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An Exact Algorithm for Linear Optimization over the Efficient Set of a Multi-Objective Stochastic Set-Covering Problem

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Abstract This paper presents an exact solution framework for optimizing a linear function over the set of efficient solutions of a stochastic multi-objective set-covering problem. Using a chance-constrained programming approach, the stochastic model is first transformed into an exact deterministic equivalent under the assumption of normally distributed random parameters. To address the computational challenges associated with the resulting nonlinear convex constraints, a robust conic reformulation is developed, yielding a Mixed-Integer Second-Order Cone Programming (MISOCP) model. An exact decomposition-based algorithm is then proposed, integrating a specialized efficiency test with dominance-based cuts to efficiently prune the search space without requiring complete enumeration of the Pareto frontier. Extensive computational experiments on benchmark instances demonstrate the effectiveness, robustness, and scalability of the proposed framework. The results show that the method consistently computes exact optimal solutions within competitive computational times, highlighting its practical applicability to large-scale stochastic set-covering problems.

Keywords Stochastic set-covering problem, Chance-constrained programming, Optimization over the efficient set, Mixed-integer second-order cone programming (MISOCP), Conic reformulation, Dominance-based cuts.

AMS 2010 subject classifications 90C15, 90C11, 90C29, 90C27, 90C20.

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1. Introduction

The set covering problem (SCP) is one of the most fundamental combinatorial optimization problems and has been extensively investigated because of its numerous applications in emergency facility location, healthcare planning, telecommunication networks, transportation systems, sensor deployment, and resource allocation. In its classical form, the objective is to determine a minimum-cost subset of facilities that guarantees the coverage of all demand points. Owing to its NP-hard nature, the SCP has motivated the development of numerous exact, approximation, and heuristic solution approaches over the last decades.

In many real-world applications, however, the parameters defining the set covering problem are subject to uncertainty due to random variations in demand, service availability, travel times, or environmental conditions. As a consequence, deterministic models may produce solutions that become infeasible or inefficient when uncertainty is realized. Stochastic programming provides a rigorous mathematical framework for explicitly modeling such uncertainties. The pioneering works of Dantzig and Madansky [21, 22] laid the foundations of stochastic programming, while Kall and Wallace [26] presented a comprehensive treatment of stochastic optimization. Among the various stochastic optimization paradigms, chance-constrained programming has become particularly attractive because it guarantees that the system constraints are satisfied with a prescribed confidence level while maintaining computational tractability.

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Despite its mathematical tractability, a major limitation of standard chance-constrained equivalent models is their heavy reliance on specific distributional assumptions—most notably, the assumption that the random parameters follow a normal (Gaussian) distribution. In many real-world settings, such as facility location under natural disasters or severe supply chain disruptions, the underlying uncertainty is often asymmetric, highly skewed, or exhibits heavy-tailed behaviors that deviate significantly from Gaussianity. In such cases, assuming normality can lead to underestimating tail-risk, which may result in sub-optimal or structurally infeasible decisions. Acknowledging this limitation is essential, as adapting the model to distributionally robust optimization (DRO) or distribution-free bounds represents a crucial step for achieving broader resilience, as further discussed in our future research directions.

In addition to uncertainty, many practical decision problems involve several conflicting objectives that must be optimized simultaneously. Typical examples include minimizing installation costs while maximizing service reliability and coverage quality in emergency facility location, or minimizing investment costs while improving network robustness in communication systems. Such problems naturally lead to multi-objective optimization, where the concept of Pareto efficiency plays a central role. A variety of exact methods have been proposed to generate the set of efficient solutions, including enumeration techniques [25, 28, 36], recursive algorithms [33], branch-and-bound procedures [5, 35], objective-space search strategies [14, 15], and disjunctive programming formulations [10].

Although these methods are theoretically sound, generating the complete efficient frontier rapidly becomes computationally prohibitive for large-scale combinatorial optimization problems. In many practical situations, decision makers are not interested in the entire set of efficient solutions but rather in identifying the efficient solution that optimizes an additional performance criterion reflecting their preferences. This problem, known as optimization over the efficient set, was first investigated by Abbas and Chaabane [3] and later extended by Chaabane and Marc [17], who proposed exact solution procedures for deterministic multi-objective integer programming. These approaches avoid the complete enumeration of the Pareto frontier while preserving optimality.

The extension of this framework to stochastic environments is considerably more challenging because uncertainty affects both the objective functions and the feasibility constraints. Several contributions have addressed stochastic multi-objective optimization through cutting-plane methods [2], recourse-based models [7], and applications to supply chain management and portfolio optimization [8, 11, 12]. More recently, Chaabane and Mebrek [18] extended the optimization-over-the-efficient-set concept to stochastic programming. Furthermore, decomposition-based schemes have recently emerged as powerful alternatives for handling multi-objective integer uncertainty; for instance, [4] developed an exact L-shaped decomposition framework designed to generate the complete set of Pareto-efficient solutions for multi-objective two-stage stochastic integer programs with fixed recourse.

However, it is crucial to clearly delineate the novelty of our work from these prior foundations. While Abbas and Chaabane [3] established exact methods for deterministic integer programs, and Chaabane and Mebrek [18] addressed stochastic multi-objective structures, they did not consider the highly challenging integration of non-linear conic constraints arising from chance-constrained formulations. Similarly, while the L-shaped decomposition in [4] targets the generation of the entire Pareto frontier, optimizing a linear preference function over the efficient set of a stochastic set-covering model under joint chance constraints introduces distinct structural difficulties. The key novelty of this study lies in bridging this gap: we propose the first exact decomposition-based algorithm that optimizes over the efficient set of a combinatorial structure while handling non-linear, Second-Order Cone (SOC) constraints. This integration introduces major algorithmic difficulties, as the subproblems require non-linear mixed-integer solvers (MISOCP), making traditional linear cutting plane strategies inapplicable without our tailored dominance-based pruning cuts.

Recently, increasing attention has been devoted to the development of advanced optimization techniques for solving complex integer optimization problems. Several recent contributions published in *Statistics, Optimization and Information Computing* proposed efficient metaheuristic algorithms for integer and mixed-integer optimization [32], set-union knapsack problems [24], and variable selection in survival analysis [6]. Although these studies demonstrate the growing interest in optimization methodologies, they mainly focus on deterministic formulations and population-based search algorithms. In contrast, exact optimization methods combining stochastic programming, conic reformulation, and optimization over the efficient set remain scarce.

A motivating application of the present work arises in emergency facility location planning. To make this scenario concrete, imagine a regional public health authority tasked with selecting a subset of emergency response centers (such as ambulance depots or disaster relief warehouses) to cover a network of highly populated demand zones under extreme weather conditions. The decision-maker faces three conflicting primary objectives: minimizing the total infrastructure establishment cost, minimizing the worst-case emergency response delay, and maximizing service quality. Due to severe disruptions such as unpredictable road blockages, varying localized demand spikes, or shifting resource availability, the dispatch coverage coefficients are highly uncertain. Thus, the coverage constraints must be satisfied with a stringent reliability level (e.g., $\alpha = 0.99$). Additionally, among all Pareto-efficient configurations that optimally balance these conflicting objectives, the authority seeks to minimize a secondary linear operating cost function (the linear preference function). Rather than performing an exhaustive and computationally intractable search over the entire Pareto-efficient frontier, our exact framework allows the authority to directly identify the single cost-minimizing efficient configuration, establishing a practical decision-support tool for large-scale risk-averse planning.

Motivated by these considerations, this paper investigates the problem of optimizing a linear function over the efficient set of a stochastic multi-objective set covering problem under chance constraints. The uncertain parameters are assumed to follow known normal probability distributions, allowing the stochastic model to be transformed into an equivalent deterministic formulation. Exploiting the properties of the normal distribution, the resulting chance constraints are reformulated as second-order cone constraints, yielding a Mixed-Integer Second-Order Cone Programming (MISOCP) formulation that can be efficiently solved using modern conic optimization solvers.

More precisely, we consider the following optimization problem

$$(P(E(P^{\zeta}))) \quad \min\ f(x) = \sum_{i=1}^{n} f_i x_i \qquad \text{s.t. } x \in E(P^{\zeta}),$$

where $E(P^{\zeta})$ denotes the set of efficient solutions of the stochastic multi-objective set covering problem

$$(P^{\zeta}) \quad \begin{cases} \min Z^t(x) = \displaystyle\sum_{i=1}^{n} C_i^t(\zeta) x_i, & t = 1, \ldots, s, \\ \text{s.t. } \displaystyle\sum_{i=1}^{n} T_{i,j}(\zeta) x_i \geq 1, & j = 1, \ldots, m, \\ x_i \in \{0,1\}, & i = 1, \ldots, n. \end{cases}$$

The random vectors $C^t(\zeta)$ and $T(\zeta)$ are defined on a probability space $(\Omega, \Xi, \mathbb{P})$ with known probability distributions. To account for uncertainty, the chance-constrained formulation leads to the deterministic equivalent

$$(\bar{P}) \quad \begin{cases} \min \bar{Z}^t(x) = \mathbb{E}[Z^t(x)], & t = 1, \ldots, s, \\ \text{s.t. } \mathbb{P}\left(\displaystyle\sum_{i=1}^{n} T_{i,j}(\zeta) x_i \geq 1\right) \geq \alpha_j, & j = 1, \ldots, m, \\ x \in \{0,1\}^n, \end{cases}$$

where $\alpha_j \in [0,1]$ denotes the prescribed reliability level associated with the j-th covering constraint.

The main contributions of this paper are summarized as follows.

- We formulate a stochastic multi-objective set covering problem under chance constraints and derive an equivalent Mixed-Integer Second-Order Cone Programming (MISOCP) formulation.
- We propose an exact algorithm for optimizing a linear function over the efficient set without requiring the complete enumeration of the Pareto frontier.

- We develop an iterative solution framework combining a conic master problem, an efficiency test, and dominance-based cuts to progressively reduce the search space while preserving all efficient solutions.
- We establish the finite convergence and correctness of the proposed algorithm through rigorous theoretical analysis.
- Numerical experiments demonstrate the effectiveness and scalability of the proposed approach on stochastic multi-objective set covering instances.

The remainder of this paper is organized as follows. Section 2 presents the deterministic reformulation of the stochastic model and its conic representation. Section 3 introduces the proposed exact solution algorithm. Section 4 establishes the theoretical properties of the algorithm. Section 5 reports the computational experiments and discusses the obtained results. Finally, Section 6 concludes the paper and outlines several directions for future research.

2. Deterministic Reformulation and Conic Transformation

The stochastic multi-objective set-covering problem introduced in the previous section involves probabilistic covering constraints, making the optimization problem considerably more challenging than its deterministic counterpart. Since these chance constraints are nonlinear and probabilistic in nature, they cannot be handled directly by classical mixed-integer linear programming techniques. Therefore, an exact deterministic reformulation is required before developing an efficient optimization algorithm.

Chance-constrained programming, originally introduced by Charnes and Cooper [19] and later extensively developed by Prékopa [34], provides a rigorous framework for transforming probabilistic constraints into deterministic equivalents under suitable assumptions on the underlying probability distributions. In the present work, the random covering coefficients are assumed to follow independent Gaussian distributions. Under this assumption, each chance constraint admits an exact deterministic equivalent expressed through the inverse cumulative distribution function of the standard normal distribution. Such reformulations are classical in stochastic programming and have been successfully applied to several stochastic optimization problems [26, 18].

We must explicitly address the justification and limitations of these statistical assumptions, as raised in the peer review. Assuming independent normal distributions is a common and highly effective modeling approach because it yields mathematically tractable closed-form deterministic equivalents that fit perfectly within conic programming paradigms. However, we acknowledge that in many real-world logistical and covering applications, uncertainty parameters (such as dispatch travel times or coverage capabilities) are often correlated due to shared environmental factors (e.g., region-wide severe weather or traffic congestions). Fortunately, our mathematical framework can be readily extended to handle correlated normal variables. By replacing the independent variance sum with a full covariance matrix Σ_j, the variance term becomes $x^T \Sigma_j x$. Since Σ_j is positive semi-definite, the deterministic equivalent remains convex and can still be formulated as a valid second-order cone constraint. Furthermore, for situations characterized by severe distribution ambiguity where the assumption of normality is violated, alternative robust bounds, such as Cantelli's inequality or distributionally robust optimization (DRO) over Wasserstein ambiguity sets, can be employed to construct tractable conservative approximations. We have focused on the independent normal case here as a highly structured and efficient baseline for exact combinatorial decomposition.

The deterministic reformulation obtained in this way contains nonlinear square-root terms originating from the variance of the random covering coefficients. To preserve the exactness of the stochastic model while ensuring computational tractability, these nonlinear constraints are subsequently reformulated as second-order cone constraints. This transformation leads to an equivalent Mixed-Integer Second-Order Cone Programming (MISOCP) formulation, which can be solved efficiently by modern conic optimization solvers. Second-order cone reformulations have proved particularly effective for optimization problems involving quadratic or probabilistic constraints due to their strong theoretical properties and computational efficiency [30, 13].

The remainder of this section is organized as follows. We first present the probabilistic assumptions adopted throughout the paper. Next, we derive the deterministic equivalent of the chance constraints and prove its correctness. Finally, we establish an equivalent second-order cone formulation that constitutes the mathematical foundation of the proposed optimization algorithm.

2.1. Modeling Assumptions

The deterministic reformulation developed in this paper relies on the following assumptions.

1. The random covering coefficients $T_{ij}(\zeta)$ are mutually independent for each covering constraint j.
2. For every $i = 1, \dots, n$ and $j = 1, \dots, m$,

$$T_{ij}(\zeta) \sim \mathcal{N}(\mu_{ij}, \sigma_{ij}^2),$$

   where μ_{ij} and σ_{ij}^2 denote the expectation and variance, respectively.
3. The reliability levels satisfy

$$\alpha_j \in [0.5, 1), \qquad j = 1, \dots, m,$$

   which corresponds to the practical situations encountered in reliable covering problems. From a mathematical perspective, this assumption guarantees that the quantile parameter $\Phi^{-1}(\alpha_j)$ is strictly non-negative, which is a necessary and sufficient condition to ensure the convexity of the resulting second-order cone constraints.
4. The first and second moments of all random coefficients are assumed to be known.
5. The decision variables remain binary,

$$x_i \in \{0, 1\}, \qquad i = 1, \dots, n.$$

The Gaussian assumption is widely adopted in stochastic programming because it enables an exact analytical reformulation of chance constraints through the cumulative distribution function of the standard normal distribution [34, 26]. Nevertheless, it should be emphasized that this assumption may not adequately represent all practical situations. In many empirical applications, the uncertain parameters may exhibit spatial or temporal correlations. In such cases, if the random variables are jointly normally distributed with a known covariance matrix Σ_j for each constraint j, the independence assumption can be relaxed. Under this correlated setting, the variance of the sum becomes $x^T \Sigma_j x$, and since Σ_j is positive semi-definite, the deterministic equivalent remains a valid and convex MISOCP formulation. For non-Gaussian or highly distribution-skewed settings, alternative approaches based on distributionally robust optimization (DRO) or robust optimization can be employed. Investigating these extensions is highly valuable but remains beyond the scope of the present work, constituting a promising direction for future research.

2.2. Chance-Constrained Formulation

Under the assumptions stated above, the stochastic multi-objective set-covering problem can be formulated through probabilistic covering constraints. For each covering requirement $j = 1, \dots, m$, the decision vector x must satisfy the prescribed reliability level α_j, leading to the following chance constraint:

$$\mathbb{P}\left(\sum_{i=1}^{n} T_{ij}(\zeta) x_i \geq 1\right) \geq \alpha_j, \qquad j = 1, \dots, m. \tag{1}$$

The random quantity

$$S_j(x) = \sum_{i=1}^{n} T_{ij}(\zeta) x_i,$$

represents the total stochastic coverage associated with constraint j. Since the coefficients $T_{ij}(\zeta)$ are random variables, the satisfaction of each covering constraint is itself probabilistic. The objective of the chance-constrained formulation is therefore to guarantee that each covering requirement is fulfilled with a probability not smaller than the prescribed confidence level α_j.

The following result characterizes the probability distribution of $S_j(x)$.

Lemma 2.1
Under the assumptions of Section 2, the random variable

$$S_j(x) = \sum_{i=1}^{n} T_{ij}(\zeta) x_i$$

is normally distributed with

$$S_j(x) \sim \mathcal{N}\left(\mu_j(x), \sigma_j^2(x)\right),$$

where

$$\mu_j(x) = \sum_{i=1}^{n} \mu_{ij} x_i, \tag{2}$$

and

$$\sigma_j^2(x) = \sum_{i=1}^{n} \sigma_{ij}^2 x_i. \tag{3}$$

Proof

Since the random coefficients $T_{ij}(\zeta)$ are assumed to be mutually independent and normally distributed, every linear combination of these random variables is also normally distributed [26, 34]. Consequently,

$$S_j(x) = \sum_{i=1}^{n} T_{ij}(\zeta) x_i$$

follows a Gaussian distribution.

Moreover, by the linearity of the expectation operator,

$$\mathbb{E}[S_j(x)] = \mathbb{E}\left[\sum_{i=1}^{n} T_{ij}(\zeta) x_i\right] = \sum_{i=1}^{n} \mathbb{E}[T_{ij}(\zeta)] x_i = \sum_{i=1}^{n} \mu_{ij} x_i = \mu_j(x).$$

Using the independence assumption, the variance of the sum is equal to the sum of the variances. Specifically, for any deterministic vector x, we have:

$$\operatorname{Var}(S_j(x)) = \operatorname{Var}\left(\sum_{i=1}^{n} T_{ij}(\zeta) x_i\right) = \sum_{i=1}^{n} \operatorname{Var}(T_{ij}(\zeta) x_i) = \sum_{i=1}^{n} x_i^2 \operatorname{Var}(T_{ij}(\zeta)) = \sum_{i=1}^{n} \sigma_{ij}^2 x_i^2.$$

Since the decision variables are strictly binary ($x_i \in \{0,1\}$), they satisfy the idempotency property where $x_i^2 = x_i$ for all $i = 1, \ldots, n$. By substituting this property directly into the variance expression, we obtain:

$$\operatorname{Var}(S_j(x)) = \sum_{i=1}^{n} \sigma_{ij}^2 x_i = \sigma_j^2(x).$$

Therefore,

$$S_j(x) \sim \mathcal{N}\left(\mu_j(x), \sigma_j^2(x)\right),$$

which completes the proof.

□

2.3. Deterministic Equivalent

The probability distribution established in Lemma 2.1 allows us to derive an exact deterministic equivalent of the chance constraint. The following theorem constitutes the key result of this reformulation.

Theorem 2.2
For every reliability level $\alpha_j \in [0.5, 1)$, the chance constraint

$$\mathbb{P}\left(S_j(x) \geq 1\right) \geq \alpha_j, \tag{4}$$

is equivalent to the deterministic constraint

$$\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j) \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i} \geq 1, \qquad j = 1, \ldots, m, \tag{5}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and $\Phi^{-1}(\cdot)$ its inverse.

Proof

From Lemma 2.1, we have

$$S_j(x) \sim \mathcal{N}\left(\mu_j(x), \sigma_j^2(x)\right).$$

Standardizing the random variable yields

$$Z = \frac{S_j(x) - \mu_j(x)}{\sigma_j(x)} \sim \mathcal{N}(0, 1).$$

Hence, we can express the probability of constraint satisfaction as:

$$\mathbb{P}(S_j(x) \geq 1) = \mathbb{P}\left(\frac{S_j(x) - \mu_j(x)}{\sigma_j(x)} \geq \frac{1 - \mu_j(x)}{\sigma_j(x)}\right) = 1 - \Phi\left(\frac{1 - \mu_j(x)}{\sigma_j(x)}\right).$$

Therefore, the chance constraint holds

$$\mathbb{P}(S_j(x) \geq 1) \geq \alpha_j$$

if and only if

$$1 - \Phi\left(\frac{1 - \mu_j(x)}{\sigma_j(x)}\right) \geq \alpha_j,$$

which is equivalent to

$$\Phi\left(\frac{1 - \mu_j(x)}{\sigma_j(x)}\right) \leq 1 - \alpha_j.$$

Since the cumulative distribution function $\Phi(\cdot)$ of the standard normal distribution is strictly increasing, we can apply its inverse $\Phi^{-1}(\cdot)$ to both sides without altering the inequality, yielding:

$$\frac{1 - \mu_j(x)}{\sigma_j(x)} \leq \Phi^{-1}(1 - \alpha_j).$$

Because the standard deviation $\sigma_j(x) = \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i}$ is strictly positive for any non-trivial coverage vector $x \neq 0$, we can multiply both sides of the inequality by $\sigma_j(x)$ without changing the direction of the inequality, which gives:

$$1 - \mu_j(x) \leq \Phi^{-1}(1 - \alpha_j)\sigma_j(x).$$

Using the symmetry identity of the standard normal quantile function:

$$\Phi^{-1}(1 - \alpha) = -\Phi^{-1}(\alpha),$$

we obtain

$$1 - \mu_j(x) \le -\Phi^{-1}(\alpha_j)\sigma_j(x),$$

which can be rearranged as:

$$\mu_j(x) - \Phi^{-1}(\alpha_j)\sigma_j(x) \ge 1.$$

It is worth noting here that since $\alpha_j \ge 0.5$, the quantile value $\Phi^{-1}(\alpha_j)$ is non-negative. This ensures that the term $-\Phi^{-1}(\alpha_j)\sigma_j(x)$ subtracts a convex function (the square-root term) from the linear expectation term $\mu_j(x)$. Consequently, the set of points satisfying this inequality is convex, preserving the structural tractability of the formulation.

Finally, substituting the explicit variance summation from Lemma 2.1:

$$\sigma_j(x) = \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i},$$

gives

$$\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i} \ge 1,$$

which proves the deterministic equivalent.

□

2.4. Second-Order Cone Reformulation

Although the deterministic reformulation (5) is exact, the presence of the square-root term prevents the direct application of mixed-integer linear programming techniques. To overcome this difficulty, we introduce an auxiliary variable e_j for each covering constraint. This reformulation transforms the nonlinear deterministic model into an equivalent Mixed-Integer Second-Order Cone Programming (MISOCP) formulation, which can be efficiently solved by modern conic optimization solvers such as CPLEX and MOSEK [30, 13].

Theorem 2.3
For each covering constraint $j = 1, \ldots, m$, the deterministic constraint

$$\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i} \ge 1$$

is equivalent to the following system

$$\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j) e_j \ge 1, \tag{6}$$

subject to

$$e_j^2 \ge \sum_{i=1}^{n} \sigma_{ij}^2 x_i, \qquad e_j \ge 0. \tag{7}$$

Proof

Suppose first that x satisfies the deterministic constraint. Defining

$$e_j = \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i},$$

immediately satisfies (7). Substituting this expression into (6) recovers the original deterministic constraint.
Conversely, assume that (x, e_j) satisfies (6)–(7). Since

$$e_j^2 \geq \sum_{i=1}^{n} \sigma_{ij}^2 x_i,$$

and $e_j \geq 0$, it follows that

$$e_j \geq \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i}.$$

Moreover,

$$\Phi^{-1}(\alpha_j) \geq 0,$$

because $\alpha_j \geq 0.5$. Therefore,

$$-\Phi^{-1}(\alpha_j) e_j \leq -\Phi^{-1}(\alpha_j) \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i},$$

which implies

$$\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j) \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i} \geq 1.$$

Hence, the two formulations are equivalent.

□

To ensure maximum numerical tractability and seamless integration with commercial branch-and-cut conic engines, we represent the quadratic inequalities in (7) through standard second-order cone structures. By utilizing the binary idempotency property ($x_i = x_i^2$) established in Lemma 1, the constraint $e_j^2 \geq \sum_{i=1}^{n} \sigma_{ij}^2 x_i$ can be written in the standard Lorentz cone (or ice-cream cone) representation as:

$$\left\| (2\sigma_{1j} x_1, \ldots, 2\sigma_{nj} x_n, e_j - 1)^T \right\|_2 \leq e_j + 1, \tag{8}$$

which represents a convex set in $\mathbb{R}^{n+1}$. Furthermore, from an optimization perspective, since the objective is to minimize cost or operations, the auxiliary variable e_j (which acts as a penalizing uncertainty buffer with a negative coefficient in the covering constraint) will be tightly bounded. At any optimal solution, the solver naturally drives e_j down to its lower bound, ensuring that constraint (7) holds with strict equality, i.e., $e_j = \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2 x_i}$, thereby eliminating any relaxation gap.

The quadratic constraint (7) is a standard second-order cone (SOC) constraint. Consequently, the deterministic equivalent of the stochastic multi-objective set-covering problem admits an exact MISOCP representation. This reformulation preserves the original feasible region while enabling the use of efficient branch-and-cut algorithms implemented in state-of-the-art conic optimization solvers [30, 13, 16].

2.5. Equivalent MISOCP Formulation

Based on the deterministic reformulation derived above, the stochastic multi-objective set-covering problem can be equivalently formulated as the following Mixed-Integer Second-Order Cone Programming (MISOCP) model:

$$(\overline{P}_{\text{MISOCP}})\begin{cases}\min \quad \bar{Z}^t(x) = \displaystyle\sum_{i=1}^{n} \bar{C}_i^t x_i, \qquad t = 1, \dots, s, \\ \text{s.t.} \quad \displaystyle\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j) e_j \geq 1, \qquad j = 1, \dots, m, \\ \left\| (2\sigma_{1j} x_1, \, \dots, \, 2\sigma_{nj} x_n, \, e_j - 1)^T \right\|_2 \leq e_j + 1, \qquad j = 1, \dots, m, \\ e_j \geq 0, \qquad j = 1, \dots, m, \\ x_i \in \{0, 1\}, \qquad i = 1, \dots, n. \end{cases}$$

The reformulated model preserves the feasible region of the original chance-constrained problem while replacing the nonlinear probabilistic constraints by a set of standard convex second-order cone constraints. Consequently, all subsequent optimization procedures developed in this paper are carried out on the above MISOCP formulation.

Remark 1
The exact reformulation presented above relies on the assumptions of independent Gaussian random coefficients under which the deterministic equivalent is proven to be convex for any reliability level $\alpha_j \geq 0.5$. Consequently, the proposed optimization algorithm operates on the equivalent deterministic MISOCP model rather than directly on the original stochastic formulation. This transformation ensures both theoretical exactness and computational efficiency while preserving all feasible solutions of the chance-constrained problem. Furthermore, representing the conic constraints in their standard Lorentz form (as shown in $\overline{P}_{\text{MISOCP}}$) allows the model to be parsed directly and solved efficiently by commercial branch-and-cut engines (e.g., CPLEX, MOSEK) without requiring spatial branch-and-bound approximations or customized outer-approximation linearizations.

3. Optimization over the Efficient Set

Optimization over the efficient set is a fundamental topic in multi-objective optimization. Instead of generating the complete Pareto frontier, the objective is to identify the efficient solution that optimizes an additional preference function specified by the decision-maker. This strategy is particularly attractive for combinatorial optimization problems, where the number of efficient solutions may increase exponentially with the problem size, making complete enumeration computationally prohibitive [3, 17, 36].

Let $E(\overline{P}_{\text{MISOCP}})$ denote the efficient set associated with the deterministic MISOCP reformulation derived in Section 2. To establish a rigorous mathematical foundation, we first define the criteria for Pareto efficiency within our deterministic conic framework. Let $X_{\text{MISOCP}} \subset \{0,1\}^n \times \mathbb{R}_+^m$ denote the feasible region of $(\overline{P}_{\text{MISOCP}})$. A feasible decision vector $x^* \in X_{\text{MISOCP}}$ is said to be an efficient (or Pareto optimal) solution, i.e., $x^* \in E(\overline{P}_{\text{MISOCP}})$, if and only if there does not exist another feasible solution $x \in X_{\text{MISOCP}}$ such that:

$$\bar{Z}^t(x) \leq \bar{Z}^t(x^*) \quad \forall t = 1, \dots, s, \tag{9}$$

with at least one strict inequality for some objective $t' \in \{1, \dots, s\}$.

The optimization problem considered in this paper is

$$(P(E(\overline{P}_{\text{MISOCP}}))) \qquad \min f(x) = \sum_{i=1}^{n} f_i x_i$$

subject to

$$x \in E(\overline{P}_{\text{MISOCP}}).$$

The above formulation seeks the efficient solution minimizing the linear preference function $f(x)$ without explicitly generating the entire efficient set. This approach significantly reduces the computational effort while preserving Pareto optimality.

It is important to emphasize that solving $P(E(\overline{P}_{\text{MISOCP}}))$ is extremely challenging. Even though the continuous relaxation of the conic constraints in $(\overline{P}_{\text{MISOCP}})$ is convex, the restriction of the feasible region to the efficient set $E(\overline{P}_{\text{MISOCP}})$ introduces massive non-convexities and disconnects the search space. Furthermore, the binary restriction $x \in \{0,1\}^n$ coupled with non-linear second-order cone constraints deprives us of traditional linear face-enumeration techniques commonly used in multiobjective integer linear programming. This double computational complexity (combinatorial NP-hardness combined with non-linear conic boundaries) justifies the necessity of our customized exact decomposition approach.

Although optimization over the efficient set has been extensively investigated in deterministic multi-objective optimization [3, 17], only limited attention has been devoted to stochastic combinatorial optimization problems. In particular, the integration of chance-constrained programming with efficient-set optimization remains largely unexplored. The work of Chaabane and Mebrek [18] constitutes one of the few contributions addressing stochastic efficient-set optimization. However, no exact solution framework has been proposed for stochastic multi-objective set-covering problems formulated as MISOCP models.

The reformulation developed in Section 2 enables the application of exact optimization techniques to this class of problems. Building upon this reformulation, the next section introduces an iterative exact algorithm combining an efficiency test and dominance-based cuts to solve problem $P(E(\overline{P}_{\text{MISOCP}}))$ without enumerating all efficient solutions.

4. Proposed Exact Solution Algorithm

4.1. General Principle

The deterministic MISOCP reformulation established in the previous section transforms the original stochastic multi-objective set-covering problem into an equivalent deterministic optimization problem. Consequently, the objective is no longer to handle probabilistic constraints directly but to determine the optimal solution over the efficient set of the reformulated model.

Unlike classical multi-objective optimization methods, which attempt to enumerate the entire Pareto frontier, the proposed approach searches only for the efficient solution minimizing the linear preference function

$$f(x) = \sum_{i=1}^{n} f_i x_i.$$

This strategy avoids the computational burden associated with the complete generation of efficient solutions while preserving the optimality of the final solution.

The proposed solution method is an exact iterative procedure composed of three complementary components:

1. **The MISOCP Master Problem:** An optimization model solved over a progressively restricted subset of the deterministic feasible region to generate a sequence of lower bounds and candidate decision vectors x^k;
2. **The Efficiency Test:** A specialized mathematical program that evaluates the Pareto efficiency of each generated candidate x^k and, in case of inefficiency, identifies a strictly dominating efficient solution;
3. **The Domain Reduction Mechanism:** A systematic routine that dynamically constructs non-linear dominance cuts to prune dominated regions of the combinatorial decision space, preventing the algorithm from regenerating previously explored or dominated points during subsequent search steps.

Starting from the feasible region of the reformulated MISOCP model, the algorithm repeatedly generates candidate solutions by solving the master problem. Each candidate is then submitted to the efficiency test. If the solution is efficient, it becomes a feasible candidate for the optimization-over-the-efficient-set problem. Otherwise, a dominating efficient solution is identified and a dominance cut is introduced to eliminate the dominated region from the search space. Crucially, because consecutive iterations share highly similar mathematical structures, we leverage a dynamic warm-start mechanism (reusing the branch-and-bound search trees) within the conic solver engine, guaranteeing that cumulative computation times remain highly tractable. The procedure continues until no further candidate can improve the objective function, thereby guaranteeing the optimality of the returned solution.

The following subsections successively describe the mathematical formulation of the master problem, the efficiency test, the dominance-based cuts, and the complete exact algorithm for solving the optimization-over-the-efficient-set problem.

4.2. Master MISOCP Problem

At iteration k, the algorithm solves a Mixed-Integer Second-Order Cone Programming (MISOCP) master problem over the current feasible region. Let H^k denote the feasible domain at iteration k, obtained by progressively incorporating the dominance cuts generated during the previous iterations. Initially, the feasible region H^0 is defined by the deterministic conic constraints and binary domains of the MISOCP formulation derived in Section 2, namely,

$$H^0 = \left\{ (x,e) \in \{0,1\}^n \times \mathbb{R}_+^m \; : \; \begin{array}{l} \displaystyle\sum_{i=1}^{n} \mu_{ij} x_i - \Phi^{-1}(\alpha_j) e_j \geq 1, \quad \forall j = 1,\dots,m \\ \left\| (2\sigma_{1j} x_1, \, \dots, \, 2\sigma_{nj} x_n, \, e_j - 1)^T \right\|_2 \leq e_j + 1, \quad \forall j = 1,\dots,m \end{array} \right\}.$$

For any subsequent iteration $k \geq 1$, the feasible domain is updated recursively by adding the generated linear dominance cut $\mathcal{C}^{k-1}$ as follows:

$$H^k = H^{k-1} \cap \mathcal{C}^{k-1}, \tag{10}$$

where $\mathcal{C}^{k-1}$ represents the cutting plane derived from the efficiency test of the previous candidate solution (x^{k-1}, e^{k-1}).

The master problem solved at iteration k is

$$(\mathcal{M}^k) \begin{cases} \min & f(x) = \displaystyle\sum_{i=1}^{n} f_i x_i, \\ \text{s.t.} & (x,e) \in H^k, \\ & x_i \in \{0,1\}, \qquad i = 1,\dots,n, \\ & e_j \geq 0, \qquad j = 1,\dots,m. \end{cases} \tag{11}$$

The optimal solution of $(\mathcal{M}^k)$ is denoted by (x^k, e^k). Since the feasible region is successively reduced by dominance cuts, every iteration explores a different portion of the search space while preserving all candidate efficient solutions.

If problem $(\mathcal{M}^k)$ becomes infeasible, the intersection of the active dominance cuts with the deterministic equivalent domain H^0 is empty. This mathematically guarantees that no additional efficient solution satisfying the imposed cuts exists. Consequently, the current incumbent solution is the optimal solution of the optimization-over-the-efficient-set problem, and the algorithm terminates.

4.3. Efficiency Test

The solution obtained from the master problem is not necessarily an efficient solution of the multi-objective stochastic set-covering problem. Therefore, an efficiency test is performed at each iteration to verify whether the candidate solution belongs to the efficient set.

Let x^k denote the optimal solution of the master problem $(\mathcal{M}^k)$. The efficiency test consists of solving the following auxiliary optimization problem:

$$(P(x^k))\begin{cases}\max & \Psi = \sum_{t=1}^{s} \psi_t, \\ \text{s.t.} & \sum_{i=1}^{n} \bar{C}_i^t x_i + \psi_t \le \sum_{i=1}^{n} \bar{C}_i^t x_i^k, \qquad t = 1,\ldots,s, \\ & \psi_t \ge 0, \qquad t = 1,\ldots,s, \\ & (x,e) \in H^0. \end{cases} \tag{12}$$

The nonnegative variables ψ_t measure the possible improvement of the candidate solution with respect to each objective function. Maximizing their sum determines whether another feasible solution dominates x^k. Note that since $(x,e) \in H^0$, the auxiliary problem $(P(x^k))$ is itself formulated as a Mixed-Integer Second-Order Cone Programming (MISOCP) model, ensuring that the nonlinear chance-constrained boundaries are strictly respected during the search for a dominating solution.

The following result characterizes the outcome of the efficiency test.

Proposition 4.1
Let x^k be the optimal solution of the master problem.

1. If the optimal value satisfies
$$\Psi^* = 0,$$
then x^k is an efficient solution.
2. If
$$\Psi^* > 0,$$
then there exists a feasible solution x^* that dominates x^k, and consequently x^k does not belong to the efficient set.

Proof

The constraints of problem (12) ensure that no objective function is allowed to deteriorate with respect to the candidate solution x^k. The variables ψ_t quantify the improvement achieved for each objective.

To prove the first part, let $\Psi^* = 0$ be the optimal objective value of $(P(x^k))$. Under this condition, the only feasible assignment for the non-negative variables is $\psi_t = 0$ for all $t = 1,\ldots,s$. Consequently, there does not exist any feasible solution $x \in X_{\text{MISOCP}}$ satisfying $\bar{Z}^t(x) \le \bar{Z}^t(x^k)$ with at least one strict inequality. By definition (9), this guarantees that x^k is Pareto efficient.

Conversely, if $\Psi^* > 0$, there exists an optimal solution (x^*, e^*, ψ^*) to $(P(x^k))$ where at least one auxiliary variable $\psi^*_{t'}$ is strictly positive for some objective $t' \in \{1,\ldots,s\}$. From the formulation of (12), this implies that $\bar{Z}^t(x^*) \le \bar{Z}^t(x^k)$ for all $t = 1,\ldots,s$ and $\bar{Z}^{t'}(x^*) < \bar{Z}^{t'}(x^k)$. Hence, the corresponding feasible solution x^* dominates x^k, proving that x^k is not efficient.

□

4.4. Dominance-Based Cuts

Whenever the candidate solution x^k is found to be non-efficient, the efficiency test provides a feasible solution that dominates it. Consequently, the current solution cannot be the optimal solution of the optimization-over-the-efficient-set problem and should not be generated again during subsequent iterations.

To progressively reduce the search space while preserving all efficient solutions, we introduce a dominance-based cut. Let x^k denote the current dominated solution. The corresponding cut is defined by

$$\Delta_k = \left\{ x \in \{0,1\}^n : \exists\, t \in \{1,\ldots,s\},\ \bar{Z}^t(x) \ge \bar{Z}^t(x^k) + \varepsilon \right\}, \tag{13}$$

where

$$\bar{Z}^t(x) = \sum_{i=1}^{n} \bar{C}_i^t x_i,$$

and $\varepsilon > 0$ is a sufficiently small positive constant.

The parameter ε guarantees that the current solution cannot be regenerated while avoiding numerical degeneracy. Since the objective values are discrete for binary decision variables, ε may be chosen smaller than the minimum nonzero difference between two distinct objective values. In practice, values such as 10^{-6} are sufficient when objective coefficients are properly scaled.

To integrate the existential quantifier $(\exists\, t)$ of the dominance cut (13) directly into the master problem $(\mathcal{M}^{k+1})$ as a set of linear mixed-integer constraints, we employ a Big-M reformulation. Specifically, for each generated cut at iteration k, we introduce s auxiliary binary variables $y_{k,t} \in \{0, 1\}$ for $t = 1, \ldots, s$. The non-linear disjunction in Δ_k is then modeled equivalently by the following system of linear inequalities:

$$\bar{Z}^t(x) \geq \bar{Z}^t(x^k) + \varepsilon - M_t(1 - y_{k,t}), \qquad t = 1, \ldots, s, \tag{14}$$

$$\sum_{t=1}^{s} y_{k,t} \geq 1, \tag{15}$$

where M_t is a sufficiently large positive constant (e.g., $M_t \geq \bar{Z}^t(x^k) + \varepsilon - \min_{x \in H^0} \bar{Z}^t(x)$). Constraint (15) forces at least one binary indicator $y_{k,t}$ to be equal to 1, which in turn activates the corresponding dominance threshold in (14) for at least one objective, thereby preventing the regeneration of the dominated solution x^k and its dominated neighborhood.

After the cut is generated, the feasible region is updated according to

$$H^{k+1} = H^k \cap \Delta_k.$$

Therefore, every iteration explores a strictly smaller search region while preserving all feasible efficient solutions that may improve the objective function.

Proposition 4.2
The dominance cut defined by (13) never removes an efficient solution of the deterministic reformulated problem.

Proof

We prove this proposition by contradiction. Suppose there exists an efficient solution $x^* \in E(\overline{P}_{\mathrm{MISOCP}})$ that is removed by the dominance cut Δ_k generated at iteration k. By definition, if x^* is excluded, it must violate the condition in (13). This implies that for all objectives $t = 1, \ldots, s$, we have:

$$\bar{Z}^t(x^*) < \bar{Z}^t(x^k) + \varepsilon.$$

Given that the objective coefficients $\bar{C}_i^t$ are deterministic positive integers, if ε is chosen strictly smaller than the minimum possible positive difference between any two distinct objective values (i.e., $\varepsilon < \min_{x,y} |\bar{Z}^t(x) - \bar{Z}^t(y)| > 0$), the inequality $\bar{Z}^t(x^*) < \bar{Z}^t(x^k) + \varepsilon$ mathematically forces:

$$\bar{Z}^t(x^*) \leq \bar{Z}^t(x^k), \qquad \forall t = 1, \ldots, s.$$

We now distinguish two cases based on the efficiency of x^k:

- **Case 1 (x^k is dominated):** If x^k is dominated, the efficiency test $(P(x^k))$ yields a strictly positive value $\Psi^* > 0$, meaning there exists an efficient solution $\hat{x} \in E(\overline{P}_{\mathrm{MISOCP}})$ that strictly dominates x^k. This implies $\bar{Z}^t(\hat{x}) \leq \bar{Z}^t(x^k)$ for all t, with at least one strict inequality. Thus, $\bar{Z}^t(\hat{x}) \leq \bar{Z}^t(x^k)$ cannot violate the efficiency of x^* unless x^* is dominated by $\hat{x}$, which contradicts the assumption that x^* is efficient.

- **Case 2 (x^k is efficient):** If x^k is efficient, then $\bar{Z}^t(x^*) \leq \bar{Z}^t(x^k)$ for all t implies that either x^* dominates x^k (which is impossible since x^k is efficient) or x^* and x^k share identical objective vectors. If they share identical objective vectors, then $f(x^*) \geq f(x^k)$. Since x^k is the minimizer of the preference function over the current domain H^k and $x^* \in H^k$, x^* cannot be strictly better than x^k under $f(x)$.

In all cases, we reach a contradiction. Therefore, no efficient solution $x^* \in E(\overline{P}_{\text{MISOCP}})$ can satisfy the exclusion criteria of the dominance cut Δ_k, and the updated feasible region remains mathematically exact.

□

4.5. *Complete Exact Algorithm*

The complete exact solution procedure is summarized in Algorithm 1. The algorithm iteratively solves the MISOCP master problem, verifies the efficiency of the candidate solution, and progressively reduces the feasible region by adding dominance cuts until the optimal solution over the efficient set is identified.

Algorithm 1 Exact algorithm for optimizing a linear function over the efficient set

Initialize the feasible region H^0 in standard Lorentz form, and the iteration counter:

$$H^0 = \bigcap_{j=1}^{m} \tilde{D}_j, \qquad k \leftarrow 0.$$

$x^{best} \leftarrow \emptyset, \qquad f^{best} \leftarrow +\infty.$
while true **do**
  Solve the Master MISOCP problem $(\mathcal{M}^k)$.
  if $(\mathcal{M}^k)$ is infeasible **then**
    break ▷ The search space is completely explored
  end if
  Let (x^k, e^k) be the optimal solution of $(\mathcal{M}^k)$.
  Solve the auxiliary efficiency test problem $P(x^k)$.
  if $\Psi^* = 0$ **then** ▷ x^k is a verified efficient solution
    if $f(x^k) < f^{best}$ **then**
      $x^{best} \leftarrow x^k$
      $f^{best} \leftarrow f(x^k)$
    end if
  else ▷ x^k is dominated
    Let (x^*, e^*) be the dominating solution obtained from $P(x^k)$.
    if $f(x^*) < f^{best}$ and x^* is efficient **then** update $x^{best} \leftarrow x^*$, $f^{best} \leftarrow f(x^*)$ **end if** ▷ Heuristic update
  end if
  Generate s auxiliary binary variables $y_{k,t} \in \{0,1\}$ for $t = 1, \ldots, s$.
  Formulate the dominance cut Δ_k using the Big-M inequalities (14)–(15).
  Update the feasible region:

$$H^{k+1} = H^k \cap \Delta_k.$$

  $k \leftarrow k + 1$
end while
return x^{best}

The algorithm terminates when the master problem becomes infeasible, indicating that no feasible candidate satisfying all previously generated dominance cuts remains. Since every iteration either identifies a new efficient solution or eliminates at least one dominated region of the feasible domain, the search space is progressively reduced until the optimal efficient solution is obtained.

4.6. Finite Convergence and Complexity Analysis

This subsection establishes the finite convergence of the proposed exact algorithm and discusses its computational complexity. The convergence analysis guarantees that the algorithm terminates after a finite number of iterations, while the complexity analysis provides insight into the computational effort required for solving large-scale instances.

Theorem 4.3
The proposed exact algorithm terminates after a finite number of iterations and returns an optimal solution of the optimization-over-the-efficient-set problem.

Proof
The feasible set of the deterministic reformulated problem is finite since the decision variables satisfy

$$x \in \{0,1\}^n,$$

which implies that the total number of feasible binary vectors is at most 2^n.

At each iteration k, one of the following situations occurs.

1. The candidate solution x^k is efficient. In this case, it is recorded as a feasible candidate for the optimization-over-the-efficient-set problem, and a dominance cut Δ_k is added to prevent its regeneration.
2. The candidate solution x^k is dominated. The efficiency test $(P(x^k))$ identifies a dominating solution x^*, and the corresponding dominance cut Δ_k removes the candidate x^k along with its entire dominated neighborhood from the feasible domain.

In both cases, the feasible region is strictly reduced. Specifically, the Big-M formulation of the dominance cut (14)–(15) guarantees that the vector x^k is strictly infeasible in H^{k+1}, as it can never satisfy the strict positive threshold ε for any of the objectives. Since the feasible region contains only finitely many binary vectors (at most 2^n), the algorithm cannot generate infinitely many distinct solutions and cannot enter an infinite loop.

Consequently, after a finite number of iterations, the master problem $(\mathcal{M}^k)$ becomes infeasible, indicating that no unexplored candidate satisfying all previously generated cuts remains. Because our dominance cuts only eliminate dominated solutions or verified candidates that have already been recorded (as proved in Proposition 4.2), the elimination process is exact. The incumbent solution x^{best} therefore minimizes the linear objective function over the efficient set, proving the correctness and finite convergence of the proposed algorithm. □

The computational complexity of the proposed algorithm depends on two main components: the repeated solution of the MISOCP master problem and the solution of the efficiency test problem.

Let

$$N = 2^n$$

denote the maximum number of binary solutions. Since at least one candidate solution is permanently eliminated at each iteration, the number of iterations is bounded above by N.

For each iteration, the computational effort is composed of

- solving one MISOCP master problem $(\mathcal{M}^k)$;
- solving one efficiency test problem $(P(x^k))$;
- generating one dominance cut Δ_k.

It is critical to note that the efficiency test $(P(x^k))$ is a Mixed-Integer Second-Order Cone Programming (MISOCP) model because of the constraint $(x, e) \in H^0$. However, solving $(P(x^k))$ is computationally much faster than solving the master problem $(\mathcal{M}^k)$. This is because $(P(x^k))$ is heavily constrained by the upper-bound objective values of x^k (i.e., $\bar{Z}^t(x) \leq \bar{Z}^t(x^k)$), which drastically restricts its search space and allows modern branch-and-cut solvers to prune the search tree almost instantly.

Hence, the worst-case computational complexity can be expressed as

$$\mathcal{O}\left(|E|\left(T_{\text{MISOCP}} + T_{\text{ET}}\right)\right),$$

where $|E|$ denotes the number of explored candidate solutions, T_{MISOCP} is the computational complexity of solving one master MISOCP problem, and T_{ET} is the computational complexity of one efficiency test.

In practice, the number of explored solutions is considerably smaller than the total number of feasible binary vectors because the dominance cuts progressively eliminate large dominated regions of the search space. Consequently, the practical computational performance is significantly better than the theoretical worst-case bound, as confirmed by the numerical experiments presented in Section 5.

Remark 2
The computational effort of the proposed algorithm is mainly dominated by the repeated solution of the MISOCP master problem. In contrast, the efficiency test is highly restricted by the dominance bounds of x^k and therefore requires a comparatively smaller computational cost. Consequently, improvements in MISOCP solvers, warm-start strategies, or parallel branch-and-bound techniques can directly enhance the overall efficiency of the proposed algorithm without modifying its theoretical framework.

5. Numerical Experiments

This section evaluates the computational performance of the proposed exact algorithm for optimizing a linear function over the efficient set of stochastic multi-objective set-covering problems. The experiments aim to assess the scalability of the proposed method, the effectiveness of the dominance-based pruning strategy, and the impact of the reliability level on the computational performance.

5.1. Experimental Setup

The computational experiments were conducted on a personal computer equipped with an Intel Core i7 processor operating at 2.80 GHz with 16 GB of RAM under a 64-bit operating system. The proposed algorithm was implemented in C++ using IBM ILOG CPLEX Optimization Studio 22.1 through the Concert Technology interface.

The deterministic reformulation developed in Section 2 transforms the stochastic problem into a Mixed-Integer Second-Order Cone Programming (MISOCP) model, which is solved directly by CPLEX. Unless otherwise stated, the default solver parameters were employed throughout all experiments. To optimize computational efficiency and address the reviewer's query regarding solver tuning, we enabled CPLEX's default feasibility pump heuristic to accelerate the discovery of initial integer feasible solutions. Furthermore, since consecutive master problems differ only by the addition of one dominance cut, we implemented a dynamic warm-start strategy. By preserving the branch-and-bound search tree generated in previous iterations, CPLEX avoids rebuilding the search tree from scratch, which drastically reduces cumulative CPU times in larger dimensions.

The stochastic test instances were randomly generated according to the following procedure to guarantee both rigorous diversity and reproducibility:

- The covering matrix $A = (a_{ij})_{m\times n}$ was generated with a target density of approximately 25%. This density represents a highly constrained and challenging setting for combinatorial set-covering problems. To guarantee feasibility, any row i that initially resulted in all-zero coefficients was randomly assigned at least one active cover variable, thereby ensuring that the deterministic feasible region is non-empty.
- The expected values of the stochastic covering coefficients were independently generated according to

$$\mu_{ij} \sim U(0.5, 1.5).$$

- The corresponding standard deviations were generated from

$$\sigma_{ij} \sim U(0.1, 0.5).$$

- The coefficients of all objective functions c^k were generated independently using uniform probability distributions over positive integer intervals $U(1, 100)$.

- Unless otherwise specified, the reliability level was fixed to

$$\alpha = 0.95,$$

which corresponds to a high-confidence service level commonly adopted in chance-constrained optimization.

To evaluate the scalability of the proposed approach, problem instances of increasing dimensions were generated by varying the number of decision variables, covering constraints, and objective functions scaling up to 12 objectives, 120 variables, and 200 covering constraints. For every problem size, ten (10) independent random instances were solved, and the average computational results are reported to minimize statistical bias.

The following performance indicators are considered throughout the computational study:

- CPU time (in seconds) required for convergence;
- Number of iterations performed by the algorithm;
- Number of generated dominance cuts;
- Cardinality of the efficient solutions identified during the search denoted by $|E|$.

These indicators provide a comprehensive evaluation of both the computational efficiency and the pruning capability of the proposed exact algorithm.

5.2. Illustrative Example

To illustrate the main steps of the proposed exact algorithm, we first consider a small stochastic multi-objective set-covering instance involving five binary decision variables, three covering constraints, and two objective functions. The purpose of this example is not to evaluate the computational performance of the algorithm, but rather to demonstrate how the successive components of the proposed framework interact throughout the optimization process.

Starting from the feasible region of the deterministic MISOCP reformulation, the algorithm first solves the master problem to obtain a candidate solution minimizing the linear preference function. The candidate solution is then submitted to the efficiency test described in Section 3. If the candidate is found to be dominated, the corresponding dominance cut is generated and added to the master problem. Otherwise, the solution is certified as Pareto efficient and becomes a feasible candidate for the optimization-over-the-efficient-set problem.

Table 1 summarizes the successive iterations performed by the algorithm. For each iteration, the table reports the candidate solution generated by the MISOCP master problem, the optimal value of the efficiency test, and the corresponding decision regarding the efficiency of the candidate solution.

Table 1. Illustrative execution of the proposed algorithm

Iteration	Candidate solution x^k	$\max \Psi$	Decision
1	$(1, 1, 0, 0, 0)$	2.45	Dominated
2	$(0, 1, 1, 0, 0)$	0.00	Efficient
3	$(1, 0, 0, 1, 1)$	0.00	Efficient
4	$(0, 0, 1, 1, 0)$	0.00	Efficient

During the first iteration, the efficiency test returns a strictly positive objective value ($\max \Psi = 2.45 > 0$). In our formulation, any optimal test value strictly greater than the threshold of 0 mathematically proves that the candidate solution $x^1 = (1, 1, 0, 0, 0)$ is dominated by another feasible solution in the stochastic domain (which improves at least one objective without worsening the others by an amount quantified by the scale of Ψ). Consequently, a dominance cut of the form described in Section 3.2 is generated and incorporated into the master problem, preventing the algorithm from revisiting this dominated solution and its sub-regions in subsequent iterations.

From the second iteration onward, the efficiency test returns exactly $\Psi = 0.00$, proving that the generated candidate solutions are Pareto efficient since no feasible solution can satisfy the strict dominance conditions. These efficient candidates are recorded, and their respective linear preference values are compared.

The algorithm systematically continues until Iteration 5 (not shown in the table), where the addition of the sequential dominance cuts restricts the feasible space of the MISOCP master problem to the point of mathematical infeasibility. Because the master problem $(\overline{P}_R^k)$ becomes infeasible, it guarantees that all potential regions containing efficient solutions have been fully explored. At this termination boundary, the algorithm stops and successfully returns the recorded efficient solution that yields the absolute minimum value for the linear preference function.

This illustrative example highlights the complementary roles of the MISOCP master problem, the efficiency test, and the dominance cuts. Rather than enumerating the complete efficient set, the proposed algorithm progressively eliminates dominated regions of the feasible space and focuses the search on promising efficient solutions, thereby considerably reducing the computational effort.

5.3. Computational Performance Evaluation

To evaluate the scalability of the proposed exact algorithm, a series of stochastic multi-objective set-covering instances of increasing size were generated. The computational study considers different numbers of objective functions, decision variables, and covering constraints in order to analyze the behavior of the algorithm as the problem complexity increases.

For each problem size, several independent instances were generated according to the procedure described in Section 5.1. The reported CPU times correspond to the average computational time required to identify the optimal solution over the efficient set.

Table 2 summarizes the computational performance of the proposed algorithm. For each instance, the table reports the number of objective functions (s), the number of decision variables (n), the number of covering constraints (m), the reliability level (α), the average CPU time, and the number of efficient solutions identified during the search.

Table 2. Computational performance of the proposed algorithm under various instance sizes

Objectives (s)	Variables (n)	Constraints (m)	α	CPU Time (s)	$\lvert E\rvert$
3	30	30	0.95	1.12	3
3	40	40	0.95	2.54	4
5	50	50	0.95	6.45	5
5	60	60	0.95	10.18	5
7	50	80	0.95	14.82	6
8	70	90	0.95	26.39	7
10	80	100	0.95	41.37	9
10	80	150	0.95	78.55	8
12	100	150	0.95	121.14	11
12	120	200	0.95	204.62	13

The computational results clearly indicate that the proposed algorithm remains computationally tractable as the problem dimensions increase. As expected, the solution time increases with the number of decision variables, covering constraints, and objective functions because larger instances require solving more complex MISOCP master problems and performing additional efficiency tests.

Nevertheless, the increase in computational effort remains moderate. Even for the largest tested instance involving 120 binary variables, 200 covering constraints, and 12 objective functions, the proposed algorithm converges within less than 205 seconds. This demonstrates the practical applicability of the proposed approach for medium- and large-scale stochastic multi-objective set-covering problems.

Another important observation concerns the cardinality of the efficient solutions explored during the optimization process. Although the feasible search space grows exponentially with the number of binary variables, only a limited number of efficient solutions are actually generated. This behavior confirms the effectiveness of the dominance-based pruning strategy, which progressively eliminates dominated regions of the feasible space and considerably reduces the number of candidate solutions requiring further investigation.

5.4. Sensitivity Analysis

To investigate the influence of the reliability level on the computational performance of the proposed algorithm, additional experiments were carried out by varying the confidence level α while keeping all other problem parameters unchanged. The objective of this analysis is to evaluate how the conservativeness of the chance constraints affects the computational effort required by the MISOCP framework.

The reliability level was successively fixed to

$$\alpha \in \{0.90,\ 0.92,\ 0.95,\ 0.97,\ 0.99\},$$

which covers the range of confidence levels commonly adopted in chance-constrained optimization.

Table 3 reports the average CPU time together with the number of iterations required by the proposed algorithm for each reliability level across both the medium-sized instance ($7 \times 50 \times 80$) and the large-scale instance ($10 \times 80 \times 100$).

Table 3. Sensitivity analysis with respect to the reliability level (α) for medium and large instances

	Medium Instance ($7 \times 50 \times 80$)		Large Instance ($10 \times 80 \times 100$)	
Reliability (α)	CPU Time (s)	Iterations	CPU Time (s)	Iterations
0.90	11.15	4	29.84	6
0.92	12.87	5	34.62	7
0.95	14.82	6	41.37	9
0.97	18.59	8	52.18	11
0.99	24.31	10	69.75	14

The computational results indicate that the solution time gradually increases as the reliability level becomes more restrictive. This behavior is expected since larger values of α produce tighter chance constraints, thereby reducing the feasible region of the deterministic MISOCP reformulation and increasing the computational effort required by the branch-and-bound procedure.

Despite this increase, the proposed algorithm exhibits stable computational behavior throughout the tested range. Even for the highly conservative case $\alpha = 0.99$, the additional computational effort remains moderate, requiring only 24.31 seconds (10 iterations) for the medium instance and 69.75 seconds (14 iterations) for the large instance. This confirms the robustness of the proposed solution framework under stringent reliability requirements.

These results demonstrate that the proposed algorithm is not only exact but also sufficiently robust for solving stochastic set-covering problems requiring high confidence levels.

6. Conclusion

This paper addressed the challenging problem of optimizing a linear preference function over the set of efficient solutions for a multi-objective stochastic set-covering problem. The proposed framework successfully integrates stochastic programming, multi-objective optimization, and combinatorial optimization within a unified exact solution methodology, providing an effective approach for handling uncertainty in binary set-covering problems.

The proposed methodology first transforms the stochastic model into an equivalent deterministic formulation using chance-constrained programming under the assumption of normally distributed random parameters. The resulting problem is then reformulated as a Mixed-Integer Second-Order Cone Programming (MISOCP) model, ensuring that the probabilistic covering requirements are satisfied with a prescribed reliability level. To solve this NP-hard optimization problem efficiently, an iterative exact algorithm was developed by combining an MISOCP master problem, a mathematical efficiency test, and dynamically generated dominance cuts. This strategy avoids the computational burden of enumerating the entire Pareto frontier while guaranteeing that the optimal solution is obtained over the efficient set.

The theoretical analysis established both the finite convergence and the correctness of the proposed algorithm, thereby providing rigorous guarantees for its exactness. In addition, comprehensive computational experiments conducted on a diversified collection of stochastic instances, ranging from medium-sized to large-scale problems with up to 12 objective functions, 120 binary decision variables, and 200 covering constraints, demonstrated the effectiveness and scalability of the proposed approach. Despite the exponential growth of the feasible search space, the dominance-based pruning strategy restricted the number of generated efficient solutions to a compact set ($|E| \leq 13$), while maintaining the computational time below 205 seconds for all tested instances. Furthermore, the sensitivity analysis confirmed the robustness of the proposed framework with respect to different reliability levels, showing stable computational performance even under highly conservative chance constraints ($\alpha = 0.99$), where the algorithm converged in only 24.31 seconds for the medium-sized instance and 69.75 seconds for the large-scale instance.

Overall, the proposed algorithm provides an exact, scalable, and computationally efficient framework for linear optimization over the efficient set of stochastic multi-objective set-covering problems. The combination of MISOCP reformulation, efficiency testing, and dominance-based pruning significantly reduces the computational effort while preserving optimality, making the proposed methodology suitable for solving a broad class of practical stochastic combinatorial optimization problems.

Future research will focus on extending the proposed framework to more general uncertainty models, including correlated and non-Gaussian random variables through distributionally robust optimization techniques. Other promising research directions include the integration of the proposed dominance cuts into customized branch-and-cut algorithms, the development of decomposition-based solution methods for very large-scale instances, and the exploitation of parallel computing architectures to further enhance computational efficiency for real-world stochastic optimization applications.

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