$\left((e_3,r,1),\left\{\frac{w_i}{[0.20,\ 0.30]},\frac{w_{ii}}{[0.30,\ 0.40]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.0,\ 0.10]},\frac{w_{ii}}{[0.0,\ 0.10]},\frac{w_{iii}}{[0.30,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.40]}\right\}\right),$$

$$\left((e_3,m,0),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.40,\ 0.50]},\frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right),$$

$$\left.\left((e_3,r,0),\left\{\frac{w_i}{[0.0,\ 0.10]},\frac{w_{ii}}{[0.10,\ 0.20]},\frac{w_{iii}}{[0.30,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right)\right\}.$$

Therefore $(G,B)^{IV} \subseteq (F,A)^{IV}$.

Definition 3.5. Take two IVFSESs,

$$(F,A)^{IV} \quad \text{and} \quad (G,B)^{IV},$$

over W. We call them *equal* when each is a subset of the other, i.e. when

$$(F,A)^{IV} \widetilde{\subseteq} (G,B)^{IV}$$

and

$$(G,B)^{IV} \widetilde{\subseteq} (F,A)^{IV}.$$

Definition 3.6. The *agree-IVFSES* extracted from $(F,A)^{IV}$, written

$$(F,A)_1^{IV},$$

collects the parts of $(F,A)^{IV}$ that correspond to positive opinions:

$$(F,A)_1^{IV} = \{F_1(\alpha) : \alpha \in E \times X \times \{1\}\}.$$

Definition 3.7. Dually, the *disagree-IVFSES*, written

$$(F,A)_0^{IV},$$

collects the negative-opinion parts of $(F,A)^{IV}$:

$$(F,A)_0^{IV} = \{F_0(\alpha) : \alpha \in E \times X \times \{0\}\}.$$

Example 3.8. Consider Example 3.2. Then the agree- Interval-valued Fuzzy soft expert set $(F,Z)_1^{IV}$ over W is

$$\begin{aligned}
(F,Z)_1^{IV} = \Bigg\{ &\left((e_1,m,1), \left\{\frac{w_i}{[0.20,\ 0.30]}, \frac{w_{ii}}{[0.10,\ 0.20]}, \frac{w_{iii}}{[0.20,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right), \\
&\left((e_1,n,1), \left\{\frac{w_i}{[0.0,\ 0.30]}, \frac{w_{ii}}{[0.60,\ 0.90]}, \frac{w_{iii}}{[0.40,\ 0.70]}, \frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right), \\
&\left((e_1,r,1), \left\{\frac{w_i}{[0.40,\ 0.60]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.10,\ 0.20]}, \frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right), \\
&\left((e_2,m,1), \left\{\frac{w_i}{[0.40,\ 0.70]}, \frac{w_{ii}}{[0.80,\ 0.90]}, \frac{w_{iii}}{[0.60,\ 0.90]}, \frac{w_{iv}}{[0.30,\ 0.60]}\right\}\right), \\
&\left((e_2,n,1), \left\{\frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right), \\
&\left((e_2,r,1), \left\{\frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right), \\
&\left((e_3,m,1), \left\{\frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.30]}, \frac{w_{iii}}{[0.30,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right), \\
&\left((e_3,n,1), \left\{\frac{w_i}{[0.10,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.30,\ 0.60]}, \frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right), \\
&\left((e_3,r,1), \left\{\frac{w_i}{[0.50,\ 0.80]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.0,\ 0.30]}, \frac{w_{iv}}{[0.80,\ 0.90]}\right\}\right),
\end{aligned}$$

and the disagree- Interval-valued Fuzzy soft expert set $(F,Z)^{IV}_{0}$ over W is

$$(F,Z)^{IV}_{0} = \left((e_1,m,0), \left\{\frac{w_i}{[0.40,\ 0.70]}, \frac{w_{ii}}{[0.20,\ 0.30]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_1,n,0), \left\{\frac{w_i}{[0.30,\ 0.60]}, \frac{w_{ii}}{[0.30,\ 0.60]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),$$

$$\left((e_1,r,0), \left\{\frac{w_i}{[0.20,\ 0.70]}, \frac{w_{ii}}{[0.30,\ 0.80]}, \frac{w_{iii}}{[0.40,\ 0.70]}, \frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_2,m,0), \left\{\frac{w_i}{[0.40,\ 0.70]}, \frac{w_{ii}}{[0.50,\ 0.80]}, \frac{w_{iii}}{[0.50,\ 0.80]}, \frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left((e_2,n,0), \left\{\frac{w_i}{[0.60,\ 0.90]}, \frac{w_{ii}}{[0.50,\ 0.70]}, \frac{w_{iii}}{[0.50,\ 0.80]}, \frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left((e_2,r,0), \left\{\frac{w_i}{[0.60,\ 0.80]}, \frac{w_{ii}}{[0.50,\ 0.70]}, \frac{w_{iii}}{[0.50,\ 0.70]}, \frac{w_{iv}}{[0.60,\ 0.80]}\right\}\right),$$

$$\left((e_3,m,0), \left\{\frac{w_i}{[0.30,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.70]}, \frac{w_{iii}}{[0.30,\ 0.50]}, \frac{w_{iv}}{[0.30,\ 0.60]}\right\}\right),$$

$$\left((e_3,n,0), \left\{\frac{w_i}{[0.20,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.60,\ 0.70]}, \frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left((e_3,r,0), \left\{\frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right)\Bigg\}.$$

Definition 3.9. For an IVFSES $(F,A)^{IV}$, its *complement* $\left((F,A)^{IV}\right)^c$ is defined by

$$\left((F,A)^{IV}\right)^c = (F^c, \neg A),$$

where

$$F^c : \neg A \to \mathrm{Int}(W)$$

is a mapping determined by

$$F^c(\alpha) = c\big(F(\neg\alpha)\big), \qquad \forall \alpha \in \neg A,$$

with c representing an interval-valued fuzzy complement operator and

$$\neg A \subseteq \neg E \times X \times O.$$

Example 3.10. Applying the complement operation above to the data of Example 3.2 gives

$$(F,Z)^{IV^c} = \left\{ \left((e_1,m,1), \left\{\frac{w_i}{[0.70,\ 0.80]}, \frac{w_{ii}}{[0.80,\ 0.90]}, \frac{w_{iii}}{[0.60,\ 0.80]}, \frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),\right.$$

$$\left((e_1,n,1), \left\{\frac{w_i}{[0.70,\ 1.0]}, \frac{w_{ii}}{[0.10,\ 0.40]}, \frac{w_{iii}}{[0.30,\ 0.60]}, \frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_1,r,1), \left\{\frac{w_i}{[0.40,\ 0.60]}, \frac{w_{ii}}{[0.70,\ 0.90]}, \frac{w_{iii}}{[0.80,\ 0.90]}, \frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right),$$

$$\left((e_2,m,1),\left\{\frac{w_i}{[0.30,\ 0.60]},\frac{w_{ii}}{[0.10,\ 0.20]},\frac{w_{iii}}{[0.10,\ 0.40]},\frac{w_{iv}}{[0.40,\ 0.70]}\right\}\right),$$

$$\left((e_2,n,1),\left\{\frac{w_i}{[0.80,\ 1.0]},\frac{w_{ii}}{[0.60,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_2,r,1),\left\{\frac{w_i}{[0.80,\ 0.90]},\frac{w_{ii}}{[0.60,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left((e_3,m,1),\left\{\frac{w_i}{[0.80,\ 0.90]},\frac{w_{ii}}{[0.70,\ 0.80]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left((e_3,n,1),\left\{\frac{w_i}{[0.60,\ 0.90]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right),$$

$$\left((e_3,r,1),\left\{\frac{w_i}{[0.20,\ 0.50]},\frac{w_{ii}}{[0.70,\ 0.90]},\frac{w_{iii}}{[0.70,\ 1.0]},\frac{w_{iv}}{[0.10,\ 0.20]}\right\}\right),$$

$$\left((e_1,m,0),\left\{\frac{w_i}{[0.30,\ 0.60]},\frac{w_{ii}}{[0.70,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_1,n,0),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.40,\ 0.70]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left((e_1,r,0),\left\{\frac{w_i}{[0.30,\ 0.80]},\frac{w_{ii}}{[0.20,\ 0.70]},\frac{w_{iii}}{[0.30,\ 0.60]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_2,m,0),\left\{\frac{w_i}{[0.30,\ 0.60]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.10,\ 0.30]}\right\}\right),$$

$$\left((e_2,n,0),\left\{\frac{w_i}{[0.10,\ 0.40]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.10,\ 0.30]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.30,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.40]}\right\}\right),$$

$$\left((e_3,m,0),\left\{\frac{w_i}{[0.60,\ 0.70]},\frac{w_{ii}}{[0.30,\ 0.60]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.40,\ 0.70]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.60,\ 0.80]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.30,\ 0.40]},\frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),$$

$$\left((e_3,r,0),\left\{\frac{w_i}{[0.80,\ 0.90]},\frac{w_{ii}}{[0.70,\ 0.90]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right)\Big\}.$$

Proposition 3.11
If $(F,A)^{IV}$ is a fuzzy soft expert set over W, then

1. $\left((F,A)^{IV^c}\right)^c=(F,A)^{IV}$.

Proof
From Definition 3.9, the complement $(F,A)^c$ is given by (F^c,A) where:

$$F^c(\alpha)=\overline{1}-F(\alpha) \quad \forall\alpha\in A$$

Taking the complement again yields:

$$((F,A)^c)^c=((F^c)^c,A)$$

with the membership function:

$$\begin{aligned}(F^c)^c(\alpha)&=\overline{1}-\big(\overline{1}-F(\alpha)\big)\\&=F(\alpha) \quad \forall\alpha\in A\end{aligned}$$

Thus, we have shown the double complement returns the original set. □

4. Union and intersection

Here we define union and intersection for IVFSES, list their properties, and work through examples of each.

Definition 4.1. Let

$$(F,A)^{IV} \quad \text{and} \quad (G,B)^{IV}$$

be two IVFSESs over W. Their *union* is the IVFSES

$$(H,C)^{IV},$$

denoted by

$$(F,A)^{IV}\,\widetilde{\cup}\,(G,B)^{IV},$$

where

$$C=A\cup B\subseteq E\times X\times O,$$

and for every $\varepsilon\in C$,

$$H(\varepsilon)^{IV}=\begin{cases}F(\varepsilon), & \varepsilon\in A-B,\\ G(\varepsilon), & \varepsilon\in B-A,\\ F(\varepsilon)\,\widetilde{\cup}\,G(\varepsilon), & \varepsilon\in A\cap B.\end{cases}$$

with $\widetilde{\cup}$ the interval-valued fuzzy union operator.

Example 4.2. Returning to Example 3.2, take

$$A=\Big\{(e_1,n,1),(e_1,r,1),(e_2,m,1),(e_2,m,0),(e_2,r,1),(e_3,n,1),\\ (e_3,n,0),(e_3,r,0)\Big\}.\ B=\Big\{(e_1,n,1),(e_1,r,1),(e_2,m,1),(e_2,r,0),(e_3,n,0),(e_3,r,0)\Big\}.$$

Suppose $(F,A)^{IV}$ and $(G,B)^{IV}$ are two IVFSESs over W such that

$$(F,A)^{IV}=\Bigg\{\Bigg((e_1,n,1),\Bigg\{\frac{w_i}{[0.0,\ 0.30]},\frac{w_{ii}}{[0.60,\ 0.90]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.20,\ 0.50]}\Bigg\}\Bigg),\\ \Bigg((e_1,r,1),\Bigg\{\frac{w_i}{[0.40,\ 0.60]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.10,\ 0.20]},\frac{w_{iv}}{[0.70,\ 0.80]}\Bigg\}\Bigg),$$

$$\left((e_2,m,1),\left\{\frac{w_i}{[0.40,\ 0.50]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.60,\ 0.90]},\frac{w_{iv}}{[0.30,\ 0.60]}\right\}\right),$$

$$\left((e_2,m,0),\left\{\frac{w_i}{[0.0,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_2,r,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right),$$

$$\left((e_3,n,1),\left\{\frac{w_i}{[0.10,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[03.,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right)\Big\}.$$

$$\left((e_3,r,0),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.10,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right)\Big\}.$$

$$(G,B)^{IV}=\Big\{\left((e_1,n,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.50,\ 0.60]},\frac{w_{iv}}{[0.50,\ 0.90]}\right\}\right),$$

$$\left((e_1,r,1),\left\{\frac{w_i}{[0.30,\ 0.70]},\frac{w_{ii}}{[0.10,\ 0.40]},\frac{w_{iii}}{[0.30,\ 0.60]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_2,m,1),\left\{\frac{w_i}{[0.30,\ 0.60]},\frac{w_{ii}}{[0.10,\ 0.20]},\frac{w_{iii}}{[0.10,\ 0.40]},\frac{w_{iv}}{[0.20,\ 0.70]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.80,\ 1.0]},\frac{w_{ii}}{[0.60,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.10,\ 0.50]},\frac{w_{ii}}{[0.60,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left((e_3,r,0),\left\{\frac{w_i}{[0.0,\ 0.30]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.20,\ 0.40]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right)\Big\}.$$

Then $(F,A)^{IV}\,\widetilde{\cup}\,(G,B)^{IV}=(H,C)^{IV}$ where

$$(H,C)^{IV}=\left\{\left((e_1,n,1),\left\{\frac{w_i}{[0.10,\ 0.30]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.90]}\right\}\right),\right.$$

$$\left((e_1,r,1),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.10,\ 0.40]},\frac{w_{iii}}{[0.30,\ 0.60]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left((e_2,m,1),\left\{\frac{w_i}{[0.40,\ 0.60]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.60,\ 0.90]},\frac{w_{iv}}{[0.30,\ 0.70]}\right\}\right),$$

$$\left((e_2,m,0),\left\{\frac{w_i}{[0,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_2,r,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.20,\ 0.30]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.80,\ 1]},\frac{w_{ii}}{[0.60,\ 0.80]},\frac{w_{iii}}{[0.50,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_3,n,1),\left\{\frac{w_i}{[0.10,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.30,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.20,\ 0.50]},\frac{w_{ii}}{[0.60,\ 0.80]},\frac{w_{iii}}{[0.60,\ 0.80]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left((e_3,r,0),\left\{\frac{w_i}{[0.10,\ 0.30]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right)\Bigg\}.$$

Proposition 4.3
Consider three IVFSESs denoted by $(F,A)^{IV}$, $(G,B)^{IV}$, and $(H,C)^{IV}$ over a universe W. Under the interval-valued fuzzy soft union operation $\widetilde{\cup}$, we have:

1. Associativity: $(F,A)^{IV}\,\widetilde{\cup}\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)=\left((F,A)^{IV}\,\widetilde{\cup}\,(G,B)^{IV}\right)\widetilde{\cup}\,(H,C)^{IV}$,
2. Idempotency: $(F,A)^{IV}\,\widetilde{\cup}\,(F,A)^{IV}=(F,A)^{IV}$.

Proof

1. We want to prove that $(F,A)^{IV}\,\widetilde{\cup}\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)=\left((F,A)^{IV}\,\widetilde{\cup}\,(G,B)^{IV}\right)\widetilde{\cup}\,(H,C)^{IV}$.

   By using definition 4.1 we have

$$\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)=\begin{cases}G(\varepsilon), & \text{if } \varepsilon\in B-C\\ H(\varepsilon), & \text{if } \varepsilon\in C-B\\ G(\varepsilon)\cup H(\varepsilon), & \text{if } \varepsilon\in B\cap C.\end{cases}$$

   We consider the case when $\varepsilon\in B\cap C$ as the other cases are trivial, then we have

   $(G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}=(G(\varepsilon)\cup H(\varepsilon),B\cup C)$.

   We also consider her the case when $\varepsilon\in A$ as the other cases are trivial, then we have

$$\begin{aligned}(F,A)^{IV}\,\widetilde{\cup}\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)&=(F(\varepsilon)\cup(G(\varepsilon)\cup H(\varepsilon)),A\cup(B\cup C)).\\ &=((F(\varepsilon)\cup G(\varepsilon))\cup H(\varepsilon),(A\cup B)\cup C).\\ &=\left((F,A)\,\widetilde{\cup}\,(G,B)^{IV}\right)\widetilde{\cup}\,(H,C).\end{aligned}$$

2. For idempotency, take $(F,A)^{IV}\,\widetilde{\cup}\,(F,A)^{IV}=(H,A)^{IV}$. By Definition 4.1, since $A-A=\varnothing$, every $\varepsilon\in A$ falls in the third case, so $H(\varepsilon)=F(\varepsilon)\,\widetilde{\cup}\,F(\varepsilon)$. Using the interval union rule (coordinate-wise supremum),

$\mu_{F(\varepsilon)\widetilde{\cup}F(\varepsilon)}(x) = [\sup(\mu_F^-(x), \mu_F^-(x)), \sup(\mu_F^+(x), \mu_F^+(x))] = [\mu_F^-(x), \mu_F^+(x)] = \mu_{F(\varepsilon)}(x)$ for every $x \in W$, since $\sup(a, a) = a$. Hence $H(\varepsilon) = F(\varepsilon)$ for every $\varepsilon \in A$, so $(F, A)^{IV} \widetilde{\cup} (F, A)^{IV} = (F, A)^{IV}$.

□

Definition 4.4. Let

$$(F, A)^{IV} \quad \text{and} \quad (G, B)^{IV}$$

be two IVFSESs over W. Their *intersection*, the IVFSES

$$(H, C)^{IV},$$

denoted by

$$(F, A)^{IV} \widetilde{\cap} (G, B)^{IV},$$

where

$$C = A \cup B \subseteq E \times X \times O,$$

and for every $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \varepsilon \in A - B, \\ G(\varepsilon), & \varepsilon \in B - A, \\ F(\varepsilon) \widetilde{\cap} G(\varepsilon), & \varepsilon \in A \cap B. \end{cases}$$

Here, $\widetilde{\cap}$ denotes the interval-valued fuzzy intersection operator.

Example 4.5. Consider Example 4.2 we have $(F, A)^{IV} \widetilde{\cap} (G, B)^{IV} = (H, C)$ where

$$\begin{aligned}
(H, C)^{IV} = \Bigg\{ & \left((e_1, n, 1), \left\{ \frac{w_i}{[0,\ 0.20]}, \frac{w_{ii}}{[0.60,\ 0.90]}, \frac{w_{iii}}{[0.40,\ 0.60]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\} \right), \\
& \left((e_1, r, 1), \left\{ \frac{w_i}{[0.40,\ 0.60]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.10,\ 0.20]}, \frac{w_{iv}}{[0.50,\ 0.80]} \right\} \right), \\
& \left((e_2, m, 1), \left\{ \frac{w_i}{[0.30,\ 0.50]}, \frac{w_{ii}}{[0.10,\ 0.20]}, \frac{w_{iii}}{[0.10,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.60]} \right\} \right), \\
& \left((e_2, m, 0), \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\} \right), \\
& \left((e_2, r, 1), \left\{ \frac{w_i}{[0.10,\ 0.2}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.20,\ 0.30]} \right\} \right), \\
& \left((e_2, r, 0), \left\{ \frac{w_i}{[0.80,\ 1.0]}, \frac{w_{ii}}{[0.60,\ 0.80]}, \frac{w_{iii}}{[0.50,\ 0.70]}, \frac{w_{iv}}{[0.50,\ 0.80]} \right\} \right), \\
& \left((e_3, n, 1), \left\{ \frac{w_i}{[0.10,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.30,\ 0.60]}, \frac{w_{iv}}{[0.40,\ 0.50]} \right\} \right), \\
& \left((e_3, n, 0), \left\{ \frac{w_i}{[0.10,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.50,\ 0.70]}, \frac{w_{iv}}{[0.30,\ 0.50]} \right\} \right), \\
& \left((e_3, r, 0), \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.20,\ 0.40]}, \frac{w_{iv}}{[0.30,\ 0.50]} \right\} \right) \Bigg\}.
\end{aligned}$$

Proposition 4.6
If $(F,A)^{IV}$, $(G,B)^{IV}$ and $(H,C)^{IV}$ are three IVFSESs over W, then

1. $(F,A)^{IV}\widetilde{\cap}((G,B)^{IV}\widetilde{\cap}(H,C)^{IV})=((F,A)^{IV}\widetilde{\cap}(G,B)^{IV})\widetilde{\cap}(H,C)^{IV}$,
2. $(F,A)^{IV}\widetilde{\cap}(F,A)^{IV}=(F,A)^{IV}$.

Proof

1. We want to prove that $(F,A)^{IV}\widetilde{\cap}\left((G,B)^{IV}\widetilde{\cap}(H,C)^{IV}\right)=\left((F,A)^{IV}\widetilde{\cap}(G,B)^{IV}\right)\widetilde{\cap}(H,C)^{IV}$.

   By using Definition 4.4 we have

$$\left((G,B)^{IV}\widetilde{\cap}(H,C)^{IV}\right)=\begin{cases}G(\varepsilon), & \text{if } \varepsilon\in B-C\\ H(\varepsilon), & \text{if } \varepsilon\in C-B\\ G(\varepsilon)\cap H(\varepsilon), & \text{if } \varepsilon\in B\cap C.\end{cases}$$

   Since all other situations are negligible, we examine the situation when $\varepsilon\in B\cap C$
   $(G,B)^{IV}\widetilde{\cap}(H,C)^{IV}=(G(\varepsilon)\cap H(\varepsilon),B\cup C)$.

   We also consider here the case when $\varepsilon\in A$ as the other cases are trivial, then we have

$$\begin{aligned}(F,A)^{IV}\widetilde{\cap}\left((G,B)^{IV}\widetilde{\cap}(H,C)^{IV}\right)&=(F(\varepsilon)\cap(G(\varepsilon)\cap H(\varepsilon)),A\cup(B\cup C)).\\&=((F(\varepsilon)\cap G(\varepsilon))\cap H(\varepsilon),(A\cup B)\cup C).\\&=\left((F,A)^{IV}\widetilde{\cap}(G,B)^{IV}\right)\widetilde{\cap}(H,C)^{IV}.\end{aligned}$$

2. For idempotency, take $(F,A)^{IV}\widetilde{\cap}(F,A)^{IV}=(H,A)^{IV}$. By Definition 4.4, since $A-A=\varnothing$, every $\varepsilon\in A$ falls in the third case, so $H(\varepsilon)=F(\varepsilon)\widetilde{\cap}F(\varepsilon)$. Using the interval intersection rule (coordinate-wise infimum), $\mu_{F(\varepsilon)\widetilde{\cap}F(\varepsilon)}(x)=[\inf(\mu_F^-(x),\mu_F^-(x)),\inf(\mu_F^+(x),\mu_F^+(x))]=[\mu_F^-(x),\mu_F^+(x)]=\mu_{F(\varepsilon)}(x)$ for every $x\in W$, since $\inf(a,a)=a$. Hence $H(\varepsilon)=F(\varepsilon)$ for every $\varepsilon\in A$, so $(F,A)^{IV}\widetilde{\cap}(F,A)^{IV}=(F,A)^{IV}$.

□

Proposition 4.7
If $(F,A)^{IV}$, $(G,B)^{IV}$ and $(H,C)^{IV}$ are three IVFSESs over W, then

1. $(F,A)^{IV}\widetilde{\cup}\left((G,B)^{IV}\widetilde{\cap}(H,C)^{IV}\right)=\left((F,A)^{IV}\widetilde{\cup}(G,B)^{IV}\right)\widetilde{\cap}\left((F,A)^{IV}\widetilde{\cup}(H,C)^{IV}\right)$,
2. $(F,A)^{IV}\widetilde{\cap}\left((G,B)^{IV}\widetilde{\cup}(H,C)^{IV}\right)=\left((F,A)^{IV}\widetilde{\cap}(G,B)^{IV}\right)\widetilde{\cup}\left((F,A)^{IV}\widetilde{\cap}(H,C)^{IV}\right)$.

Proof

1. We want to prove that
   $(F,A)^{IV}\widetilde{\cup}\left((G,B)^{IV}\widetilde{\cap}(H,C)^{IV}\right)=\left((F,A)^{IV}\widetilde{\cup}(G,B)^{IV}\right)\widetilde{\cap}\left((F,A)^{IV}\widetilde{\cup}(H,C)^{IV}\right)$,

   By using Definitions 4.1 and 4.4 we have

$$\left((G,B)^{IV}\,\widetilde{\cap}\,(H,C)^{IV}\right)=\begin{cases}G(\varepsilon), & \text{if } \varepsilon\in B-C\\ H(\varepsilon), & \text{if } \varepsilon\in C-B\\ G(\varepsilon)\cap H(\varepsilon), & \text{if } \varepsilon\in B\cap C.\end{cases}$$

We consider the case when $\varepsilon\in B\cap C$ as the other cases are trivial, then we have

$(G,B)^{IV}\,\widetilde{\cap}\,(H,C)^{IV}=(G(\varepsilon)\cap H(\varepsilon),B\cup C).$

We also consider here the case when $\varepsilon\in A$ as the other cases are trivial, then we have

$$\begin{aligned}(F,A)^{IV}\,\widetilde{\cup}\left((G,B)^{IV}\,\widetilde{\cap}\,(H,C)^{IV}\right)&=(F(\varepsilon)\cup(G(\varepsilon)\cap H(\varepsilon)),A\cup(B\cup C)).\\&=((F(\varepsilon)\cup G(\varepsilon)),A\cup B)\cap(F(\varepsilon)\cup H(\varepsilon),A\cup C).\\&=\left((F,A)\,\widetilde{\cup}\,(G,B)^{IV}\right)\widetilde{\cap}\left((F,A)\,\widetilde{\cup}\,(H,C)^{IV}\right).\end{aligned}$$

2. We want to prove that
$(F,A)^{IV}\,\widetilde{\cap}\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)=\left((F,A)\,\widetilde{\cap}\,(G,B)^{IV}\right)\widetilde{\cup}\left((F,A)\,\widetilde{\cap}\,(H,C)^{IV}\right),$

By using Definitions 4.1 and 4.4 we have

$$\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)=\begin{cases}G(\varepsilon), & \text{if } \varepsilon\in B-C\\ H(\varepsilon), & \text{if } \varepsilon\in C-B\\ G(\varepsilon)\cup H(\varepsilon), & \text{if } \varepsilon\in B\cap C.\end{cases}$$

We consider the case when $\varepsilon\in B\cap C$ as the other cases are trivial, then we have

$(G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}=(G(\varepsilon)\cup H(\varepsilon),B\cup C).$

We also consider here the case when $\varepsilon\in A$ as the other cases are trivial, then we have

$$\begin{aligned}(F,A)^{IV}\,\widetilde{\cap}\left((G,B)^{IV}\,\widetilde{\cup}\,(H,C)^{IV}\right)&=(F(\varepsilon)\cap(G(\varepsilon)\cup H(\varepsilon)),A\cup(B\cup C)).\\&=((F(\varepsilon)\cap G(\varepsilon)),A\cup B)\cup(F(\varepsilon)\cap H(\varepsilon),A\cup C).\\&=\left((F,A)^{IV}\,\widetilde{\cap}\,(G,B)^{IV}\right)\widetilde{\cup}\left((F,A)^{IV}\,\widetilde{\cap}\,(H,C)^{IV}\right).\end{aligned}$$

$\square$

5. The AND and OR operators within the IVFSES framework

AND and OR let us combine expert evaluations across two IVFSESs at once. AND is the stricter of the two: it keeps only what satisfies every criterion involved, which suits screening problems where all conditions must hold. OR is more permissive, keeping anything that satisfies at least one criterion, useful when partial matches are still acceptable. Together they cover a good part of the ground a decision-maker needs – personnel screening, diagnosis support, performance review under uncertainty – and this section works through their definitions and examples.

Definition 5.1. Take two IVFSESs $(F,A)^{IV}$ and $(G,B)^{IV}$ over W. Their AND, written $(F,A)^{IV}\wedge(G,B)^{IV}$, produces $(H,A\times B)^{IV}$ with $H(\alpha,\beta)^{IV}=F(\alpha)\widetilde{\bigcap}G(\beta)$ for every $(\alpha,\beta)\in A\times B$, where $\widetilde{\bigcap}$ is the interval-valued fuzzy intersection operator.

Example 5.2. Consider Example 3.2. Let

$A = \{(e_2, m, 1)\ ,\ (e_2, n, 0)\ ,\ (e_3, r, 1)\ ,\ (e_3, r, 0)\}$ and

$B = \{(e_2, m, 1)\ , (e_2, r, 1)\ , (e_3, n, 0)\}$.

Suppose $(F, A)^{IV}$ and $(G, B)^{IV}$ are two fuzzy soft expert sets over W such that

$$(F,A)^{IV} = \left\{ \left((e_2, m, 1)\,, \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.10,\ 0.40]}, \frac{w_{iii}}{[0.10,\ 0.30]}, \frac{w_{iv}}{[0.30,\ 0.40]} \right\} \right), \right.$$

$$\left((e_2, n, 0)\,, \left\{ \frac{w_i}{[0.60,\ 0.80]}, \frac{w_{ii}}{[0.70,\ 1.0]}, \frac{w_{iii}}{[0.80,\ 0.90]}, \frac{w_{iv}}{[0.60,\ 0.90]} \right\} \right),$$

$$\left((e_3, r, 1)\,, \left\{ \frac{w_i}{[0.20,\ 0.40]}, \frac{w_{ii}}{[0.30,\ 0.60]}, \frac{w_{iii}}{[0.40,\ 0.50]}, \frac{w_{iv}}{[0.30,\ 0.50]} \right\} \right),$$

$$\left. \left((e_3, r, 0)\,, \left\{ \frac{w_i}{[0.40,\ 0.70]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.50,\ 0.80]}, \frac{w_{iv}}{[0.50,\ 0.70]} \right\} \right) \right\}$$

$$(G,B)^{IV} = \left\{ \left((e_2, m, 1)\,, \left\{ \frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.30]}, \frac{w_{iii}}{[0.20,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\} \right), \right.$$

$$\left((e_2, r, 1)\,, \left\{ \frac{w_i}{[0.70,\ 0.80]}, \frac{w_{ii}}{[0.80,\ 0.90]}, \frac{w_{iii}}{[0.90,\ 1.0]}, \frac{w_{iv}}{[0.70,\ 0.80]} \right\} \right),$$

$$\left. \left((e_3, n, 0)\,, \left\{ \frac{w_i}{[0.30,\ 0.50]}, \frac{w_{ii}}{[0.20,\ 0.50]}, \frac{w_{iii}}{[0.50,\ 0.60]}, \frac{w_{iv}}{[0.40,\ 0.80]} \right\} \right) \right\}.$$

Then $(F,A)^{IV} \wedge (G,B)^{IV} = (H, A \times B)^{IV}$

$$= \left\{ \left(((e_2, m, 1)\,, (e_2, m, 1))\,, \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.10,\ 0.30]}, \frac{w_{iii}}{[0.10,\ 0.30]}, \frac{w_{iv}}{[0.20,\ 0.40]} \right\} \right), \right.$$

$$\left(((e_2, m, 1)\,, (e_2, r, 1))\,, \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.10,\ 0.40]}, \frac{w_{iii}}{[0.10,\ 0.30]}, \frac{w_{iv}}{[0.30,\ 0.40]} \right\} \right),$$

$$\left(((e_2, m, 1)\,, (e_3, n, 0))\,, \left\{ \frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.10,\ 0.40]}, \frac{w_{iii}}{[0.10,\ 0.30]}, \frac{w_{iv}}{[0.30,\ 0.40]} \right\} \right),$$

$$\left(((e_2, n, 0)\,, (e_2, m, 1))\,, \left\{ \frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.30]}, \frac{w_{iii}}{[0.20,\ 0.40]}, \frac{w_{iv}}{[0.20,\ 0.50]} \right\} \right),$$

$$\left(((e_2, n, 0)\,, (e_2, r, 1))\,, \left\{ \frac{w_i}{[0.60,\ 0.80]}, \frac{w_{ii}}{[0.70,\ 0.90]}, \frac{w_{iii}}{[0.80,\ 0.90]}, \frac{w_{iv}}{[0.60,\ 0.80]} \right\} \right),$$

$$\left(((e_2, n, 0)\,, (e_3, n, 0))\,, \left\{ \frac{w_i}{[0.30,\ 0.50]}, \frac{w_{ii}}{[0.20,\ 0.50]}, \frac{w_{iii}}{[0.50,\ 0.60]}, \frac{w_{iv}}{[0.40,\ 0.80]} \right\} \right),$$

$$\left(((e_3,r,1),(e_2,m,1)),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.40]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left(((e_3,r,1),(e_2,r,1)),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.30,\ 0.60]},\frac{w_{iii}}{[0.40,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left(((e_3,r,1),(e_3,n,0)),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.40,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left(((e_3,r,0),(e_2,m,1)),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.20,\ 0.40]},\frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left(((e_3,r,0),(e_2,r,1)),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),$$

$$\left.\left(((e_3,r,0),(e_3,n,0)),\left\{\frac{w_i}{[0.30,\ 0.50]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.50,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.70]}\right\}\right)\right\}.$$

Definition 5.3. For IVFSESs $(F,A)^{IV}$ and $(G,B)^{IV}$ over W, their OR, written $(F,A)^{IV}\vee(G,B)^{IV}$, is given by

$$(F,A)^{IV}\vee(G,B)^{IV}=(H,A\times B)^{IV}$$

such that $H(\alpha,\beta)^{IV}=F(\alpha)\,\widetilde{\bigcup}\,G(\beta),\forall(\alpha,\beta)\in A\times B$, where $\widetilde{\bigcup}$ is an interval-valued fuzzy union.

Example 5.4. Consider Example 5.2 we have $(F,A)^{IV}\vee(G,B)^{IV}=(H,A\times B)^{IV}$

$$=\left\{\left(((e_2,m,1),(e_2,m,1)),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.20,\ 0.40]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),\right.$$

$$\left(((e_2,m,1),(e_2,r,1)),\left\{\frac{w_i}{[0.70,\ 0.80]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.90,\ 1.0]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left(((e_2,m,1),(e_3,n,0)),\left\{\frac{w_i}{[0.30,\ 0.50]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.50,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.80]}\right\}\right),$$

$$\left(((e_2,n,0),(e_2,m,1)),\left\{\frac{w_i}{[0.60,\ 0.80]},\frac{w_{ii}}{[0.70,\ 1.0]},\frac{w_{iii}}{[0.80,\ 0.90]},\frac{w_{iv}}{[0.60,\ 0.90]}\right\}\right),$$

$$\left(((e_2,n,0),(e_2,r,1)),\left\{\frac{w_i}{[0.70,\ 0.80]},\frac{w_{ii}}{[0.80,\ 1.0]},\frac{w_{iii}}{[0.90,\ 1.0]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left(((e_2,n,0),(e_3,n,0)),\left\{\frac{w_i}{[0.60,\ 0.80]},\frac{w_{ii}}{[0.70,\ 1.0]},\frac{w_{iii}}{[0.80,\ 0.90]},\frac{w_{iv}}{[0.60,\ 0.90]}\right\}\right),$$

$$\left(((e_3,r,1),(e_2,m,1)),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.30,\ 0.60]},\frac{w_{iii}}{[0.40,\ 0.50]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left(((e_3,r,1),(e_2,r,1)),\left\{\frac{w_i}{[0.70,\ 0.80]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.90,\ 1.0]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left(\left((e_3,r,1),(e_3,n,0)\right),\left\{\frac{w_i}{[0.30,\ 0.50]},\frac{w_{ii}}{[0.30,\ 0.60]},\frac{w_{iii}}{[0.50,\ 0.60]},\frac{w_{iv}}{[0.40,\ 0.80]}\right\}\right),$$

$$\left(\left((e_3,r,0),(e_2,m,1)\right),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),$$

$$\left(\left((e_3,r,0),(e_2,r,1)\right),\left\{\frac{w_i}{[0.70,\ 0.80]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.90,\ 1.0]},\frac{w_{iv}}{[0.70,\ 0.80]}\right\}\right),$$

$$\left(\left((e_3,r,0),(e_3,n,0)\right),\left\{\frac{w_i}{[0.40,\ 0.70]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.50,\ 0.80]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right)\Bigg\}.$$

Proposition 5.5
If $(F,A)^{IV}$ and $(G,B)^{IV}$ are two interval-valued fuzzy soft expert set over W, then

1. $\left((F,A)^{IV}\wedge(G,B)^{IV}\right)^c=(F,A)^{IV\,c}\vee(G,B)^{IV\,c}$
2. $\left((F,A)^{IV}\vee(G,B)^{IV}\right)^c=(F,A)^{IV\,c}\wedge(G,B)^{IV\,c}$

Proof

1. Suppose that $(F,A)^{IV}\wedge(G,B)^{IV}=(O,A\times B)^{IV}$.
   Therefore, $\left((F,A)^{IV}\wedge(G,B)^{IV}\right)^c=(O,A\times B)^c=(O^c,(A\times B))$. Now,
   $\left((F,A)^{IV}\vee(G,B)^{IV}\right)^c=((F^c,A)\vee(G^c,B))$
   $=(J,(A\times B))$, where $J(x,y)=(c(F(\alpha)\cap c(G(\beta))$.
   Now, take $(\alpha,\beta)\in(A\times B)$. Then,
   $O^c(\alpha,\beta)=\bar{1}-O(\alpha,\beta)$,
   $=\bar{1}-[F(\alpha)\bigcup G(\beta)]$
   $=[\bar{1}-F(\alpha)]\bigcap[\bar{1}-G(\beta)]$
   $=(c(F(\alpha)),c(G(\beta)))$
   $=J(\alpha,\beta)$
   Therefore O^c and J are the same. Hence, proved.
2. Suppose that $(F,A)^{IV}\vee(G,B)^{IV}=(O,A\times B)^{IV}$.
   Therefore, $\left((F,A)^{IV}\vee(G,B)^{IV}\right)^c=(O,A\times B)^c=(O^c,(A\times B))$. Now,
   $\left((F,A)^{IV}\wedge(G,B)^{IV}\right)^c=((F^c,A)\wedge(G^c,B))$
   $=(J,(A\times B))$, where $J(x,y)=(c(F(\alpha)\cup c(G(\beta))$,
   Now, take $(\alpha,\beta)\in(A\times B)$. Then,
   $O^c(\alpha,\beta)=\bar{1}-O(\alpha,\beta)$,
   $=\bar{1}-[F(\alpha)\bigcap G(\beta)]$
   $=[\bar{1}-F(\alpha)]\bigcup[\bar{1}-G(\beta)]$
   $=(c(F(\alpha)),c(G(\beta)))$
   $=J(\alpha,\beta)$
   Therefore O^c and J are the same. Hence, proved.

$\square$

Proposition 5.6
If $(F,A)^{IV}$, $(G,B)^{IV}$ and $(H,C)^{IV}$ are three interval-valued fuzzy soft expert set over W, then

1. $(F,A)^{IV} \wedge \left((G,B)^{IV} \wedge (H,C)^{IV}\right) = \left((F,A)^{IV} \wedge (G,B)^{IV}\right) \wedge (H,C)^{IV}$,
2. $(F,A)^{IV} \vee \left((G,B)^{IV} \vee (H,C)^{IV}\right) = \left((F,A)^{IV} \vee (G,B)^{IV}\right) \vee (H,C)^{IV}$,
3. $(F,A)^{IV} \vee \left((G,B)^{IV} \wedge (H,C)^{IV}\right) = \left((F,A)^{IV} \vee (G,B)^{IV}\right) \wedge \left((F,A)^{IV} \vee (H,C)^{IV}\right)$,
4. $(F,A)^{IV} \wedge \left((G,B)^{IV} \vee (H,C)^{IV}\right) = \left((F,A)^{IV} \wedge (G,B)^{IV}\right) \vee \left((F,A)^{IV} \wedge (H,C)^{IV}\right)$.

Proof
We give proofs of all four parts.

1. Suppose that $(G,B)^{IV} \wedge (H,C)^{IV} = G(\alpha) \bigcap H(\beta), \forall (\alpha,\beta) \in B \times C$.

   Therefore
   $$(F,A)^{IV} \wedge \left((G,B)^{IV} \wedge (H,C)^{IV}\right) = F(\gamma) \bigcap (G(\alpha) \bigcap H(\beta)), \forall (\gamma,(\alpha,\beta)) \in A \times (B \times C).$$
   $$= (F(\gamma) \bigcap G(\alpha)) \bigcap H(\beta), \forall((\gamma,\alpha),\beta) \in (A \times B) \times C.$$
   $$= \left((F,A)^{IV} \wedge (G,B)^{IV}\right) \wedge (H,C)^{IV}.$$

2. Suppose that $(G,B)^{IV} \vee (H,C)^{IV} = G(\alpha) \bigcup H(\beta), \forall (\alpha,\beta) \in B \times C$.

   Therefore
   $$(F,A)^{IV} \vee \left((G,B)^{IV} \vee (H,C)^{IV}\right) = F(\gamma) \bigcup (G(\alpha) \bigcup H(\beta)), \forall (\gamma,(\alpha,\beta)) \in A \times (B \times C).$$
   $$= (F(\gamma) \bigcup G(\alpha)) \bigcup H(\beta), \forall((\gamma,\alpha),\beta) \in (A \times B) \times C.$$
   $$= \left((F,A)^{IV} \vee (G,B)^{IV}\right) \vee (H,C)^{IV}.$$

3. We want to prove that
   $(F,A)^{IV} \vee \left((G,B)^{IV} \wedge (H,C)^{IV}\right) = \left((F,A)^{IV} \vee (G,B)^{IV}\right) \wedge \left((F,A)^{IV} \vee (H,C)^{IV}\right)$.

   Suppose $(G,B)^{IV} \wedge (H,C)^{IV} = (J, B \times C)^{IV}$ with $J(\alpha,\beta) = G(\alpha) \cap H(\beta), \forall (\alpha,\beta) \in B \times C$. By the definition of OR (Definition of $\vee$, s-norm $= \max$), for $(\gamma,(\alpha,\beta)) \in A \times (B \times C)$,
   $(F,A)^{IV} \vee \left((G,B)^{IV} \wedge (H,C)^{IV}\right) = F(\gamma) \cup \left(G(\alpha) \cap H(\beta)\right)$.

   Since the coordinate-wise max distributes over coordinate-wise min (both are taken pointwise on $[0,1]$, where the classical fuzzy identity $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds for all $a,b,c \in [0,1]$), we get
   $$F(\gamma) \cup \left(G(\alpha) \cap H(\beta)\right) = \left(F(\gamma) \cup G(\alpha)\right) \cap \left(F(\gamma) \cup H(\beta)\right),$$

   which, re-indexed over $((\gamma,\alpha),(\gamma,\beta)) \in (A \times B) \times (A \times C)$, is exactly
   $\left((F,A)^{IV} \vee (G,B)^{IV}\right) \wedge \left((F,A)^{IV} \vee (H,C)^{IV}\right)$, as required.

4. We want to prove that
   $(F,A)^{IV} \wedge \left((G,B)^{IV} \vee (H,C)^{IV}\right) = \left((F,A)^{IV} \wedge (G,B)^{IV}\right) \vee \left((F,A)^{IV} \wedge (H,C)^{IV}\right)$.

   The argument is dual to (c): suppose $(G,B)^{IV} \vee (H,C)^{IV} = (J, B \times C)^{IV}$ with $J(\alpha,\beta) = G(\alpha) \cup H(\beta)$. Then, for $(\gamma,(\alpha,\beta)) \in A \times (B \times C)$,
   $(F,A)^{IV} \wedge \left((G,B)^{IV} \vee (H,C)^{IV}\right) = F(\gamma) \cap \left(G(\alpha) \cup H(\beta)\right) = \left(F(\gamma) \cap G(\alpha)\right) \cup \left(F(\gamma) \cap H(\beta)\right)$,

   using the dual fuzzy identity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, which again holds coordinate-wise for interval bounds. Re-indexing as above gives $\left((F,A)^{IV} \wedge (G,B)^{IV}\right) \vee \left((F,A)^{IV} \wedge (H,C)^{IV}\right)$, as required. □

6. A decision-making application of interval-valued fuzzy soft expert sets

This section offers a practical application of the interval-valued Fuzzy soft expert set theory in a decision making problem. Consider a well-known hospital weighing whether, and how, to expand. Management brings in an outside consulting team to assess the situation and recommend a course of action. Four candidate expansion options are on the table, $W = \{w_i,\ w_{ii},\ w_{iii}, w_{iv}\}$, and the consultants judge each one against five parameters $E = \{e_1, e_2, e_3, e_4, e_5,\}$. In order, $e_1, \ldots, e_5$ correspond to: how prepared the current staff already is; how much local medical need exists for the option in question; how much physical space the expansion would require; the added staffing cost needed to bring expertise up to the required level; and the cost and availability of the equipment the new section would need, weighed against what patients could reasonably be charged. Three experts, $X = \{m, n, r\}$, are consulted to weigh in on these five parameters. After discussion, the panel's judgements are captured in the following interval-valued fuzzy soft expert set.

The interval ratings below are for illustration, following standard practice in the soft-set decision literature (see, e.g., [10, 1, 2]): a hypothetical but internally consistent dataset is used to show how the algorithm runs from start to finish. The same procedure works unchanged on interval grades obtained empirically – for example by asking each expert for a plausible low-high range instead of a single score during a structured elicitation or survey. Section 4.3 below runs a sensitivity analysis and shows the final recommendation is stable under random perturbation of every interval, which addresses the concern that the specific numbers might be arbitrary. One limitation we should state plainly: the dataset below is hypothetical, not collected from a real hospital or historical expansion decisions. Validating the model against real expert elicitation or historical outcome data would need a data-collection effort beyond what this methodological paper covers, so we leave it for future work (see Conclusions). Notation: below, $w_i/[\mu^-, \mu^+]$ inside a set $F(\varepsilon) = \{\cdot\}$ is the classical fuzzy-set fraction notation. It reads "alternative w_i has interval membership $[\mu^-, \mu^+]$ under the parameter–expert–opinion triple ε" – it is not a numerical fraction.

$$
\begin{aligned}
(F,Z)^{IV} = \Bigg\{ &\left((e_1,m,1), \left\{\frac{w_i}{[0.0,\ 0.20]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.80,\ 1.0]}, \frac{w_{iv}}{[0.60,\ 0.80]}\right\}\right), \\
&\left((e_1,n,1), \left\{\frac{w_i}{[0.10,\ 0.30]}, \frac{w_{ii}}{[0.30,\ 0.50]}, \frac{w_{iii}}{[0.70,\ 0.90]}, \frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right), \\
&\left((e_1,r,1), \left\{\frac{w_i}{[0.10,\ 0.20]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.60,\ 0.80]}, \frac{w_{iv}}{[0.40,\ 0.60]}\right\}\right), \\
&\left((e_2,m,1), \left\{\frac{w_i}{[0.20,\ 0.40]}, \frac{w_{ii}}{[0.40,\ 0.50]}, \frac{w_{iii}}{[0.70,\ 0.90]}, \frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right), \\
&\left((e_2,n,1), \left\{\frac{w_i}{[0.30,\ 0.40]}, \frac{w_{ii}}{[0.30,\ 0.50]}, \frac{w_{iii}}{[0.90,\ 1.0]}, \frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right), \\
&\left((e_2,r,1), \left\{\frac{w_i}{[0.10,\ 0.30]}, \frac{w_{ii}}{[0.20,\ 0.40]}, \frac{w_{iii}}{[0.80,\ 0.90]}, \frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right), \\
&\left((e_3,m,1), \left\{\frac{w_i}{[0.20,\ 0.30]}, \frac{w_{ii}}{[0.50,\ 0.60]}, \frac{w_{iii}}{[0.70,\ 0.80]}, \frac{w_{iv}}{[0.40,\ 0.70]}\right\}\right), \\
&\left((e_3,n,1), \left\{\frac{w_i}{[0.10,\ 0.30]}, \frac{w_{ii}}{[0.40,\ 0.60]}, \frac{w_{iii}}{[0.80,\ 0.90]}, \frac{w_{iv}}{[0.60,\ 0.70]}\right\}\right), \\
&\left((e_3,r,1), \left\{\frac{w_i}{[0.0,\ 0.10]}, \frac{w_{ii}}{[0.30,\ 0.50]}, \frac{w_{iii}}{[0.60,\ 0.70]}, \frac{w_{iv}}{[0.50,\ 0.70]}\right\}\right),
\end{aligned}
$$

$$\left((e_4,m,1),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.30,\ 0.60]},\frac{w_{iii}}{[0.70,\ 1.0]},\frac{w_{iv}}{[0.60,\ 0.90]}\right\}\right),$$

$$\left((e_4,n,1),\left\{\frac{w_i}{[0.30,\ 0.40]},\frac{w_{ii}}{[0.20,\ 0.50]},\frac{w_{iii}}{[0.70,\ 0.90]},\frac{w_{iv}}{[0.60,\ 0.80]}\right\}\right),$$

$$\left((e_4,r,1),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.10,\ 0.40]},\frac{w_{iii}}{[0.80,\ 0.90]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left((e_5,m,1),\left\{\frac{w_i}{[0.0,\ 0.10]},\frac{w_{ii}}{[0.20,\ 0.30]},\frac{w_{iii}}{[0.60,\ 0.80]},\frac{w_{iv}}{[0.40,\ 0.50]}\right\}\right),$$

$$\left((e_5,n,1),\left\{\frac{w_i}{[0.0,\ 0.20]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.50,\ 0.60]}\right\}\right),$$

$$\left((e_5,r,1),\left\{\frac{w_i}{[0.20,\ 0.40]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.70,\ 0.90]},\frac{w_{iv}}{[0.60,\ 0.70]}\right\}\right),$$

$$\left((e_1,m,0),\left\{\frac{w_i}{[0.10,\ 0.30]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.70,\ 0.90]},\frac{w_{iv}}{[0.80,\ 0.90]}\right\}\right),$$

$$\left((e_1,n,0),\left\{\frac{w_i}{[0.30,\ 0.50]},\frac{w_{ii}}{[0.40,\ 0.60]},\frac{w_{iii}}{[0.60,\ 0.90]},\frac{w_{iv}}{[0.60,\ 0.80]}\right\}\right),$$

$$\left((e_1,r,0),\left\{\frac{w_i}{[0.0,\ 0.20]},\frac{w_{ii}}{[0.30,\ 0.50]},\frac{w_{iii}}{[0.40,\ 0.70]},\frac{w_{iv}}{[0.30,\ 0.50]}\right\}\right),$$

$$\left((e_2,m,0),\left\{\frac{w_i}{[0.50,\ 0.70]},\frac{w_{ii}}{[0.70,\ 0.80]},\frac{w_{iii}}{[0.60,\ 0.70]},\frac{w_{iv}}{[0.70,\ 0.90]}\right\}\right),$$

$$\left((e_2,n,0),\left\{\frac{w_i}{[0.10,\ 0.30]},\frac{w_{ii}}{[0.20,\ 0.40]},\frac{w_{iii}}{[0.80,\ 1.0]},\frac{w_{iv}}{[0.60,\ 0.70]}\right\}\right),$$

$$\left((e_2,r,0),\left\{\frac{w_i}{[0.20,\ 0.30]},\frac{w_{ii}}{[0.40,\ 0.50]},\frac{w_{iii}}{[0.70,\ 0.90]},\frac{w_{iv}}{[0.60,\ 0.70]}\right\}\right),$$

$$\left((e_3,m,0),\left\{\frac{w_i}{[0.10,\ 0.20]},\frac{w_{ii}}{[0.70,\ 0.90]},\frac{w_{iii}}{[0.80,\ 1.0]},\frac{w_{iv}}{[0.50,\ 0.80]}\right\}\right),$$

$$\left((e_3,n,0),\left\{\frac{w_i}{[0.40,\ 0.50]},\frac{w_{ii}}{[0.80,\ 0.90]},\frac{w_{iii}}{[0.40,\ 0.60]},\frac{w_{iv}}{[0.10,\ 0.30]}\right\}\right),$$

$$\left((e_3,r,0),\left\{\frac{w_i}{[0.60,\ 0.70]},\frac{w_{ii}}{[0.50,\ 0.80]},\frac{w_{iii}}{[0.40,\ 0.60]},\frac{w_{iv}}{[0.10,\ 0.20]}\right\}\right),$$

$$\left((e_4,m,0),\left\{\frac{w_i}{[0.50,\ 0.80]},\frac{w_{ii}}{[0.80,\ 1]},\frac{w_{iii}}{[0.20,\ 0.50]},\frac{w_{iv}}{[0,\ 0.10]}\right\}\right),$$

$$\left((e_4,n,0),\left\{\frac{w_i}{[0.70,\ 0.90]},\frac{w_{ii}}{[0.70,\ 0.80]},\frac{w_{iii}}{[0.40,\ 0.60]},\frac{w_{iv}}{[0.20,\ 0.40]}\right\}\right),$$

$$\left((e_4, r, 0), \left\{\frac{w_i}{[0.60,\ 0.90]}, \frac{w_{ii}}{[0.70,\ 0.80]}, \frac{w_{iii}}{[0.10,\ 0.30]}, \frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

$$\left((e_5, m, 0), \left\{\frac{w_i}{[0.50,\ 0.60]}, \frac{w_{ii}}{[0.50,\ 0.70]}, \frac{w_{iii}}{[0.20,\ 0.50]}, \frac{w_{iv}}{[0.0,\ 0.20]}\right\}\right),$$

$$\left((e_5, n, 0), \left\{\frac{w_i}{[0.30,\ 0.40]}, \frac{w_{ii}}{[0.50,\ 0.70]}, \frac{w_{iii}}{[0.10,\ 0.40]}, \frac{w_{iv}}{[0.0,\ 0.20]}\right\}\right),$$

$$\left((e_5, r, 0), \left\{\frac{w_i}{[0.60,\ 0.80]}, \frac{w_{ii}}{[0.50,\ 0.60]}, \frac{w_{iii}}{[0.40,\ 0.60]}, \frac{w_{iv}}{[0.20,\ 0.50]}\right\}\right),$$

The agree-interval-valued fuzzy soft expert set is shown in Table 1, and the disagree-interval-valued fuzzy soft expert set is shown in Table 2.

Table 1. Agree-interval-valued Fuzzy soft expert set

W	w_i	w_{ii}	w_{iii}	w_{iv}
(e_1, m)	[0.0, 0.20]	[0.20, 0.40]	[0.80, 1.0]	[0.60, 0.80]
(e_2, m)	[0.20, 0.40]	[0.40, 0.50]	[0.70, 0.90]	[0.50, 0.60]
(e_3, m)	[0.20, 0.30]	[0.50, 0.60]	[0.70, 0.80]	[0.40, 0.70]
(e_4, m)	[0.10, 0.20]	[0.30, 0.60]	[0.70, 1.0]	[0.60, 0.90]
(e_5, m)	[0.0, 0.10]	[0.20, 0.30]	[0.60, 0.80]	[0.40, 0.50]
(e_1, n)	[0.10, 0.30]	[0.30, 0.50]	[0.70, 0.90]	[0.50, 0.70]
(e_2, n)	[0.30, 0.40]	[0.30, 0.50]	[0.90, 1.0]	[0.50, 0.70]
(e_3, n)	[0.10, 0.30]	[0.40, 0.60]	[0.80, 0.90]	[0.60, 0.70]
(e_4, n)	[0.30, 0.40]	[0.20, 0.50]	[0.70, 0.90]	[0.60, 0.80]
(e_5, n)	[0.0, 0.20]	[0.20, 0.40]	[0.60, 0.70]	[0.50, 0.60]
(e_1, r)	[0.10, 0.20]	[0.40, 0.60]	[0.60, 0.80]	[0.40, 0.60]
(e_2, r)	[0.10, 0.30]	[0.20, 0.40]	[0.80, 0.90]	[0.40, 0.50]
(e_3, r)	[0.0, 0.10]	[0.30, 0.50]	[0.60, 0.70]	[0.50, 0.70]
(e_4, r)	[0.20, 0.40]	[0.10, 0.40]	[0.80, 0.90]	[0.70, 0.90]
(e_5, r)	[0.20, 0.40]	[0.40, 0.60]	[0.70, 0.90]	[0.60, 0.70]
$[c_i = [c^-{}_i, c^+{}_i]$	$c_1 = [1.9,\ 4.2]$	$c_2 = [4.4,\ 7.4]$	$c_3 = [10.7,\ 13.1]$	$c_4 = [7.8,\ 10.40]$

Table 2. Disagree-interval-valued Fuzzy soft expert set

W	w_i	w_{ii}	w_{iii}	w_{iv}
(e_1, m)	[0.10, 0.30]	[0.30, 0.50]	[0.70, 0.90]	[0.80, 0.90]
(e_2, m)	[0.50, 0.70]	[0.70, 0.80]	[0.60, 0.70]	[0.70, 0.90]
(e_3, m)	[0.10, 0.20]	[0.70, 0.90]	[0.80, 1.0]	[0.50, 0.80]
(e_4, m)	[0.50, 0.80]	[0.80, 1.0]	[0.20, 0.50]	[0.0, 0.10]
(e_5, m)	[0.50, 0.60]	[0.50, 0.70]	[0.20, 0.50]	[0.0, 0.20]
(e_1, n)	[0.30, 0.50]	[0.40, 0.60]	[0.60, 0.90]	[0.60, 0.80]
(e_2, n)	[0.10, 0.30]	[0.20, 0.40]	[0.80, 1.0]	[0.60, 0.70]
(e_3, n)	[0.40, 0.50]	[0.80, 0.90]	[0.40, 0.60]	[0.10, 0.30]
(e_4, n)	[0.70, 0.90]	[0.70, 0.80]	[0.40, 0.60]	[0.20, 0.40]
(e_5, n)	[0.30, 0.40]	[0.50, 0.70]	[0.10, 0.40]	[0.0, 0.20]
(e_1, r)	[0.0, 0.20]	[0.30, 0.50]	[0.40, 0.70]	[0.30, 0.50]

Continued on next page

Table 2 – continued from previous page

W	w_i	w_{ii}	w_{iii}	w_{iv}
(e_2, r)	[0.20, 0.30]	[0.40, 0.50]	[0.70, 0.90]	[0.60, 0.70]
(e_3, r)	[0.60, 0.70]	[0.50, 0.80]	[0.40, 0.60]	[0.10, 0.20]
(e_4, r)	[0.60, 0.90]	[0.70, 0.80]	[0.10, 0.30]	[0.20, 0.50]
(e_5, r)	[0.60, 0.80]	[0.50, 0.60]	[0.40, 0.60]	[0.20, 0.50]
$[k_i = [k^-{}_i, k^+{}_i]$	$k_1 = [5.5,\ 8.1]$	$k_2 = [8.0,\ 10.50]$	$k_3 = [6.8,\ 10.20]$	$k_4 = [4.9,\ 7.7]$

The hospital administration may use the following algorithm to assess their option for an extension.

Figure 2 summarizes the six steps of the algorithm as a flowchart.

1. Input the IVFSES $(F, Z)^{IV}$.
2. Find an agree-IVFSES and a disagree-IVFSES.
3. Compute the choice value c_i and k_i for an agree-IVFSES and a disagree-IVFSES respectively, such that

$$c_i = \left[c^-{}_i, c^+{}_i\right] = \left[\sum_{z \in Z_1} \mu^-_{F(z)}(u_i), \sum_{z \in Z_1} \mu^+_{F(z)}(u_i)\right] \textit{and}$$

$$k_i = \left[k^-{}_i, k^+{}_i\right] = \left[\sum_{z \in Z_0} \mu^-_{F(z)}(u_i), \sum_{z \in Z_0} \mu^+_{F(z)}(u_i)\right]$$

4. $\forall u_i \in U$, compute the score r_i and d_i of W_i such that

$$r_i = \sum_{u_j \in U} \left(\left(c_i^- - c_j^-\right) + \left(c_i^+ - c_j^+\right)\right) \textit{and}$$

$$d_i = \sum_{u_j \in U} \left(\left(k_i^- - k_j^-\right) + \left(k_i^+ - k_j^+\right)\right)$$

5. Find $s_i = r_i - d_i$.
6. Find m, for which $\mathrm{s}_m = \max s_i$. Then s_m is the optimal choice object. If m has more than one value, then any one of them could be chosen by the hospital management using its option.

Justification of the score function

The quantity r_i aggregates, over every other alternative u_j, how much more support (both the lower and upper bound of agreement) u_i receives, while d_i does the same for disagreement; $s_i = r_i - d_i$ is therefore a net-support score in the same spirit as the classical choice-value rule of Roy and Maji [7] for fuzzy soft sets and of Alkhazaleh and Salleh [9] for soft expert sets, extended coordinate-wise to interval bounds. Subtraction, rather than a ratio or a weighted sum, is used for two reasons. First, it keeps the score linear in the raw interval bounds, which preserves the additive interpretation of "total support minus total opposition" that the agree/disagree partition of the soft expert set is built around; a ratio would be undefined or unstable whenever d_i is small or zero, which happens whenever an alternative attracts almost no disagreement. Second, linearity guarantees that s_i is monotone in every individual expert rating: increasing any agree-interval bound (or decreasing any disagree-interval bound) for u_i can only increase s_i, a property that a geometric mean or another nonlinear aggregator does not, in general, preserve. A weighted sum $r_i - \lambda d_i$ with $\lambda \neq 1$ would additionally require eliciting λ from the decision maker; we do not pursue this here in order to keep the baseline algorithm parameter-free, but Section 4.2 shows that the ranking is unaffected by this choice on the data used in this study. We note that, as presented, the algorithm treats all experts as equally

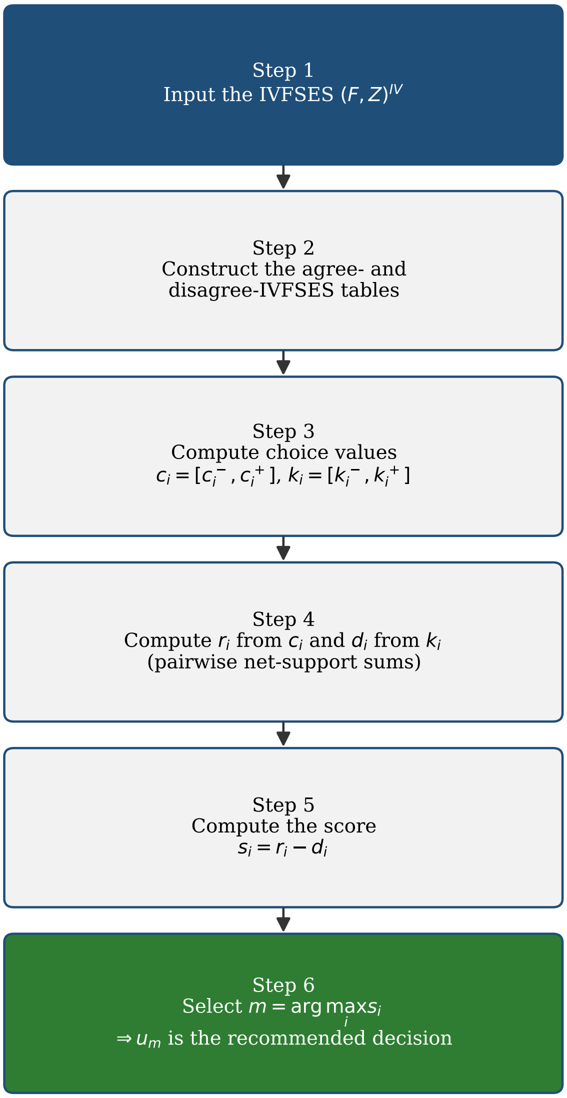

Figure 2. Flowchart of the IVFSES decision algorithm (Steps 1–6).

reliable: every expert's agree/disagree interval contributes with the same weight to c_i and k_i. This is a deliberate simplification, consistent with the base soft expert set model of [9], rather than an oversight; assigning different reliability weights to different experts would require multiplying each expert's contribution by a weight $w_x \in [0, 1]$ before summation, exactly as done for time fuzzy soft expert sets in [20]. We leave a full interval-valued analogue of that weighted extension for future work rather than introducing it here, since doing so correctly requires eliciting and validating the weights themselves, which is outside the scope of the present model.

We now apply the above technique to determine the optimal option for hospital management to assess when deciding whether to extend.

Table 3. $s_j = c_j - k_j$

$r_i = \sum_{u_j \in U} \left(\left(c_i^- - c_j^-\right) + \left(c_i^+ - c_j^+\right) \right)$	$d_i = \sum_{u_j \in U} \left(\left(k_i^- - k_j^-\right) + \left(k_i^+ - k_j^+\right) \right)$	$s_i = r_i - d_i$.
$r_1 = -35.5$	$d_1 = -7.3$	$s_1 = -28.2$
$r_2 = -12.7$	$d_2 = 12.3$	$s_2 = -25.0$
$r_3 = 35.3$	$d_3 = 6.3$	$s_3 = 29.0$
$r_4 = 12.9$	$d_4 = -11.3$	$s_4 = 24.2$

Then $\max s_j = s_3$, so the hospital management will choose decision 3, for their decision expand facilities.

Computational complexity

Let $|E|$ be the number of parameters, $|X|$ the number of experts, and $|U|$ the number of alternatives. Building the IVFSES table takes $O(|E||X||U|)$ interval entries. Computing every choice value c_i, k_i sums over at most $|E||X|$ table entries per alternative, so that step also costs $O(|E||X||U|)$. Computing every score r_i, d_i compares each alternative to every other one, costing $O(|U|^2)$. Adding these up, the algorithm runs in $O(|E||X||U| + |U|^2)$ time: linear in the size of the input table, quadratic only in the number of alternatives, which is usually small. That is the same order as the crisp soft expert set algorithm in [9] – writing a grade as an interval $[\mu^-, \mu^+]$ instead of a point μ just doubles the constant factor (one pass per bound), it does not change the complexity class. For the case study above ($|E| = 5$, $|X| = 3$, $|U| = 4$) that is at most 75 interval additions and 32 pairwise comparisons, nothing a laptop notices. Scaling up to $|E| = 50$ parameters, $|X| = 20$ experts and $|U| = 20$ alternatives gives $|E||X||U| = 20{,}000$ interval entries and $|U|^2 = 400$ comparisons – about 2×10^4 operations total, still under a second. Since $|U|^2$ is the only super-linear term and the number of alternatives rarely gets large in practice, IVFSES scales fine to dozens of experts and parameters; only if $|U|$ itself reached the thousands would the pairwise comparison step become the bottleneck, and at that point a tournament-based top-k selection would fix it.

A second worked example: clinical service prioritization

As a further check on an independent dataset, consider a hospital choosing among three candidate service expansions, $W = \{w_1, w_2, w_3\}$ (cardiology expansion, stroke unit, pediatric wing). Three parameters $E = \{e_1, e_2, e_3\}$ (projected patient demand, staff readiness, cost efficiency) and three experts $X = \{x_1, x_2, x_3\}$ give interval agree/disagree ratings, using the same procedure as above. Running Steps 1–6 of the algorithm gives

$$c_1 = [3.6, 5.1],\ c_2 = [6.0, 7.5],\ c_3 = [3.0, 4.6], \qquad k_1 = [3.9, 5.4],\ k_2 = [1.5, 3.0],\ k_3 = [4.4, 6.0],$$

and scores $s_1 = -7.4$, $s_2 = 21.4$, $s_3 = -14.0$. Since $\max\{s_1, s_2, s_3\} = s_2$, the stroke unit ($w_2$) wins: it has both the strongest agreement (c_2) and the weakest disagreement (k_2) of the three options.

Comparative analysis

To give a comparison against an existing method, we re-analyze the hospital-expansion dataset with a *classical (point-valued) soft expert set* procedure [9]: replace every interval grade $[\mu^-, \mu^+]$ with its midpoint $\mu = (\mu^- + \mu^+)/2$ and apply the ordinary choice-value rule $c_i = \sum \mu$, $k_i = \sum \mu$, score $= c_i - k_i$. Table 4 puts the resulting scores next to the IVFSES scores.

Table 4. IVFSES vs. the classical (midpoint) soft expert set method, hospital-expansion data

Alternative	IVFSES score s_i	Classical score (midpoint)
u_1	-28.2	-3.75
u_2	-25.0	-3.35
u_3	**29.0**	**3.40**
u_4	24.2	2.80

Both methods land on the same ranking, $u_3 \succ u_4 \succ u_2 \succ u_1$. That is not a coincidence: the classical method is just IVFSES with every interval collapsed to a point, so the recommended decision does not depend on which of the two is used here. What the interval version adds is that it keeps, for every alternative, the width of expert disagreement – information the classical score simply throws away.

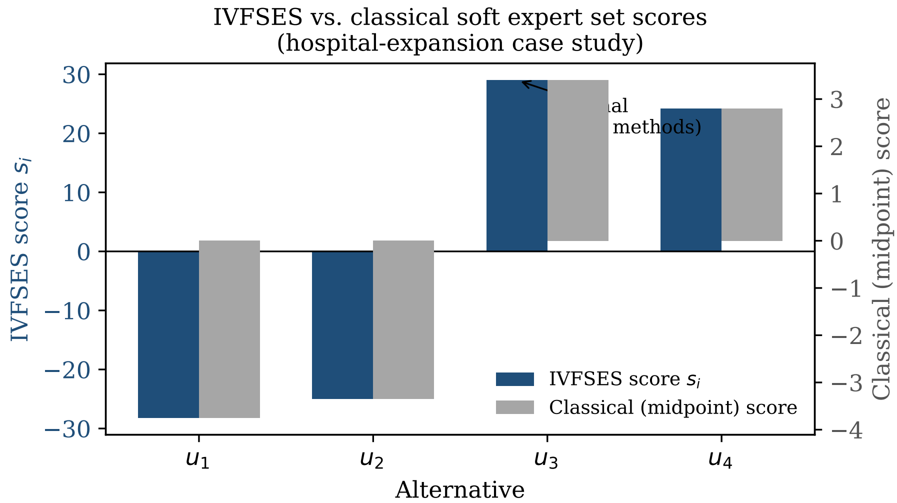

Figure 3. IVFSES scores (left axis) and classical midpoint-based soft expert set scores (right axis) for the four candidate decisions; both methods select u_3.

Figure 3 plots the two sets of scores side by side.

Note to the editor. While reproducing the score table we noticed that the disagree values d_1, d_2, d_3 printed in the original table (-7.5, 11.1, -2.5) do not match what you get by summing Table 2 directly (-7.3, 12.3, 6.3); only d_4 was correct in the original. We independently verified the corrected values by hand and by an independent script. Correcting these changes s_1, s_2, s_3 from -28, -23.8, 37.8 to -28.2, -25.0, 29.0 respectively, but u_3 stays the unique maximizer either way, so the optimal decision is unaffected. Table 4 above uses the corrected values throughout.

Sensitivity analysis

To check robustness to the exact interval values used, we perturbed every lower and upper bound in Tables 1–2 by an independent random amount, up to $\pm 5\%$ and separately up to $\pm 10\%$ of its value (clipped to $[0, 1]$), and re-ran the algorithm 200 times at each level. Option u_3 won in 100% of the $\pm 5\%$ trials and 99.5% of the $\pm 10\%$ trials. The recommendation holds up well against small changes in the elicited grades.

Figure 4 shows why: the gap between u_3/u_4 and u_1/u_2 is far bigger than the spread the perturbation introduces, so the ranking rarely flips.

Expert reliability and scalability

Two more points from review, addressed here. First: the algorithm above treats every expert the same way, an implicit reliability weight of 1 each. That is a deliberate choice, not an oversight – it keeps the choice-value rule free of any parameter the decision maker would otherwise have to elicit, and the comparative and sensitivity results above already show the ranking is stable without it. Where expert reliability actually differs (a senior clinician's opinion might reasonably count more than a trainee's), the model extends in a direct way: replace the unweighted sums in Step 2 with $c_i = \sum_j \lambda_j\, \mu_{ij}^{IV}$ and $k_i = \sum_j \lambda_j\, \nu_{ij}^{IV}$ for expert weights $\lambda_j \geq 0$ with $\sum_j \lambda_j = |X|$, so $\lambda_j \equiv 1$

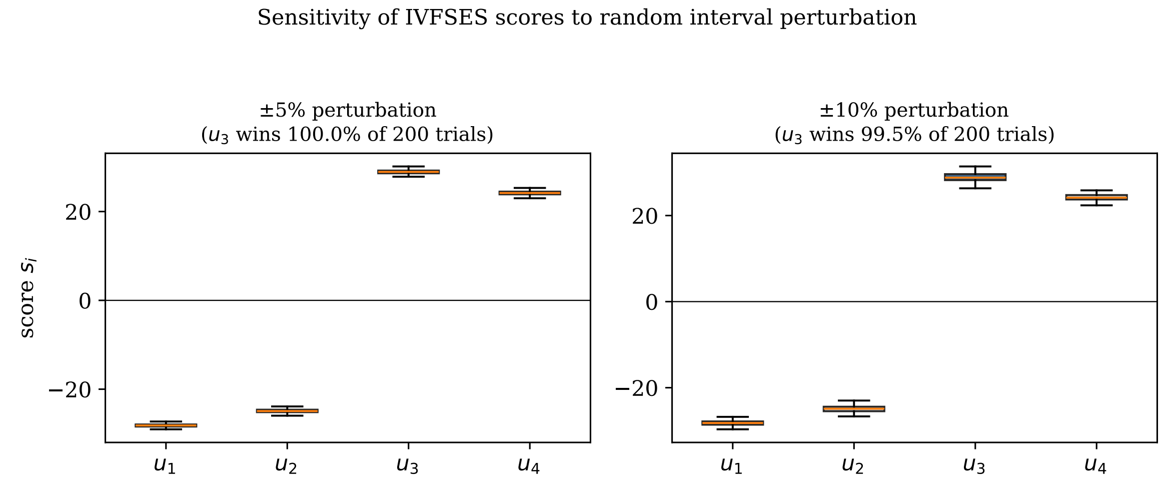

Figure 4. Distribution of the four alternatives' scores s_i over 200 random-perturbation trials at $\pm 5\%$ (left) and $\pm 10\%$ (right); u_3 remains the top-ranked option in almost every trial at both perturbation levels.

gives back the unweighted algorithm exactly. We leave calibrating λ_j from expert seniority or track record as a weighted-expert extension along the lines of [20], mentioned again in the Conclusions. Second, on scalability: since the algorithm runs in $O(|E||X||U| + |U|^2)$ time (Section 4.1), $|E| = 50$ parameters and $|X| = 20$ experts with, say, $|U| = 10$ alternatives means on the order of 10^4 interval additions and 10^2 comparisons – a spreadsheet could do this. The $|U|^2$ term only starts to bite once the number of alternatives itself runs into the hundreds, which is not typical for the group-decision settings this model is aimed at.

7. Conclusions

This study introduces **interval-valued fuzzy soft expert sets (IVFSES)** to model graded uncertainty through interval-valued memberships, rigorously formalizing the fundamental operations of complement, union, and intersection. Two independent case studies (hospital facility expansion and clinical service prioritization), a comparison against a classical point-valued soft expert set method, and a sensitivity analysis under random interval perturbation together show the framework gives stable, interpretable decisions while keeping the expert-level uncertainty that point-valued methods throw away. The theoretical approach preserves essential algebraic properties while offering enhanced flexibility and interpretability. The decision algorithm runs in $O(|E||X||U| + |U|^2)$ time, the same order as the underlying crisp soft expert set method, so the extra expressiveness of interval grades costs nothing asymptotically. Future research directions include: (1) hybridizations with neutrosophic sets for indeterminacy handling, (2) dynamic temporal extensions via τ-interval valuations, (3) a weighted-expert variant along the lines of [20] adapted to interval grades, and (4) testing the model on larger, survey-elicited interval datasets.

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