

Efficient Entropy Estimation Genetic Algorithm – Random Forest Optimization for inverted Weibull Pareto distribution using Fuzzy Ranked set Sampling: A comparative Study

Mohammed Z. Hussein¹, Asmaa A. Mahdi², Adel S. Hussain^{3,4,5}, Mohammad A. Tashtoush^{6,7,*},
Hasanain J. Alsaedi^{8,9}, Rana A. Almuttalibi¹⁰

¹ *Department of Mathematics, College of Basic Education, University of Telafer, Mousal, Iraq*

² *University of information technology and communications, Baghdad, Iraq*

³ *Department of IT, Amedi Technical Institutes, University of Duhok Polytechnic, Duhok, Iraq*

⁴ *Department of Administrative and Financial Affairs, Faculty Division, Al-Iraqi University, Baghdad, Iraq*

⁵ *Department of Computer Engineering, Al-Kitab University, Altun, Kupri, Kirkuk, Iraq*

⁶ *Department of Basic Sciences, AL-Huson University College, AL-Balqa Applied University, Salt, Jordan*

⁷ *Faculty of Education and Arts, Sohar University, Sohar, Oman*

⁸ *Business Information Technology Department, College of Business Information, University Information Technology and Communication, Baghdad, Iraq*

⁹ *Mustansiriyah university, Baghdad, Iraq*

¹⁰ *Office of University President, University of Information Technology and Communications, Baghdad, Iraq*

Abstract Entropy-based measures, give a principled measure of uncertainty and information content in lifetime and reliability models. The research formulates and contrasts estimators of four generalized entropies (Renyi, Tsallis, Havard Charvat and Arimoto) of the proposed inverted Weibull-Pareto (IWP) distribution, a flexible heavy-tailed distribution derived by a multiplicative Weibull-Pareto survival mechanism with an additive hazard structure. Estimation is done assuming simple random sampling (SRS) and ranked set sampling (RSS) with perfect and imperfect ranking. Along with the classical methods, including maximum likelihood, maximum product spacing, Kolmogorov-based, least squares, weighted least squares, Anderson-Darling, and Cramer-Mises, we propose an informed nested genetic algorithm based-random forest (GA-RF) method to enhance numerical stability and predictive power. The proposed a fuzzy Monte Carlo simulation framework to include epistemic uncertainty in model parameters and ranking quality through 8-cut representations and performance is evaluated in terms of absolute bias, mean squared error, and relative efficiency given equal measurement effort. In all the simulation situations, RSS has better estimation accuracy compared to SRS and perfect ranking is better than imperfect ranking. The tensile-strength (GPa) data of 79 carbon fibres provided by Bader and Priest is used as an example of the methodology, which indicates that the GARF method, with RSS, provides the best goodness-of-fit diagnostics, which proves the practical benefits of information-enriched sampling and smart estimation of entropy.

Keywords Maximum likelihood Estimation, Health science data, Statistical model, Genetic Algorithm – Random Forest (GA-FR)

AMS 2010 subject classifications 60J80, 60J85, 60K10

DOI: 10.19139/soic-2310-5070-3973

1. Introduction

Entropy is a basic measure of uncertainty, which is at the core of information theory, reliability analysis, statistical inference, and decision science. In addition to its classical meaning as an indication of randomness, entropy gives

*Correspondence to: Mohammad A. Tashtoush (Email: tashtoushzz@su.edu.om). Department of Basic Sciences, AL-Huson University College, AL-Balqa Applied University, Salt 19117, Jordan, and Faculty of Education and Arts, Sohar University, Sohar 311, Oman.

a quantitative measure of the amount of information in a set of probability models and forms a basis of comparison of the statistical efficiency of competing estimation procedures. Entropy has been used in lifetime and reliability modelling as a potent measure of the unpredictability in a system, tail behaviour, and structural variability.

The correct estimation of entropy functionals is thus of significant theoretical and practical interest. Nonetheless, estimating the entropy is not easy, especially when the distribution is heavy-tailed and skewed and a conventional sampling scheme can be highly variable. Although the majority of the available literature addresses the issue of entropy estimation when the sampling is a simple random sampling (SRS), there have been few studies on the effect that other sampling designs have on the information content and statistical efficiency of entropy estimators. Ranked set sampling (RSS) is a more affordable and information-enriched alternative to SRS especially in an environment where all measurements are costly, yet ranking is cheap. RSS has been demonstrated to enhance the efficiency of parameter estimation in many statistical models by adding the ordering information before measurement. Although this is the case, the effect of ordered sampling schemes on the estimation of entropy is not well studied in terms of information-theoretic view.

How to accurately estimate generalized entropy measures of the IWP distribution is the research problem addressed in this study when data are sampled using various sampling designs and when information on ranking may be imperfect. This is a theoretical problem that is important because the estimation of entropy for flexible heavy-tailed lifetime models can be influenced by sampling variability, tail sensitivity, ranking errors and numerical instability. It is also practically useful since in tests of reliability and strength an exact measurement may be expensive or difficult to obtain, whereas an approximate ranking may be easier to obtain. Thus, the comparison of SRS and RSS under the assumption of perfect and imperfect ranking guides efficiently quantifying uncertainty in real applications of lifetime-data.

The aim of this paper is to present accurate estimation of entropy measures for flexible heavy-tailed lifetime models when the data available are sampled according to different sampling schemes. The estimation of entropy becomes very difficult when the distribution of interest is heavy-tailed or skewed; sampling variability and numerical instability of the estimators, as well as ranking errors, can affect the performance of the estimators. There has been extensive research on the estimation of entropy for several lifetime distributions, but most of the literature has been devoted to the simple random sampling or censoring approaches. Ranked set sampling impact on the estimation of generalized entropy is still not studied enough and is particularly not well understood for the Inverse Weibull–Pareto (IWP) distribution.

This study fills this void by proposing and comparing estimators of four generalized entropy measures namely, Rényi, Tsallis, Havrda–Charvát and Arimoto entropies for the IWP distribution in both simple random sampling (SRS) and ranked set sampling (RSS). The novelty of the work is the combination of entropy-based inference for the IWP distribution, information-enriched sampling, imperfect ranking, fuzzy ranking-quality modelling and a nested approach based on the Genetic Algorithm–Random Forest (GA–RF) estimation strategy. This study provides a comparison between the quality of sampling design and ranking quality and the impact on parameter and entropy estimation under equal measurement effort, unlike the studies that looked at the problem only in the context of entropy estimation under the conventional sampling. The following are the major contributions of this study:

- We obtain and compute generalized entropy functions of IWP distribution.
- We construct plug-in estimators for entropy based on a number of classical estimation methods: ML, MPS, KM, LS, WLS, AD and CVM.
- Absolute bias, mean squared error and relative efficiency are compared for the SRS and RSS designs for both perfect and imperfect ranking.
- We introduce the uncertainty of ranking by using the fuzzy ranking quality mechanism and α -cut representation.
- A nested GA–RF estimation approach is proposed, which improves the numerical stability and offers competitive estimation performance.

- The proposed method is applied to the data on the tensile strength of carbon-fibre.

The theoretical development, simulation results and examples with real data indicate that in many of the situations considered ranked set sampling can yield better performance in the estimation of entropy, especially for high-ranking quality.

Estimating entropy of lifetime distributions has been the subject of much interest as entropy measures offer helpful summaries of uncertainty, information content, and tail behaviour. In addition to the well-known Shannon entropy, there are other measures of uncertainty that are flexible and can be applied in these problems, for instance classical and generalized entropy measures such as Rényi entropy [1, 2, 3], Havrda–Charvát entropy [3, 4] and Tsallis entropy [4, 5] and Arimoto entropy [5]. Much of the literature has been on entropy estimation when sampling is done with censoring or under record selection. In the past, entropy estimators were developed for some lifetime distributions such as double exponential, Rayleigh, Lomax, Burr XII, log-logistic, inverse Weibull, generalized inverse exponential, generalized hybrid censoring, generalized Bilal, power-function and Pareto-type distributions under the Type-II censoring, progressive censoring, hybrid censoring, generalized hybrid censoring or record-value censoring [6]–[25]. The studies have provided important likelihood based, Bayesian and interval estimation procedures. Their primary focus generally has been on the mechanisms of censoring, record structures or traditional sampling design, whereas the use of alternative sampling designs to enhance generalized entropy estimation has been relatively neglected.

The second stream of research focuses on ranked set sampling (RSS), which was developed to enhance the estimation efficiency for the case in which it is very expensive to measure a variable but relatively cheap to rank the units [26]. The efficiency properties of RSS have been further investigated in the statistical sampling literature [27]–[35]. Some other extensions have been suggested like extreme ranked set sampling, median ranked set sampling, and entropy related RSS methods [37]–[45]. There have been a number of previous statistical efficiency studies on ranking information, which show that ranking can be useful if ranking is sufficiently accurate, but the application of such information to estimating entropy is relatively scarce. There has been a considerable amount of literature on the estimation of the parameters of a distribution or on distributional inference, with the studies involving entropy functions focusing on particular entropy functionals or on relatively conventional lifetime settings. Thus, the impact of RSS over a number of generalized entropy metrics for flexible heavy-tails distributions has not been well explored. Another restriction is that most studies on RSS are based on an ideal ranking, while in actual applications it can be measured and/or classified incorrectly. This assumption can cause the user to have a positive view of the efficiency he can achieve through RSS. Furthermore, the effects of sampling design, quality of ranking and numerical estimation on entropy estimation have seldom been studied together in one framework. The present study overcomes these drawbacks by estimating Rényi, Tsallis, Havrda–Charvát and Arimoto entropies of the four-parameter IWP distribution for both SRS and RSS with identical measurement effort explicitly using fuzzy ranking-quality and α -cut representation of the imperfect ranking. It also contrasts several classical estimation procedures with a nested GA–RF approach to examine if computational stabilization can be used to enhance the estimation in challenging heavy-tailed scenarios. The studies indicate that prior to measurement, using ordering information can enhance statistical inference in a variety of situations. In spite of this, the majority of the studies in this area have dealt with the estimation of parameters or distributional inference, but not with generalized entropy estimation for lifetime models of heavy tails. Furthermore, although it is known that ranking is often imperfect in practice, many studies either assume perfection in ranking or consider ranking errors in a restricted manner. This justifies the development of entropy estimation procedures, which take into consideration the uncertainty of the ranking, such as the fuzzy ranking mechanisms [46, 47, 48].

The next area of relevance is the creation of flexible heavy-tailed lifetime models, like the Weibull–Pareto-type models, that are applicable to skewed reliability, fatigue, survival, and strength data [49]–[53]. These models can model both early-life behaviour and heavy-tail behaviour, but can pose numerical problems for the estimation due to the multi-parameter structure. There are classical methods that have proven useful, like maximum likelihood estimation, maximum product spacing, least squares, weighted least squares, Anderson–Darling estimation, or

Cramér–von Mises estimation, and they may be effective, but with regard to them, there are some things that need to be considered: How they are initialized, the number of samples used, and the shape of the objective function. This encourages the application of numerically stable estimation methods such as hybrid Genetic Algorithm–Random Forest techniques to get competitive estimates [54]–[58].

The present study focuses on generalized entropy estimation of IWP distribution under both SRS and RSS, while other entropy-estimation studies [6]–[26] are mostly concerned with entropy estimation of SRS, censoring schemes, record values, or other conventional sampling structures. Hence, it provides a measure of the impact of information-rich sampling with the same measurement effort. The study also generalizes the existing work on RSS [34]–[45] in that it does not consider only distributional inference or parameter estimation, but generalized entropy estimation of the lifetime model, which is a heavy tailed model.

Furthermore, instead of many RSS studies that are based on perfect ranking, this one considers fuzzy ranking quality in terms of α -cut representations [47, 48, 57, 58]. This is a more realistic estimation of the robustness of the estimator with imperfect ranking. Finally, the proposed GA–RF estimator compares classical estimators with a hybrid data-driven and optimization-based approach [59,66] to add a computational contribution. Thus, the main contribution of this study is the incorporation of generalized entropy estimation, RSS, fuzzy ranking uncertainty and computationally robust estimation in the IWP model.

In contrast to past entropy estimation research focused either on SRS or censoring schemes, record values, or traditional sampling designs, the current study investigates the generalized estimation of entropy for the IWP distribution under both an SRS and RSS. Therefore, it assesses the impact of information-rich sampling during a constant measurement effort. The present study also extends the research on RSS by considering a generalized entropy estimation problem for a heavy-tailed lifetime model instead of only parameter estimation or distributional inference. Moreover, to address the existence of ranking uncertainty, the framework of fuzzy ranking-quality is proposed with α -cut representation, which gives a more realistic evaluation of the robustness of the estimators under imperfect ranking.

Lastly, the proposed GA–RF estimator includes a contribution from the computations by comparing classical estimators with a hybrid data-driven optimization-based approach. The key contribution of this research is thus the incorporation of generalized entropy estimation, fuzzy ranking uncertainty, RSS and robust estimation that is computationally feasibly implemented in the IWP model.

The current study is an extension of these streams of research in a comprehensive approach. In particular, we introduce and study estimators of Rényi, Tsallis, Havrda–Charvát and Arimoto entropies for simple random sampling and ranked set sampling for the Inverse Weibull–Pareto (IWP) distribution [2, 3, 4, 5, 26] and compare these estimators. This work compares the impact of information-rich sampling to entropy estimation under equal measurement effort, unlike the studies that only discuss conventional sampling or censoring. Moreover, the study also takes into account imperfect ranking represented by fuzzy ranking quality mechanism [47, 48] and α -cut representation [57, 58] and compares the classical estimators with a nested GA–RF estimation approach. Hence, the contribution is not just in deriving and estimating entropy measures for the IWP model, but also in estimating the effect of sampling design, quality of ranking, and computational estimation on the reliability inference based on entropy. It is proposed that entropy estimation and sampling design, ranking uncertainty and numerical optimization can be linked in one methodology. First, the IWP distribution is selected as a parent lifetime distribution since it has four parameters that can be used to control the scale, shape, skewness and tail behaviour. Second, the entropy measures are estimated by plug-in procedures since entropy functions are functions of the unknown parameters of the IWP - after the parameter estimate is obtained from the procedure, the entropy estimate is calculated by substituting the parameter estimates into the entropy formula. Third, because of its ability to enhance the efficiency of estimation when it is expensive to obtain the actual measurement but relatively inexpensive to obtain it by ranking, RSS is considered. The methodology on the other hand assumes

imperfect ranking, which is not always possible in practice, with the help of a fuzzy ranking mechanism. This is because this way it is possible to take the uncertainty in ranking into consideration, instead of assuming the ideal ranking. Finally, the GA–RF estimator is suggested to overcome the numerical instability due to the nonlinear and possibly flat objective function in the four parameter IWP model. Therefore, from the statistical efficiency and computational stability points of view the methodological framework is justified.

The methodological framework is given in a step-by-step manner for better readability. The sampling designs for RSS and SRS are first discussed. Second, IWP model and entropy measures are introduced. Third, the classical and the GA–RF estimation methods are described. Lastly, the fuzzy Monte Carlo simulation procedure and performance measures are shown.

1.1. Ranked Set Sampling Method

Simple random sampling (SRS) is a common method in the statistical inference, but in case of small sample size or expensive measurements, SRS can provide inefficient estimators. Another sampling method presented in [26] is the ranked set sampling (RSS), which was created to enhance efficiency in estimations since the ranking information is inexpensive and is obtained before the measurement is made. RSS is especially helpful when precise measurement is costly (financially, time-wise, or otherwise), but the units can be ranked by visual inspection, expert judgment, pre-liminary data or any other relatively inexpensive auxiliary methods. RSS was mathematically developed by [34], and its high efficiency in comparison to SRS, even in the case of imperfect ranking was proved by [35]. A number of RSS extensions have been suggested such as extreme RSS [36], median RSS [37], as well as entropy-based RSS methodologies [39]–[45]. The conventional RSS design is as follows:

Step 1: Selection of Sets: The first step involves drawing t independent simple random samples, each of size t , from the target population. These sets are denoted as:

$$\{Y_{11}, Y_{12}, \dots, Y_{1t}\}, \{Y_{21}, Y_{22}, \dots, Y_{2t}\}, \dots, \{Y_{t1}, Y_{t2}, \dots, Y_{tt}\}.$$

At this stage, no precise or costly measurements are taken on any of the sampled units. Only the units themselves are identified for potential ranking.

Step 2: Ranking Within Sets: Within each of the t sets of size t , the units are ranked from smallest to largest using an inexpensive ranking mechanism. This ranking can be based on visual inspection, expert judgment, historical data, or any other auxiliary information that does not require precise measurement. The ranked units in the h_2 -th set are represented as:

$$Y_{h_2(1)}, Y_{h_2(2)}, \dots, Y_{h_2(t)}, \quad h_2 = 1, 2, \dots, t.$$

Under the assumption of perfect ranking, this ordering accurately reflects the true underlying values of the units. However, in practice, ranking may be imperfect, which can affect the efficiency of the resulting estimators.

Step 3: Measurement of Selected Units: From the h_1 -th set, the unit with the h_1 -th smallest rank is selected and its true value is precisely measured. The measured unit in the h_2 -th cycle is denoted by:

$$X_{h_2 h_1} = Y_{h_2(h_1)}, \quad h_1 = 1, 2, \dots, t, \quad h_2 = 1, 2, \dots, d. \quad (1)$$

Thus, in each cycle, exactly one unit from each of the t sets is measured. This means that for each cycle, a total of t units are precisely measured, although t^2 units were initially drawn and ranked.

Step 4: Repetition Over Cycles: Steps 1 through 3 are repeated for d independent cycles (or repetitions) to obtain a total RSS sample. The total number of precisely measured units in the final sample is: Repeating Steps (1)–(3) for d cycles yield an RSS sample of size

$$\tau^* = s \times \tau$$

The final measured observations, denoted as:

$$\{X_{h_2 h_1} : h_1 = 1, \dots, t; h_2 = 1, \dots, d\},$$

The measured observations are independent (they are based on different sets) but they are not identically distributed, as they are associated with different order statistics. The probability density of under the hypothesis of perfect ranking is given by $X_{h_2 h_1}$ is:

$$f_{X_{h_2 h_1}}(x) = \frac{\tau!}{(h_1 - 1)! (\tau - h_1)!} f(x) [F(x)]^{h_1 - 1} [1 - F(x)]^{\tau - h_1} \quad (2)$$

the PDF of the h_1 -th order statistic of a sample size τ . Practically, perfect ranking is not often achievable because of subjectivity, observational constraints or measurement error. In order to simulate this realistic situation, we add a fuzzy imperfect ranking mechanism to the RSS design. As an alternative to the assumption that ranking is based on the actual value X_i , we make the assumption that ranking is done based on a fuzzy-observed variable which is described as [47, 48].

$$\tilde{X}_i = X_i + \varepsilon_i, \quad i = 1, 2, \dots, t, \quad (3)$$

where:

$$\varepsilon_i \sim N(0, \sigma_r^2)$$

represents ranking uncertainty and $\sigma_r \geq 0$ limits the amount of fuzziness of the ranking. Step (2) ranking is thus done based on $\tilde{X}_i$, and the actual measurement in Step (3) is done using the true value X_i . The ranking quality is controlled by the parameter σ_r :

- If $\sigma_r = 0$, the ranking is perfect and the procedure reduces to classical RSS.
- As σ_r increases, ranking errors increase.
- For sufficiently large σ_r , the procedure approaches simple random sampling behavior.

1.2. Inverse Weibull Pareto distribution IWP (Proposed Model)

Flexible lifetime distributions have a central role in the modelling of reliability data, survival times, financial risks as well as heavy-tailed phenomena. In most practical uses, classical lifetime distributions, like the Weibull or Pareto distributions, might not be sufficiently flexible to simultaneously be able to both models early-life behaviour and heavy-tail behaviour. To overcome this shortcoming, composite models that entail a combination of exponential-type and Pareto-type structures have been constructed [49]. One such model is the Weibull–Pareto distribution. The Weibull-Pareto distribution is an integrated form of the Weibull and Pareto mechanisms in a multiplicative survival process; thus, it provides more flexibility to model skewness, tail heaviness and the behaviour of the hazard rate. Model-selection rationale. The motivation for the choice of the four parameter IWP distribution is that they can be used to describe early-life behaviour, heavy-tail behaviour and give independent control over the scale, shape, skewness and tail behaviour. The main question addressed in this particular paper is whether IWP is a natural and generally superior entropy estimator under this flexible lifetime model and whether it is naturally superior to some of the simpler and alternative heavy-tailed models. While the real-data analysis is complemented by KS, Anderson–Darling and Cramér–von Mises goodness-of-fit diagnostics, a formal AIC/BIC comparison with simpler models (Inverse Weibull, Pareto distributions) was not performed. For the present, this comparative model-selection analysis is beyond scope and will be kept as an important direction for future research. Probability Density Function (PDF) The probability density function of the Inverse Weibull Pareto distribution is given by:

$$f(x) = e^{-\lambda x^{-\alpha}} (1 + \beta x)^{-k} \left[\lambda \alpha x^{-\alpha-1} + \frac{k\beta}{1 + \beta x} \right], \quad x, \lambda, \alpha, \beta, k > 0. \quad (4)$$

where $\lambda > 0$: Weibull scale parameter, $\alpha > 0$: Weibull shape parameter, $\beta > 0$: Pareto scale parameter, $k > 0$: Pareto shape parameter. The model incorporates two competing mechanisms:

- Weibull component $e^{-\lambda x^\alpha}$, which controls early-life and aging behavior.
- Pareto component $(1 + \beta x)^{-k}$, which introduces heavy-tail behavior.

This hybrid form allows an exponential decay, as well as power-law tail behaviour to be modelled simultaneously.

$$F(x) = 1 - e^{-\lambda x^\alpha} (1 + \beta x)^{-k}, \quad x > 0 \quad (5)$$

As $x \rightarrow 0$, we have $F(x) \rightarrow 0$, and as $x \rightarrow \infty$, we obtain $F(x) \rightarrow 1$, confirming that it defines a proper distribution. The Survival Function is: The survival function is

$$S(x) = e^{-\lambda x^\alpha} (1 + \beta x)^{-k} \quad (6)$$

The combination of Weibull and Pareto survival behaviours is evident in this representation. The Hazard Rate Function (HRF) Hazard rate function is:

$$h(x) = \frac{f(x)}{S(x)} = \lambda \alpha x^{\alpha-1} + \frac{k\beta}{1 + \beta x} \quad (7)$$

This simple additive hazard structure is one of the principal advantages of the Inverse Weibull Pareto model [50]-[53].

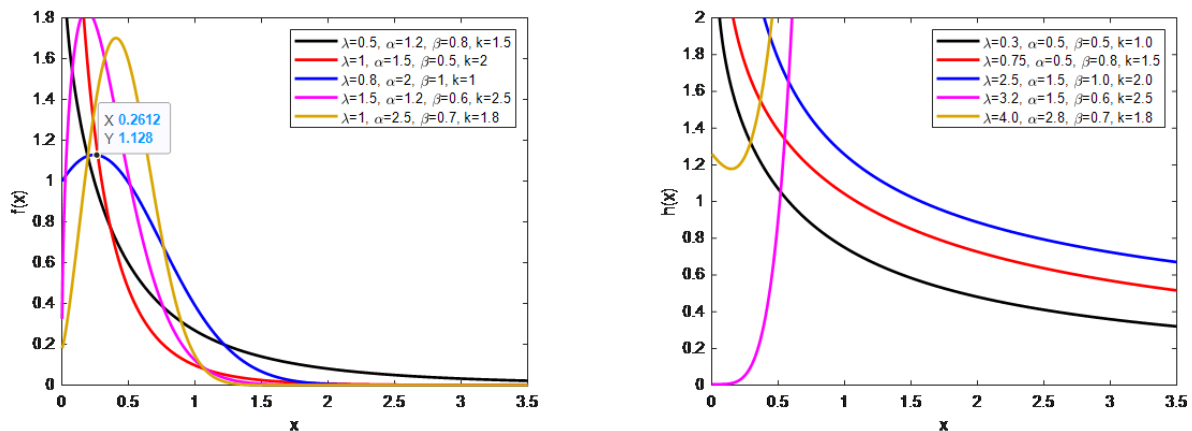


Figure 1. Plots of the PDF and HRF for the IWP distribution

1.3. Motivation of the Study

As far as we understand, no current study has taken the systematic approach of studying the estimation of the uncertainty quanta of the Inverse Weibull-Pareto (IWP) distribution on the basis of ranked set sampling (RSS). Even though entropy-based measures and RSS methodologies have been extensively investigated separately, the combination of both has not been investigated with respect to the WP distribution. The Weibull Pareto distribution is a versatile lifetime model that can be used to explain aging behaviour and heavy-tail behaviour. Due to this flexibility, it finds useful applications in mortality analysis, fatigue failure modelling, degradation testing and reliability assessment of mechanical and electrical systems. In this case, it is important to quantify uncertainty in order to make sound inferences and decisions. Entropy measures give a strict mathematical procedure of evaluating uncertainty and information content of probabilistic models. Based on such considerations, the current research seeks to formulate and compare entropy estimators of the IWP distribution when using simple random sampling (SRS) or ranked set sampling (RSS) plans. Inverse Weibull-Pareto (IWP) distribution is selected because of its capability of modelling lifetime and reliability data with nonstandard hazard behaviour, heavy tails and skewness.

It is a Weibull-Pareto survival structure that has both an exponential-type and a Pareto-type component, which makes it suitable to model early-life behaviour and heavy-tail behaviour. The additive hazard-rate form is also convenient for applications in reliability: in such cases, multiple risk mechanisms can be working at the same time. Hence, the distribution of the IWP can be used in strength data, fatigue failure, degradation modelling and reliability assessment etc.

Four entropy measures are examined in this work, some researchers generalized and complementary descriptions of uncertainty. These measures, unlike a single entropy index, enable an analysis of the uncertainty and information content at different entropy orders and sensitivity structures. This is of particular interest for heavy-tailed lifetime distributions where tail behavior can have a significant impact on uncertainty quantification.

Ranked set sampling (RSS) is warranted in practice where the cost or time of exact measurement is high, but the cost or time of ranking the units is not. If so, RSS can utilize additional ordering information to enhance the performance of the estimators under the same measurement effort. But in reality, it is difficult to guarantee a perfect rank. Hence, this study adopts fuzzy ranking mechanism as a representation of the uncertainty of ranking. Imperfect ranking quality can be modelled more realistically with α -cut representations and the fuzzy ranking approach gives more flexibility in assessing the robustness of the estimators when uncertainty is present. The present study extends the existing literature in various aspects, by combining three fields which are mostly investigated separately: generalized entropy estimation, ranked set sampling, and flexible heavy-tailed lifetime modelling. More specifically, it introduces the idea of entropy-based inference to the IWP distribution in both SRS and RSS designs, and can assess the impact of information enriched sampling under equal measurement effort. It also includes imperfect ranking using the fuzzy ranking-quality framework, which offers a more realistic measure of the performance of the estimators when there is uncertainty in ranking. Besides that, several classical estimation methods are compared with a nested GA-RF estimator, thus testing the statistical accuracy as well as numerical stability. The contribution of this work is thus not just on the derivation of entropy estimators, but also in the development of a systematic understanding of the implications of sampling design, ranking uncertainty and computational estimation techniques for entropy-based inference for heavy-tailed reliability data.

The research contribution is this work could be as follows:

- To come up with parameter estimators of the IWP distribution and to produce plug-in estimators of four information measures, viz. the Rényi, Tsallis, Havard Charvat, and Arimoto entropies.
- To apply and compare classical estimation methods, namely maximum likelihood (ML), maximum product spacing (MPS), Kolmogorov-based (KM), least squares (LS), weighted least squares (WLS), and Anderson Darling (AD) and Cramer von Mises (CVM) in both SRS and RSS designs.
- To suggest and test a smart hybrid estimator on the basis of a nested genetic algorithm-random forest (GA-RF) design and to measure its practical performance in comparison with the traditional ones in the real-data analysis [Table 4 -Table 6](#)
- To study the impact of imperfect ranking using a fuzzy ranking-quality mechanism in the simulation study, through modelling the ranking correlation as a fuzzy variable and presenting the performance via α -cut envelopes.

To examine finite sample performance with SRS and RSS in the presence of perfect and imperfect ranking in terms of bias, mean squared error (MSE) and relative efficiency (RE), and to illustrate the usefulness of the procedures for real data with the routine goodness-of-fit tests. Thus, the main research question of this study is: How does ranked set sampling (RSS) help to improve the quality of estimation of generalized entropy measures for the distribution of IWP, compared with simple random sampling (SRS), under equal measurement effort, and how is this gain influenced by the quality of the ranking and the type of estimation method? Another question is whether the proposed GA-RF approach can achieve competitive estimation accuracy and numerical stability compared with the conventional estimators in the same sampling and parameter conditions.

Table 1. Differentiation of the present study from closely related prior work sharing overlapping authorship [58]

Study	Distribution / process	Estimand	Sampling design	Ranking uncertainty	Estimation methods compared	AI / hybrid component
This Study	Inverse Weibull–Pareto (4-parameter, heavy-tailed)	Rényi, Tsallis, Havrda–Charvát, Arimoto entropy (plug-in)	SRS and RSS	Fuzzy ranking-quality mechanism, α -cut, $\rho = 0, 0.7, 0.9, 1$	ML, MPS, KM, LS, WLS, AD, CVM, GA–RF	Nested GA–RF (GA-tuned random-forest surrogate + inner GA refinement)
[42]	Rayleigh process (occurrence-rate parameter)	Point/interval parameter estimates	SRS only	Not considered	Classical (ML/LS-type) vs. “intelligent” methods	Single meta-heuristic, not GA–RF
[50]	Non-homogeneous Poisson process (Gompertz)	Process-rate parameters, reliability function	SRS only	Not considered	Classical vs. AI methods (comparative)	AI method used individually, no RF surrogate
[36]	Truncated Nadarajah–Haghighi inverse Rayleigh (NHPP)	Process parameters	SRS only	Not considered	Not the focus (single hybrid estimator)	Hybrid SVM–GA (no random-forest stage)
[28]	DUS-transformed generalized polynomial quadratic failure-rate model	Distribution parameters	SRS only	Not considered	Not the focus (single hybrid estimator)	Hybrid SVM–GA (no random-forest stage)
[32]	Maxwell distribution (alpha-series process)	Occurrence-rate parameter	SRS only	Not considered	Classical vs. artificial bee colony algorithm	Single metaheuristic (ABC), no GA–RF
[55]	Logistic extreme value distribution	Distribution parameters	SRS only	Not considered	Classical vs. machine-learning estimation	ML used individually, no nested GA–RF
[58]	Inverted exponentiated Pareto	Rényi-type entropy (comparative study)	RSS (perfect ranking) vs. SRS	Not modelled (ranking assumed perfect)	ML and classical distance-based methods	None (no hybrid/AI estimator)

2. Entropy Metrics

The analytical expressions of the Rényi entropy, denoted by β_1 , Havard Charvat entropy, denoted by β_2 , Tsallis entropy, denoted by β_3 , and Arimoto entropy, denoted by β_4 , for Inverse Weibull–Pareto distribution. The Rényi entropy of order u , where $u > 0$, and $u \neq 1$, is defined in Equation 8 as follows:

$$\beta_1 = \frac{1}{1-u} \log \left[\int_{-\infty}^{\infty} f(x)^u dx \right] \quad (8)$$

By inserting Equation 4 in Equation 8, the entropy of IWP distribution, based on the Rényi entropy, is obtained as follows:

$$\beta_1 = \frac{1}{1-u} \log \left[\int_0^{\infty} e^{-u\lambda x^{-\alpha}} (1+\beta x)^{-uk} \left[\lambda \alpha x^{-\alpha-1} + \frac{k\beta}{1+\beta x} \right]^u dy \right] \quad (9)$$

The generalized binomial expansion of the integrand in the equation below, (9) provides:

$$f(x)^u = e^{-u\lambda x^{-\alpha}} (1+\beta x)^{-ku} \sum_{j=0}^{\infty} \binom{u}{j} (\lambda \alpha)^{u-j} x^{(-\alpha-1)(u-j)} (k\beta)^j (1+\beta x)^{-j}. \quad (10)$$

Combining powers of $(1+\beta x)$ gives $(1+\beta x)^{-ku-j}$. Thus:

$$I(u) = \sum_{j=0}^{\infty} C_j \int_0^{\infty} x^{(-\alpha-1)(u-j)} e^{-u\lambda x^{-\alpha}} (1+\beta x)^{-ku-j} dx \quad (11)$$

Where:

$$C_j = \binom{u}{j} (\lambda \alpha)^{u-j} (k\beta)^j \quad (12)$$

After transformation, we obtain

$$I(u) = \sum_{j=0}^{\infty} C_j (u\lambda)^{-\frac{(\alpha-1)(u-j)+1}{\alpha}} \int_0^{\infty} t^{\frac{(\alpha-1)(u-j)+1}{\alpha}-1} e^{-t} \left(1 + \beta \left(\frac{t}{u\lambda} \right)^{1/\alpha} \right)^{-ku-j} dt \quad (13)$$

Define

$$J_j(u, \lambda, \alpha, \beta, k) = (u\lambda)^{-\frac{(\alpha-1)(u-j)+1}{\alpha}} \int_0^{\infty} t^{\frac{(\alpha-1)(u-j)+1}{\alpha}-1} e^{-t} \left(1 + \beta \left(\frac{t}{u\lambda} \right)^{1/\alpha} \right)^{-ku-j} dt \quad (14)$$

Then

$$I(u) = \sum_{j=0}^{\infty} C_j J_j(u, \lambda, \alpha, \beta, k)$$

By replacing the form of the equation in (14) in the equation in (9) the Rényi entropy of the Weibull-Pareto (WP) distribution will be obtained as:

$$\beta_1 = \frac{1}{1-u} \log[I(u)] = \frac{1}{1-u} \log \left(\sum_{j=0}^{\infty} C_j J_j \right) \quad (15)$$

The Tsallis entropy of order $u, u > 0$, and $u \neq 1$, is defined by:

$$\beta_2 = \frac{1}{u-1} [1 - I(u)] \quad (16)$$

The Havard and Charvát Entropy of order u , where $u > 0$, and $u \neq 1$, is defined by:

$$\beta_3 = \frac{1}{2^{1-u} - 1} [I(u) - 1] \quad (17)$$

The Arimoto entropy of order $u, u > 0$, and $u \neq 1$, is defined by:

$$\beta_4 = \frac{u}{1-u} \left[I(u)^{1/u} - 1 \right] \quad (18)$$

3. Maximum Likelihood Estimator

3.1. Estimators Based on RSS

Let the RSS design has set size m and cycles r , and measured observations

$$\left\{ X_{i:m}^{(j)} : i = 1, \dots, m; j = 1, \dots, r \right\} \quad (19)$$

with observed values x_{ij} for $X_{i:m}^{(j)}$. Total measured sample size is $n = mr$. Under perfect ranking, $X_{i:m}^{(j)}$ has the distribution of the i th order statistic from an SRS of size m from the parent model. For $\theta = (\lambda, \alpha, \beta, k)$, the i th order statistic density is

$$g_{i:m}(x; \theta) = \frac{m!}{(i-1)!(m-i)!} [F(x; \theta)]^{i-1} [1 - F(x; \theta)]^{m-i} f(x; \theta) \quad (20)$$

Substitute $F(x) = 1 - S(x)$ and $f(x) = S(x) \left[\lambda \alpha x^{-\alpha-1} + \frac{k\beta}{1+\beta x} \right]$:

$$g_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} [1 - S(x)]^{i-1} [S(x)]^{m-i} S(x) \left[\lambda \alpha x^{-\alpha-1} + \frac{k\beta}{1+\beta x} \right] \quad (21)$$

Assuming independence across cycles and sets:

$$\begin{aligned} L(\lambda, \alpha, \beta, k) &= \prod_{j=1}^r \prod_{i=1}^m g_{i:m}(x_{ij}; \lambda, \alpha, \beta, k) \\ &= \prod_{j=1}^r \prod_{i=1}^m \left[\frac{m!}{(i-1)!(m-i)!} \left(1 - e^{-\lambda x_{ij}^\alpha} (1 + \beta x_{ij})^{-k} \right)^{i-1} \left(e^{-\lambda x_{ij}^\alpha} (1 + \beta x_{ij})^{-k} \right)^{m-i+1} \lambda \alpha x_{ij}^{\alpha-1} + \frac{k\beta}{1 + \beta x_{ij}} \right] \end{aligned} \quad (22)$$

Taking logs

$$\begin{aligned} \uparrow(\lambda, \alpha, \beta, k) = \log L &= \sum_{j=1}^r \sum_{i=1}^m \left[\log \frac{m!}{(i-1)!(m-i)!} + (i-1) \log \left(1 - e^{-\lambda x_{ij}^\alpha} (1 + \beta x_{ij})^{-k} \right) \right. \\ &\quad \left. + (m-i+1) \log \left(e^{-\lambda x_{ij}^\alpha} (1 + \beta x_{ij})^{-k} \right) + \log \left(\lambda \alpha x_{ij}^{\alpha-1} + \frac{k\beta}{1 + \beta x_{ij}} \right) \right] \end{aligned} \quad (23)$$

we can write

$$\begin{aligned} \updownarrow = & \sum_{j=1}^r \sum_{i=1}^m \log \frac{m!}{(i-1)!(m-i)!} + \sum_{j=1}^r \sum_{i=1}^m \left[(i-1) \log \left(1 - e^{-\lambda x_{ij}^\alpha} (1 + \beta x_{ij})^{-k} \right) \right. \\ & \left. + (m-i+1) \left(-\lambda x_{ij}^\alpha - k \log (1 + \beta x_{ij}) \right) + \log \left(\lambda \alpha x_{ij}^{\alpha-1} + \frac{k\beta}{1 + \beta x_{ij}} \right) \right] \quad (24) \end{aligned}$$

Taking partial derivatives of the log-likelihood with respect to λ, α, β and k yields the score equations.

$$\frac{\partial \updownarrow}{\partial \lambda} = \sum_{j=1}^r \sum_{i=1}^m \left[(i-1) \frac{x_{ij}^\alpha S_{ij}}{1 - S_{ij}} - (m-i+1) x_{ij}^\alpha + \frac{\alpha x_{ij}^{\alpha-1}}{H_{ij}} \right] = 0 \quad (25)$$

$$\frac{\partial \updownarrow}{\partial \alpha} = \sum_{j=1}^r \sum_{i=1}^m \left[(i-1) \frac{\lambda x_{ij}^\alpha \ln(x_{ij}) S_{ij}}{1 - S_{ij}} - (m-i+1) \lambda x_{ij}^\alpha \ln(x_{ij}) + \frac{\lambda x_{ij}^{\alpha-1} (1 + \alpha \ln(x_{ij}))}{H_{ij}} \right] = 0 \quad (26)$$

$$\frac{\partial \updownarrow}{\partial \beta} = \sum_{j=1}^r \sum_{i=1}^m \left[(i-1) \frac{k x_{ij} S_{ij}}{(1 + \beta x_{ij})(1 - S_{ij})} - (m-i+1) \frac{k x_{ij}}{1 + \beta x_{ij}} + \frac{k}{(1 + \beta x_{ij})^2 H_{ij}} \right] = 0 \quad (27)$$

$$\frac{\partial \updownarrow}{\partial k} = \sum_{j=1}^r \sum_{i=1}^m \left[(i-1) \frac{\ln(1 + \beta x_{ij}) S_{ij}}{1 - S_{ij}} - (m-i+1) \ln(1 + \beta x_{ij}) + \frac{\beta}{(1 + \beta x_{ij}) H_{ij}} \right] = 0 \quad (28)$$

The MLEs $(\hat{\lambda}, \hat{\alpha}, \hat{\beta}, \hat{k})$ are obtained by solving the nonlinear system, $\frac{\partial \updownarrow}{\partial \lambda} = 0$, $\frac{\partial \updownarrow}{\partial \alpha} = 0$, $\frac{\partial \updownarrow}{\partial \beta} = 0$, $\frac{\partial \updownarrow}{\partial k} = 0$, or equivalently by numerically maximizing $\updownarrow(\lambda, \alpha, \beta, k)$ (e.g., Newton-Raphson, BFGS, etc.).

3.2. Estimators Based on SRS

Let $x_1, \dots, x_n$ be an i.i.d. simple random sample from the IWP model. The likelihood is

$$\begin{aligned} L(\lambda, \alpha, \beta, k) &= \prod_{i=1}^n f(x_i; \lambda, \alpha, \beta, k) \\ &= \prod_{i=1}^n \left[e^{-\lambda x_i^\alpha} (1 + \beta x_i)^{-k} \left(\lambda \alpha x_i^{\alpha-1} + \frac{k\beta}{1 + \beta x_i} \right) \right]. \quad (29) \end{aligned}$$

Taking logs gives

$$\updownarrow(\lambda, \alpha, \beta, k) = \log L(\lambda, \alpha, \beta, k) = \sum_{i=1}^n \left[-\lambda x_i^\alpha - k \log(1 + \beta x_i) + \log \left(\lambda \alpha x_i^{\alpha-1} + \frac{k\beta}{1 + \beta x_i} \right) \right]. \quad (30)$$

Differentiate $\updownarrow$ and set the derivatives to zero.

$$\frac{\partial \updownarrow}{\partial \lambda} = \sum_{i=1}^n \left[\frac{\partial}{\partial \lambda} (-\lambda x_i^\alpha) + \frac{\partial}{\partial \lambda} \log H_i \right] = \sum_{i=1}^n \left[-x_i^\alpha + \frac{1}{\lambda \alpha x_i^{\alpha-1} + \frac{k\beta}{1 + \beta x_i}} (\alpha x_i^{\alpha-1}) \right] = 0 \quad (31)$$

Differentiate α and set the derivatives to zero.

$$\frac{\partial \updownarrow}{\partial \alpha} = \sum_{i=1}^n \left[-\lambda x_i^\alpha \ln x_i + \frac{1}{\lambda \alpha x_i^{\alpha-1} + \frac{k\beta}{1 + \beta x_i}} (\lambda x_i^{\alpha-1} (1 + \alpha \ln x_i)) \right] = 0 \quad (32)$$

The parameters β and k and setting these derivatives to zero:

$$\frac{\partial \downarrow}{\partial \beta} = \sum_{i=1}^n \left[-\frac{kx_i}{1 + \beta x_i} + \frac{k}{(1 + \beta x_i)^2 \left(\lambda \alpha x_i^{\alpha-1} + \frac{k\beta}{1 + \beta x_i} \right)} \right] = 0, \quad (33)$$

$$\frac{\partial \downarrow}{\partial k} = \sum_{i=1}^n \left[-\log(1 + \beta x_i) + \frac{\beta}{(1 + \beta x_i) \left(\lambda \alpha x_i^{\alpha-1} + \frac{k\beta}{1 + \beta x_i} \right)} \right] = 0. \quad (34)$$

Further, the ML estimators of entropy measures $\tilde{\beta}_1^{ML}$ of β_1 , $\tilde{\beta}_2^{ML}$ of β_2 , $\tilde{\beta}_3^{ML}$ of β_3 , and $\tilde{\beta}_4^{ML}$ of β_4 are obtained after inserting $\tilde{\lambda}^{ML}$, $\tilde{\alpha}^{ML}$, $\tilde{\beta}^{ML}$ and $\tilde{k}^{ML}$ in Equations the invariance property of the ML estimators is used in Equation 9 and Equation 16 - Equation 18 respectively.

4. Maximum Product Spacing

The maximum product spacing (MPS) method is used instead of maximum likelihood estimation. We find that, in MPS $\tilde{\lambda}^{MPS}$, $\tilde{\alpha}^{MPS}$, $\tilde{\beta}^{MPS}$ and $\tilde{k}^{MPS}$ for the IWP distribution under both RSS and SRS. These estimates are then used to construct MPS-based estimators of $\beta_1, \beta_2, \beta_3$ and β_4 [55, 56].

4.1. Estimators Based on RSS

Assume that $X_{(1:t^*)}, \dots, X_{(t^*:t^*)}$ is an RSS of size $t^* = t \times d$, where d is the number of cycles and t is the set size. Let $H(\cdot | \tau) \equiv F(\cdot | \tau)$. The spacings for a sample of size t^* are defined by

$$\Lambda_k(\tau) = H(x_{(k:t^*)} | \tau) - H(x_{(k-1:t^*)} | \tau), \quad k = 1, 2, \dots, t^* + 1,$$

with boundary conditions $H(x_{(0)} | \tau) = 0$, $H(x_{(t^*+1:t^*)} | \tau) = 1$. For the IWP CDF, an explicit form of the spacings is

$$\begin{aligned} \Lambda_k(\tau) &= \left[1 - e^{-\lambda y_{(k)}^\alpha} (1 + \beta y_{(k)})^{-k} \right] - \left[1 - e^{-\lambda y_{(k-1)}^\alpha} (1 + \beta y_{(k-1)})^{-k} \right] \\ &= e^{-\lambda y_{(k-1)}^\alpha} (1 + \beta y_{(k-1)})^{-k} - e^{-\lambda y_{(k)}^\alpha} (1 + \beta y_{(k)})^{-k}. \end{aligned}$$

The MPS criterion is

$$M^*(\tau) = \frac{1}{t^* + 1} \sum_{k=1}^{t^*+1} \ln(\Lambda_k(\tau)) \quad (35)$$

and the MPS estimators are

$$(\hat{\lambda}^{MPS}, \hat{\alpha}^{MPS}, \hat{\beta}^{MPS}, \hat{k}^{MPS}) = \arg \max_{\lambda, \alpha, \beta, k} M^*(\lambda, \alpha, \beta, k).$$

For any $\theta \in \{\lambda, \alpha, \beta, k\}$:

$$\frac{\partial M^*}{\partial \theta} = \frac{1}{t^* + 1} \sum_{k=1}^{t^*+1} \frac{1}{\Lambda_k(\tau)} \frac{\partial \Lambda_k(\tau)}{\partial \theta} = 0 \quad (36)$$

Since $\Lambda_k(\tau) = H(y_{(k)} | \tau) - H(y_{(k-1)} | \tau)$, define

$$\pi_\theta(y | \tau) \equiv \frac{\partial}{\partial \theta} H(y | \tau) \quad (37)$$

So that:

$$\frac{\partial \Lambda_k(\tau)}{\partial \theta} = \pi_\theta(y_{(k)} | \tau) - \pi_\theta(y_{(k-1)} | \tau)$$

Hence the four score equations are:

$$\frac{\partial M^*}{\partial \lambda} = \frac{1}{t^* + 1} \sum_{k=1}^{t^*+1} \frac{\pi_\lambda(y_{(k)} | \tau) - \pi_\lambda(y_{(k-1)} | \tau)}{\Lambda_k(\tau)} = 0 \quad (38)$$

$$\frac{\partial M^*}{\partial \alpha} = \frac{1}{t^* + 1} \sum_{k=1}^{t^*+1} \frac{\pi_\alpha(y_{(k)} | \tau) - \pi_\alpha(y_{(k-1)} | \tau)}{\Lambda_k(\tau)} = 0 \quad (39)$$

$$\frac{\partial M^*}{\partial \beta} = \frac{1}{t^* + 1} \sum_{k=1}^{t^*+1} \frac{\pi_\beta(y_{(k)} | \tau) - \pi_\beta(y_{(k-1)} | \tau)}{\Lambda_k(\tau)} = 0 \quad (40)$$

$$\frac{\partial M^*}{\partial k} = \frac{1}{t^* + 1} \sum_{k=1}^{t^*+1} \frac{\pi_k(y_{(k)} | \tau) - \pi_k(y_{(k-1)} | \tau)}{\Lambda_k(\tau)} = 0 \quad (41)$$

Closed forms of $\pi_\lambda, \pi_\alpha, \pi_\beta, \pi_k$. For $H(y | \tau) = 1 - S(y | \tau)$ with $S(y | \tau) = e^{-\lambda y^\alpha} (1 + \beta y)^{-k}$, we have $\pi_\theta(y | \tau) = \frac{\partial}{\partial \theta} H(y | \tau) = -\frac{\partial}{\partial \theta} S(y | \tau)$. Thus:

$$\pi_\lambda(y | \tau) = \frac{\partial H(y | \tau)}{\partial \lambda} = y^\alpha e^{-\lambda y^\alpha} (1 + \beta y)^{-k} \quad (42)$$

$$\pi_\alpha(y | \tau) = \frac{\partial H(y | \tau)}{\partial \alpha} = \lambda y^\alpha \ln(y) e^{-\lambda y^\alpha} (1 + \beta y)^{-k} \quad (43)$$

$$\pi_\beta(y | \tau) = \frac{\partial H(y | \tau)}{\partial \beta} = \frac{ky}{1 + \beta y} e^{-\lambda y^\alpha} (1 + \beta y)^{-k} \quad (44)$$

$$\pi_k(y | \tau) = \frac{\partial H(y | \tau)}{\partial k} = \ln(1 + \beta y) e^{-\lambda y^\alpha} (1 + \beta y)^{-k} \quad (45)$$

The quantities $\pi_\theta(y_{(k-1)} | \tau)$ have the same expressions with $y_{(k)}$ replaced by $y_{(k-1)}$. Under the standard MPS boundary convention, $\pi_\theta(y_{(0)} | \tau) = \pi_\theta(y_{(t^*+1)} | \tau) = 0$. The MPS estimators $(\hat{\lambda}^{\text{MPS}}, \hat{\alpha}^{\text{MPS}}, \hat{\beta}^{\text{MPS}}, \hat{k}^{\text{MPS}})$ are obtained by solving (S_λ) (S_k) numerically, or equivalently maximizing $M^*(\tau)$.

4.2. Estimators Based on SRS

To obtain the MPS estimators, let $x_{(1)}, \dots, x_{(t^*)}$ be an SRS of size t^* from the IWP model, arranged in increasing order: $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(t^*)}$. The spacing of a random sample of size t^* is defined by

$$\Lambda_j(\tau) = H(x_{(j)} | \tau) - H(x_{(j-1)} | \tau), \quad j = 1, 2, \dots, t^* + 1,$$

with boundary conditions $H(x_{(0)} | \tau) = 0$, $H(x_{(t^*+1)} | \tau) = 1$. For the IWP model, this can be written explicitly as:

$$\Lambda_i(\tau) = [1 - S(y_{(i)} | \tau)] - [1 - S(y_{(i-1)} | \tau)] = e^{-\lambda y_{(i-1)}^\alpha} (1 + \beta y_{(i-1)})^{-k} - e^{-\lambda y_{(i)}^\alpha} (1 + \beta y_{(i)})^{-k} \quad (46)$$

The MPS estimators $(\tilde{\lambda}^{\text{MPS}}, \tilde{\alpha}^{\text{MPS}}, \tilde{\beta}^{\text{MPS}}, \tilde{k}^{\text{MPS}})$ are obtained by maximizing the average log-spacing function. Differentiating $M_1^*(\tau)$ with respect to each parameter $\theta \in \{\lambda, \alpha, \beta, k\}$. Hence the 4-MPS estimating equations are:

$$\frac{\partial M_1^*}{\partial \lambda} = \frac{1}{t^* + 1} \sum_{i=1}^{t^*+1} \frac{\pi_\lambda(y_{(i)} | \tau) - \pi_\lambda(y_{(i-1)} | \tau)}{\Lambda_i(\tau)} = 0, \quad (47)$$

$$\frac{\partial M_1^*}{\partial \alpha} = \frac{1}{t^* + 1} \sum_{i=1}^{t^*+1} \frac{\pi_\alpha(y_{(i)} | \tau) - \pi_\alpha(y_{(i-1)} | \tau)}{\Lambda_i(\tau)} = 0, \quad (48)$$

$$\frac{\partial M_1^*}{\partial \beta} = \frac{1}{t^* + 1} \sum_{i=1}^{t^*+1} \frac{\pi_\beta(y_{(i)} | \tau) - \pi_\beta(y_{(i-1)} | \tau)}{\Lambda_i(\tau)} = 0, \quad (49)$$

$$\frac{\partial M_1^*}{\partial k} = \frac{1}{t^* + 1} \sum_{i=1}^{t^*+1} \frac{\pi_k(y_{(i)} | \tau) - \pi_k(y_{(i-1)} | \tau)}{\Lambda_i(\tau)} = 0. \quad (50)$$

Because $H(x | \tau) = 1 - S(x | \tau)$, we have $\pi_\theta(x | \tau) = -\frac{\partial}{\partial \theta} S(x | \tau)$. With $S(x | \tau) = e^{-\lambda x^\alpha} (1 + \beta x)^{-k}$, it follows:

$$\pi_\lambda(y | \tau) = \frac{\partial H(y | \tau)}{\partial \lambda} = y^\alpha e^{-\lambda y^\alpha} (1 + \beta y)^{-k}, \quad (51)$$

$$\pi_\alpha(y | \tau) = \frac{\partial H(y | \tau)}{\partial \alpha} = \lambda y^\alpha \ln(y) e^{-\lambda y^\alpha} (1 + \beta y)^{-k}, \quad (52)$$

$$\pi_\beta(y | \tau) = \frac{\partial H(y | \tau)}{\partial \beta} = \frac{ky}{1 + \beta y} e^{-\lambda y^\alpha} (1 + \beta y)^{-k}, \quad (53)$$

$$\pi_k(y | \tau) = \frac{\partial H(y | \tau)}{\partial k} = \ln(1 + \beta y) e^{-\lambda y^\alpha} (1 + \beta y)^{-k}. \quad (54)$$

The quantities $\pi_\theta(y_{(i-1)} | \tau)$ are obtained by replacing $y_{(i)}$ with $y_{(i-1)}$ in the above expressions.

5. Least Squares and Weighted Least Squares

Suppose that $X_{(1:t^*)}, \dots, X_{(t^*:t^*)}$ is the ordered measured RSS sample of size $t^* = t \times d$, where d is the number of cycles and t is the set size. Let $x_{(u:t^*)}$ denote the u th ordered observation, $u = 1, \dots, t^*$. LS and WLS estimators of the unknown parameters of the IWP distribution are obtained by minimizing the weighted sum of squared errors.

$$\downarrow_3(\tau) = \sum_{u=1}^{t^*} \phi_u \left[H(y_{(u:t^*)} | \tau) - \frac{u}{t^* + 1} \right]^2. \quad (55)$$

Equivalently, using the IWP CDF explicitly:

$$\downarrow_3(\tau) = \sum_{u=1}^{t^*} \phi_u \left[1 - e^{-\lambda y_{(u:t^*)}^\alpha (1 + \beta y_{(u:t^*)})^{-k}} - \frac{u}{t^* + 1} \right]^2. \quad (56)$$

The LS estimators correspond to $\phi_u = 1$ for all u , whereas the WLS estimators use

$$\phi_u = \frac{(t^* + 1)^2 (t^* + 2)}{u (t^* - u + 1)}. \quad (57)$$

Define the residual

$$r_u(\tau) = H(y_{(u:t^*)} | \tau) - \frac{u}{t^* + 1}. \quad (58)$$

Minimization of $\downarrow_3(\tau)$ yields the normal equations

$$\frac{\partial \downarrow_3(\tau)}{\partial \theta} = 2 \sum_{u=1}^{t^*} \phi_u r_u(\tau) \frac{\partial}{\partial \theta} H(y_{(u:t^*)} | \tau) = 0, \theta \in \{\lambda, \alpha, \beta, k\} \quad (59)$$

we obtain the system

$$\sum_{u=1}^{t^*} \phi_u \left[H(y_{(u:t^*)} | \tau) - \frac{u}{t^* + 1} \right] \pi_\lambda(y_{(u:t^*)} | \tau) = 0, \quad (60)$$

$$\sum_{u=1}^{t^*} \phi_u \left[H(y_{(u:t^*)} | \tau) - \frac{u}{t^* + 1} \right] \pi_\alpha(y_{(u:t^*)} | \tau) = 0, \quad (61)$$

$$\sum_{u=1}^{t^*} \phi_u \left[H(y_{(u:t^*)} | \tau) - \frac{u}{t^* + 1} \right] \pi_\beta(y_{(u:t^*)} | \tau) = 0, \quad (62)$$

$$\sum_{u=1}^{t^*} \phi_u \left[H(y_{(u:t^*)} | \tau) - \frac{u}{t^* + 1} \right] \pi_k(y_{(u:t^*)} | \tau) = 0. \quad (63)$$

These equations are solved numerically to obtain the LS estimates $(\hat{\lambda}^{LS}, \hat{\alpha}^{LS}, \hat{\beta}^{LS}, \hat{k}^{LS})$ (for $\phi_u = 1$) or the WLS estimates $(\hat{\lambda}^{WS}, \hat{\alpha}^{WS}, \hat{\beta}^{WS}, \hat{k}^{WS})$ (for weighted ϕ_u). Since $H(y | \tau) = 1 - S(y | \tau)$ with $S(y | \tau) = e^{-\lambda y^\alpha} (1 + \beta y)^{-k}$, we have $\pi_\theta(y | \tau) = -\frac{\partial}{\partial \theta} S(y | \tau)$, leading to

$$\pi_\lambda(y | \tau) = \frac{\partial H(y | \tau)}{\partial \lambda} = y^\alpha e^{-\lambda y^\alpha} (1 + \beta y)^{-k}, \quad (64)$$

$$\pi_\alpha(y | \tau) = \frac{\partial H(y | \tau)}{\partial \alpha} = \lambda y^\alpha \ln(y) e^{-\lambda y^\alpha} (1 + \beta y)^{-k}, \quad (65)$$

$$\pi_\beta(y | \tau) = \frac{\partial H(y | \tau)}{\partial \beta} = \frac{ky}{1 + \beta y} e^{-\lambda y^\alpha} (1 + \beta y)^{-k}, \quad (66)$$

$$\pi_k(y | \tau) = \frac{\partial H(y | \tau)}{\partial k} = \ln(1 + \beta y) e^{-\lambda y^\alpha} (1 + \beta y)^{-k} \quad (67)$$

If entropy measures $\beta_1(\tau), \dots, \beta_4(\tau)$ are available for the IWP model, then

$$\begin{aligned} \hat{\beta}_j^{LS} &= \beta_j (\hat{\lambda}^{LS}, \hat{\alpha}^{LS}, \hat{\beta}^{LS}, \hat{k}^{LS}), \\ \hat{\beta}_j^{WS} &= \beta_j (\hat{\lambda}^{WS}, \hat{\alpha}^{WS}, \hat{\beta}^{WS}, \hat{k}^{WS}), \quad j = 1, 2, 3, 4. \end{aligned}$$

The LS and WLS estimators under SRS are obtained by minimizing

$$\downarrow_3^\bullet(\tau) = \sum_{i=1}^{t^*} \phi_i \left[1 - e^{-\lambda y_{(i)}^\alpha} (1 + \beta y_{(i)})^{-k} - \frac{i}{t^* + 1} \right]^2 \quad (68)$$

The LS estimators correspond to $\phi_i = 1$, whereas the WLS estimators use

$$\phi_i = \frac{(t^* + 1)^2 (t^* + 2)}{i (t^* - i + 1)} \quad (69)$$

Then for $\theta \in \{\lambda, \alpha, \beta, k\}$

$$\sum_{i=1}^{t^*} \phi_i \left[H(y_{(i)} | \tau) - \frac{i}{t^* + 1} \right] \pi_\theta(y_{(i)} | \tau) = 0 \quad (70)$$

where $\pi_\theta(y | \tau)$ is given above. These equations are solved numerically to obtain $(\tilde{\lambda}^{LS}, \tilde{\alpha}^{LS}, \tilde{\beta}^{LS}, \tilde{k}^{LS})$ or $(\tilde{\lambda}^{WS}, \tilde{\alpha}^{WS}, \tilde{\beta}^{WS}, \tilde{k}^{WS})$.

6. Estimators of Minimum Distance for the IWP Model

We consider AD- and CVM-based minimum-distance estimators of the IWP parameters $\tau = (\lambda, \alpha, \beta, k)$, which are obtained by minimizing goodness-of-fit statistics of the distance between the sample ECDF and the fitted IWP CDF. Let $X_{(1:t^*)}, \dots, X_{(t^*:t^*)}$ be the ordered measured RSS sample of size $t^* = t \times d$. The CVM objective function is:

$$\downarrow_{CVM}(\tau) = \frac{1}{12t^*} + \sum_{j=1}^{t^*} \left[H(x_{(j:t^*)} | \tau) - \frac{2j-1}{2t^*} \right]^2 \quad (71)$$

Substituting the IWP CDF $H(x | \tau) = 1 - e^{-\lambda x^{-\alpha}} (1 + \beta x)^{-k}$ yields:

$$\downarrow_{CVM}(\tau) = \frac{1}{12t^*} + \sum_{j=1}^{t^*} \left[1 - e^{-\lambda x_{(j:t^*)}^{-\alpha}} (1 + \beta x_{(j:t^*)})^{-k} - \frac{2j-1}{2t^*} \right]^2 \quad (72)$$

The RSS-CVM estimators are obtained by: $(\hat{\lambda}^{CVM}, \hat{\alpha}^{CVM}, \hat{\beta}^{CVM}, \hat{k}^{CVM}) = \arg \min_{\lambda, \alpha, \beta, k} \downarrow_{CVM}(\tau)$. Let $x_{(1)}, \dots, x_{(t^*)}$ be an ordered SRS of size t^* . The SRS-CVM objective is:

$$\downarrow_{CVM}^{\bullet}(\tau) = \frac{1}{12t^*} + \sum_{j=1}^{t^*} \left[1 - e^{-\lambda x_{(j)}^{-\alpha}} (1 + \beta x_{(j)})^{-k} - \frac{2j-1}{2t^*} \right]^2. \quad (73)$$

The SRS-CVM estimators are: $(\tilde{\lambda}^{CVM}, \tilde{\alpha}^{CVM}, \tilde{\beta}^{CVM}, \tilde{k}^{CVM}) = \arg \min_{\lambda, \alpha, \beta, k} \downarrow_{CVM}^{\bullet}(\tau)$. If entropy measures $\beta_1(\tau), \dots, \beta_4(\tau)$ are available for the IWP model, then $\hat{\beta}_j^{CVM} = \beta_j(\hat{\lambda}^{CVM}, \hat{\alpha}^{CVM}, \hat{\beta}^{CVM}, \hat{k}^{CVM})$, $j = 1, 2, 3, 4$. Let $n = t^*$ and let $x_{(1)} \leq \dots \leq x_{(n)}$. The Anderson-Darling objective is:

$$\downarrow_{AD}(\tau) = -n - \frac{1}{n} \sum_{j=1}^n (2j-1) [\ln H(x_{(j)} | \tau) + \ln (1 - H(x_{(n+1-j)} | \tau))] \quad (74)$$

Since $1 - H(y | \tau) = S(y | \tau)$ for the IWP model, where $S(y | \tau) = e^{-\lambda y^{-\alpha}} (1 + \beta y)^{-k}$, the objective can also be written as:

$$\downarrow_{AD}(\tau) = -n - \frac{1}{n} \sum_{j=1}^n (2j-1) \left[\ln \left(1 - e^{-\lambda y_{(j)}^{-\alpha}} (1 + \beta y_{(j)})^{-k} \right) + \ln \left(e^{-\lambda y_{(n+1-j)}^{-\alpha}} (1 + \beta y_{(n+1-j)})^{-k} \right) \right]. \quad (75)$$

Let $Y_{(1:t^*)}, \dots, Y_{(t^*:t^*)}$ be the ordered measured RSS sample and set $n = t^*$. Using Equation 9 and Equation 10, the RSS-AD estimators are $(\hat{\lambda}^{AD}, \hat{\alpha}^{AD}, \hat{\beta}^{AD}, \hat{k}^{AD}) = \arg \min_{\lambda, \alpha, \beta, k} \downarrow_{AD}(\tau)$ [57].

7. Estimation of Genetic Algorithm-Random Forest (GA-RF) Proposed Method

Classical inferential procedures that will be addressed in this research (ML, MPS, LS, WLS, AD, CVM, and KM) are all parametric and based on objective functions based on the assumed distribution. In the case of flexible multi-parameter models like the inverted Weibull-Pareto (IWP), numerical optimization might be initialization-sensitive, and can have flat or multi-modal objective surfaces. We will solve these problems by suggesting a

smart hybrid model that combines (i) the Random Forest (RF) surrogate model and (ii) a Genetic Algorithm (GA). The RF offers a rapid, information-based prediction, and serves as a global surrogate and GA offers a derivative-free global optimization mechanism. Notably, the suggested framework uses the nested GA: an outer GA optimizes RF hyperparameters, and an inner GA optimizes the IWP parameter estimates on the RF-guided initializer. Given observed data $y = (y_1, \dots, y_n)$, The reason for using the GA–RF approach is the computational problems that stem from the parameter estimation of the IWP distribution. There are a number of parameters in the IWP model that collaboratively determine scale, shape, skewness and tail behaviour. The consequence is that the likelihood-based, spacing based, least-squares, and minimum-distance objective functions can be very non-linear and have flat regions and/or multiple local optima. Thus, conventional numerical optimization techniques (like Newton–Raphson, quasi-Newton, or minimization by gradient) can be sensitive to the choice of initial values and can fail to converge to a stable solution, particularly in small sample sizes, the presence of heavy tailed data, or imperfectly ranked samples.

The suggested GA–RF method is designed to alleviate such limitations. The latter is employed as a surrogate learner which maps informative sample features like moments or empirical quantiles to approximate the values of the parameters of IWP. This gives a data driven initial estimate, rather than a guess. The Genetic Algorithm component, in turn, performs derivative-free global optimization for refining estimates of the parameters in the log-parameter space. GA does not need to have a derivative of the objective function, and is not as likely to get stuck in poor local optima as conventional gradient-based methods are. Therefore, the GA–RF scheme is a global stochastic optimization with the use of a predictive initialization.

However, there are some drawbacks to each estimation method. Under regularity conditions, ML can be efficient, however, it can be sensitive to the starting values and extreme observations. MPS can provide more stability in certain scenarios of irregular likelihoods, but may be sensitive to the very small spacings near the sample extremes. LS and WLS are not difficult to compute, but require plots and weights to be chosen. Both AD and CVM are minimum-distance estimates that are useful, but the estimate from AD may be more sensitive to the observations in the tails of the distribution, and CVM may be less sensitive toward differences in the tails. Even though it is easy to interpret, KM could be affected by the largest empirical distribution discrepancy. In RSS, another source of instability can be introduced when the rankings are not perfect, since errors in ranking will diminish the information advantage of RSS, and increase the variability of the estimation procedure.

This means that the GA–RF estimator is not intended to be a general solution for the classical estimators, but rather a method to supplement them when they do not perform well in certain situations. It comes with a significant improvement in numerical stability because of data-driven initialisation and derivative-free global refinement. It is, however, much more computationally intensive than classical approach, and relies on the training ranges, the hyperparameters of the RF, the settings of the GA, and the representativeness of the simulated training data. This is the reason why its performance is compared to that of ML, MPS, KM, LS, WLS, AD and CVM in both SRS and RSS design. The log-likelihood is [57, 58]:

$$\ell(\tau \mid \mathbf{y}) = \sum_{i=1}^n \log f(y_i \mid \tau). \quad (76)$$

The target criterion based on likelihood is the negative log-likelihood $\mathcal{J}(\tau) = -\ell(\tau \mid \mathbf{y})$. The goal is to obtain $\hat{\tau}$ that minimizes $\mathcal{J}(\tau)$, subject to $\lambda, \alpha, \beta, k > 0$. To link the samples to the parameters, each dataset y is mapped to a feature vector $s(y)$. A robust, tail-sensitive choice is $s(\mathbf{y}) = (\bar{y}, \text{Var}(y), \text{Skew}(y), \text{Kurt}(y), Q_{0.10}, Q_{0.25}, Q_{0.50}, Q_{0.75}, Q_{0.90}, Q_{0.95}, \log y, \text{Var}(\log y))$, where Q_p denotes the empirical p -quantile. To ensure positivity during learning and optimization, we use the log-parameterization $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (\log \lambda, \log \alpha, \log \beta, \log k)$, $\tau = \exp(\theta) = (e^{\theta_1}, e^{\theta_2}, e^{\theta_3}, e^{\theta_4})$. The RF is trained to predict θ from $s(\mathbf{y})$. Since the real parameters are not known in real data, the RF is trained with synthetic data produced using the IWP model in a prescribed parameter space Ω_{train} . For $m = 1, \dots, M$:

1. Sample $\tau^{(m)} = (\lambda^{(m)}, \alpha^{(m)}, \beta^{(m)}, k^{(m)}) \in \Omega_{\text{train}}$.

2. Generate a dataset $\mathbf{y}^{(m)}$ of size n from $\text{IWP}(\tau^{(m)})$.
3. Compute features $s^{(m)} = s(\mathbf{y}^{(m)})$.
4. Store the training pair $(s^{(m)}, \theta^{(m)})$, where $\theta^{(m)} = \log(\tau^{(m)})$.

The RF model is then fitted to approximate $\theta \approx g_{\text{RF}}(s)$. Let the RF hyperparameter vector be

$$h = (N_{\text{trees}}, \max_{\text{depth}}, \min_{\text{split}}, \min_{\text{leaf}}, \max_{\text{features}}). \quad (77)$$

The outer GA finds the hyperparameter setting h which minimizes a validation loss:

$$\mathcal{L}(h) = \frac{1}{M_{\text{val}}} \sum_{m \in \mathcal{V}} \left\| \theta^{(m)} - \hat{\theta}^{(m)}(h) \right\|_2^2, \quad \hat{\theta}^{(m)}(h) = g_{\text{RF}}(s^{(m)}; h), \quad (78)$$

where $\mathcal{V}$ is a set of validation subsets of simulated datasets. The hyperparameters that are tuned are $h^* = \arg \min_h \mathcal{L}(h)$. The RF is then refit with h^* on the full training set (or training-validation) with h . Given observed data $\mathbf{y}$, compute $s_{\text{obs}} = s(\mathbf{y})$. Using the tuned RF, obtain an initial estimate

$$\hat{\theta}_0 = g_{\text{RF}}(s_{\text{obs}}; h^*), \quad \hat{\tau}_0 = \exp(\hat{\theta}_0). \quad (79)$$

The inner GA conducts a limited global search in the log-parameter space about $\hat{\theta}_0$ by minimizing

$$\mathcal{J}(\theta) = - \sum_{i=1}^n \log f(y_i | e^{\theta_1}, e^{\theta_2}, e^{\theta_3}, e^{\theta_4}) + \mathcal{P}(\theta), \quad (80)$$

where $\mathcal{P}(\theta)$ is a weak penalty which may be optional (e.g., quadratic regularization of the extreme log-parameters) to aid stability. The final estimator is $\hat{\theta} = \arg \min_{\theta \in \Theta} \mathcal{J}(\theta)$, $\hat{\tau} = \exp(\hat{\theta})$.

Algorithm: Nested GA-RF estimation of IWP parameters

Input: Observed Data $\mathbf{y}$; training range Ω_{train} ; number of simulated datasets M ; RF hyperparameter search space $\mathcal{H}$; GA settings (population size, generations, crossover/mutation); log-parameter bounds Θ .

1. Simulate training datasets: generate $(\mathbf{y}^{(m)}, \tau^{(m)})$ from IWP over Ω_{train} .
2. Compute features and targets: $s^{(m)} = s(\mathbf{y}^{(m)})$ and $\theta^{(m)} = \log(\tau^{(m)})$.
3. Outer GA (hyperparameter tuning):
 - Initialize a GA population over $h \in \mathcal{H}$.
 - For each candidate h , train an RF on the training subset and compute $\mathcal{L}(h)$ on a validation subset.
 - Evolve the GA until convergence and return h^* .
4. Fit final RF: train $g_{\text{RF}}(\cdot; h^*)$ on all simulated training data.
5. Warm-start for observed data: Compute $s_{\text{obs}} = s(\mathbf{y})$, then $\hat{\theta}_0 = g_{\text{RF}}(s_{\text{obs}}; h^*)$.
 - Initialize a GA population in θ -space around $\hat{\theta}_0$.
 - Minimize $\mathcal{J}(\theta)$ to obtain $\hat{\theta}$.
6. Return estimates: $\hat{\tau} = \exp(\hat{\theta})$.

7.1. Computational Settings for the GA-RF Estimator

In order to guarantee the reproducibility of the proposed GA-RF estimator, the computational parameters, employed in the numerical study, are fixed as follows. The synthetic datasets from the IWP distribution in the parameter space were used to train the RF surrogate model.

$$\lambda \in [0.10, 5.00], \alpha \in [0.20, 5.00], \beta \in [0.05, 3.00], k \in [0.20, 5.00].$$

The ranges chosen to represent these parameters were chosen to represent the parameters considered in the simulation study and also to have moderate variation in scale, shape, and tail behaviour. The positive nature of the IWP parameters was ensured by training the RF to predict the log-parameter vector.

$$\eta = (\log \lambda, \log \alpha, \log \beta, \log k).$$

For each sample-size configuration, $N_{sim}=10000$ synthetic datasets were generated. The data were divided into 80% training data and 20% validation data. For each generated dataset, the feature vector included the sample mean, standard deviation, median, interquartile range, coefficient of variation, skewness, kurtosis, minimum, maximum, and empirical quantiles at probabilities.

$$0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95$$

The GA–RF estimator was trained and validated with different synthetic datasets to avoid overfitting. The hyperparameters of the RF surrogate were optimized on the validation subset to minimize prediction error, while it was fitted on the training subset. Then, the final GA–RF estimates were tested on Monte Carlo scenarios that were not included in the RF training process. The separation between training, tuning and evaluation helps to ensure that the performance reported is a general predictive ability rather than a memorization of simulated training samples. The usefulness of the GA–RF method is evaluated against the other estimation methods, namely ML, MPS, KM, LS, WLS, AD and CVM, based on estimation accuracy, computational stability performance and goodness-of-fit performance. It is useful if it yields values of AB and MSE similar to or lower than those of GA, doesn't result in estimations in the boundary regions, has better convergence stability, or yields good goodness-of-fit values. It thus improves the accuracy of the points in specific situations, while at the same time providing the numerical robustness of the flexible four-parameter IWP model.

The features were chosen to reflect location, dispersion, skewness and tail behaviour in the samples of the IWP. The hyperparameters of the RF were tuned using the outer GA. The values of the hyperparameter space for the RF model were $n_{trees} \in \{200, 500, 1000\}$, $m_{try} \in \{\sqrt{p}, 0.5p, p\}$, $d_{max} \in \{5, 10, 20, \text{None}\}$, $n_{min} \in \{1, 3, 5, 10\}$.

Where p is the number of input features, n_{trees} is the number of trees, m_{try} is the number of candidate predictors considered at each split, d_{max} is the maximum tree depth, and n_{min} is the minimum number of observations in a terminal node. The RF model was selected by minimizing the validation mean squared error of the predicted log-parameters.

The outer GA used a population size of 40 chromosomes and was run for a maximum of 50 generations. The tournament selection with tournament size 3 was employed. The crossover probability was 0.80, the mutation probability was 0.10 and elitism was implemented by keeping the 2 best chromosomes at each generation. The outer GA terminated when the maximum number of generations was reached or if the validation loss was not decreased by 10^{-5} for 10 generations in a row. Once the RF was tuned, the final RF model was trained on the entire synthetic training set with the hyperparameters that were selected. For an observed dataset, the fitted RF was used to give the initial log-parameter estimate, η_{RF} . The inner GA then fine-tuned this estimate by maximizing the negative log-likelihood in the space of the log-parameters. The GA search was limited to the inner bounds on the log-parameters corresponding to the training ranges above for the inner GA search. Initial population was generated around the value of η_{RF} with a normal perturbation of standard deviation 0.10 on the log scale.

The inner GA had 60 individuals and a maximum number of generations of 100. The number of generations was set at 100, the probability of crossover was 0.85, the probability of mutation was 0.15, and elitism was set to keep the top two candidate solutions at each generation. A Gaussian perturbation was added to the chosen log-parameter to generate mutation. The inner GA stopped if the relative difference log-likelihood in the negative log-likelihood was less than 10^{-6} for the last 20 generations or after the maximum number of generations was

reached. The optimized log-parameter vector was back-transformed to get the final estimate $\hat{\theta} = (e^{\hat{\eta}_1}, e^{\hat{\eta}_2}, e^{\hat{\eta}_3}, e^{\hat{\eta}_4})$.

To ensure reproducible results, a fixed random seed was used for all random number generation in the simulation study and training of the GA-RF. The computational settings were identical for the SRS and RSS, with the differences in the sampling designs being the differences in the sampling information only.

7.2. Computational Efficiency and Running-Time Comparison

The computational cost of the proposed GA–RF estimator was compared with the running times of the conventional estimators, and the same simulation environment was used for these times. All methods were performed on the same programming platform and on the same computer to make it easier to compare methods. The reported running time for each simulation scenario was the average CPU time of the Monte Carlo replications. The classical estimators such as ML, MPS, KM, LS, WLS, AD or CVM generally were less time consuming, since each of them entailed the numerical optimization of just one objective function. The GA–RF estimator, on the other hand, consists of two computational steps: (1) training the RF surrogate model with synthetic datasets generated from the IWP distribution; (2) derivative-free refinement of the parameter estimates by the GA in the log-parameter space. Thus, it is likely that the GA–RF method will involve a higher start-up computational cost compared to classical methods.

This extra complexity in computation is somewhat compensated for by the greater numerical stability of the GA–RF procedure. The RF component gives an initial estimate based on the data, which avoids the need to use arbitrary initial values, and the GA component is designed to refine the solution stochastically at a global level, and less sensitive to non-smooth, flat, or multimodal objective surfaces. Therefore, GA–RF is an alternative that is more stable and robust, especially when the samples are heavy tailed, have a small number of samples, and/or the RSS conditions are not optimal, and should be considered a more robust and computationally demanding alternative. A summary of the running-time comparison is given in Table 2. The average CPU time for each estimator in representative SRS and RSS cases is reported. The results indicate that the classical methods are faster; however, the training stage and the global search stage are extra sources of computational time for GA–RF. However, GA–RF is still computationally feasible for the sample size sizes that are being discussed in this study and yields competitive estimation and goodness-of-fit performance.

Table 2. Average computational time of the estimation methods under representative simulation settings

Sampling Design	Sample Size (t^*)	ML	MPS	KM	LS	WLS	AD	CVM	GA–RF
SRS	20	0.046	0.020	0.015	0.022	0.027	0.032	0.022	0.079
SRS	80	0.017	0.019	0.020	0.028	0.027	0.033	0.026	0.132
SRS	200	0.025	0.022	0.021	0.025	0.033	0.043	0.030	0.242
RSS ($\rho = 0.7$)	20	0.043	0.022	0.012	0.019	0.024	0.030	0.021	0.085
RSS ($\rho = 0.9$)	80	0.029	0.022	0.012	0.024	0.028	0.033	0.028	0.139
RSS ($\rho = 1$)	200	0.037	0.023	0.021	0.024	0.029	0.039	0.028	0.256

8. Numerical Evaluation and Fuzzy Ranking Quality

The finite-sample performance of the proposed estimators of the inverted Weibull-Pareto (IWP) model under Ranked Set Sampling (RSS) and Simple Random Sampling (SRS) is tested in this section with the help of fuzzy-Monte Carlo simulation framework. Besides variability in sampling, the design has (i) epistemic uncertainty in the model parameters and (ii) uncertainty in the quality of ranking. Absolute Bias (AB), Mean Squared Error (MSE), and Relative Efficiency (RE) are used to summarize estimator performance, reported as interval-valued quantities, in the form of interval-valued α -cuts. The following flowchart gives a step-by-step process of creating

imperfectly fuzzy ranked set samples of the IWP distribution with the Probability Integral Transform (PIT) and the RSS concomitant based ranking mechanism, as outlined in Steps 1-6 of the simulation design.

Step 1. Sampling configurations

To explore the influence of sample size with equal measurement costs, we examine a fixed set size t and different cycle numbers d . Specifically:

$$t = 3, 4, 5, 6, d \in \{5, 10, 20, 30, 50\} \quad (81)$$

So that the number of measured units is

$$t^* = t \times d \in \{20, 40, 80, 120, 200\} \quad (82)$$

Step 2. IWP parameter scenarios (Sets 1-4)

To ensure that we cover a range of different distributional shapes, we consider four typical parameter settings for the IWP model:

$$\text{Set 1} \equiv (\lambda = 0.60, \alpha = 1.20, \beta = 0.50, k = 1.00),$$

$$\text{Set 2} \equiv (\lambda = 0.35, \alpha = 2.00, \beta = 0.80, k = 2.00),$$

$$\text{Set 3} \equiv (\lambda = 0.10, \alpha = 0.70, \beta = 0.40, k = 0.80),$$

$$\text{Set 4} \equiv (\lambda = 0.25, \alpha = 1.50, \beta = 1.20, k = 3.00)$$

The parameter settings were chosen not for any particular numerical combination, but so as to reflect a variety of distributional and reliability conditions. The parameters related to Weibull have the main influence on the scale and shape of the early-life and aging component, while the parameters related to Pareto influence the scale and heaviness of the upper tail. The configurations chosen are therefore relatively moderate and more extreme tail behaviours and varying amounts of skewness and hazard-rate variation. These scenarios are meant to illustrate reliability settings where the reliability is relatively stable across the lifetime through to cases where a few extreme observations are recorded in the lifetime, and have significant impact on estimation and entropy. The combinations of parameters are also consistent with the representative PDF and HRF patterns shown in Figure 2. In this context this design is important because besides comparing the performance of estimators for various sample sizes and ranking qualities, it is also important to study their numerical and statistical properties when different distributional shapes are employed. Sensitivity to the tail parameters is crucial since the generalized entropies used in this study are highly sensitive to tail behaviour and distributional spread. These sets are chosen to represent different regimes (e.g. slower vs. faster decay, different impact of the Pareto-type component $(1 + \beta y)^{-k}$), thus providing a meaningful stress-test of the estimators under different data-generating conditions. The way these sets are utilized in the simulation. The entire simulation experiment is carried out under each set $s \in \{1, 2, 3, 4\}$. Specifically, we simulate each set in:

- sample-design configurations

$$(t, d) \Rightarrow t^* = t \times d \in \{20, 40, 80, 120, 200\}$$

- ranking regimes (crisp or fuzzy)

$$\rho \in \{1, 0.9, 0.7, 0\} \text{ or } \tilde{\rho},$$

where $\rho = 1$ denotes perfect ranking, $\rho \in \{0.9, 0.7\}$ denotes imperfect ranking, and $\rho = 0$ corresponds to the SRS-equivalent baseline,

- estimation methods (ML/MPS/LS/WLS/AD/CVM/KM).

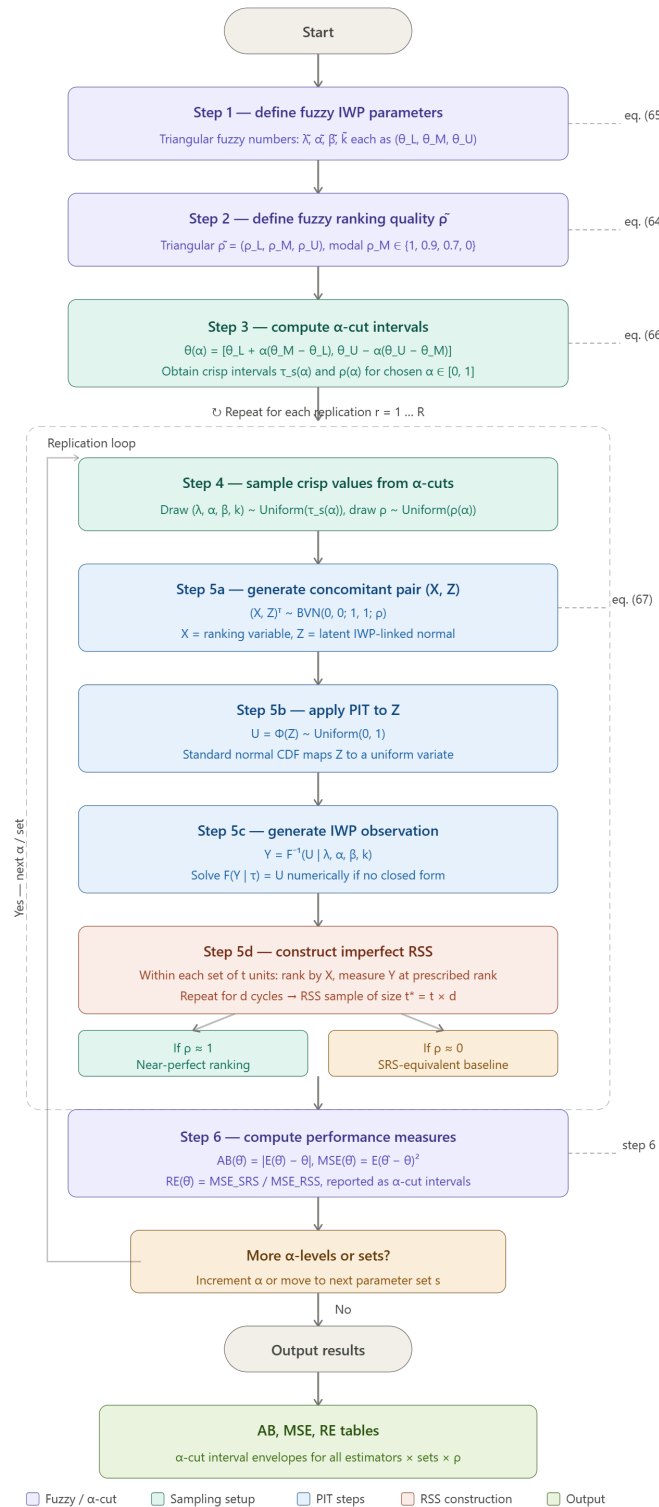


Figure 2. Flowchart for generating imperfectly fuzzy (RSS) from the (IWP) distribution using the (PIT)

Step 3. Fuzzy ranking quality

The imperfect ranking in RSS is controlled by a correlation parameter $\rho \in [0, 1]$. In the fuzzy design, ranking quality is represented by a fuzzy number $\tilde{\rho}$ (e.g., triangular):

$$\tilde{\rho} = (\rho_L, \rho_M, \rho_U), \quad 0 \leq \rho_L \leq \rho_M \leq \rho_U \leq 1,$$

where ρ_M is the modal ranking level and $[\rho_L, \rho_U]$ captures uncertainty. For comparability with the classical simulation design, we consider fuzzy regimes centred at $\rho_M \in \{1, 0.9, 0.7\}$, while $\rho = 0$ is retained as the SRS baseline. The α -cut interval of the fuzzy ranking-quality parameter is a range of ranking-quality values that can be considered possible to accommodate the given membership degree within the proposed framework. These intervals should not be interpreted as confidence intervals for the entropy estimators. The fuzzy framework does not only vary a fixed value of the sensitivity parameter, but also provides a quantification of the impact of imprecise ranking information, resulting in an uncertainty envelope for measures like AB, MSE and RE, for a given value of ρ when it is represented by a fuzzy number. It is a different kind of sensitivity analysis since the uncertainty in quality ranking is clearly reflected and carried forward using the fuzzy Monte Carlo process. The ranking uncertainty is added by using a fuzzy ranking-quality parameter $\tilde{\rho}$. For imperfect RSS, ρ is the magnitude of the correlation between the true and auxiliary ranking variables. The ranking-quality parameter is fuzzy as represented by

$$\tilde{\rho} = (\rho_L, \rho_0, \rho_U),$$

where ρ_0 is the modal ranking quality and ρ_L and ρ_U define the uncertainty range around it. Its α -cut interval is

$$[\rho]_\alpha = [\rho_L + \alpha(\rho_0 - \rho_L), \rho_U - \alpha(\rho_U - \rho_0)].$$

The cases $\rho = 1$, $\rho = 0.9$, $\rho = 0.7$, and $\rho = 0$ correspond to perfect RSS, high-quality imperfect RSS, moderate imperfect RSS, and the SRS-equivalent baseline, respectively. A value of ρ is chosen from the relevant α -cut interval and the value of the concomitant ranking variable is obtained by a correlated normal process, for each replication.

$$Y = \rho Z + \sqrt{1 - \rho^2} \varepsilon$$

where Z and ε are independent standard normal variables. The units are sorted by the accompanying variable Y , and the actual values measured are obtained by applying the probability integral transform to the IWP distribution. So, $\rho = 1$ indicates perfect ranking, lower positive values of ρ indicate ranking errors and $\rho = 0$ indicates that there is no useful ranking information. Absolute bias, mean squared error and relative efficiency are calculated for each α -level and each scenario of simulation.

Step 4. Fuzzy parameters and α -cut implementation

The parameter vector is considered to be fuzzy in order to add epistemic uncertainty in the IWP parameters. We set a set of fuzzy neighbourhoods $\tilde{\tau}_s$ around the modal value τ_s . As an illustration, with triangular fuzzy numbers:

$$\begin{aligned} \tilde{\lambda}_s &= (\lambda_{s,L}, \lambda_{s,M}, \lambda_{s,U}), & \tilde{\alpha}_s &= (\alpha_{s,L}, \alpha_{s,M}, \alpha_{s,U}), \\ \tilde{\beta}_s &= (\beta_{s,L}, \beta_{s,M}, \beta_{s,U}), & \tilde{k}_s &= (k_{s,L}, k_{s,M}, k_{s,U}). \end{aligned} \quad (83)$$

We propagate fuzziness using the α -cut method. For a triangular fuzzy number $\tilde{\theta} = (\theta_L, \theta_M, \theta_U)$, the α -cut interval is

$$\tilde{\theta}^{(\alpha)} = [\theta_L + \alpha(\theta_M - \theta_L), \theta_U - \alpha(\theta_U - \theta_M)], \quad \alpha \in [0, 1]. \quad (84)$$

At each α -level, we obtain crisp intervals $\tilde{\tau}_s^{(\alpha)}$ and $\tilde{\rho}^{(\alpha)}$; parameters are then sampled from these intervals to generate datasets. The epistemic uncertainty of the IWP parameters and ranking quality is captured as triangular fuzzy numbers in order to be able to fully reproduce the fuzzy Monte Carlo framework. If the value of an uncertain

quantity is denoted as θ , then the triangular fuzzy number is represented as:

$$\tilde{\theta} = (\theta_L, \theta_0, \theta_U),$$

where θ_L, θ_0 , and θ_U The lower bound, the mode and the upper bound, respectively. It has the following membership function:

$$\mu_{\tilde{\theta}}(\theta) = \begin{cases} 0, & \theta < \theta_L \\ \frac{\theta - \theta_L}{\theta_0 - \theta_L}, & \theta_L \leq \theta \leq \theta_0 \\ \frac{\theta_U - \theta}{\theta_U - \theta_0}, & \theta_0 \leq \theta \leq \theta_U \\ 0, & \theta > \theta_U \end{cases}$$

For each model parameter $\theta_j \in \{\lambda, \alpha, \beta, k\}$, the fuzzy parameter is specified as

$$\tilde{\theta}_j = (\theta_j^0 (1 - \delta_j), \theta_j^0, \theta_j^0 (1 + \delta_j)),$$

where θ_j^0 is the modal parameter value in the corresponding simulation set and δ_j controls the degree of epistemic uncertainty. The α -cut interval of $\tilde{\theta}_j$ is given by

$$[\theta_j]_{\alpha} = [\theta_j^0 (1 - \delta_j) + \alpha \{\theta_j^0 - \theta_j^0 (1 - \delta_j)\}, \theta_j^0 (1 + \delta_j) - \alpha \{\theta_j^0 (1 + \delta_j) - \theta_j^0\}],$$

for $0 \leq \alpha \leq 1$. In the simulation study, the α -cut levels are taken as

$$\alpha \in \{0, 0.1, 0.2, \dots, 0.9, 1.0\}.$$

At each α -level, crisp parameter values are generated from the corresponding α -cut intervals and are then used to simulate IWP samples.

Step 5. Data generation and imperfect ranking mechanism (IWP)

For each replication, each set s , each configuration (t, d) , each ranking regime, and each α -level:

1. Sample crisp values from α -cuts. Draw $(\lambda, \alpha, \beta, k)$ from $\tilde{\tau}_s^{(\alpha)}$ and draw ρ from $\tilde{\rho}^{(\alpha)}$.
2. Generate correlated normal (concomitant mechanism). Generate (X, Z) from

$$\begin{pmatrix} X \\ Z \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right), \quad (85)$$

where X is the concomitant variable (used for ranking) and Z is a latent normal variable.

3. Apply probability integral transform (PIT). Set

$$U = \Phi(Z) \sim Uniform(0, 1)$$

where $\Phi(\cdot)$ is the standard normal CDF.

4. Generate IWP observations. Obtain

$$Y = F^{-1}(U | \tau).$$

If a closed form of F^{-1} is unavailable, compute Y numerically as the unique solution of $F(y | \tau) = U$ for $y > 0$.

5. Construct RSS in imperfect ranking. Design RSS with set size t and cycles d where each unit is ranked on X and the rank obtained in the desired order is noted, thus generating an RSS sample of size $t^* = td$. Ranking is perfect when $\rho \approx 1$ and becomes more imperfect as ρ decreases. The SRS baseline is generated by sampling t^* independent Y -values from the IWP model (equivalently $\rho = 0$).

Step 6. Performance measures under fuzziness

For any estimate and $\theta \in \{\lambda, \alpha, \beta, k, \beta_1, \beta_2, \beta_3, \beta_4\}$, performance is assessed using

$$AB(\hat{\theta}) = |E(\hat{\theta}) - \theta|, \quad MSE(\hat{\theta}) = E \left[(\hat{\theta} - \theta)^2 \right],$$

and relative efficiency

$$RE(\theta) = \frac{MSE_{SRS}(\hat{\theta})}{MSE_{RSS}(\hat{\theta})}$$

In the fuzzy framework, AB, MSE, and RE are evaluated across α -cuts and reported as interval envelopes, e.g.

$$MSE^{(\alpha)}(\hat{\theta}) \in \left[MSE_L^{(\alpha)}(\hat{\theta}), MSE_U^{(\alpha)}(\hat{\theta}) \right],$$

with analogous expressions for AB and RE. The interval-valued summaries of the fuzzy performance measures are reported across the α -cuts. If only one representative value is needed, defuzzification is done by the centroid method. Let α be a triangular fuzzy number. $A = (A_L, A_0, A_U)$, the centroid defuzzified value is:

$$A^* = \frac{A_L + A_0 + A_U}{3}$$

For interval-valued α -cut results, the final summary may also be reported as the midpoint

$$A_{\alpha}^{mid} = \frac{A_{\alpha}^L + A_{\alpha}^U}{2},$$

together with the interval bounds to preserve the uncertainty information. The results from the simulations are presented as point estimates and as the α -cut interval summaries for AB, MSE and RE. These intervals represent the change in performance of the estimators when the ranking-quality parameter and the model parameters of IWP are uncertain. Larger intervals mean a greater sensitivity to fuzzy ranking uncertainty, while smaller intervals mean more stable performance. The fuzzy-based results thus offer more than just the crisp ranking of the cases, and can demonstrate the tolerance of the estimators to uncertain ranking situations.

- The above criteria offer quantifiable measures of the estimators' accuracy and precision. Taken together, they allow for a detailed examination of the performance of the estimators and provide compelling evidence of the efficiency and practical feasibility of each of the estimation techniques for the model at hand.
- Steps (1) and (2) are repeated 5,000 times to obtain robust empirical estimates of the performance measures.
- The results of the numerical study are reported in [Table 13 - Table 24](#) and [Figure 3 - Figure 9](#), which altogether give a full picture of the performance of the estimators under the studied scenarios.
- The full numerical simulation results on the performance of parameter estimation are provided in the Appendix to preserve the flow of the main text and to focus on interpretation. In particular, this study compares the absolute bias and mean squared error of the IWP parameter estimates of the various estimation methods outlined for the four sets (Sets 1–4) of data for SRS, imperfect RSS and perfect RSS. The tables in the appendix give the details of the numbers that form the basis for the main findings discussed in this section. In the main text, the predominant trends across these tables are summarized, but the numerical entries are not reproduced. In general, the results indicate that the more samples that are measured, the more accurate the estimates will be, that RSS will outperform SRS when ranking quality is high, and that the advantages of RSS will decrease as ranking quality decreases.
- As illustrated in [Table 13 - Table 24](#) and [Figure 3 - Figure 9](#), we note the following:
 1. Consistency with increasing measurement effort. For both SRS ($\rho = 0$) and RSS ($\rho \in \{1, 0.9, 0.7\}$), the absolute bias (AB) and mean squared error (MSE) are decreasing with the increase of the effective measured sample size t^* , suggesting the expected convergence to the true parameter/entropy values.

2. Entropy estimators show the same monotone improvement. For all four entropy measures ($\beta_1, \beta_2, \beta_3, \beta_4$), AB and MSE generally decline with t^* under both perfect and imperfect RSS and under SRS (See the Appendix).
3. Different targets (parameters vs. entropies) lead to different rankings. In the tables of entropies, the KM method is not always the best; in a number of cases, likelihood/distance-based methods (most often AD, ML, or MPS) are competitive for some β_j and t^* . Thus, KM can be considered competitive in many, but not all, cases.
4. The ranking quality is important as expected. Comparing SRS ($\rho = 0$), imperfect RSS ($\rho = 0.7, 0.9$) and perfect RSS ($\rho = 1$), we generally observe increased accuracy (smaller AB/MSE), with improved ranking resulting in more efficient estimation for the same cost.
5. Relative efficiency (RE) confirms gains of RSS. The RE summaries confirm the potential of RSS over SRS at fixed sample size (measured), with the greatest gains under perfect ranking (parameters: [Table 9](#) - [Table 10](#), entropies: [Table 5](#) - [Table 11](#)).
6. Error magnitude scales with the entropy level. Across sets with larger entropy values, MSEs tend to be larger, while smaller true entropy levels are associated with smaller MSEs; this pattern appears consistently across ρ regimes and sample sizes in the entropy AB/MSE tables.

Within-set directional claims should be tied to the simulation sets. More granular statements (e.g., how $\hat{\alpha}$ and $\hat{\beta}$ change as α or β vary, or differences between $\nu = 0.7$ and $\nu = 2$) should be reported only where the corresponding set definitions and tables directly support them.

8.1. Analytical interpretation of the simulation results

A combination of the interaction of the four factors sample size, ranking quality, tail behaviour and numerical optimization stability can explain the observed simulation patterns. The general decrease in AB and MSE as the sample size n measured is increased. The expectation is that it will increase as samples get larger, as this will give more information about the IWP distribution and will decrease the random sampling variability in the IWP distribution. Because of the plug-in nature of the entropy estimators as a function of the estimated model parameters, better estimates of the model parameters result directly in better estimates of the entropy. This is why the estimates of the parameters and the entropy tend to be more stable, as the measurement is increased. The improvement of RSS over SRS is primarily because of ordering information added prior to measuring. If the ranking is perfect or very accurate, the observations that are measured by RSS are more evenly distributed across the distribution than the ordinary SRS observations. This enhances the information for the central part of the IWP distribution, and particularly for heavy-tailed models, the information for the tail of the distribution. Therefore, in certain instances, parameter estimates and entropy estimates may be lower under RSS than under SRS when the quality of the samples is good. If the quality of the unranked units decreases, however, the rank positions of the units selected no longer closely match the positions for which they were ranked. This reduces the information advantage of RSS and why ranking's efficiency decline under sub-optimal ranking.

The objective functions of each estimation method are also of different structure, which affects the difference between the methods as well. When the model is correctly specified, likelihood-based methods can do a good job, but can be sensitive to high outliers and to the initial values due to the heavy tails of the IWP distribution and the fact that several of the parameters in the model interact. MPS can yield more numerical stability in certain situations because it is computed based on spacings of the distribution instead of directly on the likelihood; however, very small spacings in the tails of the sample can still have an impact on MPS. Both LS and WLS methods are relatively easy to compute but their accuracy relies on the approximation of the empirical distribution function by the fitted distribution function, which can be poor. The minimum-distance methods are the AD and CVM estimators, the latter emphasizing the fit more globally over the distribution, and the former more toward the tails. Hence, there is no classical estimators which outperform all others in any setting of the parameters. The behaviour of the entropy estimates shows sensitivity of the entropy measures to tail behaviour and parameter uncertainty. In this regard, the reasons why MSE may be larger with higher entropy settings for some parameter settings are due to the fact that the estimated tail parameters can vary significantly for small changes in the

parameter settings. This type of pattern is in line with the idea that measures of generalized entropy should reflect uncertainty and distributional spread, the latter of which is heavily affected by the tail of the IWP distribution.

The performance of the GA–RF estimator is competitive, as it includes data-driven initialization and derivative-free global refinement. The RF component gives a very stable initial estimate, derived from simulated IWP samples, and informative summary features, and the GA component fine tunes the estimate in the log-parameter space. This will minimize sensitivity to bad starting points and to inconsistent objective surfaces. The advantage of the GA–RF method over the classical estimators is however due to the additional computational effort needed and not to its universal superiority. In general, the results of the simulations suggest that a compromise between statistical information and numerical stability control the performance of the estimator. When ranking is reliable RSS enhances the statistical information; when the sample size measured is large, sampling variability is decreased and when the situation is heavy-tailed and the number of objects is large, a computationally powerful approach like GA–RF can assist in stabilization of the estimation. In general, RSS is useful only in the case of reliable ranking information and large sample size. If the quality of the data drops, the efficiency benefit of RSS can become less pronounced or even go away. Likewise, GA–RF is claimed to be a numerically stable and competitive estimator rather than a general alternative to ML, MPS, KM, LS, WLS, AD, and CVM methods, particularly for the unstable optimization settings.

8.2. Discussion of findings, comparison, implications, and limitations

The key results of this study indicate that RSS generally is superior under high-quality ranking but fails to do the same under imperfect ranking. Further, it is seen that the performance of the estimators depends on the sample size, the setting of the parameters, the quality of the ranking and the method of optimization. In no case is one classical estimator uniformly best and in some cases GA–RF is a more accurate and numerically more stable estimator.

This study deals with generalized entropy estimation of the IWP distribution in both SRS and RSS-based censoring schemes compared with the earlier studies on entropy estimation where the focus was exclusively on SRS, censoring schemes, record values, and conventional sampling structures. It also improves the concept of RSS research in two aspects: (1) It adopts the point of view of generalized entropy estimation for a heavy-tailed lifetime model; (2) It considers the fuzzy ranking uncertainty by using α -cut representation.

The results suggest that under the assumption of reliable ranking information, and of a high cost of exact measurement, RSS may be helpful in reliability inference related to entropy. But it is not universally agreed that RSS is better than SRS, as it is only efficient when it comes to ranking accuracy and estimator stability. Likewise, GA–RF can be employed as a numerically stable and robust complementary estimator when the classical objective functions are flat or unstable, or when they are sensitive to initial values.

The significant advantages of this study are the synthesis of an entropy estimation, RSS, fuzzy ranking uncertainty and GA–RF estimation in one framework. The limitations are small number of simulation settings, potential for numerical instability of the four-parameter IWP model, incomplete validation of GA–RF, and the use of a single real dataset.

9. Real Data Analysis

This section presents an analysis of real data to confirm the applicability of the estimation methodologies proposed. We use an empirical dataset to show how the competing estimators can be used in practice and under what circumstances each of them can be beneficial. The data include tensile strength (GPa) measurements of 79 carbon fibres that were investigated by [58]. The values of these estimated entropies in reliability engineering are considered as measures of uncertainty with regard to lifetime or strength distribution. A reliable estimate of

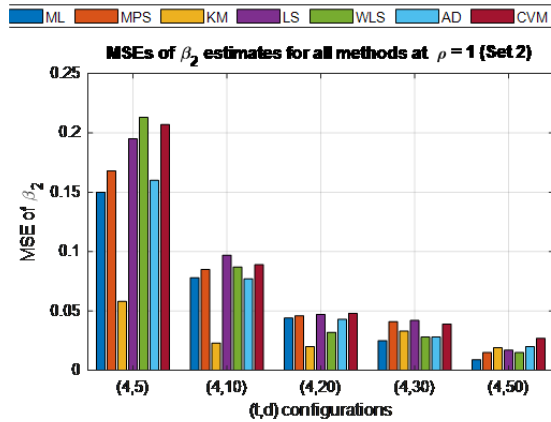


Figure 3. MSEs of β_2 estimates for all methods at $\rho = 1$ for Set 2

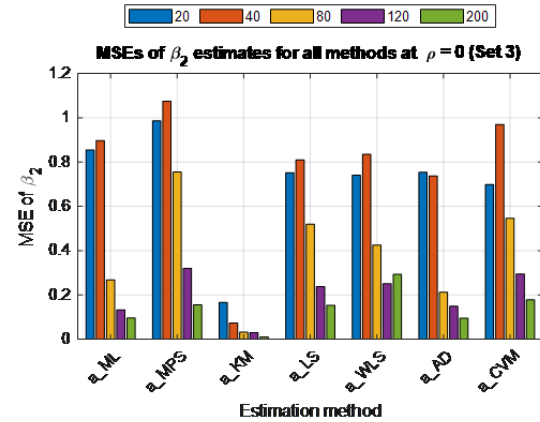


Figure 4. MSEs of λ estimates for all methods at $\rho = 0$ (SRS) for Set 3

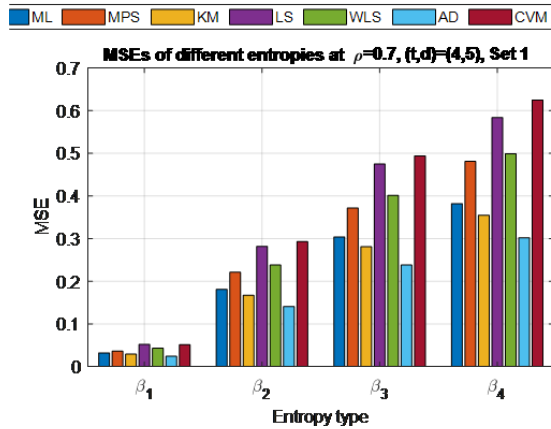


Figure 5. MSEs of different entropies at $\rho = 0.7$, (t,d)=(4,5) for Set 1

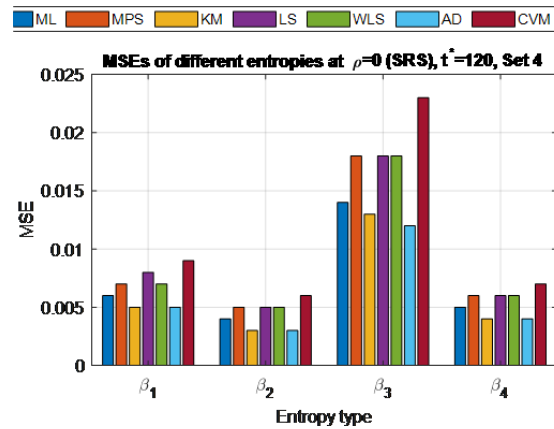


Figure 6. MSEs of different entropies at $\rho = 0$ (SRS), $t^* = 120$ for Set 4

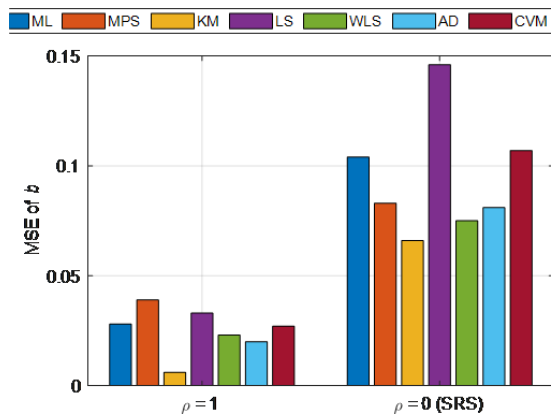


Figure 7. MSEs of λ estimates for all methods

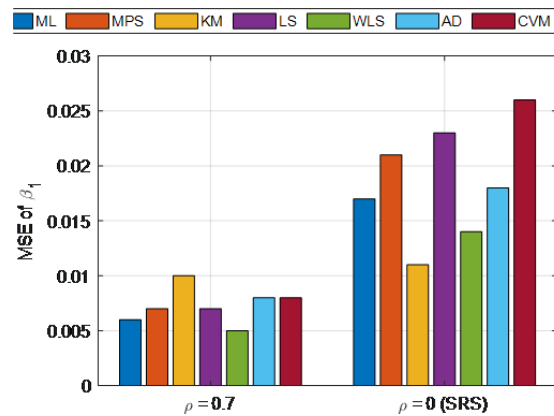
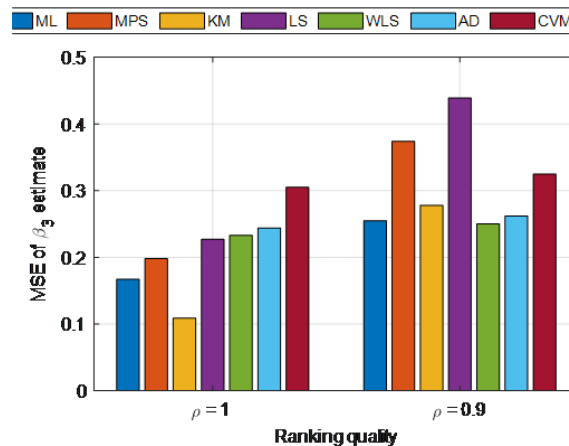


Figure 8. MSEs of β_1 estimates at $t^* = 200$ for Set 3

Figure 9. MSEs of β_3 estimates at $t^*=20$ for Set 1

entropy can, therefore, be used in conjunction with the traditional parameter estimates, goodness of fit, etc., to evaluate the predictability of the material failure behaviour when considering the tensile strength data from carbon fibre. When testing individual components is expensive, but it is possible to rank the units comparatively, the use of RSS can reduce the number of measurements while maintaining information that is relevant to the uncertainty quantification. Therefore, the estimates of entropy under RSS can help in decisions related to reliability screening, prioritizing inspection, degradation monitoring and allocation of experiments. The present analysis proves that this is statically feasible but it is not claimed that a specific maintenance or clinical decision can be made based on entropy alone.

Table 3. Summary descriptive measures and exploratory visualizations for the tensile-strength dataset.

Dataset	Sample size	Mean	Median	Variance	Skewness	Kurtosis	Range	Minimum	Maximum
	79	3.544	3.5870	0.3561	-0.0393	3.9508	3.374	1.4231	3.6961

To supplement the exploratory analysis summarized in Table 3, descriptive statistics and the graphical diagnostics presented in Figure 10 are analysed to support the exploratory analysis. The graphical representations, such as histograms, kernel density estimates, quantile-quantile (Q-Q) plots, box plots, and the violin plots together with the total time on test (TTT) plot, complement the distributional characteristics of the tensile-strength data. The dataset in general seems to be about symmetric, and has at most weak skew. There are no evident outliers or structural aberrations. Moreover, the TTT plot indicates that the hazard rate increases with hazard rate function (HRF), which is in line with the modelling assumptions taken into account in this study. All these diagnostic plots suggest that the data is well-behaved and can be used in parametric modelling and other inferential analysis. In order to formally determine the goodness of the IWP model, we also use the Kolmogorov-Smirnov (KS) goodness-of-fit test. The test value is 0.0403 with the p-value of 1.000, which does not give any indication against the IWP distribution and instead indicates that it can be used to model the current dataset. The competing estimation methods were evaluated using the Kolmogorov-Smirnov (KS), Anderson-Darling (AD) and Cramér-von Mises (CvM) goodness-of-fit statistics to test the adequacy of the fitted IWP model. These are reported in Table 6. The p values generated by the IWP parameters are approximate goodness of fit indicators, because they are based on the same data set. The p values calculated with the IWP parameters should not be interpreted as distribution-free p values because these are computed with the same set of data. The KS statistic of the fitted IWP model is small, which suggests that the fitted distribution is close to the empirical distribution. It is recommended, however, that the goodness-of-fit evidence be considered together based on the KS, AD, and CvM statistics along with their p-values.

As can be seen in the results tabulated in Table 6, several of the estimation methods are acceptable fits for the tensile-strength data of the carbon-fibres. In particular, the distance-based methods generally yield small values of KS, AD, and CvM, and relatively large p-values, suggesting that the fitted IWP distribution is well-fit to the empirical distribution. However, some methods yield higher AD or CvM statistics, indicating less tail fit. Hence, the real-data analysis validates the ability of the IWP model to be used in this data set, and should be interpreted comparatively from one estimation method to another rather than based on one KS p-value. This is further supported by the fitted-distribution diagnostics as shown in Figure 11. The point estimates of the four

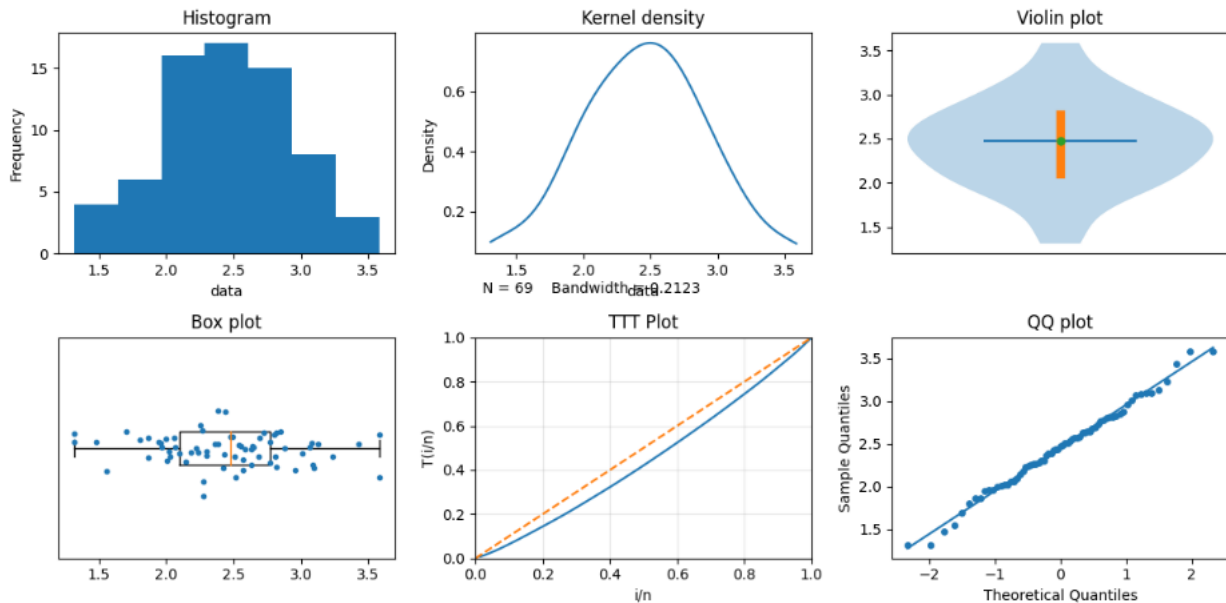


Figure 10. Exploratory graphical diagnostics for the real dataset

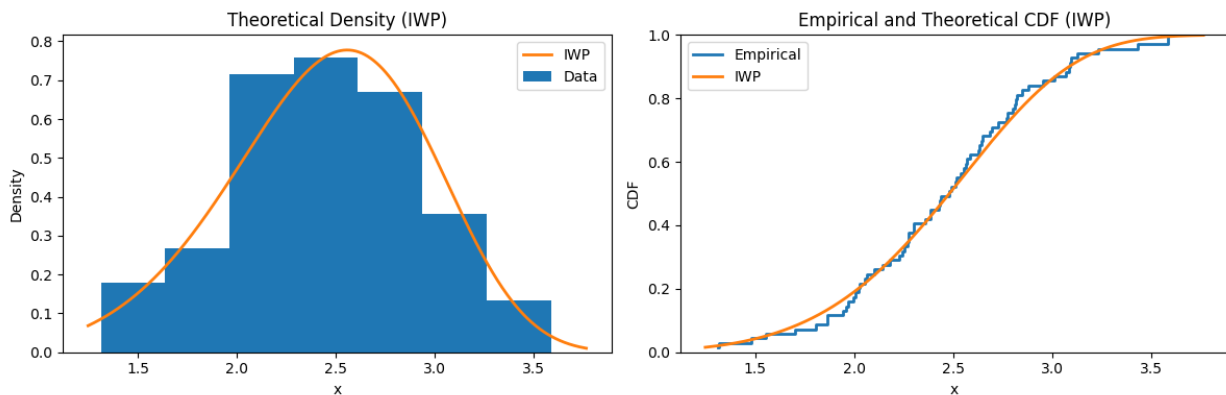


Figure 11. Estimated PDF and CDF for IWP

generalized entropy measures are provided in Table 15 for the various estimation procedures. Reported values are not standard errors or confidence intervals. The GA-RF entries are in particular the resulting point-estimates after the application of the suggested hybrid estimator. As the GA-RF procedure is not based on the traditional maximum-likelihood asymptotic theory, standard errors and confidence intervals are not reported in this table.

Table 4. Estimated parameter values of the IWP distribution calculated through different estimation approaches utilizing the RSS and SRS dataset.

ρ	Method	Parameters			
		λ	α	β	k
1	ML	0.028289	3.486549	3.000e-6	2728.135000
	MPS	0.005935	5.220615	6.420e-4	6.000e-5
	KM	0.002068	6.130633	0.662289	0.102260
	LS	0.002422	6.006269	1.100e-5	2464.145000
	WLS	0.004613	5.429869	5.000e-6	1132.000000
	AD	0.005000	5.368757	7.080e-4	7.000e-5
	CVM	0.002264	6.092147	1.000e-5	2431.940000
	GA-RF	0.028319	3.485774	0.002876	2.682243
0.9	ML	0.012279	4.394984	0.001944	6.000e-5
	MPS	0.005935	5.220615	6.420e-4	6.000e-5
	KM	0.002068	6.130633	0.662289	0.102260
	LS	0.002422	6.006269	1.100e-5	2464.145000
	WLS	0.004613	5.429869	5.000e-6	1132.000000
	AD	0.005000	5.368757	7.080e-4	7.000e-5
	CVM	0.002264	6.092147	1.000e-5	2431.940000
	GA-RF	0.012258	4.396349	0.002495	0.049133
0.7	ML	0.007333	4.952268	0.009334	6.000e-5
	MPS	0.005935	5.220615	6.420e-4	6.000e-5
	KM	0.002068	6.130633	0.662289	0.102260
	LS	0.002422	6.006269	1.100e-5	2464.145000
	WLS	0.004613	5.429869	5.000e-6	1132.000000
	AD	0.005000	5.368757	7.080e-4	7.000e-5
	CVM	0.002264	6.092147	1.000e-5	2431.940000
	GA-RF	0.007322	4.953492	0.002495	0.049133
0 (SRS)	ML	0.004832	5.396842	0.011500	6.000e-5
	MPS	0.005935	5.220615	6.420e-4	6.000e-5
	KM	0.002068	6.130633	0.662289	0.102260
	LS	0.002422	6.006269	1.100e-5	2464.145000
	WLS	0.004613	5.429869	5.000e-6	1132.000000
	AD	0.005000	5.368757	7.080e-4	7.000e-5
	CVM	0.002264	6.092147	1.000e-5	2431.940000
	GA-RF	0.004827	5.397660	0.002495	0.049133

It remains to be seen how to perform formal uncertainty quantification of GA–RF, e.g., parametric bootstrap or Bayesian approach, which is left for future investigations.

The carbon-fibre tensile-strength dataset were analysed using two sampling designs: the simple random sampling (SRS) and ranked set sampling (RSS) with perfect and imperfect ranking to understand the theoretical development above. The IWP model was estimated using ML, KM, LS, WLS, AD, CVM, and the intelligent GA-RF method proposed. This dataset was not analysed using the MPS method, due to the possible occurrence of repeated (tied) observations, which can result in zero spacings, and hence the MPS criterion is unstable or undefined, in both RSS ($\rho=0.9, 0.7, 0.1$) and SRS ($\rho=0$) cases. If tied observations occur, standard MPS yields zero spacings, and the definition of the logarithmic spacing criterion is not applicable. Other options might be to use tie adjusted spacing constructions or to calculate the spacing criterion after combining/aggregating tied observations, as long as this is

Table 5. Estimated entropy values for the IWP distribution calculated through different estimation approaches utilising the RSS and SRS dataset at $v = 0.7$ and 2.

ρ	u	Method	Entropy Measures			
			β_1	β_2	β_3	β_4
1	0.7	ML	1.2875	1.5716	2.0397	1.7182
		MPS	0.8808	1.0082	1.3085	1.0701
		KM	0.9960	1.1607	1.5065	1.2423
		LS	0.9380	1.0833	1.4060	1.1546
		WLS	0.9008	1.0343	1.3424	1.0994
		AD	0.8614	0.9830	1.2758	1.0420
		CVM	0.9140	1.0516	1.3649	1.1189
		GA-RF	1.2864	1.5699	2.0376	1.7163
	2	ML	1.0713	0.6574	1.3149	0.8294
		MPS	0.6516	0.4788	0.9576	0.5561
		KM	0.6782	0.4925	0.9850	0.5752
		LS	0.6325	0.4688	0.9375	0.5423
		WLS	0.6418	0.4737	0.9473	0.5490
		AD	0.6306	0.4677	0.9354	0.5408
		CVM	0.6076	0.4553	0.9107	0.5240
		GA-RF	1.0704	0.6571	1.3143	0.8289
0.9	0.7	ML	1.0470	1.2300	1.5964	1.3213
		MPS	0.8808	1.0082	1.3085	1.0701
		KM	0.9960	1.1607	1.5065	1.2423
		LS	0.9380	1.0833	1.4060	1.1546
		WLS	0.9008	1.0343	1.3424	1.0994
		AD	0.8614	0.9830	1.2758	1.0420
		CVM	0.9140	1.0516	1.3649	1.1189
		GA-RF	1.0481	1.2315	1.5984	1.3230
	2	ML	0.8279	0.5630	1.1260	0.6779
		MPS	0.6516	0.4788	0.9576	0.5561
		KM	0.6782	0.4925	0.9850	0.5752
		LS	0.6325	0.4688	0.9375	0.5423
		WLS	0.6418	0.4737	0.9473	0.5490
		AD	0.6306	0.4677	0.9354	0.5408
		CVM	0.6076	0.4553	0.9107	0.5240
		GA-RF	0.8280	0.5631	1.1262	0.6780
0.7	0.7	ML	0.9368	1.0816	1.4039	1.1527
		MPS	0.8808	1.0082	1.3085	1.0701
		KM	0.9960	1.1607	1.5065	1.2423
		LS	0.9380	1.0833	1.4060	1.1546
		WLS	0.9008	1.0343	1.3424	1.0994
		AD	0.8614	0.9830	1.2758	1.0420
		CVM	0.9140	1.0516	1.3649	1.1189
		GA-RF	0.9383	1.0837	1.4065	1.1550
	2	ML	0.7107	0.5087	1.0174	0.5981
		MPS	0.6516	0.4788	0.9576	0.5561
		KM	0.6782	0.4925	0.9850	0.5752
		LS	0.6325	0.4688	0.9375	0.5423
		WLS	0.6418	0.4737	0.9473	0.5490
		AD	0.6306	0.4677	0.9354	0.5408
		CVM	0.6076	0.4553	0.9107	0.5240
		GA-RF	0.7109	0.5088	1.0176	0.5983
0 (SRS)	0.7	ML	0.8581	0.9787	1.2702	1.0372
		MPS	0.8808	1.0082	1.3085	1.0701
		KM	0.9960	1.1607	1.5065	1.2423
		LS	0.9380	1.0833	1.4060	1.1546
		WLS	0.9008	1.0343	1.3424	1.0994
		AD	0.8614	0.9830	1.2758	1.0420
		CVM	0.9140	1.0516	1.3649	1.1189
		GA-RF	0.8600	0.9812	1.2734	1.0399
	2	ML	0.6269	0.4658	0.9315	0.5382
		MPS	0.6516	0.4788	0.9576	0.5561
		KM	0.6782	0.4925	0.9850	0.5752
		LS	0.6325	0.4688	0.9375	0.5423
		WLS	0.6418	0.4737	0.9473	0.5490
		AD	0.6306	0.4677	0.9354	0.5408
		CVM	0.6076	0.4553	0.9107	0.5240
		GA-RF	0.6272	0.4659	0.9319	0.5384

Table 6. Goodness of fit measures by different estimation methods for IWP distribution RSS and SRS dataset.

ρ	Method	KS		AD		CvM	
		KS	P-Val	AD	P-Val	CvM	P-Val
1	ML	0.1491	0.1264	2.6943	0.0010	0.4086	0.0010
	MPS	0.0737	0.8884	0.3684	0.4291	0.0659	0.3743
	KM	0.0555	0.9909	0.5828	0.1294	0.0654	0.3794
	LS	0.0677	0.9385	0.4210	0.3235	0.0534	0.5447
	WLS	0.0645	0.9586	0.3459	0.4832	0.0565	0.4990
	AD	0.0668	0.9445	0.3389	0.5027	0.0572	0.4900
	CVM	0.0728	0.8971	0.4210	0.3236	0.0524	0.5616
	GA-RF	0.1494	0.1251	2.6877	0.0010	0.4076	0.0010
0.9	ML	0.0948	0.6319	0.9070	0.0209	0.1376	0.0397
	MPS	0.0737	0.8884	0.3684	0.4291	0.0659	0.3743
	KM	0.0555	0.9909	0.5828	0.1294	0.0654	0.3794
	LS	0.0677	0.9385	0.4210	0.3235	0.0534	0.5447
	WLS	0.0645	0.9586	0.3459	0.4832	0.0565	0.4990
	AD	0.0668	0.9445	0.3389	0.5027	0.0572	0.4900
	CVM	0.0728	0.8971	0.4210	0.3236	0.0524	0.5616
	GA-RF	0.0948	0.6319	0.9075	0.0208	0.1376	0.0397
0.7	ML	0.0708	0.9150	0.4412	0.2898	0.0733	0.2983
	MPS	0.0737	0.8884	0.3684	0.4291	0.0659	0.3743
	KM	0.0555	0.9909	0.5828	0.1294	0.0654	0.3794
	LS	0.0677	0.9385	0.4210	0.3235	0.0534	0.5447
	WLS	0.0645	0.9586	0.3459	0.4832	0.0565	0.4990
	AD	0.0668	0.9445	0.3389	0.5027	0.0572	0.4900
	CVM	0.0728	0.8971	0.4210	0.3236	0.0524	0.5616
	GA-RF	0.0707	0.9152	0.4416	0.2892	0.0733	0.2981
0 (SRS)	ML	0.0694	0.9257	0.3404	0.4972	0.0567	0.4968
	MPS	0.0737	0.8884	0.3684	0.4291	0.0659	0.3743
	KM	0.0555	0.9909	0.5828	0.1294	0.0654	0.3794
	LS	0.0677	0.9385	0.4210	0.3235	0.0534	0.5447
	WLS	0.0645	0.9586	0.3459	0.4832	0.0565	0.4990
	AD	0.0668	0.9445	0.3389	0.5027	0.0572	0.4900
	CVM	0.0728	0.8971	0.4210	0.3236	0.0524	0.5616
	GA-RF	0.0693	0.9266	0.3403	0.4974	0.0567	0.4971

theoretically justified for the sampling design. Alternatively, the use of estimation criteria with well-determined ties (e.g., ML, LS, WLS, AD, CVM, and the proposed GA-RF estimator) is another practical possibility. For this reason, in the present study, we do not consider the standard MPS in the real-data analysis, but rather consider it to drop out of the analysis altogether. We therefore do not add an ad hoc tie correction to the real-data analysis of the present study, but we let the standard MPS drop out of it altogether. When using MPS on a reliability data set with repeated measurements, this restriction should be taken into account. The estimates made under SRS and RSS (when the ranking is perfect and imperfect) were obtained as follows and summarised in [Table 4](#) and [Table 5](#). To achieve a fair comparison, the two designs were compared with the same measurement budget: an SRS sample of size 40 ($\rho=0$) was compared with an RSS design with set size $t = 4$ and $d = 10$ cycles (so that the total amount of measured units is the same). This equal-cost model gives the opportunity to any seen increase

in under RSS to be ascribed to the extra information acquired in the form of ranking and not to more observed measurements. To further underscore the practical superiority of RSS- under perfect and imperfect ranking- over SRS among the competing estimation methods, we tested model adequacy with various goodness-of-fit measures, viz. the Kolmogorov-Smirnov (KS) statistic, the Anderson-Darling (AD) statistic, and the Cramer-von Mises (CVM) statistic, along with their respective p. Such measures are a quantitative measure of the agreement between the observed data and the fitted IWP model and give an objective test of the comparison of sampling designs and estimation procedures. Table 16 compares SRS ($\rho=0$) to RSS under perfect ($\rho=0$) and imperfect ($\rho=0.9, 0.7$) ranking, and reports the comparative goodness-of-fit. Overall, the results in Table 4, Table 5 and Table 6 indicate that RSS provides improved or competitive fit and estimation performance relative to SRS for the carbon-fibre dataset under the ranking conditions considered. However, the simulation results show that this advantage is not universal and depends on ranking quality, sample size, parameter configuration, and estimation method. Notably, the suggested GA-RF methodology shows the most competitive results among the methods under consideration, providing the parameter estimates and entropy values that are generally characterized by lower error indices and good goodness-of-fit results. These results confirm the practical usefulness of integrating a sample design (RSS) with an intelligent estimation strategy (GA-RF) to make robust inferences based on the IWP model.

10. Statistical Uncertainty and Significance Testing for the Real-Data Application

The six classical estimators (ML, KM, LS, WLS, AD, CVM) and the proposed GA-RF estimator were re-fitted for each of the six sets of $B=200$ bootstrap resamples of size n drawn with replacement from a dataset of tensile-strength (GPa) carbon fibres, with $n=69$, the same number employed by [58]. The same $n=69$ observations were employed for each set of $B=200$ resamples so that the comparisons between the different methods were paired. The MPS method was not included because tied observations give zero spacings for a non-negligible number of resamples, making the MPS objective undefined for a non-negligible number of resamples. The standard errors are the sample standard deviations of each of the estimates across the 200 refits and the 2.5th and 97.5th percentiles are the 95% confidence intervals.

11. Systematic Analysis of the Factors Influencing Performance

This section reports a dedicated 4-way factorial Monte Carlo simulation, designed specifically to isolate and rank the contribution of estimation method, sampling design, sample size and entropy order to estimation accuracy, complementing the bias/MSE/RE summaries already reported in Section 6 for the main simulation study.

12. Conclusions

This paper explored the inference of entropy of lifetime data with information-enriched sampling. In particular, we presented the inverted Weibull-Pareto (IWP) model and constructed plug-in estimators of four generalized entropy measures (Renyi, Tsallis, Havard Charvat and Arimoto) assuming both simple random sampling (SRS) and ranked set sampling (RSS). The thorough comparison has been made against classical estimation procedures (ML, MPS, KM, LS, WLS, AD, and CVM) and an intelligent nested genetic algorithm-random forest (GA-RF) methodology. The quantitative findings indicate that RSS is evident to be much more efficient than SRS when the quantity of units measured is kept constant. RSS generally has lower absolute bias and means squared error to the estimate of the parameters, and entropy, in both the sample size and settings of parameter settings that are explored, and perfect ranking performs better than imperfect ranking. The suggested fuzzy framework of ranking-quality further confirmed the existence of these benefits in ranking uncertainty in accuracy, offering a more realistic picture of estimator robustness. The results of the real-data analysis conducted using carbon-fibre tensile-strength measurements confirmed the results of the simulation. In equal effort of measurement, RSS-based estimation tended to achieve good goodness-of-fit compared with SRS, and the GA-RF approach generated quite

Table 7. Bootstrap point estimate, standard error (SE) and 95% percentile confidence interval (CI) for the IWP parameters, real-data application ($n = 69$, $B = 200$ resamples), including the GA–RF estimator.

Method	Parameter	Estimate	Bootstrap SE	95% CI	
				Lower	Upper
ML	λ	4.482	2.863e-08	4.482	4.482
	α	2.428	0.07297	2.319	2.587
	β	7.142e-05	8.342e-04	3.365e-07	0.001869
	k	2.172e-04	1227	3.863e-07	18.01
KM	λ	1.303e-04	4.663e-06	1.235e-04	1.392e-04
	α	9.959	0.0545	9.776	9.973
	β	2.240e-05	8.956e-07	2.089e-05	2.405e-05
	k	1.331e+04	17.07	1.33e+04	1.336e+04
LS	λ	1.236e-04	1.194e-06	1.234e-04	1.262e-04
	α	9.973	0.01302	9.929	9.974
	β	2.224e-05	6.025e-07	2.133e-05	2.347e-05
	k	1.336e+04	6.225	1.334e+04	1.336e+04
WLS	λ	1.235e-04	2.990e-07	1.234e-04	1.244e-04
	α	9.972	0.00514	9.955	9.974
	β	2.357e-05	6.698e-07	2.231e-05	2.490e-05
	k	1.336e+04	4.554	1.334e+04	1.336e+04
AD	λ	1.309e-04	5.532e-06	1.235e-04	1.425e-04
	α	9.868	0.08436	9.653	9.97
	β	2.238e-05	6.205e-07	2.137e-05	2.379e-05
	k	1.333e+04	68.73	1.307e+04	1.336e+04
CVM	λ	1.237e-04	8.538e-07	1.234e-04	1.261e-04
	α	9.974	0.01471	9.925	9.974
	β	2.228e-05	6.023e-07	2.133e-05	2.349e-05
	k	1.334e+04	7.092	1.333e+04	1.336e+04
GA-RF	λ	5	2.576e-04	5	5
	α	2.557	0.07767	2.427	2.715
	β	0.05115	0.00948	0.05	0.08247
	k	0.2021	0.008591	0.2	0.2317

competitive fits and consistent estimates. In general, the findings justify the use of information-enriched sampling (RSS) and intelligent estimation (GA–RF) to incur reliable entropy-based inference in the framework of the IWP model. The future work could take place in a number of directions:

- Future research should include an extension of the proposed estimation scheme to both censored and truncated lifetime data, typical in reliability and survival studies.
- The likelihood-based or Bayesian quantification of uncertainty in GA–RF (e.g. parametric bootstrap, simulation-based posterior approximations) should be developed to give standard errors and interval estimates.
- The next step is to consider other imperfect-ranking mechanisms and concomitant-variable structures such as covariate-dependent ranking and heteroscedastic ranking errors.
- Deriving and analysing more information measures (e.g. Shannon entropy, cumulative residual entropy, cumulative past entropy) of the IWP family, and understanding how sensitive they are to tail behavior

Table 8. Bootstrap point estimate (SE) [95% CI] for the four generalised entropy measures of the IWP distribution at $u = 2$, real-data application ($n = 69$, $B = 200$ resamples), including the GA–RF estimator.

Method	Entropy Measures			
	β_1 (Rényi)	β_2 (Tsallis)	β_3 (Havrda–Charvát)	β_4 (Arimoto)
ML	1.178 (0.04857) [1.075, 1.253]	0.6921 (0.01515) [0.6588, 0.7144]	1.384 (0.0303) [1.318, 1.429]	0.8902 (0.02712) [0.8317, 0.9312]
KM	-1.608 (0.007726) [-1.626, -1.597]	-3.993 (0.0387) [-4.081, -3.937]	-7.987 (0.0774) [-8.163, -7.874]	-2.469 (0.01729) [-2.508, -2.444]
LS	-1.615 (0.004771) [-1.622, -1.605]	-4.028 (0.02395) [-4.064, -3.979]	-8.055 (0.04791) [-8.129, -7.957]	-2.484 (0.01069) [-2.501, -2.463]
WLS	-1.604 (0.005373) [-1.614, -1.593]	-3.974 (0.02673) [-4.024, -3.921]	-7.948 (0.05345) [-8.049, -7.841]	-2.461 (0.01198) [-2.483, -2.437]
AD	-1.608 (0.005957) [-1.621, -1.598]	-3.995 (0.02978) [-4.06, -3.943]	-7.99 (0.05956) [-8.12, -7.887]	-2.47 (0.01332) [-2.499, -2.447]
CVM	-1.615 (0.004785) [-1.623, -1.604]	-4.026 (0.02403) [-4.066, -3.974]	-8.053 (0.04805) [-8.132, -7.948]	-2.484 (0.01072) [-2.502, -2.461]
GA-RF	1.146 (0.04897) [1.049, 1.233]	0.682 (0.01564) [0.6497, 0.7085]	1.364 (0.03128) [1.299, 1.417]	0.8723 (0.02767) [0.8163, 0.9202]

Table 9. Paired bootstrap comparison of methods for the Rényi entropy β_1 at $u = 2$ (all seven methods, including GA–RF).

Comparison	Mean diff.	SE (diff.)	z	p (bootstrap z-test)	p (Wilcoxon)
ML vs KM	2.785	0.05329	52.26	0	1.44e-34
ML vs LS	2.787	0.05323	52.36	0	1.44e-34
ML vs WLS	2.777	0.05334	52.07	0	1.44e-34
ML vs AD	2.783	0.0525	53.02	0	1.44e-34
ML vs CVM	2.787	0.05328	52.3	0	1.44e-34
ML vs GA-RF	0.02255	0.008516	2.648	0.00809	3.43e-33
KM vs LS	0.002128	0.00642	0.3314	0.74	2.88e-05
KM vs WLS	-0.007674	0.005078	-1.511	0.131	2.1e-30
KM vs AD	-0.001398	0.008491	-0.1647	0.869	0.0219
KM vs CVM	0.002048	0.006245	0.3279	0.743	6.05e-05
KM vs GA-RF	-2.762	0.05368	-51.45	0	1.44e-34
LS vs WLS	-0.009802	0.002645	-3.706	0.000211	1.6e-34
LS vs AD	-0.003526	0.004668	-0.7554	0.45	1.3e-27
LS vs CVM	-8.014e-05	0.001144	-0.07003	0.944	0.00192
LS vs GA-RF	-2.764	0.05359	-51.58	0	1.44e-34
WLS vs AD	0.006276	0.005807	1.081	0.28	6.74e-25
WLS vs CVM	0.009722	0.002482	3.917	8.96e-05	1.44e-34
WLS vs GA-RF	-2.754	0.05373	-51.26	0	1.44e-34
AD vs CVM	0.003446	0.004744	0.7265	0.468	1.41e-27
AD vs GA-RF	-2.761	0.05287	-52.22	0	1.44e-34
CVM vs GA-RF	-2.764	0.05365	-51.52	0	1.44e-34

Table 10. AIC, BIC and AICc for the real-data IWP fits ($n = 69$, $k = 4$ parameters).

Method	logL	AIC	BIC	AICc	Δ AIC	Akaike weight
GA-RF	-82.7048	173.4096	182.3460	174.0346	0.0000	0.922
ML	-85.1793	178.3586	187.2951	178.9836	4.9491	0.0777
WLS	-132.9944	273.9889	282.9253	274.6139	100.5793	1.33e-22
AD	-133.9215	275.8430	284.7794	276.4680	102.4334	5.27e-23
KM	-133.9328	275.8656	284.8020	276.4906	102.4560	5.21e-23
CVM	-133.9910	275.9820	284.9184	276.6070	102.5724	4.92e-23
LS	-133.9966	275.9932	284.9296	276.6182	102.5836	4.89e-23

Table 11. Type II ANOVA of log-squared-error of the Rényi entropy estimate, factorial simulation (7 methods $\times$ 3 designs $\times$ 2 sample sizes $\times$ 2 entropy orders, $R = 8$ replicates/cell, 672 simulated datasets).

Factor	df	Sum of squares	% of total SS	F	p -value
Estimation method	6	1707.00	41.59	83.86	1.99×10^{-77}
Method $\times$ Entropy order	6	98.38	2.398	4.834	7.71×10^{-5}
Entropy order (u)	1	84.31	2.055	24.85	7.96×10^{-7}
Method $\times$ Sampling design	12	18.17	0.443	0.446	0.944
Sample size (n)	1	14.11	0.344	4.159	0.0418
Sampling design	2	0.493	0.012	0.073	0.930
Residual	643	2181.00	53.16	—	—

Table 12. Mean squared error of the Rényi entropy estimate by method, averaged over all design $\times$ $n \times u$ cells (96 replicates per method), with Tukey HSD groupings on the log scale ($\alpha = 0.05$).

Method	Mean SE	Median SE	SE of mean
GA-RF	0.374	0.187	0.059
LS	0.591	0.349	0.076
WLS	0.627	0.252	0.093
AD	0.696	0.404	0.096
CVM	0.844	0.316	0.183
KM	5.055	5.042	0.118
ML	7.309	7.880	0.282

- Take into account model extensions (regression structures (accelerated failure time or proportional hazards formulations) and multivariate/generalized IWP families) to more complicated applications.

Appendix

Table 13. ABs and MSEs of IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 1 at $t = 3$.

t^*	d	Par.	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	λ	1	0.345	0.143	0.375	0.164	0.129	0.023	0.327	0.131	0.315	0.122	0.281	0.099	0.319	0.121
		α		0.438	0.531	0.432	0.319	0.266	0.099	0.395	0.321	0.299	0.190	0.227	0.101	0.373	0.339
40	10	λ		0.308	0.115	0.302	0.119	0.102	0.013	0.273	0.089	0.254	0.083	0.231	0.066	0.286	0.098
		α		0.373	0.357	0.342	0.229	0.188	0.051	0.287	0.181	0.264	0.152	0.198	0.076	0.287	0.182
80	20	λ		0.292	0.114	0.350	0.143	0.077	0.008	0.277	0.091	0.286	0.095	0.259	0.081	0.262	0.083
		α		0.328	0.303	0.315	0.219	0.163	0.034	0.258	0.156	0.205	0.087	0.207	0.091	0.227	0.119
120	30	λ		0.219	0.065	0.257	0.081	0.075	0.008	0.232	0.067	0.210	0.058	0.197	0.052	0.222	0.064
		α		0.178	0.069	0.146	0.035	0.110	0.016	0.175	0.053	0.162	0.046	0.154	0.039	0.182	0.061
200	50	λ		0.219	0.064	0.262	0.082	0.091	0.012	0.184	0.049	0.198	0.056	0.190	0.051	0.180	0.047
		α		0.144	0.041	0.133	0.027	0.097	0.013	0.132	0.034	0.142	0.041	0.134	0.033	0.121	0.029
20	5	λ	0.9	0.368	0.159	0.382	0.167	0.114	0.018	0.357	0.146	0.341	0.136	0.327	0.129	0.325	0.129
		α		0.931	1.408	0.501	0.415	0.285	0.109	0.481	0.369	0.421	0.316	0.341	0.229	0.442	0.341
40	10	λ		0.392	0.168	0.392	0.172	0.096	0.012	0.283	0.101	0.283	0.099	0.255	0.079	0.275	0.098
		α		0.643	0.633	0.430	0.305	0.198	0.054	0.278	0.153	0.300	0.179	0.243	0.109	0.370	0.321
80	20	λ		0.310	0.114	0.313	0.122	0.079	0.008	0.260	0.083	0.258	0.080	0.236	0.069	0.261	0.083
		α		0.450	0.299	0.247	0.135	0.179	0.040	0.192	0.087	0.181	0.085	0.159	0.056	0.179	0.077
120	30	λ		0.342	0.129	0.278	0.095	0.087	0.010	0.227	0.066	0.216	0.060	0.193	0.050	0.249	0.078
		α		0.378	0.203	0.180	0.068	0.134	0.026	0.133	0.042	0.126	0.039	0.127	0.039	0.147	0.052
200	50	λ		0.322	0.117	0.286	0.093	0.088	0.010	0.227	0.066	0.213	0.062	0.207	0.058	0.222	0.065
		α		0.337	0.136	0.160	0.044	0.128	0.021	0.166	0.053	0.137	0.037	0.132	0.032	0.167	0.056
20	5	λ	0.7	0.382	0.163	0.362	0.155	0.131	0.022	0.369	0.158	0.341	0.140	0.324	0.130	0.363	0.153
		α		0.815	0.920	0.435	0.335	0.322	0.130	0.595	0.747	0.517	0.529	0.378	0.273	0.629	0.856
40	10	λ		0.367	0.146	0.366	0.153	0.116	0.019	0.284	0.102	0.314	0.122	0.299	0.111	0.263	0.089
		α		0.649	0.569	0.354	0.247	0.208	0.064	0.272	0.144	0.339	0.262	0.242	0.136	0.279	0.189
80	20	λ		0.357	0.140	0.304	0.111	0.094	0.011	0.268	0.087	0.266	0.093	0.251	0.084	0.287	0.103
		α		0.627	0.485	0.236	0.110	0.162	0.034	0.189	0.069	0.213	0.093	0.186	0.070	0.266	0.156
120	30	λ		0.369	0.142	0.257	0.085	0.070	0.007	0.214	0.061	0.210	0.059	0.194	0.053	0.224	0.065
		α		0.574	0.408	0.159	0.058	0.124	0.020	0.154	0.059	0.134	0.046	0.130	0.042	0.155	0.060
200	50	λ		0.344	0.123	0.224	0.061	0.073	0.007	0.170	0.042	0.153	0.031	0.159	0.037	0.170	0.039
		α		0.445	0.240	0.092	0.014	0.096	0.013	0.110	0.028	0.079	0.013	0.080	0.013	0.103	0.024
20	5	λ	0	0.428	0.204	0.423	0.194	0.145	0.028	0.380	0.166	0.386	0.170	0.358	0.148	0.366	0.156
		α		1.069	2.089	0.606	0.604	0.303	0.127	0.504	0.450	0.490	0.426	0.414	0.387	0.570	0.673
40	10	λ		0.310	0.126	0.381	0.167	0.114	0.021	0.305	0.115	0.294	0.108	0.274	0.100	0.323	0.123
		α		0.435	0.489	0.426	0.319	0.213	0.056	0.280	0.159	0.226	0.104	0.182	0.068	0.259	0.140
80	20	λ		0.301	0.114	0.309	0.117	0.088	0.011	0.247	0.079	0.277	0.094	0.246	0.079	0.235	0.071
		α		0.332	0.280	0.258	0.139	0.156	0.033	0.262	0.145	0.236	0.119	0.203	0.088	0.238	0.119
120	30	λ		0.261	0.089	0.265	0.091	0.095	0.013	0.206	0.054	0.236	0.072	0.212	0.057	0.205	0.055
		α		0.260	0.181	0.182	0.065	0.139	0.024	0.140	0.039	0.167	0.056	0.148	0.043	0.128	0.035
200	50	λ		0.226	0.067	0.239	0.072	0.077	0.008	0.212	0.060	0.198	0.052	0.202	0.054	0.199	0.053
		α		0.159	0.062	0.126	0.034	0.100	0.015	0.103	0.022	0.104	0.025	0.104	0.025	0.103	0.022

Table 14. ABs and MSEs of IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 2 at $t = 4$.

t^*	d	Par.	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	λ	1	0.150	0.042	0.212	0.084	0.068	0.007	0.168	0.046	0.167	0.045	0.171	0.039	0.179	0.060
		α		0.869	0.553	0.476	0.819	0.092	0.042	0.992	0.695	0.820	0.673	0.795	0.957	0.102	0.610
40	10	λ		0.076	0.010	0.123	0.023	0.052	0.005	0.121	0.023	0.098	0.016	0.105	0.018	0.118	0.022
		α		0.425	0.306	0.719	0.960	0.118	0.073	0.582	0.582	0.473	0.397	0.519	0.434	0.612	0.686
80	20	λ		0.074	0.009	0.089	0.013	0.038	0.002	0.086	0.011	0.069	0.007	0.065	0.007	0.075	0.009
		α		0.398	0.265	0.490	0.400	0.098	0.035	0.418	0.286	0.324	0.199	0.341	0.192	0.369	0.260
120	30	λ		0.059	0.005	0.065	0.007	0.032	0.002	0.065	0.007	0.060	0.005	0.053	0.005	0.067	0.006
		α		0.290	0.133	0.327	0.192	0.074	0.033	0.318	0.181	0.272	0.112	0.259	0.105	0.312	0.155
200	50	λ		0.051	0.004	0.049	0.004	0.026	0.001	0.060	0.006	0.048	0.004	0.044	0.003	0.043	0.003
		α		0.262	0.110	0.248	0.097	0.051	0.009	0.270	0.122	0.226	0.082	0.204	0.075	0.232	0.090
20	5	λ	0.9	0.192	0.056	0.246	0.099	0.092	0.014	0.175	0.050	0.168	0.047	0.163	0.043	0.192	0.056
		α		0.940	0.698	0.623	0.857	0.173	0.212	1.016	0.711	0.840	0.695	0.799	0.965	0.940	0.698
40	10	λ		0.118	0.023	0.146	0.035	0.067	0.007	0.132	0.033	0.125	0.023	0.110	0.022	0.118	0.023
		α		0.585	0.707	0.878	0.972	0.127	0.103	0.768	1.826	0.610	0.632	0.543	0.529	0.585	0.707
80	20	λ		0.071	0.008	0.087	0.013	0.042	0.003	0.100	0.015	0.078	0.010	0.075	0.009	0.071	0.008
		α		0.340	0.204	0.459	0.381	0.076	0.024	0.435	0.296	0.379	0.227	0.364	0.210	0.340	0.204
120	30	λ		0.059	0.006	0.081	0.010	0.037	0.002	0.073	0.008	0.064	0.006	0.060	0.006	0.059	0.006
		α		0.282	0.136	0.409	0.301	0.071	0.026	0.325	0.183	0.301	0.146	0.259	0.126	0.282	0.136
200	50	λ		0.047	0.003	0.048	0.004	0.028	0.001	0.062	0.006	0.043	0.003	0.042	0.003	0.047	0.003
		α		0.216	0.071	0.218	0.083	0.065	0.015	0.284	0.125	0.196	0.063	0.201	0.073	0.216	0.071
20	5	λ	0.7	0.167	0.058	0.260	0.113	0.095	0.016	0.199	0.070	0.179	0.059	0.135	0.047	0.167	0.058
		α		0.973	0.754	0.601	0.860	0.165	0.103	1.160	0.729	0.900	0.707	0.690	0.977	0.973	0.754
40	10	λ		0.110	0.021	0.145	0.035	0.070	0.008	0.121	0.025	0.134	0.027	0.113	0.020	0.110	0.021
		α		0.590	0.703	0.752	0.994	0.116	0.066	0.633	0.665	0.655	0.791	0.523	0.463	0.590	0.703
80	20	λ		0.075	0.009	0.100	0.017	0.052	0.004	0.088	0.013	0.094	0.014	0.080	0.010	0.075	0.009
		α		0.354	0.226	0.570	0.636	0.075	0.022	0.444	0.370	0.450	0.359	0.318	0.179	0.354	0.226
120	30	λ		0.057	0.006	0.084	0.011	0.037	0.002	0.066	0.007	0.070	0.008	0.065	0.006	0.057	0.006
		α		0.279	0.126	0.414	0.280	0.058	0.013	0.331	0.160	0.354	0.222	0.304	0.146	0.279	0.126
200	50	λ		0.046	0.004	0.051	0.004	0.035	0.002	0.056	0.005	0.055	0.005	0.052	0.004	0.046	0.004
		α		0.208	0.079	0.238	0.096	0.048	0.008	0.276	0.132	0.257	0.100	0.245	0.092	0.208	0.079
20	5	λ	0	0.170	0.060	0.322	0.165	0.106	0.018	0.203	0.089	0.194	0.062	0.196	0.072	0.265	0.092
		α		0.996	0.855	0.811	0.986	0.181	0.166	1.175	0.752	0.968	0.741	0.754	1.000	1.408	0.699
40	10	λ		0.124	0.025	0.149	0.036	0.075	0.009	0.134	0.030	0.142	0.029	0.126	0.026	0.145	0.043
		α		0.702	0.897	0.799	1.075	0.125	0.073	0.636	0.810	0.673	0.835	0.639	0.737	0.780	0.970
80	20	λ		0.090	0.013	0.286	0.021	0.062	0.006	0.106	0.019	0.096	0.020	0.089	0.010	0.117	0.019
		α		0.387	0.268	0.650	0.755	0.073	0.032	0.547	0.519	0.486	0.425	0.354	0.212	0.495	0.546
120	30	λ		0.060	0.007	0.087	0.019	0.047	0.003	0.080	0.010	0.089	0.010	0.075	0.007	0.074	0.009
		α		0.286	0.133	0.449	0.320	0.083	0.030	0.403	0.238	0.394	0.251	0.392	0.149	0.378	0.295
200	50	λ		0.047	0.005	0.062	0.006	0.044	0.003	0.060	0.006	0.058	0.006	0.053	0.004	0.068	0.008
		α		0.241	0.096	0.301	0.156	0.055	0.010	0.295	0.153	0.287	0.293	0.269	0.095	0.324	0.178

Table 15. ABs and MSEs of IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 3 at $t = 5$.

t^*	d	Par.	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	λ	1	0.090	0.011	0.098	0.013	0.098	0.019	0.117	0.019	0.111	0.016	0.107	0.015	0.106	0.015
		α		0.641	0.674	0.495	0.398	0.306	0.138	0.542	0.461	0.477	0.373	0.473	0.419	0.695	0.731
40	10	λ		0.092	0.011	0.086	0.010	0.083	0.012	0.085	0.010	0.086	0.010	0.085	0.010	0.084	0.009
		α		0.526	0.409	0.406	0.233	0.202	0.060	0.408	0.254	0.354	0.218	0.382	0.223	0.448	0.306
80	20	λ		0.100	0.014	0.079	0.009	0.087	0.013	0.085	0.010	0.077	0.008	0.082	0.009	0.086	0.010
		α		0.368	0.225	0.285	0.118	0.154	0.037	0.320	0.156	0.318	0.154	0.306	0.141	0.311	0.152
120	30	λ		0.075	0.007	0.066	0.006	0.057	0.006	0.073	0.008	0.072	0.007	0.072	0.007	0.064	0.005
		α		0.338	0.190	0.236	0.105	0.152	0.038	0.328	0.158	0.283	0.128	0.281	0.132	0.358	0.185
200	50	λ		0.068	0.006	0.064	0.005	0.055	0.005	0.075	0.008	0.068	0.007	0.068	0.007	0.076	0.008
		α		0.324	0.193	0.242	0.113	0.134	0.035	0.287	0.138	0.298	0.148	0.249	0.111	0.299	0.147
20	5	λ	0.9	0.078	0.008	0.066	0.006	0.098	0.017	0.070	0.007	0.069	0.006	0.071	0.006	0.079	0.008
		α		0.524	0.419	0.433	0.241	0.300	0.150	0.468	0.352	0.452	0.292	0.438	0.249	0.518	0.435
40	10	λ		0.084	0.009	0.069	0.006	0.078	0.010	0.091	0.010	0.075	0.007	0.084	0.009	0.090	0.010
		α		0.498	0.386	0.356	0.216	0.236	0.122	0.476	0.351	0.376	0.256	0.381	0.240	0.497	0.410
80	20	λ		0.092	0.011	0.080	0.008	0.053	0.005	0.078	0.007	0.085	0.009	0.085	0.009	0.086	0.011
		α		0.439	0.290	0.343	0.186	0.184	0.060	0.412	0.244	0.387	0.226	0.323	0.177	0.386	0.243
120	30	λ		0.078	0.008	0.065	0.006	0.085	0.012	0.083	0.008	0.073	0.007	0.075	0.007	0.086	0.009
		α		0.355	0.203	0.303	0.150	0.143	0.042	0.371	0.234	0.318	0.161	0.319	0.164	0.391	0.250
200	50	λ		0.060	0.005	0.059	0.005	0.062	0.007	0.071	0.007	0.067	0.006	0.064	0.006	0.073	0.007
		α		0.262	0.153	0.217	0.099	0.103	0.020	0.303	0.163	0.262	0.133	0.240	0.103	0.305	0.166
20	5	λ	0.7	0.105	0.015	0.082	0.008	0.077	0.010	0.088	0.011	0.077	0.008	0.087	0.010	0.098	0.013
		α		0.598	0.560	0.496	0.341	0.294	0.130	0.439	0.257	0.457	0.283	0.469	0.295	0.510	0.349
40	10	λ		0.091	0.011	0.071	0.008	0.081	0.014	0.102	0.015	0.105	0.017	0.096	0.013	0.110	0.021
		α		0.541	0.509	0.430	0.304	0.223	0.081	0.504	0.396	0.440	0.329	0.437	0.345	0.520	0.426
80	20	λ		0.077	0.007	0.073	0.007	0.058	0.006	0.075	0.006	0.079	0.008	0.077	0.007	0.078	0.007
		α		0.407	0.289	0.326	0.192	0.129	0.034	0.412	0.248	0.352	0.210	0.323	0.185	0.426	0.273
120	30	λ		0.078	0.007	0.078	0.008	0.069	0.007	0.074	0.008	0.074	0.007	0.075	0.007	0.079	0.009
		α		0.426	0.272	0.313	0.163	0.155	0.052	0.357	0.197	0.307	0.142	0.310	0.149	0.386	0.220
200	50	λ		0.059	0.005	0.061	0.005	0.063	0.006	0.081	0.009	0.075	0.007	0.074	0.007	0.081	0.009
		α		0.228	0.105	0.224	0.082	0.099	0.018	0.286	0.139	0.285	0.136	0.282	0.131	0.292	0.143
20	5	λ	0	0.101	0.015	0.089	0.012	0.092	0.016	0.100	0.014	0.107	0.016	0.103	0.014	0.096	0.014
		α		0.605	0.584	0.469	0.304	0.279	0.122	0.476	0.429	0.400	0.252	0.392	0.220	0.501	0.529
40	10	λ		0.100	0.012	0.081	0.008	0.063	0.008	0.091	0.010	0.088	0.009	0.090	0.009	0.104	0.014
		α		0.474	0.350	0.443	0.252	0.244	0.084	0.277	0.149	0.350	0.197	0.341	0.182	0.316	0.185
80	20	λ		0.085	0.009	0.071	0.006	0.054	0.005	0.071	0.007	0.074	0.007	0.076	0.007	0.065	0.005
		α		0.391	0.238	0.332	0.162	0.151	0.042	0.306	0.153	0.350	0.181	0.351	0.186	0.301	0.147
120	30	λ		0.075	0.009	0.073	0.008	0.055	0.006	0.080	0.009	0.075	0.009	0.071	0.007	0.083	0.009
		α		0.296	0.170	0.237	0.095	0.168	0.041	0.392	0.206	0.332	0.162	0.332	0.171	0.383	0.206
200	50	λ		0.063	0.005	0.058	0.005	0.066	0.007	0.071	0.007	0.062	0.005	0.058	0.004	0.068	0.006
		α		0.262	0.134	0.192	0.071	0.115	0.028	0.320	0.188	0.271	0.120	0.254	0.100	0.308	0.183

Table 16. ABs and MSEs of IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 4 at $t = 6$.

t^*	d	Par.	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	λ	1	1.443	7.341	0.726	1.446	0.179	0.079	1.459	6.168	1.146	4.016	1.113	2.884	2.157	9.553
		α		5.281	38.705	1.839	7.422	0.969	1.477	1.125	2.141	1.397	5.322	1.513	5.719	1.573	6.671
40	10	λ		0.640	1.234	0.575	0.865	0.386	0.574	1.242	3.634	1.227	3.161	0.955	1.745	1.594	6.099
		α		4.533	32.262	1.838	10.110	1.403	6.539	1.200	3.993	1.018	3.624	1.462	8.169	1.094	2.444
80	20	λ		0.407	0.328	0.287	0.152	0.308	0.355	0.746	1.197	0.686	0.970	0.505	0.544	0.823	1.403
		α		3.275	18.302	1.271	3.220	0.766	0.907	0.888	2.196	0.801	2.015	0.895	2.695	1.096	4.449
120	30	λ		0.295	0.185	0.359	0.325	0.261	0.247	0.395	0.380	0.284	0.188	0.316	0.254	0.514	0.617
		α		2.833	11.744	1.548	4.516	0.727	1.806	1.490	6.959	1.452	6.502	0.973	2.329	1.481	7.128
200	50	λ		0.336	0.214	0.338	0.254	0.185	0.096	0.642	0.848	0.424	0.391	0.408	0.355	0.708	1.000
		α		2.753	13.607	1.873	7.419	0.866	3.059	0.740	0.892	0.813	1.833	0.919	2.188	0.675	0.854
20	5	λ	0.9	1.315	5.715	0.501	1.066	0.507	1.148	1.149	3.213	0.705	1.554	0.946	2.487	1.768	7.238
		α		4.549	32.005	2.083	9.626	0.907	1.451	1.223	4.037	1.550	6.928	1.548	6.464	1.190	3.621
40	10	λ		0.363	0.406	0.309	0.161	0.436	0.654	1.062	2.306	0.591	0.758	0.666	0.906	1.505	3.730
		α		4.823	35.106	2.384	14.411	0.545	0.451	1.561	9.229	1.006	4.144	0.842	2.321	1.593	9.858
80	20	λ		0.299	0.174	0.360	0.279	0.272	0.225	0.721	0.975	0.620	0.650	0.593	0.601	0.730	1.080
		α		3.764	20.664	1.691	7.988	0.864	2.067	1.164	2.961	0.816	1.062	1.042	3.446	1.349	5.598
120	30	λ		0.355	0.314	0.344	0.228	0.126	0.056	0.534	0.595	0.562	0.622	0.502	0.568	0.735	1.061
		α		2.952	14.555	1.935	6.915	0.612	0.615	0.924	1.945	1.170	5.821	1.006	3.249	1.205	4.761
200	50	λ		0.444	0.420	0.251	0.104	0.185	0.095	0.525	0.583	0.424	0.377	0.580	0.741	0.619	0.773
		α		1.928	6.808	1.174	2.640	0.391	0.226	0.774	1.213	0.952	2.233	0.815	2.062	1.018	3.695
20	5	λ	0.7	1.773	10.393	0.548	1.004	0.265	0.235	0.768	1.787	1.027	3.950	1.228	4.021	1.467	5.177
		α		5.775	42.338	1.971	10.258	1.694	8.110	1.029	1.926	1.007	1.799	0.953	1.837	1.068	2.905
40	10	λ		0.551	0.670	0.512	0.544	0.391	0.486	0.999	2.382	0.980	1.939	0.840	1.530	1.463	4.330
		α		3.239	21.165	1.330	3.650	1.062	3.781	1.127	2.176	0.805	0.929	1.409	5.657	1.121	3.067
80	20	λ		0.444	0.459	0.400	0.410	0.167	0.061	0.840	1.227	0.628	0.785	0.577	0.649	1.029	1.837
		α		2.677	14.119	1.328	4.833	0.930	1.522	0.867	1.972	1.044	2.846	0.936	3.121	0.673	0.905
120	30	λ		0.433	0.381	0.365	0.344	0.218	0.179	0.841	1.222	0.590	0.724	0.670	0.873	0.884	1.286
		α		3.034	16.533	1.571	5.464	0.621	0.791	0.924	1.878	0.791	1.168	0.670	0.954	0.803	1.711
200	50	λ		0.449	0.486	0.309	0.259	0.170	0.096	0.682	0.975	0.551	0.678	0.592	0.793	0.738	0.986
		α		1.602	5.070	0.869	1.101	0.618	0.518	0.846	2.183	0.560	0.604	0.865	2.085	1.017	3.130
20	5	λ	0	1.656	8.650	1.213	5.127	0.908	2.930	1.391	5.162	1.537	6.886	1.572	6.507	2.620	13.033
		α		5.270	38.828	1.587	6.836	0.702	0.681	1.090	2.618	0.973	2.494	0.872	1.584	1.093	3.643
40	10	λ		0.315	0.205	0.450	0.600	0.448	0.684	0.697	1.261	0.818	1.484	0.481	0.594	0.887	1.915
		α		4.767	32.658	2.487	14.327	1.125	2.782	0.751	0.827	0.787	1.094	2.046	10.911	0.850	2.075
80	20	λ		0.345	0.210	0.358	0.326	0.391	0.418	0.938	2.696	0.569	0.668	0.519	0.537	1.049	3.272
		α		2.652	14.085	1.242	3.618	0.536	0.605	0.793	1.532	0.722	0.808	0.984	3.263	0.874	2.223
120	30	λ		0.340	0.201	0.322	0.214	0.128	0.063	0.484	0.467	0.534	0.604	0.492	0.534	0.559	0.526
		α		1.921	7.059	0.868	1.513	0.798	0.965	0.775	1.135	0.837	1.596	0.703	0.965	0.964	1.991
200	50	λ		0.402	0.384	0.452	0.584	0.172	0.089	0.743	1.052	0.702	0.886	0.643	0.855	0.807	1.097
		α		2.152	8.496	1.150	2.775	0.691	0.818	0.567	0.720	0.823	3.170	0.813	2.555	0.573	0.853

Table 17. ABs and MSEs of different entropy measures for IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 1 at $t = 4$.

t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	β_1	1	0.1349	0.0375	0.1339	0.0356	0.1979	0.0669	0.1429	0.0403	0.1362	0.0368	0.1342	0.0372	0.1540	0.0470
		β_2	1	0.0766	0.0145	0.0755	0.0130	0.1171	0.0263	0.0816	0.0150	0.0770	0.0136	0.0761	0.0142	0.0890	0.0182
		β_3	1	0.1532	0.0580	0.1510	0.0518	0.2342	0.1052	0.1632	0.0601	0.1540	0.0542	0.1521	0.0567	0.1781	0.0728
		β_4	1	0.1013	0.0230	0.1003	0.0213	0.1518	0.0416	0.1077	0.0244	0.1021	0.0221	0.1007	0.0227	0.1167	0.0289
40	10	β_1	1	0.0910	0.0109	0.0883	0.0107	0.0864	0.0117	0.0936	0.0111	0.0926	0.0108	0.0907	0.0107	0.0892	0.0104
		β_2	1	0.0455	0.0028	0.0443	0.0028	0.0450	0.0033	0.0469	0.0028	0.0461	0.0027	0.0454	0.0027	0.0449	0.0027
		β_3	1	0.0910	0.0112	0.0885	0.0110	0.0900	0.0132	0.0938	0.0114	0.0922	0.0109	0.0907	0.0109	0.0897	0.0108
		β_4	1	0.0643	0.0055	0.0625	0.0054	0.0623	0.0062	0.0662	0.0056	0.0653	0.0054	0.0641	0.0054	0.0632	0.0053
80	20	β_1	1	0.0741	0.0077	0.0751	0.0081	0.0858	0.0117	0.0865	0.0104	0.0800	0.0089	0.0760	0.0078	0.0873	0.0108
		β_2	1	0.0377	0.0021	0.0384	0.0022	0.0452	0.0034	0.0447	0.0029	0.0409	0.0024	0.0386	0.0021	0.0452	0.0031
		β_3	1	0.0753	0.0082	0.0769	0.0089	0.0904	0.0136	0.0893	0.0116	0.0817	0.0097	0.0772	0.0083	0.0904	0.0122
		β_4	1	0.0528	0.0040	0.0537	0.0042	0.0623	0.0063	0.0621	0.0055	0.0571	0.0046	0.0541	0.0040	0.0628	0.0057
120	30	β_1	1	0.0443	0.0027	0.0463	0.0029	0.0486	0.0036	0.0484	0.0031	0.0471	0.0029	0.0471	0.0029	0.0487	0.0031
		β_2	1	0.0219	0.0007	0.0230	0.0007	0.0246	0.0010	0.0243	0.0008	0.0234	0.0007	0.0234	0.0007	0.0244	0.0008
		β_3	1	0.0439	0.0026	0.0459	0.0029	0.0493	0.0038	0.0485	0.0031	0.0468	0.0029	0.0468	0.0029	0.0489	0.0032
		β_4	1	0.0312	0.0013	0.0326	0.0014	0.0346	0.0019	0.0343	0.0015	0.0332	0.0014	0.0332	0.0014	0.0345	0.0016
200	50	β_1	1	0.0437	0.0035	0.0441	0.0033	0.0437	0.0035	0.0469	0.0035	0.0449	0.0032	0.0443	0.0031	0.0470	0.0036
		β_2	1	0.0219	0.0008	0.0220	0.0008	0.0219	0.0009	0.0236	0.0008	0.0225	0.0008	0.0222	0.0008	0.0237	0.0009
		β_3	1	0.0437	0.0034	0.0440	0.0033	0.0438	0.0036	0.0471	0.0035	0.0449	0.0032	0.0443	0.0031	0.0473	0.0036
		β_4	1	0.0309	0.0016	0.0311	0.0015	0.0309	0.0017	0.0332	0.0017	0.0317	0.0015	0.0313	0.0015	0.0334	0.0017
20	5	β_1	0.9	0.1538	0.0359	0.1492	0.0333	0.1809	0.0509	0.1557	0.0370	0.1497	0.0340	0.1495	0.0336	0.1616	0.0402
		β_2	0.9	0.0877	0.0116	0.0852	0.0109	0.1063	0.0178	0.0890	0.0121	0.0855	0.0111	0.0853	0.0110	0.0927	0.0130
		β_3	0.9	0.1753	0.0587	0.1704	0.0545	0.2147	0.0836	0.1773	0.0609	0.1708	0.0558	0.1702	0.0551	0.1845	0.0665
		β_4	0.9	0.1157	0.0199	0.1123	0.0184	0.1393	0.0327	0.1171	0.0208	0.1127	0.0189	0.1124	0.0187	0.1217	0.0222
40	10	β_1	0.9	0.1070	0.0150	0.1062	0.0144	0.1310	0.0221	0.1122	0.0163	0.1082	0.0152	0.1073	0.0149	0.1122	0.0160
		β_2	0.9	0.0541	0.0039	0.0540	0.0038	0.0676	0.0060	0.0572	0.0043	0.0550	0.0040	0.0546	0.0039	0.0572	0.0042
		β_3	0.9	0.1077	0.0152	0.1073	0.0147	0.1358	0.0236	0.1129	0.0166	0.1089	0.0155	0.1080	0.0152	0.1131	0.0164
		β_4	0.9	0.0760	0.0076	0.0754	0.0072	0.0941	0.0110	0.0796	0.0083	0.0768	0.0076	0.0761	0.0075	0.0796	0.0082
80	20	β_1	0.9	0.0643	0.0073	0.0641	0.0071	0.0761	0.0104	0.0718	0.0089	0.0656	0.0076	0.0643	0.0072	0.0720	0.0089
		β_2	0.9	0.0326	0.0018	0.0325	0.0018	0.0393	0.0026	0.0365	0.0023	0.0332	0.0019	0.0326	0.0018	0.0366	0.0023
		β_3	0.9	0.0648	0.0075	0.0645	0.0074	0.0793	0.0112	0.0728	0.0093	0.0662	0.0079	0.0649	0.0074	0.0729	0.0093
		β_4	0.9	0.0457	0.0036	0.0455	0.0035	0.0550	0.0051	0.0512	0.0044	0.0467	0.0038	0.0457	0.0036	0.0513	0.0044
120	30	β_1	0.9	0.0557	0.0047	0.0545	0.0044	0.0628	0.0062	0.0596	0.0053	0.0564	0.0047	0.0560	0.0046	0.0601	0.0054
		β_2	0.9	0.0281	0.0012	0.0274	0.0011	0.0318	0.0016	0.0302	0.0014	0.0285	0.0012	0.0282	0.0012	0.0305	0.0014
		β_3	0.9	0.0558	0.0047	0.0544	0.0044	0.0633	0.0064	0.0599	0.0054	0.0565	0.0048	0.0560	0.0047	0.0604	0.0055
		β_4	0.9	0.0394	0.0023	0.0384	0.0022	0.0446	0.0031	0.0422	0.0026	0.0399	0.0024	0.0395	0.0023	0.0426	0.0027
200	50	β_1	0.9	0.0372	0.0021	0.0369	0.0021	0.0429	0.0031	0.0415	0.0025	0.0385	0.0022	0.0381	0.0021	0.0421	0.0026
		β_2	0.9	0.0187	0.0005	0.0185	0.0005	0.0216	0.0008	0.0209	0.0006	0.0193	0.0006	0.0191	0.0005	0.0212	0.0007
		β_3	0.9	0.0371	0.0021	0.0369	0.0021	0.0433	0.0031	0.0416	0.0025	0.0386	0.0022	0.0380	0.0021	0.0422	0.0026
		β_4	0.9	0.0263	0.0010	0.0261	0.0010	0.0305	0.0015	0.0294	0.0013	0.0273	0.0011	0.0269	0.0010	0.0299	0.0013
20	5	β_1	0.7	0.1527	0.0397	0.1476	0.0364	0.2281	0.0789	0.1617	0.0432	0.1535	0.0393	0.1507	0.0374	0.1692	0.0462
		β_2	0.7	0.0873	0.0129	0.0843	0.0118	0.1330	0.0278	0.0924	0.0140	0.0876	0.0127	0.0860	0.0121	0.0971	0.0150
		β_3	0.7	0.1742	0.0647	0.1685	0.0594	0.2643	0.1262	0.1847	0.0698	0.1749	0.0637	0.1717	0.0608	0.1931	0.0751
		β_4	0.7	0.1149	0.0220	0.1112	0.0202	0.1725	0.0504	0.1218	0.0240	0.1154	0.0218	0.1133	0.0208	0.1276	0.0258
40	10	β_1	0.7	0.1466	0.0258	0.1444	0.0249	0.1622	0.0359	0.1534	0.0287	0.1482	0.0262	0.1471	0.0255	0.1540	0.0284
		β_2	0.7	0.0748	0.0068	0.0738	0.0066	0.0837	0.0099	0.0784	0.0077	0.0757	0.0070	0.0752	0.0068	0.0787	0.0076

Continued on next page

t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
80	20	β_3	0.7	0.1482	0.0264	0.1460	0.0254	0.1682	0.0387	0.1551	0.0294	0.1496	0.0267	0.1486	0.0260	0.1556	0.0290
		β_4	0.7	0.1041	0.0130	0.1027	0.0125	0.1177	0.0188	0.1090	0.0145	0.1051	0.0132	0.1045	0.0129	0.1094	0.0143
		β_1	0.7	0.0906	0.0113	0.0887	0.0109	0.0953	0.0127	0.0957	0.0124	0.0915	0.0115	0.0893	0.0110	0.0964	0.0125
		β_2	0.7	0.0460	0.0030	0.0451	0.0029	0.0487	0.0034	0.0487	0.0033	0.0465	0.0031	0.0454	0.0029	0.0491	0.0034
	30	β_3	0.7	0.0913	0.0115	0.0893	0.0110	0.0972	0.0131	0.0966	0.0127	0.0922	0.0117	0.0900	0.0111	0.0972	0.0128
		β_4	0.7	0.0643	0.0057	0.0630	0.0055	0.0682	0.0065	0.0681	0.0063	0.0650	0.0058	0.0635	0.0055	0.0685	0.0064
		β_1	0.7	0.0852	0.0102	0.0840	0.0099	0.0942	0.0127	0.0904	0.0113	0.0867	0.0104	0.0858	0.0101	0.0908	0.0113
		β_2	0.7	0.0432	0.0026	0.0426	0.0025	0.0479	0.0032	0.0459	0.0028	0.0440	0.0026	0.0435	0.0026	0.0461	0.0029
120	30	β_3	0.7	0.0856	0.0103	0.0843	0.0100	0.0956	0.0131	0.0909	0.0114	0.0871	0.0104	0.0862	0.0102	0.0913	0.0114
		β_4	0.7	0.0603	0.0051	0.0595	0.0050	0.0672	0.0064	0.0641	0.0057	0.0614	0.0052	0.0608	0.0051	0.0644	0.0057
		β_1	0.7	0.0476	0.0034	0.0467	0.0032	0.0514	0.0042	0.0513	0.0038	0.0487	0.0035	0.0477	0.0033	0.0519	0.0038
		β_2	0.7	0.0240	0.0009	0.0236	0.0008	0.0260	0.0011	0.0260	0.0010	0.0246	0.0009	0.0241	0.0009	0.0263	0.0010
	50	β_3	0.7	0.0477	0.0034	0.0467	0.0033	0.0520	0.0043	0.0515	0.0039	0.0489	0.0035	0.0479	0.0033	0.0522	0.0039
		β_4	0.7	0.0337	0.0017	0.0330	0.0016	0.0366	0.0021	0.0363	0.0019	0.0345	0.0017	0.0338	0.0016	0.0368	0.0019
		β_1	0	0.1875	0.0549	0.1817	0.0507	0.2487	0.0992	0.1985	0.0603	0.1897	0.0550	0.1855	0.0527	0.2058	0.0659
		β_2	0	0.1071	0.0177	0.1038	0.0165	0.1440	0.0337	0.1134	0.0196	0.1083	0.0179	0.1060	0.0171	0.1176	0.0215
200	5	β_3	0	0.2128	0.0898	0.2063	0.0830	0.2827	0.1625	0.2254	0.0985	0.2153	0.0901	0.2107	0.0862	0.2338	0.1080
		β_4	0	0.1410	0.0311	0.1366	0.0288	0.1851	0.0612	0.1490	0.0344	0.1424	0.0314	0.1394	0.0300	0.1547	0.0378
		β_1	0	0.1162	0.0197	0.1145	0.0188	0.1138	0.0198	0.1240	0.0220	0.1186	0.0205	0.1152	0.0192	0.1222	0.0213
		β_2	0	0.0590	0.0051	0.0582	0.0049	0.0579	0.0052	0.0632	0.0058	0.0604	0.0054	0.0586	0.0050	0.0623	0.0056
	10	β_3	0	0.1165	0.0198	0.1147	0.0189	0.1147	0.0202	0.1246	0.0223	0.1191	0.0207	0.1155	0.0193	0.1228	0.0215
		β_4	0	0.0822	0.0099	0.0810	0.0095	0.0808	0.0101	0.0879	0.0112	0.0840	0.0103	0.0815	0.0096	0.0867	0.0108
		β_1	0	0.1123	0.0211	0.1101	0.0203	0.1166	0.0256	0.1160	0.0218	0.1120	0.0208	0.1100	0.0201	0.1166	0.0217
		β_2	0	0.0572	0.0055	0.0560	0.0052	0.0597	0.0067	0.0592	0.0056	0.0570	0.0053	0.0560	0.0051	0.0594	0.0056
400	20	β_3	0	0.1135	0.0215	0.1112	0.0206	0.1187	0.0264	0.1171	0.0220	0.1132	0.0210	0.1111	0.0203	0.1177	0.0219
		β_4	0	0.0797	0.0106	0.0782	0.0101	0.0832	0.0132	0.0823	0.0110	0.0796	0.0105	0.0782	0.0101	0.0828	0.0110
		β_1	0	0.0777	0.0111	0.0762	0.0107	0.0803	0.0120	0.0826	0.0119	0.0793	0.0112	0.0778	0.0109	0.0829	0.0120
		β_2	0	0.0394	0.0029	0.0386	0.0028	0.0408	0.0031	0.0420	0.0031	0.0403	0.0029	0.0395	0.0028	0.0422	0.0031
	30	β_3	0	0.0781	0.0113	0.0764	0.0108	0.0813	0.0123	0.0831	0.0121	0.0797	0.0113	0.0781	0.0110	0.0834	0.0121
		β_4	0	0.0550	0.0056	0.0539	0.0054	0.0572	0.0061	0.0585	0.0060	0.0561	0.0056	0.0550	0.0055	0.0587	0.0061
		β_1	0	0.0604	0.0054	0.0591	0.0051	0.0578	0.0050	0.0643	0.0063	0.0613	0.0056	0.0604	0.0054	0.0652	0.0064
		β_2	0	0.0306	0.0014	0.0299	0.0013	0.0292	0.0013	0.0326	0.0016	0.0310	0.0014	0.0306	0.0014	0.0330	0.0016
800	50	β_3	0	0.0608	0.0055	0.0594	0.0052	0.0584	0.0051	0.0648	0.0064	0.0617	0.0057	0.0608	0.0055	0.0657	0.0065
		β_4	0	0.0428	0.0027	0.0419	0.0026	0.0411	0.0025	0.0456	0.0032	0.0435	0.0028	0.0428	0.0027	0.0463	0.0033

Table 18. ABs and MSEs of different entropy measures for IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 2 at $t = 4$.

t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	β_1	1	0.1223	0.0234	0.1294	0.0270	0.1540	0.0385	0.1356	0.0268	0.1277	0.0267	0.1146	0.0207	0.1426	0.0289
		β_2	1	0.1005	0.0170	0.1052	0.0193	0.1345	0.0316	0.1130	0.0200	0.1065	0.0201	0.0935	0.0148	0.1193	0.0218
		β_3	1	0.2010	0.0679	0.2103	0.0770	0.2690	0.1266	0.2260	0.0799	0.2130	0.0804	0.1869	0.0593	0.2386	0.0871
		β_4	1	0.1107	0.0198	0.1164	0.0226	0.1437	0.0348	0.1236	0.0230	0.1164	0.0230	0.1034	0.0174	0.1303	0.0250
40	10	β_1	1	0.0917	0.0145	0.0973	0.0159	0.1153	0.0205	0.0906	0.0127	0.0886	0.0131	0.0897	0.0131	0.0930	0.0133
		β_2	1	0.0736	0.0100	0.0786	0.0112	0.0951	0.0147	0.0722	0.0086	0.0708	0.0091	0.0717	0.0090	0.0743	0.0090
		β_3	1	0.1473	0.0398	0.1573	0.0446	0.1901	0.0586	0.1444	0.0346	0.1416	0.0363	0.1435	0.0362	0.1487	0.0360
		β_4	1	0.0821	0.0119	0.0873	0.0132	0.1046	0.0173	0.0808	0.0104	0.0791	0.0108	0.0801	0.0108	0.0831	0.0109
80	20	β_1	1	0.0709	0.0069	0.0718	0.0071	0.0825	0.0099	0.0701	0.0069	0.0683	0.0067	0.0694	0.0068	0.0696	0.0068
		β_2	1	0.0555	0.0043	0.0562	0.0044	0.0665	0.0067	0.0549	0.0044	0.0534	0.0042	0.0542	0.0042	0.0546	0.0043
		β_3	1	0.1110	0.0173	0.1123	0.0177	0.1330	0.0266	0.1099	0.0174	0.1067	0.0167	0.1083	0.0169	0.1091	0.0170
		β_4	1	0.0627	0.0055	0.0635	0.0056	0.0741	0.0081	0.0620	0.0055	0.0604	0.0053	0.0613	0.0054	0.0616	0.0054
120	30	β_1	1	0.0567	0.0045	0.0560	0.0044	0.0536	0.0047	0.0547	0.0041	0.0566	0.0045	0.0564	0.0045	0.0549	0.0041
		β_2	1	0.0443	0.0028	0.0437	0.0028	0.0426	0.0030	0.0427	0.0025	0.0443	0.0028	0.0441	0.0028	0.0428	0.0026
		β_3	1	0.0886	0.0112	0.0875	0.0110	0.0852	0.0121	0.0854	0.0102	0.0885	0.0113	0.0882	0.0113	0.0856	0.0102
		β_4	1	0.0501	0.0035	0.0495	0.0035	0.0478	0.0038	0.0483	0.0032	0.0500	0.0036	0.0499	0.0036	0.0485	0.0032
200	50	β_1	1	0.0368	0.0021	0.0370	0.0021	0.0424	0.0026	0.0381	0.0022	0.0372	0.0021	0.0368	0.0021	0.0380	0.0021
		β_2	1	0.0284	0.0013	0.0285	0.0012	0.0332	0.0016	0.0295	0.0013	0.0287	0.0012	0.0284	0.0012	0.0294	0.0013
		β_3	1	0.0567	0.0050	0.0570	0.0049	0.0664	0.0065	0.0590	0.0052	0.0573	0.0049	0.0568	0.0049	0.0588	0.0052
		β_4	1	0.0323	0.0016	0.0325	0.0016	0.0375	0.0020	0.0335	0.0017	0.0326	0.0016	0.0324	0.0016	0.0334	0.0017
20	5	β_1	0.9	0.1773	0.0471	0.1792	0.0481	0.2262	0.0791	0.2028	0.0737	0.2017	0.0663	0.1724	0.0473	0.1946	0.0703
		β_2	0.9	0.1520	0.0382	0.1529	0.0381	0.2078	0.0732	0.1836	0.0687	0.1775	0.0570	0.1482	0.0386	0.1770	0.0666
		β_3	0.9	0.3051	0.1528	0.3058	0.1523	0.4157	0.2929	0.3673	0.2748	0.3550	0.2281	0.2964	0.1544	0.3540	0.2663
		β_4	0.9	0.1641	0.0421	0.1651	0.0425	0.2162	0.0755	0.1922	0.0704	0.1885	0.0608	0.1594	0.0424	0.1848	0.0677
40	10	β_1	0.9	0.0981	0.0173	0.0975	0.0169	0.1002	0.0162	0.0960	0.0143	0.0922	0.0139	0.0972	0.0164	0.0959	0.0144
		β_2	0.9	0.0789	0.0123	0.0780	0.0119	0.0824	0.0115	0.0770	0.0096	0.0730	0.0091	0.0771	0.0113	0.0771	0.0097
		β_3	0.9	0.1578	0.0492	0.1559	0.0477	0.1648	0.0462	0.1540	0.0384	0.1460	0.0365	0.1543	0.0451	0.1542	0.0389
		β_4	0.9	0.0879	0.0145	0.0870	0.0141	0.0908	0.0136	0.0859	0.0117	0.0820	0.0112	0.0865	0.0135	0.0859	0.0118
80	20	β_1	0.9	0.0949	0.0130	0.0975	0.0135	0.1043	0.0158	0.0997	0.0144	0.0958	0.0132	0.0953	0.0129	0.0982	0.0140
		β_2	0.9	0.0749	0.0081	0.0772	0.0086	0.0847	0.0108	0.0794	0.0093	0.0758	0.0084	0.0753	0.0082	0.0782	0.0091
		β_3	0.9	0.1497	0.0325	0.1544	0.0343	0.1694	0.0431	0.1587	0.0371	0.1515	0.0335	0.1506	0.0326	0.1564	0.0363
		β_4	0.9	0.0842	0.0102	0.0867	0.0107	0.0939	0.0130	0.0889	0.0115	0.0851	0.0105	0.0846	0.0102	0.0875	0.0112
120	30	β_1	0.9	0.0687	0.0082	0.0698	0.0082	0.0765	0.0093	0.0727	0.0091	0.0721	0.0089	0.0713	0.0088	0.0729	0.0089
		β_2	0.9	0.0544	0.0054	0.0553	0.0054	0.0612	0.0062	0.0578	0.0060	0.0571	0.0058	0.0564	0.0058	0.0579	0.0059
		β_3	0.9	0.1089	0.0216	0.1106	0.0217	0.1223	0.0249	0.1156	0.0240	0.1143	0.0234	0.1129	0.0230	0.1158	0.0236
		β_4	0.9	0.0611	0.0066	0.0621	0.0067	0.0684	0.0076	0.0648	0.0074	0.0642	0.0072	0.0634	0.0071	0.0649	0.0072
200	50	β_1	0.9	0.0537	0.0046	0.0537	0.0046	0.0569	0.0057	0.0531	0.0047	0.0524	0.0044	0.0530	0.0045	0.0534	0.0047
		β_2	0.9	0.0417	0.0028	0.0417	0.0028	0.0450	0.0037	0.0413	0.0029	0.0407	0.0027	0.0412	0.0028	0.0416	0.0029
		β_3	0.9	0.0834	0.0112	0.0834	0.0111	0.0901	0.0148	0.0827	0.0114	0.0814	0.0107	0.0824	0.0111	0.0831	0.0115
		β_4	0.9	0.0473	0.0036	0.0473	0.0036	0.0506	0.0046	0.0468	0.0036	0.0462	0.0034	0.0467	0.0035	0.0471	0.0036
20	5	β_1	0.7	0.1614	0.0368	0.1693	0.0401	0.1749	0.0492	0.1352	0.0289	0.1473	0.0315	0.1463	0.0311	0.1403	0.0292
		β_2	0.7	0.1300	0.0256	0.1355	0.0274	0.1519	0.0405	0.1091	0.0204	0.1179	0.0218	0.1159	0.0208	0.1130	0.0204
		β_3	0.7	0.2600	0.1024	0.2709	0.1098	0.3037	0.1622	0.2182	0.0818	0.2359	0.0871	0.2318	0.0831	0.2261	0.0815
		β_4	0.7	0.1445	0.0304	0.1510	0.0328	0.1626	0.0443	0.1212	0.0241	0.1315	0.0260	0.1299	0.0252	0.1257	0.0242
40	10	β_1	0.7	0.1034	0.0180	0.1054	0.0200	0.1255	0.0257	0.1157	0.0187	0.1104	0.0204	0.1081	0.0199	0.1164	0.0194
		β_2	0.7	0.0834	0.0126	0.0854	0.0143	0.1038	0.0188	0.0927	0.0127	0.0895	0.0147	0.0877	0.0143	0.0942	0.0135
		β_3	0.7	0.1668	0.0505	0.1708	0.0573	0.2076	0.0750	0.1854	0.0506	0.1789	0.0586	0.1755	0.0573	0.1885	0.0541

Continued on next page

t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
80	20	β_4	0.7	0.0927	0.0150	0.0947	0.0168	0.1139	0.0218	0.1035	0.0153	0.0992	0.0172	0.0972	0.0168	0.1046	0.0161
		β_1	0.7	0.1004	0.0139	0.1027	0.0149	0.1165	0.0192	0.1083	0.0170	0.1027	0.0150	0.1024	0.0148	0.1082	0.0170
		β_2	0.7	0.0826	0.0097	0.0847	0.0104	0.0974	0.0138	0.0900	0.0120	0.0848	0.0105	0.0844	0.0103	0.0900	0.0121
		β_3	0.7	0.1651	0.0387	0.1694	0.0416	0.1947	0.0550	0.1801	0.0481	0.1695	0.0421	0.1689	0.0413	0.1800	0.0483
		β_4	0.7	0.0910	0.0116	0.0932	0.0124	0.1064	0.0162	0.0987	0.0143	0.0932	0.0126	0.0930	0.0123	0.0986	0.0143
120	30	β_1	0.7	0.0734	0.0096	0.0729	0.0098	0.0810	0.0103	0.0731	0.0094	0.0727	0.0095	0.0731	0.0095	0.0728	0.0093
		β_2	0.7	0.0581	0.0062	0.0578	0.0064	0.0647	0.0068	0.0580	0.0062	0.0576	0.0062	0.0579	0.0062	0.0578	0.0061
		β_3	0.7	0.1161	0.0250	0.1156	0.0258	0.1295	0.0271	0.1160	0.0247	0.1151	0.0247	0.1158	0.0248	0.1157	0.0244
		β_4	0.7	0.0652	0.0077	0.0649	0.0079	0.0723	0.0083	0.0650	0.0076	0.0646	0.0076	0.0650	0.0076	0.0648	0.0075
		β_1	0.7	0.0585	0.0047	0.0560	0.0045	0.0587	0.0055	0.0535	0.0044	0.0550	0.0044	0.0555	0.0044	0.0535	0.0044
200	50	β_2	0.7	0.0456	0.0029	0.0436	0.0028	0.0463	0.0035	0.0418	0.0027	0.0428	0.0027	0.0432	0.0027	0.0417	0.0027
		β_3	0.7	0.0912	0.0117	0.0872	0.0111	0.0927	0.0139	0.0836	0.0108	0.0856	0.0107	0.0864	0.0108	0.0834	0.0108
		β_4	0.7	0.0517	0.0037	0.0494	0.0035	0.0521	0.0044	0.0473	0.0034	0.0485	0.0034	0.0490	0.0034	0.0472	0.0034
		β_1	0	0.2469	0.0882	0.2362	0.0843	0.2950	0.1177	0.2484	0.0898	0.2445	0.0893	0.2505	0.0880	0.2573	0.1002
		β_2	0	0.2161	0.0791	0.2065	0.0759	0.2722	0.1143	0.2209	0.0789	0.2150	0.0799	0.2179	0.0771	0.2351	0.0959
20	5	β_3	0	0.4321	0.3163	0.4129	0.3037	0.5444	0.4573	0.4417	0.3156	0.4300	0.3197	0.4358	0.3085	0.4702	0.3835
		β_4	0	0.2297	0.0818	0.2196	0.0782	0.2819	0.1141	0.2332	0.0830	0.2280	0.0829	0.2324	0.0807	0.2447	0.0964
		β_1	0	0.2023	0.0622	0.2083	0.0665	0.2096	0.0644	0.1982	0.0626	0.1991	0.0641	0.2001	0.0628	0.1946	0.0609
		β_2	0	0.1685	0.0497	0.1736	0.0532	0.1779	0.0526	0.1662	0.0495	0.1668	0.0512	0.1667	0.0497	0.1634	0.0485
		β_3	0	0.3371	0.1989	0.3472	0.2128	0.3559	0.2102	0.3324	0.1981	0.3336	0.2048	0.3333	0.1988	0.3267	0.1939
40	10	β_4	0	0.1838	0.0546	0.1892	0.0583	0.1923	0.0573	0.1807	0.0548	0.1813	0.0562	0.1818	0.0548	0.1775	0.0534
		β_1	0	0.1155	0.0183	0.1176	0.0189	0.1156	0.0184	0.1198	0.0196	0.1173	0.0188	0.1164	0.0185	0.1181	0.0191
		β_2	0	0.0899	0.0114	0.0915	0.0118	0.0914	0.0119	0.0940	0.0126	0.0914	0.0118	0.0905	0.0115	0.0925	0.0121
		β_3	0	0.1799	0.0456	0.1830	0.0472	0.1828	0.0478	0.1880	0.0503	0.1827	0.0470	0.1811	0.0461	0.1851	0.0485
		β_4	0	0.1018	0.0144	0.1036	0.0149	0.1027	0.0148	0.1060	0.0156	0.1034	0.0148	0.1025	0.0145	0.1044	0.0151
80	20	β_1	0	0.0734	0.0097	0.0741	0.0098	0.0815	0.0110	0.0750	0.0100	0.0732	0.0096	0.0740	0.0098	0.0748	0.0099
		β_2	0	0.0570	0.0059	0.0577	0.0061	0.0643	0.0071	0.0586	0.0062	0.0569	0.0059	0.0576	0.0060	0.0584	0.0062
		β_3	0	0.1139	0.0237	0.1154	0.0242	0.1287	0.0283	0.1172	0.0250	0.1138	0.0234	0.1151	0.0241	0.1168	0.0246
		β_4	0	0.0646	0.0075	0.0653	0.0077	0.0723	0.0088	0.0662	0.0079	0.0645	0.0075	0.0652	0.0076	0.0661	0.0078
		β_1	0	0.0593	0.0059	0.0627	0.0061	0.0674	0.0071	0.0630	0.0062	0.0630	0.0062	0.0623	0.0061	0.0626	0.0062
120	30	β_2	0	0.0468	0.0038	0.0495	0.0039	0.0538	0.0047	0.0497	0.0040	0.0497	0.0040	0.0491	0.0039	0.0494	0.0040
		β_3	0	0.0937	0.0151	0.0989	0.0157	0.1076	0.0188	0.0994	0.0161	0.0993	0.0160	0.0982	0.0157	0.0989	0.0160
		β_4	0	0.0527	0.0047	0.0557	0.0049	0.0602	0.0058	0.0559	0.0050	0.0559	0.0050	0.0553	0.0049	0.0556	0.0050
		β_1	0	0.0593	0.0059	0.0627	0.0061	0.0674	0.0071	0.0630	0.0062	0.0630	0.0062	0.0623	0.0061	0.0626	0.0062
		β_2	0	0.0468	0.0038	0.0495	0.0039	0.0538	0.0047	0.0497	0.0040	0.0497	0.0040	0.0491	0.0039	0.0494	0.0040
200	50	β_3	0	0.0937	0.0151	0.0989	0.0157	0.1076	0.0188	0.0994	0.0161	0.0993	0.0160	0.0982	0.0157	0.0989	0.0160
		β_4	0	0.0527	0.0047	0.0557	0.0049	0.0602	0.0058	0.0559	0.0050	0.0559	0.0050	0.0553	0.0049	0.0556	0.0050

Table 19. ABs and MSEs of different entropy measures for IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 3 at $t = 4$.

t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	β_1	1	0.1957	0.0577	0.2107	0.0715	0.2369	0.0830	0.2434	0.0903	0.2083	0.0783	0.1988	0.0702	0.2235	0.0741
		β_2	1	0.0293	0.0012	0.0295	0.0012	0.0387	0.0025	0.0347	0.0017	0.0290	0.0014	0.0280	0.0013	0.0322	0.0015
		β_3	1	0.0586	0.0050	0.0589	0.0049	0.0773	0.0098	0.0693	0.0069	0.0580	0.0055	0.0559	0.0051	0.0645	0.0059
		β_4	1	0.0754	0.0083	0.0784	0.0092	0.0951	0.0139	0.0913	0.0122	0.0773	0.0101	0.0742	0.0093	0.0844	0.0102
40	10	β_1	1	0.2145	0.0614	0.2225	0.0625	0.2198	0.0613	0.2367	0.0718	0.2244	0.0678	0.2224	0.0634	0.2178	0.0605
		β_2	1	0.0348	0.0018	0.0353	0.0016	0.0363	0.0018	0.0367	0.0018	0.0347	0.0016	0.0345	0.0016	0.0341	0.0015
		β_3	1	0.0695	0.0072	0.0705	0.0066	0.0725	0.0073	0.0734	0.0071	0.0694	0.0066	0.0689	0.0062	0.0681	0.0062
		β_4	1	0.0860	0.0103	0.0882	0.0100	0.0890	0.0105	0.0928	0.0110	0.0879	0.0104	0.0872	0.0098	0.0858	0.0095
80	20	β_1	1	0.1171	0.0197	0.1102	0.0174	0.1307	0.0250	0.1477	0.0296	0.1347	0.0239	0.1267	0.0218	0.1384	0.0263
		β_2	1	0.0189	0.0005	0.0176	0.0005	0.0216	0.0007	0.0234	0.0008	0.0216	0.0006	0.0202	0.0006	0.0219	0.0007
		β_3	1	0.0377	0.0021	0.0353	0.0019	0.0431	0.0029	0.0468	0.0030	0.0431	0.0026	0.0405	0.0023	0.0439	0.0027
		β_4	1	0.0460	0.0032	0.0440	0.0028	0.0530	0.0042	0.0580	0.0047	0.0530	0.0039	0.0500	0.0035	0.0550	0.0042
120	30	β_1	1	0.0846	0.0097	0.0858	0.0095	0.0916	0.0106	0.0857	0.0105	0.0790	0.0089	0.0814	0.0094	0.0843	0.0099
		β_2	1	0.0128	0.0002	0.0129	0.0002	0.0142	0.0003	0.0128	0.0002	0.0118	0.0002	0.0121	0.0002	0.0126	0.0002
		β_3	1	0.0256	0.0009	0.0258	0.0009	0.0284	0.0010	0.0257	0.0009	0.0236	0.0008	0.0243	0.0008	0.0252	0.0009
		β_4	1	0.0329	0.0015	0.0333	0.0014	0.0360	0.0017	0.0331	0.0015	0.0305	0.0013	0.0314	0.0014	0.0326	0.0015
200	50	β_1	1	0.0692	0.0084	0.0711	0.0090	0.0665	0.0082	0.0684	0.0084	0.0714	0.0087	0.0709	0.0085	0.0667	0.0080
		β_2	1	0.0108	0.0002	0.0111	0.0002	0.0106	0.0002	0.0107	0.0002	0.0112	0.0002	0.0110	0.0002	0.0104	0.0002
		β_3	1	0.0216	0.0009	0.0222	0.0009	0.0211	0.0009	0.0214	0.0009	0.0223	0.0009	0.0221	0.0009	0.0208	0.0008
		β_4	1	0.0274	0.0013	0.0281	0.0014	0.0265	0.0013	0.0270	0.0013	0.0282	0.0014	0.0280	0.0014	0.0264	0.0013
20	5	β_1	0.9	0.2428	0.0941	0.2416	0.1008	0.2100	0.0714	0.2454	0.1160	0.2418	0.1113	0.2467	0.1016	0.2477	0.1026
		β_2	0.9	0.0357	0.0020	0.0347	0.0020	0.0325	0.0018	0.0363	0.0025	0.0340	0.0020	0.0352	0.0019	0.0362	0.0021
		β_3	0.9	0.0714	0.0080	0.0694	0.0079	0.0651	0.0071	0.0726	0.0100	0.0680	0.0081	0.0703	0.0078	0.0724	0.0086
		β_4	0.9	0.0925	0.0133	0.0908	0.0136	0.0822	0.0109	0.0934	0.0163	0.0899	0.0145	0.0924	0.0136	0.0940	0.0143
40	10	β_1	0.9	0.2152	0.0634	0.2228	0.0655	0.2368	0.0809	0.2261	0.0755	0.2258	0.0711	0.2231	0.0676	0.2210	0.0712
		β_2	0.9	0.0333	0.0015	0.0344	0.0016	0.0397	0.0025	0.0354	0.0019	0.0346	0.0017	0.0340	0.0016	0.0344	0.0018
		β_3	0.9	0.0665	0.0062	0.0689	0.0065	0.0794	0.0101	0.0707	0.0078	0.0692	0.0068	0.0680	0.0064	0.0688	0.0072
		β_4	0.9	0.0843	0.0097	0.0872	0.0102	0.0965	0.0140	0.0890	0.0119	0.0880	0.0108	0.0867	0.0102	0.0868	0.0111
80	20	β_1	0.9	0.1542	0.0386	0.1564	0.0390	0.1607	0.0434	0.1648	0.0418	0.1730	0.0445	0.1699	0.0435	0.1629	0.0405
		β_2	0.9	0.0235	0.0010	0.0236	0.0005	0.0259	0.0008	0.0254	0.0011	0.0263	0.0011	0.0257	0.0011	0.0250	0.0010
		β_3	0.9	0.0470	0.0039	0.0472	0.0038	0.0518	0.0051	0.0508	0.0044	0.0526	0.0045	0.0515	0.0043	0.0501	0.0042
		β_4	0.9	0.0600	0.0060	0.0606	0.0060	0.0643	0.0073	0.0645	0.0066	0.0672	0.0069	0.0659	0.0067	0.0637	0.0064
120	30	β_1	0.9	0.1162	0.0210	0.1154	0.0210	0.1203	0.0221	0.1195	0.0205	0.1152	0.0209	0.1142	0.0204	0.1150	0.0195
		β_2	0.9	0.0174	0.0005	0.0173	0.0005	0.0184	0.0005	0.0180	0.0005	0.0172	0.0005	0.0170	0.0004	0.0172	0.0004
		β_3	0.9	0.0347	0.0018	0.0345	0.0018	0.0369	0.0021	0.0359	0.0018	0.0344	0.0018	0.0340	0.0018	0.0344	0.0017
		β_4	0.9	0.0448	0.0031	0.0446	0.0031	0.0470	0.0034	0.0463	0.0030	0.0444	0.0030	0.0440	0.0030	0.0444	0.0029
200	50	β_1	0.9	0.0954	0.0128	0.0954	0.0134	0.1106	0.0175	0.1011	0.0147	0.1045	0.0155	0.1012	0.0145	0.0997	0.0142
		β_2	0.9	0.0150	0.0003	0.0150	0.0003	0.0177	0.0005	0.0159	0.0004	0.0165	0.0004	0.0159	0.0004	0.0156	0.0004
		β_3	0.9	0.0299	0.0013	0.0299	0.0014	0.0354	0.0019	0.0317	0.0015	0.0329	0.0016	0.0318	0.0015	0.0313	0.0014
		β_4	0.9	0.0378	0.0020	0.0378	0.0021	0.0442	0.0029	0.0400	0.0023	0.0414	0.0025	0.0401	0.0023	0.0394	0.0023
20	5	β_1	0.7	0.2772	0.1073	0.2822	0.1144	0.2926	0.1259	0.2792	0.1166	0.2659	0.1013	0.2708	0.1053	0.2676	0.1031
		β_2	0.7	0.0455	0.0030	0.0457	0.0031	0.0514	0.0044	0.0462	0.0036	0.0419	0.0026	0.0434	0.0029	0.0436	0.0030
		β_3	0.7	0.0910	0.0122	0.0914	0.0122	0.1029	0.0176	0.0925	0.0142	0.0838	0.0103	0.0869	0.0114	0.0872	0.0119
		β_4	0.7	0.1116	0.0176	0.1128	0.0181	0.1218	0.0230	0.1127	0.0197	0.1048	0.0157	0.1077	0.0168	0.1073	0.0170
40	10	β_1	0.7	0.2275	0.0750	0.2348	0.0776	0.2333	0.0867	0.2216	0.0745	0.2354	0.0810	0.2383	0.0832	0.2161	0.0705
		β_2	0.7	0.0366	0.0022	0.0381	0.0023	0.0395	0.0028	0.0365	0.0024	0.0378	0.0024	0.0377	0.0023	0.0355	0.0022
		β_3	0.7	0.0731	0.0087	0.0761	0.0093	0.0789	0.0110	0.0729	0.0096	0.0756	0.0095	0.0754	0.0091	0.0711	0.0089
		β_4	0.7	0.0907	0.0125	0.0940	0.0131	0.0954	0.0151	0.0893	0.0131	0.0937	0.0135	0.0942	0.0134	0.0871	0.0122

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t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
80	20	β_1	0.7	0.1270	0.0325	0.1275	0.0303	0.1531	0.0416	0.1444	0.0434	0.1391	0.0398	0.1400	0.0394	0.1423	0.0418
		β_2	0.7	0.0176	0.0006	0.0178	0.0005	0.0217	0.0008	0.0197	0.0007	0.0189	0.0007	0.0191	0.0007	0.0194	0.0007
		β_3	0.7	0.0353	0.0023	0.0356	0.0022	0.0434	0.0032	0.0393	0.0029	0.0378	0.0027	0.0383	0.0027	0.0388	0.0028
		β_4	0.7	0.0472	0.0043	0.0475	0.0040	0.0574	0.0057	0.0531	0.0056	0.0511	0.0051	0.0516	0.0051	0.0524	0.0054
120	30	β_1	0.7	0.1216	0.0237	0.1184	0.0224	0.1543	0.0347	0.1343	0.0272	0.1312	0.0260	0.1247	0.0247	0.1306	0.0258
		β_2	0.7	0.0200	0.0007	0.0194	0.0006	0.0260	0.0010	0.0222	0.0008	0.0216	0.0007	0.0205	0.0007	0.0215	0.0007
		β_3	0.7	0.0400	0.0027	0.0388	0.0025	0.0520	0.0041	0.0444	0.0031	0.0432	0.0030	0.0411	0.0028	0.0430	0.0030
		β_4	0.7	0.0492	0.0040	0.0479	0.0038	0.0633	0.0060	0.0545	0.0046	0.0532	0.0044	0.0505	0.0042	0.0529	0.0043
200	50	β_1	0.7	0.0825	0.0118	0.0831	0.0119	0.0828	0.0109	0.0734	0.0091	0.0825	0.0113	0.0827	0.0113	0.0735	0.0092
		β_2	0.7	0.0127	0.0003	0.0128	0.0003	0.0128	0.0003	0.0111	0.0002	0.0127	0.0003	0.0127	0.0003	0.0111	0.0002
		β_3	0.7	0.0254	0.0011	0.0257	0.0012	0.0255	0.0010	0.0223	0.0008	0.0254	0.0011	0.0254	0.0011	0.0223	0.0008
		β_4	0.7	0.0324	0.0018	0.0326	0.0018	0.0325	0.0017	0.0286	0.0014	0.0323	0.0017	0.0324	0.0017	0.0286	0.0014
20	5	β_1	0	0.3994	0.2361	0.3668	0.2157	0.4004	0.2371	0.3634	0.2201	0.3776	0.2345	0.3807	0.2250	0.3616	0.2165
		β_2	0	0.0584	0.0051	0.0549	0.0052	0.0648	0.0072	0.0536	0.0051	0.0548	0.0051	0.0546	0.0047	0.0530	0.0048
		β_3	0	0.1168	0.0204	0.1098	0.0207	0.1297	0.0286	0.1072	0.0203	0.1095	0.0203	0.1091	0.0186	0.1059	0.0192
		β_4	0	0.1503	0.0323	0.1396	0.0310	0.1584	0.0384	0.1372	0.0309	0.1413	0.0318	0.1419	0.0301	0.1361	0.0299
40	10	β_1	0	0.2384	0.0820	0.2423	0.0849	0.2366	0.0862	0.2493	0.0848	0.2432	0.0825	0.2359	0.0784	0.2403	0.0800
		β_2	0	0.0418	0.0028	0.0421	0.0028	0.0428	0.0031	0.0434	0.0028	0.0416	0.0026	0.0405	0.0025	0.0418	0.0026
		β_3	0	0.0836	0.0111	0.0843	0.0111	0.0856	0.0123	0.0869	0.0111	0.0832	0.0103	0.0810	0.0099	0.0836	0.0104
		β_4	0	0.0995	0.0149	0.1007	0.0152	0.1003	0.0162	0.1037	0.0152	0.1002	0.0144	0.0974	0.0138	0.0999	0.0143
80	20	β_1	0	0.1684	0.0496	0.1637	0.0453	0.1475	0.0375	0.1731	0.0569	0.1738	0.0539	0.1748	0.0527	0.1679	0.0525
		β_2	0	0.0278	0.0016	0.0268	0.0015	0.0241	0.0012	0.0286	0.0019	0.0286	0.0018	0.0285	0.0017	0.0275	0.0017
		β_3	0	0.0555	0.0066	0.0536	0.0059	0.0482	0.0047	0.0572	0.0075	0.0571	0.0071	0.0571	0.0068	0.0549	0.0067
		β_4	0	0.0680	0.0089	0.0660	0.0080	0.0594	0.0066	0.0700	0.0102	0.0701	0.0096	0.0703	0.0093	0.0676	0.0092
120	30	β_1	0	0.1236	0.0223	0.1236	0.0225	0.1178	0.0219	0.1244	0.0210	0.1202	0.0215	0.1226	0.0216	0.1227	0.0206
		β_2	0	0.0196	0.0006	0.0196	0.0006	0.0189	0.0006	0.0193	0.0005	0.0190	0.0005	0.0192	0.0005	0.0190	0.0005
		β_3	0	0.0391	0.0023	0.0392	0.0023	0.0379	0.0023	0.0386	0.0020	0.0379	0.0022	0.0385	0.0021	0.0381	0.0020
		β_4	0	0.0491	0.0035	0.0491	0.0036	0.0472	0.0035	0.0489	0.0032	0.0477	0.0034	0.0485	0.0034	0.0483	0.0032
200	50	β_1	0	0.1015	0.0146	0.1016	0.0145	0.1109	0.0171	0.1176	0.0204	0.1081	0.0165	0.1093	0.0169	0.1152	0.0196
		β_2	0	0.0155	0.0003	0.0155	0.0003	0.0171	0.0004	0.0178	0.0005	0.0164	0.0004	0.0165	0.0004	0.0174	0.0004
		β_3	0	0.0309	0.0013	0.0310	0.0013	0.0342	0.0016	0.0355	0.0018	0.0328	0.0015	0.0331	0.0015	0.0348	0.0017
		β_4	0	0.0396	0.0022	0.0396	0.0022	0.0435	0.0026	0.0456	0.0030	0.0421	0.0025	0.0425	0.0025	0.0447	0.0029

Table 20. ABs and MSEs of different entropy measures for IWP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 4 at $t = 4$.

t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	β_1	1	0.1578	0.0445	0.1922	0.0551	0.1774	0.0547	0.1726	0.0504	0.1679	0.0493	0.1677	0.0493	0.1791	0.0522
		β_2	1	0.2688	0.1449	0.3205	0.1694	0.3211	0.2076	0.2928	0.1591	0.2830	0.1549	0.2826	0.1533	0.3045	0.1637
		β_3	1	0.5376	0.5795	0.6409	0.6775	0.6422	0.8306	0.5857	0.6363	0.5661	0.6196	0.5652	0.6132	0.6090	0.6548
		β_4	1	0.2051	0.0790	0.2471	0.0948	0.2377	0.1050	0.2239	0.0880	0.2171	0.0859	0.2168	0.0854	0.2326	0.0909
40	10	β_1	1	0.0992	0.0132	0.0977	0.0143	0.1077	0.0159	0.0993	0.0152	0.1022	0.0144	0.0976	0.0132	0.1042	0.0157
		β_2	1	0.1699	0.0399	0.1684	0.0440	0.1871	0.0492	0.1713	0.0474	0.1750	0.0436	0.1661	0.0395	0.1803	0.0488
		β_3	1	0.3398	0.1596	0.3369	0.1759	0.3741	0.1967	0.3426	0.1896	0.3500	0.1743	0.3322	0.1580	0.3606	0.1950
		β_4	1	0.1298	0.0229	0.1282	0.0251	0.1419	0.0279	0.1303	0.0268	0.1336	0.0250	0.1272	0.0228	0.1370	0.0276
80	20	β_1	1	0.0851	0.0109	0.0814	0.0096	0.0976	0.0159	0.0981	0.0153	0.0888	0.0119	0.0898	0.0118	0.0924	0.0140
		β_2	1	0.1439	0.0329	0.1362	0.0283	0.1684	0.0506	0.1662	0.0464	0.1492	0.0359	0.1510	0.0353	0.1575	0.0428
		β_3	1	0.2878	0.1318	0.2725	0.1132	0.3367	0.2024	0.3323	0.1858	0.2985	0.1435	0.3021	0.1412	0.3150	0.1714
		β_4	1	0.1106	0.0189	0.1053	0.0165	0.1280	0.0283	0.1275	0.0265	0.1150	0.0206	0.1164	0.0203	0.1205	0.0244
120	30	β_1	1	0.0587	0.0051	0.0601	0.0055	0.0690	0.0064	0.0672	0.0062	0.0634	0.0068	0.0625	0.0065	0.0652	0.0057
		β_2	1	0.0939	0.0128	0.0957	0.0138	0.1123	0.0170	0.1075	0.0156	0.0996	0.0159	0.0981	0.0155	0.1045	0.0144
		β_3	1	0.1877	0.0514	0.1915	0.0550	0.2246	0.0680	0.2151	0.0623	0.1992	0.0638	0.1963	0.0620	0.2089	0.0576
		β_4	1	0.0742	0.0081	0.0758	0.0087	0.0880	0.0104	0.0850	0.0098	0.0794	0.0104	0.0782	0.0100	0.0825	0.0091
200	50	β_1	1	0.0543	0.0044	0.0544	0.0044	0.0586	0.0050	0.0548	0.0048	0.0521	0.0042	0.0521	0.0041	0.0556	0.0048
		β_2	1	0.0891	0.0120	0.0890	0.0120	0.0965	0.0137	0.0901	0.0134	0.0851	0.0115	0.0849	0.0112	0.0914	0.0134
		β_3	1	0.1783	0.0482	0.1781	0.0482	0.1930	0.0550	0.1801	0.0535	0.1703	0.0459	0.1698	0.0447	0.1827	0.0535
		β_4	1	0.0696	0.0073	0.0696	0.0073	0.0752	0.0083	0.0702	0.0080	0.0666	0.0069	0.0665	0.0068	0.0712	0.0080
20	5	β_1	0.9	0.1657	0.0440	0.1537	0.0388	0.2032	0.0643	0.1878	0.0505	0.1575	0.0394	0.1745	0.0459	0.1719	0.0456
		β_2	0.9	0.3001	0.1605	0.2754	0.1333	0.3850	0.2519	0.3232	0.1589	0.2757	0.1296	0.3077	0.1523	0.3099	0.1617
		β_3	0.9	0.6003	0.6420	0.5508	0.5333	0.7700	1.0078	0.6463	0.6358	0.5514	0.5185	0.6153	0.6094	0.6197	0.6466
		β_4	0.9	0.2224	0.0832	0.2053	0.0714	0.2790	0.1265	0.2456	0.0885	0.2079	0.0709	0.2311	0.0828	0.2303	0.0852
40	10	β_1	0.9	0.1540	0.0415	0.1480	0.0387	0.1370	0.0330	0.1381	0.0356	0.1435	0.0366	0.1446	0.0383	0.1381	0.0359
		β_2	0.9	0.2451	0.1084	0.2347	0.0996	0.2260	0.0985	0.2204	0.0942	0.2271	0.0941	0.2271	0.0964	0.2207	0.0952
		β_3	0.9	0.4902	0.4334	0.4693	0.3982	0.4520	0.3940	0.4407	0.3766	0.4542	0.3763	0.4543	0.3855	0.4415	0.3807
		β_4	0.9	0.1935	0.0658	0.1857	0.0610	0.1754	0.0560	0.1739	0.0569	0.1799	0.0577	0.1806	0.0597	0.1740	0.0575
80	20	β_1	0.9	0.0970	0.0165	0.0945	0.0161	0.1122	0.0197	0.1048	0.0185	0.0999	0.0180	0.0986	0.0165	0.1039	0.0183
		β_2	0.9	0.1656	0.0519	0.1607	0.0500	0.1900	0.0599	0.1789	0.0583	0.1700	0.0562	0.1665	0.0505	0.1774	0.0574
		β_3	0.9	0.3312	0.2075	0.3213	0.1998	0.3800	0.2396	0.3578	0.2330	0.3399	0.2247	0.3330	0.2019	0.3549	0.2298
		β_4	0.9	0.1266	0.0291	0.1230	0.0282	0.1458	0.0341	0.1367	0.0327	0.1301	0.0316	0.1280	0.0287	0.1356	0.0322
120	30	β_1	0.9	0.0624	0.0067	0.0616	0.0064	0.0582	0.0057	0.0543	0.0053	0.0567	0.0056	0.0550	0.0052	0.0549	0.0053
		β_2	0.9	0.1021	0.0183	0.1008	0.0175	0.0953	0.0154	0.0879	0.0137	0.0917	0.0148	0.0886	0.0136	0.0889	0.0137
		β_3	0.9	0.2041	0.0733	0.2016	0.0700	0.1907	0.0616	0.1758	0.0548	0.1834	0.0592	0.1772	0.0542	0.1778	0.0548
		β_4	0.9	0.0798	0.0111	0.0788	0.0105	0.0744	0.0094	0.0691	0.0085	0.0720	0.0091	0.0698	0.0084	0.0699	0.0085
200	50	β_1	0.9	0.0570	0.0051	0.0557	0.0054	0.0599	0.0058	0.0628	0.0058	0.0610	0.0056	0.0612	0.0058	0.0633	0.0058
		β_2	0.9	0.0931	0.0139	0.0903	0.0142	0.0972	0.0154	0.1023	0.0157	0.0991	0.0148	0.0990	0.0151	0.1031	0.0157
		β_3	0.9	0.1861	0.0554	0.1806	0.0568	0.1944	0.0615	0.2047	0.0628	0.1981	0.0590	0.1980	0.0604	0.2061	0.0626
		β_4	0.9	0.0728	0.0084	0.0709	0.0087	0.0763	0.0095	0.0801	0.0096	0.0777	0.0091	0.0778	0.0093	0.0807	0.0095
20	5	β_1	0.7	0.1761	0.0526	0.2155	0.0715	0.2171	0.0747	0.1919	0.0677	0.1855	0.0628	0.1906	0.0633	0.1830	0.0639
		β_2	0.7	0.3135	0.1881	0.3801	0.2557	0.4121	0.3386	0.3528	0.2760	0.3339	0.2448	0.3442	0.2484	0.3372	0.2626
		β_3	0.7	0.6269	0.7525	0.7601	1.0227	0.8241	1.3545	0.7056	1.1039	0.6679	0.9791	0.6884	0.9935	0.6744	1.0503
		β_4	0.7	0.2341	0.0981	0.2847	0.1327	0.2975	0.1560	0.2587	0.1343	0.2475	0.1217	0.2547	0.1231	0.2470	0.1274
40	10	β_1	0.7	0.1624	0.0374	0.1464	0.0313	0.1509	0.0319	0.1408	0.0295	0.1517	0.0329	0.1515	0.0326	0.1390	0.0293
		β_2	0.7	0.2617	0.1002	0.2281	0.0749	0.2485	0.0905	0.2234	0.0741	0.2359	0.0771	0.2367	0.0781	0.2213	0.0737
		β_3	0.7	0.5233	0.4008	0.4562	0.2996	0.4970	0.3620	0.4469	0.2966	0.4718	0.3083	0.4735	0.3124	0.4426	0.2950

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t^*	d	Entropy	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
80	20	β_4	0.7	0.2056	0.0605	0.1824	0.0480	0.1932	0.0532	0.1770	0.0463	0.1888	0.0499	0.1890	0.0500	0.1750	0.0460
		β_1	0.7	0.1166	0.0215	0.1109	0.0201	0.1230	0.0236	0.1133	0.0207	0.1090	0.0205	0.1081	0.0205	0.1130	0.0210
		β_2	0.7	0.1905	0.0594	0.1809	0.0549	0.2030	0.0680	0.1813	0.0553	0.1749	0.0537	0.1734	0.0536	0.1822	0.0569
		β_3	0.7	0.3809	0.2375	0.3618	0.2197	0.4061	0.2720	0.3626	0.2210	0.3497	0.2148	0.3467	0.2146	0.3644	0.2276
		β_4	0.7	0.1488	0.0354	0.1414	0.0329	0.1577	0.0396	0.1430	0.0335	0.1378	0.0329	0.1366	0.0328	0.1432	0.0342
120	30	β_1	0.7	0.1125	0.0151	0.1101	0.0144	0.1084	0.0143	0.1006	0.0126	0.1065	0.0138	0.1107	0.0144	0.1021	0.0130
		β_2	0.7	0.1875	0.0429	0.1814	0.0399	0.1821	0.0416	0.1680	0.0360	0.1759	0.0385	0.1830	0.0400	0.1707	0.0371
		β_3	0.7	0.3749	0.1718	0.3627	0.1597	0.3643	0.1662	0.3360	0.1441	0.3518	0.1541	0.3661	0.1598	0.3414	0.1483
		β_4	0.7	0.1451	0.0254	0.1412	0.0239	0.1404	0.0243	0.1299	0.0213	0.1367	0.0230	0.1423	0.0239	0.1320	0.0219
200	50	β_1	0.7	0.0793	0.0081	0.0789	0.0082	0.0771	0.0078	0.0726	0.0071	0.0790	0.0076	0.0764	0.0072	0.0732	0.0072
		β_2	0.7	0.1289	0.0214	0.1282	0.0218	0.1254	0.0207	0.1174	0.0187	0.1273	0.0196	0.1233	0.0188	0.1182	0.0189
		β_3	0.7	0.2578	0.0856	0.2565	0.0872	0.2509	0.0830	0.2348	0.0749	0.2546	0.0785	0.2466	0.0751	0.2365	0.0756
		β_4	0.7	0.1010	0.0131	0.1005	0.0134	0.0983	0.0127	0.0923	0.0115	0.1002	0.0122	0.0970	0.0116	0.0930	0.0117
20	5	β_1	0	0.2433	0.1017	0.2248	0.0887	0.2622	0.1217	0.2599	0.1162	0.2580	0.1132	0.2523	0.1131	0.2578	0.1179
		β_2	0	0.4318	0.3687	0.3867	0.2931	0.4965	0.5505	0.4599	0.4484	0.4545	0.4137	0.4507	0.4283	0.4681	0.4739
		β_3	0	0.8635	1.4749	0.7733	1.1724	0.9929	2.2022	0.9199	1.7936	0.9090	1.6547	0.9013	1.7131	0.9361	1.8956
		β_4	0	0.3212	0.1855	0.2923	0.1540	0.3570	0.2481	0.3419	0.2168	0.3389	0.2061	0.3336	0.2097	0.3435	0.2251
40	10	β_1	0	0.1561	0.0400	0.1559	0.0432	0.1546	0.0436	0.1489	0.0355	0.1538	0.0415	0.1496	0.0373	0.1455	0.0358
		β_2	0	0.2620	0.1186	0.2581	0.1239	0.2612	0.1299	0.2445	0.0961	0.2511	0.1123	0.2479	0.1068	0.2406	0.1000
		β_3	0	0.5239	0.4742	0.5163	0.4958	0.5223	0.5197	0.4890	0.3845	0.5022	0.4494	0.4959	0.4273	0.4812	0.3999
		β_4	0	0.2015	0.0678	0.1998	0.0718	0.2002	0.0739	0.1902	0.0576	0.1958	0.0671	0.1920	0.0621	0.1866	0.0590
80	20	β_1	0	0.0934	0.0136	0.0896	0.0130	0.0898	0.0124	0.0945	0.0135	0.0901	0.0118	0.0886	0.0120	0.0935	0.0132
		β_2	0	0.1490	0.0347	0.1433	0.0333	0.1461	0.0336	0.1498	0.0339	0.1416	0.0286	0.1395	0.0293	0.1489	0.0336
		β_3	0	0.2980	0.1388	0.2865	0.1333	0.2922	0.1343	0.2996	0.1357	0.2832	0.1144	0.2790	0.1171	0.2977	0.1346
		β_4	0	0.1179	0.0216	0.1132	0.0207	0.1144	0.0203	0.1189	0.0213	0.1128	0.0183	0.1111	0.0187	0.1179	0.0210
120	30	β_1	0	0.0901	0.0118	0.0858	0.0117	0.0923	0.0123	0.0963	0.0126	0.0932	0.0121	0.0922	0.0119	0.0944	0.0124
		β_2	0	0.1472	0.0319	0.1395	0.0311	0.1512	0.0331	0.1575	0.0344	0.1513	0.0322	0.1498	0.0317	0.1545	0.0337
		β_3	0	0.2944	0.1276	0.2790	0.1244	0.3025	0.1325	0.3150	0.1375	0.3026	0.1288	0.2997	0.1269	0.3091	0.1347
		β_4	0	0.1151	0.0193	0.1093	0.0190	0.1181	0.0201	0.1230	0.0208	0.1187	0.0197	0.1175	0.0194	0.1207	0.0203
200	50	β_1	0	0.0661	0.0071	0.0645	0.0072	0.0696	0.0087	0.0695	0.0078	0.0642	0.0069	0.0651	0.0070	0.0690	0.0078
		β_2	0	0.1105	0.0207	0.1078	0.0208	0.1177	0.0262	0.1166	0.0229	0.1070	0.0199	0.1083	0.0202	0.1158	0.0228
		β_3	0	0.2209	0.0830	0.2155	0.0834	0.2353	0.1049	0.2332	0.0916	0.2139	0.0798	0.2166	0.0810	0.2315	0.0911
		β_4	0	0.0854	0.0122	0.0833	0.0122	0.0904	0.0151	0.0900	0.0134	0.0828	0.0117	0.0839	0.0119	0.0894	0.0133

Table 21. Relative Efficiency (RE) of the parameter estimates for IWP distribution for Set 1 and Set 2 ($t = 4$).

Set	t^*	Parameter	ρ	ML	MPS	KM	LS	WLS	AD	CVM
Set 1	20	λ	1	1.427	1.183	1.217	1.267	1.393	1.495	1.289
			0.9	1.283	1.162	1.556	1.137	1.250	1.147	1.209
			0.7	1.252	1.252	1.273	1.051	1.214	1.138	1.020
		α	1	3.934	1.893	1.283	1.402	2.242	3.832	1.985
			0.9	1.484	1.455	1.165	1.220	1.348	1.690	1.974
			0.7	2.271	1.803	0.977	0.602	0.805	1.418	0.786
	40	λ	1	1.096	1.403	1.615	1.292	1.301	1.515	1.255
			0.9	0.750	0.971	1.750	1.139	1.091	1.266	1.255
			0.7	0.863	1.092	1.105	1.127	0.885	0.901	1.382
		α	1	1.370	1.393	1.098	0.878	0.684	0.895	0.769
			0.9	0.773	1.046	1.037	1.039	0.581	0.624	0.436
			0.7	0.859	1.291	0.875	1.104	0.397	0.500	0.741
	80	λ	1	1.000	0.818	1.375	0.868	0.989	0.975	0.855
			0.9	1.000	0.959	1.375	0.952	1.175	1.145	0.855
			0.7	0.814	1.054	1.000	0.908	1.011	0.940	0.689
		α	1	0.924	0.635	0.971	0.929	1.368	0.967	1.000
			0.9	0.936	1.030	0.825	1.667	1.400	1.571	1.545
			0.7	0.577	1.264	0.971	2.101	1.280	1.257	0.763
	120	λ	1	1.369	1.123	1.625	0.806	1.241	1.096	0.859
			0.9	0.690	0.958	1.300	0.818	1.200	1.140	0.705
			0.7	0.627	1.071	1.857	0.885	1.220	1.075	0.846
		α	1	2.623	1.857	1.500	0.736	1.217	1.103	0.574
			0.9	0.892	0.956	0.923	0.929	1.436	1.103	0.673
			0.7	0.444	1.121	1.200	0.661	1.217	1.024	0.583
	200	λ	1	1.047	0.878	0.667	1.224	0.929	1.059	1.128
			0.9	0.573	0.774	0.800	0.909	0.839	0.931	0.815
			0.7	0.545	1.180	1.143	1.429	1.677	1.459	1.359
		α	1	1.512	1.259	1.154	0.647	0.610	0.758	0.759
			0.9	0.456	0.773	0.714	0.415	0.676	0.781	0.393
			0.7	0.258	2.429	1.154	0.786	1.923	1.923	0.917
	20	λ	1	1.429	1.964	2.571	1.935	1.378	1.846	1.533
			0.9	1.071	1.667	1.286	1.780	1.319	1.674	1.643
			0.7	1.034	1.460	1.125	1.271	1.051	1.532	1.586
		α	1	1.546	1.204	3.952	1.082	1.101	1.045	1.146
			0.9	1.225	1.151	0.783	1.058	1.066	1.036	1.001
			0.7	1.134	1.147	1.612	1.032	1.048	1.024	0.927
	40	λ	1	2.500	1.565	1.800	1.304	1.812	1.444	1.955
			0.9	1.087	1.029	1.286	0.909	1.261	1.182	1.870
			0.7	1.190	1.029	1.125	1.200	1.074	1.300	2.048
		α	1	2.931	1.120	1.000	1.392	2.103	1.698	1.414
			0.9	1.269	1.106	0.709	0.444	1.321	1.393	1.372
			0.7	1.276	1.081	1.106	1.218	1.056	1.592	1.380
Set 2	80	λ	1	1.444	1.615	3.000	1.727	2.857	1.429	2.111
			0.9	1.625	1.615	2.000	1.267	2.000	1.111	2.375
			0.7	1.444	1.235	1.500	1.462	1.429	1.000	2.111
		α	1	1.011	1.887	0.914	1.815	2.136	1.104	2.100
			0.9	1.314	1.982	1.333	1.753	1.872	1.010	2.676
			0.7	1.186	1.187	1.455	1.403	1.184	1.184	2.416
	120	λ	1	1.400	2.714	1.500	1.429	2.000	1.400	1.500
		α								

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Set	t^*	Parameter	ρ	ML	MPS	KM	LS	WLS	AD	CVM
200		α	0.9	1.167	1.900	1.500	1.250	1.667	1.167	1.500
			0.7	1.167	1.727	1.500	1.429	1.250	1.167	1.500
			1	1.000	1.667	0.909	1.315	2.241	1.419	1.903
			0.9	0.978	1.063	1.154	1.301	1.719	1.183	2.169
			0.7	1.056	1.143	2.308	1.487	1.131	1.021	2.341
		λ	1	1.250	1.500	3.000	1.000	1.500	1.333	2.667
			0.9	1.667	1.500	3.000	1.000	2.000	1.333	2.667
			0.7	1.250	1.500	1.500	1.200	1.200	1.000	2.000
		α	1	0.873	1.608	1.111	1.254	3.573	1.267	1.978
			0.9	1.352	1.880	0.667	1.224	4.651	1.301	2.507
			0.7	1.215	1.625	1.250	1.159	2.930	1.033	2.253

Table 22. Relative Efficiency (RE) of the parameter estimates for IWP distribution for Set 3 and Set 4 ($t = 4$).

Set	t^*	Par.	ρ	ML	MPS	KM	LS	WLS	AD	CVM
Set 3	20	λ	1	1.364	0.923	0.842	0.737	1.000	0.933	0.933
			0.9	1.875	2.000	0.941	2.000	2.667	2.333	1.750
			0.7	1.000	1.500	1.600	1.273	2.000	1.400	1.077
	20	α	1	0.866	0.764	0.884	0.931	0.676	0.525	0.724
			0.9	1.394	1.261	0.813	1.219	0.863	0.884	1.216
			0.7	1.043	0.891	0.938	1.669	0.890	0.746	1.516
	40	λ	1	1.091	0.800	0.667	1.000	0.900	0.900	1.556
			0.9	1.333	1.333	0.800	1.000	1.286	1.000	1.400
			0.7	1.091	1.000	0.571	0.667	0.529	0.692	0.667
	40	α	1	0.856	1.082	1.400	0.587	0.904	0.816	0.605
			0.9	0.907	1.167	0.689	0.425	0.770	0.758	0.451
			0.7	0.688	0.829	1.037	0.376	0.599	0.528	0.434
	80	λ	1	0.643	0.667	0.385	0.700	0.875	0.778	0.500
			0.9	0.818	0.750	1.000	1.000	0.778	0.778	0.455
			0.7	1.286	0.857	0.833	1.167	0.875	1.000	0.714
	80	α	1	1.058	1.373	1.135	0.981	1.175	1.319	0.967
			0.9	0.821	0.871	0.700	0.627	0.801	1.051	0.605
			0.7	0.824	0.844	1.235	0.617	0.862	1.005	0.538
	120	λ	1	1.286	1.333	1.000	1.125	1.286	1.000	1.800
			0.9	1.125	1.333	0.500	1.125	1.286	1.000	1.000
			0.7	1.286	1.000	0.857	1.125	1.286	1.000	1.000
	120	α	1	0.895	0.905	1.079	1.304	1.266	1.295	1.114
			0.9	0.837	0.633	0.976	0.880	1.006	1.043	0.824
			0.7	0.625	0.583	0.788	1.046	1.141	1.148	0.936
	200	λ	1	0.833	1.000	1.400	0.875	0.714	0.571	0.750
			0.9	1.000	1.000	1.000	1.000	0.833	0.667	0.857
			0.7	1.000	1.000	1.167	0.778	0.714	0.571	0.667
	200	α	1	0.694	0.628	0.800	1.362	0.811	0.901	1.245
			0.9	0.876	0.717	1.400	1.153	0.902	0.971	1.102
			0.7	1.276	0.866	1.556	1.353	0.882	0.763	1.280
Set 4	20	λ	1	1.178	3.546	> 9.99	0.837	1.715	2.256	1.364
			0.9	1.514	4.810	2.552	1.607	4.431	2.616	1.801
			0.7	0.832	5.107	> 9.99	2.889	1.743	1.618	2.517
	20	α	1	1.003	0.921	0.461	1.223	0.469	0.277	0.546
			0.9	1.213	0.710	0.469	0.649	0.360	0.245	1.006
			0.7	0.917	0.666	0.084	1.359	1.386	0.862	1.254
	40	λ	1	0.166	0.694	1.192	0.347	0.469	0.340	0.314
			0.9	0.505	3.727	1.046	0.547	1.958	0.656	0.513
			0.7	0.306	1.103	1.407	0.529	0.765	0.388	0.442
	40	α	1	1.012	1.417	0.425	0.207	0.302	1.336	0.849
			0.9	0.930	0.994	6.169	0.090	0.264	4.701	0.210
			0.7	1.543	3.925	0.736	0.380	1.178	1.929	0.677
	80	λ	1	0.640	2.145	1.177	2.252	0.689	0.987	2.332
			0.9	1.207	1.168	1.858	2.765	1.028	0.894	3.030
			0.7	0.458	0.795	6.852	2.197	0.851	0.827	1.781
	80	α	1	0.770	1.124	0.667	0.698	0.401	1.211	0.500
			0.9	0.682	0.453	0.293	0.517	0.761	0.947	0.397
			0.7	0.998	0.749	0.398	0.777	0.284	1.045	2.456
	120	λ	1	1.086	0.658	0.255	1.229	3.213	2.102	0.853
			0.9	0.640	0.939	1.125	0.785	0.971	0.940	0.496
			0.7	0.528	0.622	0.352	0.382	0.834	0.612	0.409
	120	α	1	0.601	0.335	0.534	0.163	0.245	0.414	0.279
			0.9	0.485	0.219	1.569	0.584	0.274	0.297	0.418
			0.7	0.427	0.277	1.220	0.604	1.366	1.012	1.164

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Table 22 – continued from previous page

Set	t^*	Par.	ρ	ML	MPS	KM	LS	WLS	AD	CVM
200	λ		1	1.794	2.299	0.927	1.241	2.266	2.408	1.097
			0.9	0.914	5.615	0.937	1.804	2.350	1.154	1.419
			0.7	0.790	2.255	0.927	1.079	1.307	1.078	1.113
200	α		1	0.624	0.374	0.267	0.807	1.729	1.168	0.999
			0.9	1.248	1.051	3.619	0.594	1.420	1.239	0.231
			0.7	1.676	2.520	1.579	0.330	5.248	1.225	0.273

Table 23. Relative Efficiency (RE) of entropy measure estimates for IWP distribution Set 1.

t^*	Entropy	ρ	ML	MPS	KM	LS	WLS	AD	CVM
20		1	1.464	1.517	1.483	1.385	1.530	1.366	1.434
	β_1 (Rényi)		1.228	1.331	1.532	1.233	1.324	1.148	1.264
	β_2 (Tsallis)		1.231	1.340	1.531	1.235	1.327	1.152	1.264
	β_3 (H-C)		1.343	1.413	1.505	1.299	1.416	1.251	1.346
	β_4 (Arimoto)								
	β_1 (Rényi)	0.9	1.529	1.521	1.949	1.585	1.711	1.578	1.668
	β_2 (Tsallis)		1.712	1.784	2.303	1.796	1.978	1.852	1.901
	β_3 (H-C)		1.720	1.784	2.305	1.810	1.970	1.845	1.905
	β_4 (Arimoto)		1.618	1.645	2.122	1.686	1.830	1.701	1.776
	β_1 (Rényi)	0.7	1.383	1.406	1.257	1.130	1.337	1.280	1.140
	β_2 (Tsallis)		1.483	1.491	1.334	1.114	1.452	1.315	1.133
	β_3 (H-C)		1.484	1.492	1.335	1.121	1.444	1.319	1.132
	β_4 (Arimoto)		1.437	1.447	1.304	1.124	1.385	1.303	1.141
40	β_1 (Rényi)	1	1.807	1.570	1.692	1.423	1.389	1.561	1.490
	β_2 (Tsallis)		2.000	1.679	1.818	1.571	1.556	1.704	1.593
	β_3 (H-C)		2.009	1.691	1.818	1.544	1.523	1.679	1.602
	β_4 (Arimoto)		1.909	1.630	1.758	1.482	1.463	1.611	1.547
	β_1 (Rényi)	0.9	1.313	1.113	0.896	0.709	0.802	1.050	0.657
	β_2 (Tsallis)		1.366	1.175	0.909	0.688	0.808	1.070	0.623
	β_3 (H-C)		1.364	1.155	0.913	0.688	0.794	1.052	0.629
	β_4 (Arimoto)		1.346	1.128	0.908	0.697	0.806	1.048	0.646
	β_1 (Rényi)	0.7	0.764	0.592	0.552	0.598	0.595	0.668	0.550
	β_2 (Tsallis)		0.812	0.618	0.545	0.587	0.618	0.676	0.518
	β_3 (H-C)		0.809	0.612	0.544	0.583	0.606	0.670	0.520
	β_4 (Arimoto)		0.789	0.603	0.551	0.593	0.603	0.669	0.539
80	β_1 (Rényi)	1	2.740	2.716	2.188	2.260	2.539	2.821	2.269
	β_2 (Tsallis)		3.000	2.955	2.441	2.517	2.792	3.143	2.484
	β_3 (H-C)		3.061	2.921	2.449	2.526	2.773	3.181	2.525
	β_4 (Arimoto)		2.850	2.833	2.302	2.364	2.652	2.975	2.386
	β_1 (Rényi)	0.9	2.890	2.750	2.462	3.052	2.935	2.785	3.267
	β_2 (Tsallis)		3.316	2.955	2.594	3.318	3.190	3.000	3.667
	β_3 (H-C)		3.218	2.989	2.622	3.330	3.165	3.034	3.624
	β_4 (Arimoto)		3.000	2.902	2.544	3.171	3.050	2.902	3.400
	β_1 (Rényi)	0.7	1.867	1.849	2.016	1.865	2.132	2.056	2.025
	β_2 (Tsallis)		2.032	2.031	2.306	2.086	2.310	2.276	2.265
	β_3 (H-C)		2.041	2.016	2.329	2.078	2.360	2.276	2.299
	β_4 (Arimoto)		1.932	1.919	2.164	1.940	2.218	2.164	2.125
120	β_1 (Rényi)	1	4.111	3.931	3.333	3.581	3.724	3.724	3.516
	β_2 (Tsallis)		3.571	3.571	2.800	3.125	3.429	3.429	3.125
	β_3 (H-C)		3.808	3.517	3.000	3.258	3.345	3.310	3.094
	β_4 (Arimoto)		4.000	3.857	3.053	3.533	3.643	3.643	3.250
	β_1 (Rényi)	0.9	2.362	2.327	1.935	2.467	2.634	2.634	2.422
	β_2 (Tsallis)		2.273	2.083	1.750	2.273	2.400	2.667	2.273
	β_3 (H-C)		2.302	2.217	1.810	2.349	2.487	2.526	2.250
	β_4 (Arimoto)		2.364	2.250	1.871	2.409	2.550	2.550	2.364
	β_1 (Rényi)	0.7	1.088	1.096	0.945	1.047	1.069	1.069	1.019
	β_2 (Tsallis)		0.926	0.893	0.778	0.893	0.889	0.889	0.862
	β_3 (H-C)		0.917	0.919	0.803	0.894	0.907	0.897	0.861
	β_4 (Arimoto)		1.000	1.000	0.866	0.964	0.981	0.981	0.945
	β_1 (Rényi)	1	2.348	2.250	1.429	2.192	2.333	2.292	2.111
	β_2 (Tsallis)		2.167	2.333	1.300	2.000	2.333	2.333	2.143

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t^*	Entropy	ρ	ML	MPS	KM	LS	WLS	AD	CVM
	β_3 (H-C)		2.250	2.250	1.368	2.148	2.333	2.200	2.071
	β_4 (Arimoto)		2.250	2.250	1.444	2.231	2.333	2.250	2.071
	β_1 (Rényi)	0.9	2.571	2.455	1.613	2.591	2.947	2.750	2.591
	β_2 (Tsallis)		2.600	2.333	1.625	2.333	2.800	2.800	2.500
	β_3 (H-C)		2.571	2.455	1.576	2.417	2.800	2.619	2.522
	β_4 (Arimoto)		2.455	2.455	1.625	2.417	2.800	2.700	2.636
	β_1 (Rényi)	0.7	1.588	1.500	1.190	1.676	1.750	1.667	1.629
	β_2 (Tsallis)		1.444	1.400	1.083	1.556	1.750	1.556	1.667
	β_3 (H-C)		1.500	1.421	1.130	1.568	1.647	1.571	1.526
	β_4 (Arimoto)		1.588	1.421	1.182	1.611	1.750	1.588	1.611

Table 24. Relative Efficiency (RE) of entropy measure estimates for IWP distribution Set 2.

t^*	Entropy	ρ	ML	MPS	KM	LS	WLS	AD	CVM
20	β_1 (Rényi)	1	3.769	3.122	3.057	3.351	3.345	4.251	3.467
	β_2 (Tsallis)		4.653	3.933	3.617	3.945	3.975	5.209	4.399
	β_3 (H-C)		4.658	3.944	3.612	3.950	3.976	5.202	4.403
	β_4 (Arimoto)		4.131	3.460	3.279	3.609	3.604	4.638	3.856
	β_1 (Rényi)	0.9	1.873	1.753	1.488	1.218	1.347	1.860	1.425
	β_2 (Tsallis)		2.071	1.992	1.561	1.148	1.402	1.997	1.440
	β_3 (H-C)		2.070	1.994	1.561	1.148	1.402	1.998	1.440
	β_4 (Arimoto)		1.943	1.840	1.511	1.179	1.363	1.903	1.424
	β_1 (Rényi)	0.7	2.397	2.102	2.392	3.107	2.835	2.830	3.432
	β_2 (Tsallis)		3.090	2.770	2.822	3.868	3.665	3.707	4.701
	β_3 (H-C)		3.089	2.766	2.819	3.858	3.670	3.712	4.706
	β_4 (Arimoto)		2.691	2.384	2.576	3.444	3.188	3.202	3.983
40	β_1 (Rényi)	1	4.290	4.182	3.141	4.929	4.893	4.794	4.579
	β_2 (Tsallis)		4.970	4.750	3.578	5.756	5.626	5.522	5.389
	β_3 (H-C)		4.997	4.771	3.587	5.725	5.642	5.492	5.386
	β_4 (Arimoto)		4.588	4.417	3.312	5.269	5.204	5.074	4.899
	β_1 (Rényi)	0.9	3.595	3.935	3.975	4.378	4.612	3.829	4.229
	β_2 (Tsallis)		4.041	4.471	4.574	5.156	5.626	4.398	5.000
	β_3 (H-C)		4.043	4.461	4.550	5.159	5.611	4.408	4.985
	β_4 (Arimoto)		3.766	4.135	4.213	4.684	5.018	4.059	4.525
	β_1 (Rényi)	0.7	3.456	3.325	2.506	3.348	3.142	3.156	3.139
	β_2 (Tsallis)		3.944	3.720	2.798	3.898	3.483	3.476	3.593
	β_3 (H-C)		3.939	3.714	2.803	3.915	3.495	3.469	3.584
	β_4 (Arimoto)		3.640	3.470	2.628	3.582	3.267	3.262	3.317
80	β_1 (Rényi)	1	2.652	2.662	1.859	2.841	2.806	2.721	2.809
	β_2 (Tsallis)		2.651	2.682	1.776	2.864	2.810	2.738	2.814
	β_3 (H-C)		2.636	2.667	1.797	2.891	2.814	2.728	2.853
	β_4 (Arimoto)		2.618	2.661	1.827	2.836	2.792	2.685	2.796
	β_1 (Rényi)	0.9	1.408	1.400	1.165	1.361	1.424	1.434	1.364
	β_2 (Tsallis)		1.407	1.372	1.102	1.355	1.405	1.402	1.330
	β_3 (H-C)		1.403	1.376	1.109	1.356	1.403	1.414	1.336
	β_4 (Arimoto)		1.412	1.393	1.138	1.357	1.410	1.422	1.348
	β_1 (Rényi)	0.7	1.317	1.268	0.958	1.153	1.253	1.250	1.124
	β_2 (Tsallis)		1.175	1.135	0.862	1.050	1.124	1.117	1.000
	β_3 (H-C)		1.178	1.135	0.869	1.046	1.116	1.116	1.004
	β_4 (Arimoto)		1.241	1.202	0.914	1.091	1.175	1.179	1.056
120	β_1 (Rényi)	1	2.156	2.227	2.340	2.439	2.133	2.178	2.415
	β_2 (Tsallis)		2.107	2.179	2.367	2.480	2.107	2.143	2.385
	β_3 (H-C)		2.116	2.200	2.339	2.451	2.071	2.133	2.412
	β_4 (Arimoto)		2.143	2.200	2.316	2.469	2.083	2.111	2.437
	β_1 (Rényi)	0.9	1.183	1.195	1.183	1.099	1.079	1.114	1.112
	β_2 (Tsallis)		1.093	1.130	1.145	1.033	1.017	1.034	1.051
	β_3 (H-C)		1.097	1.115	1.137	1.042	1.000	1.048	1.042
	β_4 (Arimoto)		1.136	1.149	1.158	1.068	1.042	1.070	1.083
	β_1 (Rényi)	0.7	1.010	1.000	1.068	1.064	1.011	1.032	1.065
	β_2 (Tsallis)		0.952	0.953	1.044	1.000	0.952	0.968	1.016
	β_3 (H-C)		0.948	0.938	1.044	1.012	0.947	0.972	1.008
	β_4 (Arimoto)		0.974	0.975	1.060	1.039	0.987	1.000	1.040
	β_1 (Rényi)	1	2.810	2.905	2.731	2.818	2.952	2.905	2.952
	β_2 (Tsallis)		2.923	3.250	2.938	3.077	3.333	3.250	3.077

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Table 24 – continued from previous page

t^*	Entropy	ρ	ML	MPS	KM	LS	WLS	AD	CVM
	β_3 (H-C)		3.020	3.204	2.892	3.096	3.265	3.204	3.077
	β_4 (Arimoto)		2.938	3.062	2.900	2.941	3.125	3.062	2.941
	β_1 (Rényi)	0.9	1.283	1.326	1.246	1.319	1.409	1.356	1.319
	β_2 (Tsallis)		1.357	1.393	1.270	1.379	1.481	1.393	1.379
	β_3 (H-C)		1.348	1.414	1.270	1.412	1.495	1.414	1.391
	β_4 (Arimoto)		1.306	1.361	1.261	1.389	1.471	1.400	1.389
	β_1 (Rényi)	0.7	1.255	1.356	1.291	1.409	1.409	1.386	1.409
	β_2 (Tsallis)		1.310	1.393	1.343	1.481	1.481	1.444	1.481
	β_3 (H-C)		1.291	1.414	1.353	1.491	1.495	1.454	1.481
	β_4 (Arimoto)		1.270	1.400	1.318	1.471	1.471	1.441	1.471

Acknowledgment

The authors would like to express their gratitude to the management of Duhok Polytechnic University for providing the necessary facilities, which helped in collecting data and thus contributed to the higher quality of the current work.

Ethical Approval

All the authors demonstrate that they have adhered to the accepted ethical standards of a genuine research study.

Competing Interests

No conflict of interest is declared by authors.

Author contributions

All authors have sufficiently contributed to the study and agree with the results and conclusions.

Funding

No funding source is reported for this study.

Availability of Data and Materials

The data used to support the findings of this study are included in the article.

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