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Application of Model Predictive Control on Flow Line Production System in Sugar Factory at Milling Station Using Max-Plus Algebra

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Abstract Efficient timing and synchronization are crucial elements in flow-line production systems, as delays in a single processing stage can propagate to subsequent operations. This study applies a Model Predictive Control (MPC) strategy to a sugar factory milling station modeled as a Max-Plus Linear (MPL) system. The main objective is to develop a mathematical representation of the milling production flow and determine feasible input-event time sequences under operational constraints. The 12-station milling production system is represented using constrained Max-Plus algebraic state equations. The MPC formulation is then constructed by incorporating input-spacing constraints, output deadlines, prediction horizons, and control horizons to minimize the makespan while suppressing Work-in-Process (WIP). The model was tested using realistic configurations for processing and transition time parameters. Computational results demonstrate that the MPC-MPL controller successfully generates input-time sequences that satisfy output deadline limits and operational constraints. Compared to the FIFO Baseline method, MPC reduces the makespan by up to $\sim 7.6\%$ and decreases WIP accumulation in buffers/tanks by over 70%. These findings prove the applicability and robustness of the MPC-MPL framework as a structured solution for optimizing production scheduling in sugar milling operations.

Keywords Max-Plus Linear Systems, Model Predictive Control, Flow-Line Production, Sugar Factory, Milling Station, Production Scheduling

AMS 2010 subject classifications 93B45, 15A80.

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1. Introduction

Sugar is a strategic commodity in the food industry, used extensively both for direct consumption and as a raw material for further processing. Increasing demand for sugar requires industrial production systems to operate efficiently without compromising timing reliability. In multi-stage manufacturing systems, efficiency is not determined solely by processing durations at individual machines, but also by the synchronization between consecutive processing stages. Delays at one stage can propagate to subsequent stages (a domino effect) and disrupt the timely completion of the final product. Therefore, determining precise production-event times is a crucial issue in flow-line production management.

Flow-line production systems can be categorized as discrete-event systems, where system dynamics advance based on the occurrence of specific events [1]. Max-Plus algebra provides an ideal mathematical framework for modeling these systems, as precedence and event-synchronization constraints can be expressed linearly using maximum and addition operations [2]. Seleim and ElMaraghy [3] demonstrated the effectiveness of Max-Plus

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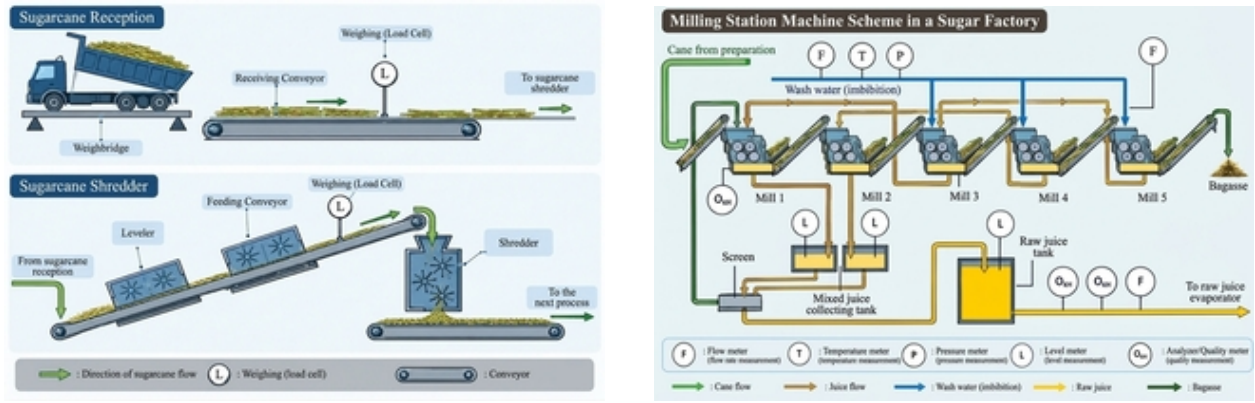
algebra in analyzing both serial and parallel manufacturing lines. In the sugar industry, Indriyani and Subiono [4] successfully applied Max-Plus algebra to schedule crystal sugar production. However, conventional Max-Plus approaches generally function as passive modeling tools and lack dynamic predictive control mechanisms under flexible operational constraints.

To address these limitations, Model Predictive Control (MPC) can be integrated with Max-Plus modeling [5, 6]. MPC is a model-based control strategy that predicts future system behavior over a finite horizon and calculates optimal control actions while explicitly considering input and output variable constraints. De Schutter and van den Boom [7, 8] pioneered the MPC framework for Max-Plus Linear (MPL) systems. The application of MPC-MPL has since expanded across various manufacturing and logistics sectors, including Work-in-Process (WIP) optimization on production lines, fuel distribution terminals [9], and energy savings in assembly lines.

The sugar factory milling station is a vital stage that extracts raw juice from sugarcane before it is directed to the purification and evaporation stations [10, 11, 12]. This process involves sequential machine interactions (ranging from reception, shredders, and milling tandems to screens and juice tanks) constrained by residence times and imbibition water circulation. These characteristics make the milling station uniquely suited for representation as an MPL system. Nevertheless, most existing literature keeps Max-Plus algebraic modeling in sugar factories [4] separate from constrained MPC applications, which primarily focus on general manufacturing industries. A practical gap remains regarding how the constrained MPC-MPL framework can be explicitly applied to a multi-stage sugar milling line to regulate raw material feed times (input-spacing) while satisfying raw juice deadlines (output-deadline). Furthermore, this constrained optimization-based approach aligns with developments in manufacturing modeling and optimal control [13, 14].

2. Literature Review

2.1. Production System Scheme of the Milling Station



(a) (b)

Figure 1. (a) Preparation station and (b) Milling station [12]

2.2. Max-Plus Algebra for Discrete-Event Production Systems

2.2.1. Mathematical Formulation of Max-Plus Algebra

Max-Plus algebra is an algebraic structure $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ denotes the set of real numbers. Its two primary operations are defined as follows [1, 2]:

- Max-Plus Addition Operation ($\oplus$): $a \oplus b = \max(a, b)\, \forall a, b \in \mathbb{R}_{\max}$.

- Max-Plus Multiplication Operation ($\otimes$): $a \otimes b = a + b \, \forall a, b \in \mathbb{R}_{\max}$.

The zero element (identity element with respect to $\oplus$) is denoted by ε, where $\varepsilon = -\infty$. The unit element (identity element with respect to $\otimes$) is denoted by e, where $e = 0$ [1].

2.2.2. *Max-Plus Matrix Multiplication*

For matrices $A \in \mathbb{R}_{\max}^{m \times p}$ and $B \in \mathbb{R}_{\max}^{p \times n}$, the (i, j)-th entry of the product $C = A \otimes B$ is calculated as [2]:

$$C_{ij} = \bigoplus_{k=1}^{p} (A_{ik} \otimes B_{kj}) = \max_{k=1,\ldots,p} (A_{ik} + B_{kj}).$$

2.2.3. *Dynamic Representation of Flow-Line Production Systems*

Consider a production line consisting of n sequential machine stages. For the k-th event cycle/part, the time variables are defined as follows [3]:

- $x_i(k)$: Start time of processing on machine i for cycle k.
- d_i: Processing time on machine i.
- $t_{(i,i-1)}$: Transportation/synchronization time from machine i to machine $i-1$.

a) Event Precedence Equations

Processing on machine i for the k-th part can commence only after the preceding machine $(i-1)$ has finished processing the k-th part and machine i has completed processing the previous part $(k-1)$ [3]:

$$x_i(k) = \max \left(x_{i-1}(k) + d_{i-1} + t_{(i,i-1)}, ; x_i(k-1) + d_i \right)$$

In Max-Plus algebraic notation, this precedence condition becomes linear:

$$x_i(k) = \left((d_{i-1} + t_{(i,i-1)}) \otimes x_{i-1}(k) \right) \oplus (d_i \otimes x_i(k-1))$$

b) Max-Plus Linear (MPL) State-Space Representation

Overall, a flow-line system with input $u(k)$ (raw material arrival time) and output $y(k)$ (product completion time) is expressed in state-space form as follows [7]:

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus B_0 \otimes u(k) \tag{1}$$

$$y(k) = C \otimes x(k) \tag{2}$$

By eliminating the implicit dependency on $x(k)$ using the Kleene star operator, the system can be expressed in explicit state-space form [7]:

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \tag{3}$$

$$y(k) = C \otimes x(k) \tag{4}$$

where $A = A_0^* \otimes A_1$ and $B = A_0^* \otimes B_0$ represent the system dynamic matrices containing processing time and inter-stage delay information [1].

In Max-Plus algebra, $A^* = I \oplus A \oplus A^{\otimes 2} \oplus \cdots \oplus A^{\otimes m}$. The (i, j)-th entry of A^* represents the weight (duration) of the longest path from state/machine j to state/machine i within the same event cycle.

2.3. *Model Predictive Control for Production Systems*

2.3.1. *State and Output Prediction Equations*

The production system dynamics are represented in explicit Max-Plus Linear (MPL) state-space form, as shown in Equations (3) and (4), where $u(k) \in \mathbb{R}^m_{\max}$ is the k-th input event-time vector, $x(k) \in \mathbb{R}^n_{\max}$ is the k-th process start-time vector across the working machines, and $y(k) \in \mathbb{R}^p_{\max}$ is the k-th output event-time vector. For a prediction horizon N_p, the future input vector $\tilde{U}(k)$ and predicted output vector $\tilde{Y}(k)$ are defined as [7]:

$$\tilde{U}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N_p-1) \end{bmatrix}, \qquad \tilde{X}(k) = \begin{bmatrix} x(k) \\ x(k+1) \\ \vdots \\ x(k+N_p-1) \end{bmatrix}, \qquad \tilde{Y}(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+N_p-1) \end{bmatrix}$$

The future output prediction equations are obtained by recursively substituting the state equations:

$$\tilde{Y}(k) = H \otimes x(k-1) \oplus G \otimes \tilde{U}(k) \tag{5}$$

with the Max-Plus structural matrices:

$$H = \begin{bmatrix} C \otimes A \\ C \otimes A^{\otimes 2} \\ \vdots \\ C \otimes A^{\otimes N_p} \end{bmatrix}, \qquad G = \begin{bmatrix} C \otimes B & \varepsilon & \cdots & \varepsilon \\ C \otimes A \otimes B & C \otimes B & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes (N_p-1)} \otimes B & C \otimes A^{\otimes (N_p-2)} \otimes B & \cdots & C \otimes B \end{bmatrix}$$

2.3.2. *Operational Constraints of the Production System*

- Input-spacing constraints: Ensure that time intervals between consecutive raw material arrivals do not fall below the minimum threshold $\Delta u_{\min}$ in order to control Work-in-Process (WIP) and prevent raw material accumulation along the production line [3]:

  $$u(k) - u(k-1) \geq \Delta u_{\min}(k)$$

  In matrix linear inequality form, this constraint is expressed as [7, 8]:

  $$E_u \tilde{u}(k) \leq f_u(k)$$

  $$y(k) \leq r(k)$$

- Output-deadline constraints: Ensure that the product completion time $y(k)$ does not exceed the specified deadline $r(k)$ [7, 9]:

  Expanded into a set of conventional linear inequalities based on the input matrix [7, 6]:

  $$E_y \tilde{u}(k) \leq f_y(k)$$

2.3.3. *MPC Optimization Problem (Linear Programming)*

The MPC cost function minimizes the production completion time (makespan) while maximizing the input time $u(k)$, ensuring that raw materials are fed as late as possible in alignment with a just-in-time (JIT) schedule [7, 9]:

$$J = \sum_{j=1}^{N_p} \lambda_y \hat{y}(k+j|k) - \sum_{j=1}^{N_c} \lambda_u u(k+j-1) \tag{6}$$

where λ_1 and λ_2 are weighting parameters. This optimization problem is formulated as a standard Linear Programming (LP) problem [7, 6]:

$$\min_{\tilde{u}(k)} c^T \tilde{u}(k) \qquad \text{with constraints} \qquad A_{\text{lp}} \tilde{u}(k) \leq b_{\text{lp}}(k)$$

2.4. Sugar Production and Milling Process Flow

2.4.1. Max-Plus Linear (MPL) Modeling of the Sugar Milling Flow

The milling station consists of N sequential milling machines $(P_1, P_2, \ldots, P_N)$.

- Event-Time Variables:
  - $u(k)$: Time at which the k-th batch of sugarcane is fed into the milling station.
  - $x_i(k)$: Time at which processing of the k-th sugarcane batch begins on Machine i $(i = 1, 2, \ldots, N)$.
  - $y(k)$: Time at which the raw juice extracted from the k-th sugarcane batch completes processing and is ready for transfer to the purification/evaporation station.
- System Parameters:
  - d_i: Processing time (milling time) on Machine i
  - $t_{(i,i+1)}$: Transfer/conveyance time of bagasse from Machine i to Machine $i+1$.

2.4.2. Event Dynamics Equations

- First Machine (P_1): Commences processing when the k-th sugarcane batch arrives and Machine 1 finishes processing the previous batch $(k-1)$.

$$x_1(k) = \max\left(x_1(k-1) + d_1, ; u(k)\right)$$

  In Max-Plus algebra:

$$x_1(k) = d_1 \otimes x_1(k-1) \oplus u(k)$$

- Sequential Machines $(i = 2, \ldots, N)$: Commences processing when Machine $i-1$ finishes processing the k-th sugarcane batch and transfers it, and Machine i has finished processing the previous batch $(k-1)$.

$$x_i(k) = \max\left(x_{i-1}(k) + d_{i-1} + t_{(i-1,i)}, \quad x_i(k-1) + d_i\right)$$

  In Max-Plus algebra:

$$x_i(k) = (d_{i-1} + t_{(i-1,i)}) \otimes x_{i-1}(k) \oplus d_i \otimes x_i(k-1)$$

- Raw Juice Output $(y(k))$: Determined by the completion time at the final milling machine (P_N).

$$y(k) = d_N \otimes x_N(k)$$

2.4.3. Explicit State-Space Equations of the Milling Station

In matrix form, the recursive state equations are expressed as implicit equations, as shown in Equations (1) and (2). By utilizing the Kleene Star operator, they are transformed into explicit form, as presented in Equations (3) and (4):

2.4.4. Constrained MPC Integration for the Milling Line Flow

For a prediction horizon N_p, the predicted raw juice output vector $\tilde{Y}(k) = \begin{bmatrix} y(k) & y(k+1) & \cdots & y(k+N_p-1) \end{bmatrix}^T$ is calculated using Equation (5).
a) Operational Milling Constraint Inequalities:

- Restricts the sugarcane feeding rate to prevent overcrowding at the preparation station and conveyors (τ_{in}).

$$u(k+j) - u(k+j-1) \geq \tau_{\text{in}} \quad \Longrightarrow \quad M_{\text{in}}\tilde{U}(k) \leq h_{\text{in}}(k)$$

- Raw Juice Output-Deadline: Ensures a timely supply of raw juice to the evaporation station (evaporators) so that the downstream crystallization process is not interrupted.

$$y(k+j) \leq d_{\text{out}}(k+j) \quad \Longrightarrow \quad M_{\text{out}}\tilde{U}(k) \leq h_{\text{out}}(k)$$

Optimasi Linear Programming (LP):

$$\min_{\tilde{U}(k)} J(k) = \sum_{j=0}^{N_p-1} y(k+j) - \lambda \sum_{j=0}^{N_p-1} u(k+j)$$

$$\text{Subject to: } \begin{bmatrix} M_{\text{in}} \\ M_{\text{out}} \end{bmatrix} \tilde{U}(k) \leq \begin{bmatrix} h_{\text{in}}(k) \\ h_{\text{out}}(k) \end{bmatrix}$$

2.5. Optimization and Scheduling Perspectives

2.5.1. Components of the MPC-MPL Optimization Problem

Under the Model Predictive Control (MPC) framework, the scheduling problem is formulated as a finite-horizon optimization problem at each event cycle k [7, 6].

- Decision Variable: Vector of future input-event time sequences over the prediction horizon N_p:

$$\tilde{U}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N_p-1) \end{bmatrix} \in \mathbb{R}^{N_p}$$

- Predicted Output Vector:

$$\tilde{Y}(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+N_p-1) \end{bmatrix} = H \otimes x(k-1) \oplus G \otimes \tilde{U}(k)$$

2.5.2. Objective Function Formulation

The optimization objective is to minimize the production completion time (makespan) while simultaneously preventing premature raw material release (in alignment with Just-In-Time principles to reduce Work-In-Process) [7, 9]:

$$J(\tilde{u}(k)) = \lambda_y \sum_{j=1}^{N_p} y(k+j-1|k) - \lambda_u \sum_{j=1}^{N_p} u(k+j-1) \tag{7}$$

In conventional matrix form:

$$J(\tilde{u}(k)) = \lambda_y 1^T \tilde{y}(k) - \lambda_u 1^T \tilde{u}(k) \tag{8}$$

where:

- $1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^{N_p}$
- $\lambda_y, \lambda_u > 0$ are weighting factors used to balance input delay and output completion time.

2.5.3. Constraint Formulations

a) Minimum Input-Spacing Constraint:

Restricts raw material arrival intervals to prevent overloading the receiving station [3]:

$$u(k+j) - u(k+j-1) \geq \Delta u_{\min}(k+j), \qquad \forall j \in 0, 1, \ldots, N_p - 1$$

In matrix inequality form:

$$E_u \tilde{u}(k) \leq f_u(k)$$

where E_u and $f_u(k)$ represent the difference matrix and the minimum time-interval threshold vector, respectively [7].

b) Output-Deadline Constraint:

Ensures that the extraction process is completed prior to the processing deadline to prevent quality degradation of the raw juice [10, 4]:

$$\hat{y}(k+j-1|k) \leq r(k+j-1), \qquad \forall j \in 1, 2, \ldots, N_p$$

Expanded into conventional algebraic linear inequalities via elimination of the Max-Plus operators [7, 8]:

$$E_y \tilde{u}(k) \leq f_y(k)$$

2.5.4. Reduction to Standard Linear Programming (LP) Form

Since the system equations are linear with respect to the variable vector $\tilde{u}(k)$, the MPC-MPL optimization problem is efficiently solved using Linear Programming (LP) [7, 6]:

$$\min_{\tilde{u}(k)} c^T \tilde{u}(k) \qquad \text{dengan kendala} \qquad A_{\text{lp}} \tilde{u}(k) \leq b_{\text{lp}}(k) \tag{9}$$

where:

$$c^T = \lambda_y 1^T \tilde{G} - \lambda_u 1^T$$

$$A_{\text{lp}} \begin{bmatrix} E_u \\ E_y \end{bmatrix}$$

$$b_{\text{lp}}(k) = \begin{bmatrix} f_u(k) \\ f_y(k) - \lambda_y 1^T (\tilde{F} \otimes x(k-1)) \end{bmatrix}$$

2.6. FIFO Baseline

First-In, First-Out (FIFO) is a scheduling rule in which materials or entities that arrive first at the production line are processed and dispatched first. In production system modeling and Max-Plus algebra, FIFO serves as a baseline or benchmark to compare the performance of a system without predictive control against a system controlled via MPC [1, 15].

In practice, FIFO executes processes sequentially based on the arrival sequence of materials. A new material is processed as soon as the machine or workstation becomes available, whereas materials that cannot yet be processed wait in the buffer. There is no predictive mechanism regarding future system states, nor are there planned input delays designed to control Work-In-Process (WIP) or anticipate bottlenecks [7, 3].

From a Max-Plus algebra perspective, the start time of a process is fundamentally determined by machine availability and material arrival time, which can generally be expressed as:

$$x_i(k) = \max\left[x_{i-1}(k) + d_{i-1} x_i(k-1) + d_i\right]$$

where $x_i(k)$ represents the event time at station i for the k-th cycle, and d_i denotes the processing time at that station. Because FIFO releases materials according to arrival order without optimizing for downstream conditions, the system may generate waiting times and accumulate WIP when downstream station capacities are lower or when bottlenecks occur [1, 2].

Accordingly, FIFO is utilized in this study as a comparative benchmark scenario to evaluate the capability of MPC in regulating feeding times and reducing queues and WIP along the milling station production line.

3. Results and Discussion

3.1. Milling Station Data

The processing time required for each processing machine (d_i) as listed in Table 1, and the transit time required for transferring intermediate material from one machine to the next is presented in Table 2 [11].

The following is the detailed formulation of the Max-Plus algebraic state-space equations based on the 12 state variables (X_1 to X_12) and the time parameters from Table 1 and Table 2:

Table 1. Processing Times of the Processing Machines

No.	d_i	Stage	Initial Value (min)	Optimization Range (min)	Remarks
1	d_1	Receiving Conveyor	2.00	1.50–3.00	L/V; e.g., a 23.5–24.48 m carrier at 9–17 m/min yields approximately 1.4–2.6 min
2	d_2	Feeding Conveyor	1.50	1.00–2.50	Variable-speed conveyor and steady feeding principles
3	d_3	Cane Leveller	0.30	0.20–0.50	Effective contact time; leveller maintains feed layer thickness
4	d_4	Prepared Cane	0.80	0.50–1.20	Buffer/transfer after cane preparation
5	d_5	Cane Shredder	0.25	0.15–0.40	Effective residence time in cane preparation; Preparation Index (PI) serves as a constraint
6	d_6	Mill 1	0.50	0.35–0.70	Effective milling residence time; tandem dikontrol melalui first-mill speed
7	d_7	Mill 2	0.50	0.35–0.70	Tandem milling
8	d_8	Mill 3	0.50	0.35–0.70	Tandem milling
9	d_9	Mill 4	0.50	0.35–0.70	Tandem milling
10	d_{10}	Mill 5	0.50	0.35–0.70	Tandem milling; industrial practice typically uses a 5-mill tandem operating at ~ 6 rpm
11	d_{11}	Juice Screen	2.00	1.00–3.00	Mixed-juice system design data; screen and drawing tank operating at the milling station
12	d_{12}	Juice Tank	2.00	1.00–3.00	Representative mixed juice drawing tank with a 2-minute retention time

Table 2. Inter-stage Transfer Times of Intermediate Materials

No.	Inter-Stage Transfer	Flow Function	Initial Value (min)	Optimization Range (min)	Remarks
1	$U_1 \to P_1$	Sugarcane → Receiving Conveyor	2.00	1.50–3.00	Carrier transport time, image
2	$P_1 \to P_2$	Receiving Conveyor → Feeding Conveyor	1.00	0.70–1.50	Transfer conveyor
3	$P_2 \to P_3$	Feeding Conveyor → Cane Leveller	0.30	0.20–0.50	Transfer/feeding time
4	$P_3 \to P_4$	Cane Leveller → Prepared Cane	0.30	0.20–0.50	Effective residence/transfer
5	$P_4 \to P_5$	Prepared Cane → Cane Shredder	0.30	0.20–0.50	Transfer to cane preparation
6	$P_5 \to P_6$	Cane Shredder → Mill 1	0.40	0.25–0.60	Prepared cane transfer
7	$P_6 \to P_7$	Mill 1 → Mill 2	0.40	0.25–0.60	Bagasse transfer
8	$P_7 \to P_8$	Mill 2 → Mill 3	0.40	0.25–0.60	Bagasse transfer
9	$P_8 \to P_9$	Mill 3 → Mill 4	0.40	0.25–0.60	Bagasse transfer
10	$P_9 \to P_{10}$	Mill 4 → Mill 5	0.40	0.25–0.60	Bagasse transfer
11	$P_{10} \to P_8$	Mill 5 Juice → Imbibition Line to Mill 3/Mill 4	0.60	0.40–1.00	Return juice/compound imbibition
12	$P_9 \to P_7$	Mill 4 Juice → Imbibition Line	0.60	0.40–1.00	Return juice
13	$P_8 \to P_6$	Mill 3 Juice → Imbibition Line	0.60	0.40–1.00	Return juice
14	$P_6 \to P_{11}$	Mill 1 Juice → Juice Screen	0.50	0.30–0.80	Juice transfer
15	$P_7 \to P_{11}$	Mill 2 Juice → Juice Screen	0.50	0.30–0.80	Juice transfer
16	$P_{11} \to P_{12}$	Juice Screen → Juice Tank	0.30	0.20–0.50	Screen discharge
17	$P_{12} \to P_6$	Juice → Juice Return to Imbibition System	0.70	0.40–1.20	Juice return
18	$U_2 \to P_8$	Imbibition Water → Mill 3	0.50	0.30–0.80	Imbibition application
19	$U_2 \to P_9$	Imbibition Water → Mill 4	0.50	0.30–0.80	Imbibition application
20	$U_2 \to P_{10}$	Imbibition Water → Mill 5	0.50	0.30–0.80	Imbibition application
21	$P_{12} \to Y$	Juice Tank → Raw Juice Tank	0.50	0.30–0.80	Discharge/transfer

3.2. Formulation of Max-Plus State Equations

The dynamic equations for each machine are formulated using the general expression:

$$X_i(k+1) = \left[\bigoplus_j \left(d_j + t_{(j\to i)}\right) \otimes X_j(k+1)\right] \oplus [d_i \otimes X_i(k)] \oplus \left[\bigoplus_m t_{(U_m\to i)} \otimes U_m(k+1)\right] \tag{10}$$

These equations are subsequently assembled into the implicit Max-Plus algebra state-space representation, as defined in Equations (1) and (2), with full-dimension matrices A, B, C, and D full-dimension matrices (12×12, 1×12, dan 12×2) in Max-Plus algebra, explicitly retaining the notation ε (which equals $-\infty$) to denote the absence of a direct precedence coupling between corresponding variables.

1) Matrix $A \in \mathbb{R}_{\max}^{12\times 12}$

Matrix A captures the inter-machine precedence dependencies occurring within the same event cycle $(k+1)$.

$$A = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3.00 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1.80 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.60 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1.10 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.65 & \varepsilon & \varepsilon & 1.10 & \varepsilon & \varepsilon & \varepsilon & 2.70 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.90 & \varepsilon & \varepsilon & 1.10 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.90 & \varepsilon & \varepsilon & 1.10 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.90 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.90 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1.00 & 1.00 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2.30 & \varepsilon \end{bmatrix}$$

2) Matrix $B \in \mathbb{R}_{\max}^{12\times 12}$

Matrix B represents the internal processing durations of each machine from the preceding event cycle (k).

$$B = \begin{bmatrix} 2.00 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1.50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.30 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.80 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.25 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.50 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.50 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2.00 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2.00 \end{bmatrix}$$

3) Matrix $D \in \mathbb{R}_{\max}^{12\times 2}$

Matrix D connects the input variables (U_1: Raw Sugarcane, U_2: Imbibition Water) to their respective initial entry stations.

$$D^T = \begin{bmatrix} 2.00 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.50 & 0.50 & 0.50 & \varepsilon & \varepsilon \end{bmatrix}^T$$

4) Matrix $C \in \mathbb{R}_{\max}^{1\times 12}$

Matrix C epresents the mapping from the state variable of the Juice Tank (X_{12}) to the output release time $Y(k)$ (Raw Juice Tank).

$$C = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2.50 \end{bmatrix}$$

3.3. Explicit Reduced State-Space Model

To enable direct recursive computation of the equations without implicit dependencies on $X(k+1)$ on the right-hand side, the Kleene Star operator is utilized. By defining $A_{\text{exp}} = A^* \otimes B$ and $B_{\text{exp}} = A^* \otimes D$, the standard explicit state-space formulation is obtained:

$$X(k+1) = A_{\text{exp}} \otimes X(k) \oplus B_{\text{exp}} \otimes U(k+1) \tag{11}$$

$$Y(k) = C \otimes X(k) \tag{12}$$

To calculate the explicit state matrix A_{exp} it is first necessary to compute the Kleene Star matrix A^*.

1) Matriks Kleene Star A^{*12}

   The non-ε elements of matrix A^* are computed based on the cumulative transit times between machines:

$$A^* = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3.00 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 4.80 & 1.80 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 5.40 & 2.40 & 0.60 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 6.50 & 3.50 & 1.70 & 1.10 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 7.15 & 4.15 & 2.35 & 1.75 & 0.65 & 0 & \varepsilon & 1.10 & 1.10 & 1.10 & \varepsilon & 2.70 \\ 8.05 & 5.05 & 3.25 & 2.65 & 1.55 & 0.90 & 0 & 2.00 & 1.10 & 2.00 & \varepsilon & 3.60 \\ 8.95 & 5.95 & 4.15 & 3.55 & 2.45 & 1.80 & 0.90 & 0 & 2.00 & 1.10 & \varepsilon & 4.50 \\ 9.85 & 6.85 & 5.05 & 4.45 & 3.35 & 2.70 & 1.80 & 0.90 & 0 & 2.00 & \varepsilon & 5.40 \\ 10.75 & 7.75 & 5.95 & 5.35 & 4.25 & 3.60 & 2.70 & 1.80 & 0.90 & 0 & \varepsilon & 6.30 \\ 9.05 & 6.05 & 4.25 & 3.65 & 2.55 & 1.00 & 1.00 & 2.10 & 2.10 & 2.10 & 0 & 3.70 \\ 11.35 & 8.35 & 6.55 & 5.95 & 4.85 & 3.30 & 3.30 & 4.40 & 4.40 & 2.30 & 0 & 0 \end{bmatrix}$$

2) Matriks Eksplisit A_{exp}

   The explicit state-transition matrix A is formulated as $A_{\text{exp}} = A \otimes B \in \mathbb{R}_{\max}^{12\times 2}$, yielding:

$$A_{\text{exp}} = \begin{bmatrix} 2.00 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 5.00 & 1.50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 6.80 & 3.30 & 0.30 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 7.40 & 3.90 & 0.90 & 0.80 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 8.50 & 5.00 & 2.00 & 1.90 & 0.25 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 9.15 & 5.65 & 2.65 & 2.55 & 0.90 & 0.50 & \varepsilon & 1.60 & 1.60 & 1.60 & \varepsilon & 4.70 \\ 10.05 & 6.55 & 3.55 & 3.45 & 1.80 & 1.40 & 0.50 & 2.50 & 1.60 & 2.50 & \varepsilon & 5.60 \\ 10.95 & 7.45 & 4.45 & 4.35 & 2.70 & 2.30 & 1.40 & 0.50 & 2.50 & 1.60 & \varepsilon & 6.50 \\ 11.85 & 8.35 & 5.35 & 5.25 & 3.60 & 3.20 & 2.30 & 1.40 & 0.50 & 2.50 & \varepsilon & 7.40 \\ 12.75 & 9.25 & 6.25 & 6.15 & 4.50 & 4.10 & 3.20 & 2.30 & 1.40 & 0.50 & \varepsilon & 8.30 \\ 11.05 & 7.55 & 4.55 & 4.45 & 2.80 & 1.50 & 1.50 & 2.60 & 2.60 & 2.60 & 2.00 & 5.70 \\ 13.35 & 9.85 & 6.85 & 6.75 & 5.10 & 3.80 & 3.80 & 4.90 & 4.90 & 4.90 & 4.30 & 2.00 \end{bmatrix}$$

3) Explicit Input Matrix B_{exp}

   The explicit input matrix B is formulated as $B_{\text{exp}} = A \otimes D \in \mathbb{R}_{\max}^{12\times 2}$. This matrix measures the total propagation time from the inputs U_1 (raw sugarcane) and U_2 (imbibition water) to each respective state variable:

$$B_{\text{exp}} = \begin{bmatrix} 2.00 & 5.00 & 6.80 & 7.40 & 8.50 & 9.15 & 10.05 & 10.95 & 11.85 & 12.75 & 11.05 & 13.35 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1.60 & 2.50 & 2.50 & 2.50 & 2.30 & 2.60 & 4.90 \end{bmatrix}^T$$

With matrices A_{exp} and B_{exp}, explicitly established, this state-space representation is directly implemented within the Model Predictive Control (MPC) algorithm.

3.4. Objective Function Formulation

Based on the system dynamic equations defined in Equations (11) and (12), state and output predictions are executed over a prediction horizon $N_p = 2$ (predicting two future cycles at step $k = 1$), under the following initial conditions and input vectors:

- Initial State ($X(0)$): The initial system condition is assumed to be fully prepared and normalized at $X(0) = 0_{12\times 1}$ (0 minutes).
- Input Decision Vector ($\tilde{U}$): $\tilde{U} = \begin{bmatrix} U_1(1) & U_2(1) & U_1(2) & U_2(2) \end{bmatrix}^T$

where U_1 denotes the raw sugarcane feeding time and U_2 represents the imbibition water application time.

3.4.1. Mathematical Formulation of the MPC Objective Function

The primary objective of the Max-Plus Linear Model Predictive Control (MPC) scheme is to minimize the completion time of raw juice production (Y) to ensure early completion, while simultaneously postponing the arrival of input raw materials (U) as late as possible to prevent work-in-process (WIP) buffer accumulation in accordance with Just-In-Time (JIT) principles:

$$\min_{\tilde{U}} J = \sum_{j=1}^{N_p} Y(j) - \lambda \sum_{j=1}^{N_p} 1^T U(j)$$

By selecting the queue penalty weight $\lambda = 0.1$ and prediction horizon $N_p = 2$, the objective function is expressed as:

$$J = (Y(1) + Y(2)) - 0.1 \times (U_1(1) + U_2(1) + U_1(2) + U_2(2))$$

3.4.2. Derivation of Output Prediction Equations $Y(1)$ and $Y(2)$

- First-Cycle Prediction ($k = 1$)

$$Y(1) = C \otimes X(1) = 2.50 + X_{12}(1) = 2.50 + 13.35 + U_1(1) = 15.85 + U_1(1)$$

- Second-Cycle Prediction ($k = 2$)

$$Y(2) = C \otimes X(2) = \max\left(17.85 + U_1(1), 15.85 + U_1(2)\right)$$

3.4.3. Implementation of Operational Constraints

To solve the objective function, physical operational constraints derived from the system data in Table 1 are enforced:

- Initial Sugarcane Arrival Time ($U_1(1)$): Set at the start of the operating shift:

$$U_1(1) = 0.00 \text{ minutes}$$

- Cycle 1 Imbibition Water ($U_2(1)$): Scheduled synchronously with the operational readiness of Mill 3 ($B_{\text{exp}}(8, 1) = 10.95$):

$$U_2(1) = 10.95 \text{ minutes}$$

- Input-Spacing Constraints ($U_1(2)$):

$$U_1(2) - U_1(1) \geq \tau_{\text{in}} \Longrightarrow U_1(2) \geq 0.00 + 3.00 = 3.00 \text{ minutes}$$

To achieve Just-In-Time (JIT) optimization with minimal Work-in-Process (WIP) buffer accumulation, the optimal decision values are evaluated at the lower boundary of the feasible constraint space:

$$U_1(2) = 3.00 \text{ minutes}$$

- Cycle 2 Imbibition Water 2 ($U_2(2)$):

$$U_2(2) = U_2(1) + \tau_{\text{in}} = 10.95 + 3.00 = 13.95 \text{ minutes}$$

3.4.4. Numerical Computation of the Objective Function (J)

Substituting the constrained input values into the output prediction equations yields:

- Cycle 1 Output Value 1:

$$Y(1) = 15.85 + 0.00 = 15.85 \text{ minutes}$$

- Nilai Output Siklus 2:

$$Y(2) = \max(17.85 + 0.00, 15.85 + 3.00) = \max(17.85, 18.85) = 18.85 \text{ minutes}$$

- Cycle 2 Output Value ($\sum Y$):

$$\sum_{j=1}^{2} Y(j) = 15.85 + 18.85 = 34.70 \text{ minutes}$$

- Total Input Penalty Term ($\lambda \sum U$):

$$\lambda \sum_{j=1}^{2} U(j) = 0.1 \times (0.00 + 10.95 + 3.00 + 13.95) = 0.1 \times 27.90 = 2.79 \text{ minutes}$$

- Final Objective Function Value (J):

$$J = 34.70 - 2.79 = 31.91$$

Through the explicit calculation steps detailed above, the optimal objective function value for $N_p = 2$ is evaluated as $J = 31.91$. This value demonstrates that the MPC controller successfully predicts a total output completion time of 34,70 minutes across two batches of raw juice, while maintaining efficient sugarcane input postponement to minimize intermediate bagasse/juice accumulation.

3.5. Solution Using the FIFO Method

Based on the production system workflow and time parameters, the baseline First-In, First-Out (FIFO) principle operates by sequentially introducing raw material inputs (U_1) without planned delays. The system processes incoming entities on a first-come, first-served basis, utilizing the minimum feeding interval determined by the operational readiness of the initial processing station (P_1).

3.5.1. Baseline FIFO Parameters based on Flowchart

- Sugarcane Input (U_1): Processed initially at P_1 with a processing duration $d_1 = 2.00$ minutes and a transfer time $t_{(U1 \to P1)} = 2.00$ minutes.
- FIFO Feeding Interval (τ_{FIFO}): Since FIFO operates under continuous feeding conditions, input U_1 $(k+1)$ is fed immediately once P_1 becomes operational for the next cycle.
- $\tau_{\text{FIFO}} = d_1 + t_{(U1 \to P1)} = 2.00 + 2.00 = 4.00$ minutes
- Initial Condition ($k = 1$): All processing stations are assumed to be idle/empty at $t = 0.00$ minutes.

3.5.2. Baseline FIFO Sequential Execution Results

The completion time for the first batch ($Y(1)$) under the baseline FIFO rule is calculated as:

$$Y(1) = 14.45 + 0.50 = 14.95 \text{ minutes}$$

3.5.3. Subsequent Batch Execution and Work-In-Process (WIP) Accumulation

In the baseline FIFO method, the second batch ($U_1(2)$ is introduced at a fixed interval of $t = 4.00$ minutes. Queue time accumulation—or Work-In-Process (WIP) buildup—occurs at processing stations with longer processing times:

- Juice Tank Queue (P_{12}): Because $d_{12} = 2.00$ minutes while the arrival interval from mill circulation is faster, batch $k + 2$ juice must wait in the buffer holding tank before P_{12} finishes processing batch $k + 1$.
- FIFO Output Calculation ($Y(k)$):

$$Y(k+1) = \max\left(Y(k) + d_{12},\ U_1(k+1) + 14.95\right)$$

Because FIFO does not adjust or delay the input feeding schedule $U_1(k + 1)$ when downstream bottlenecks occur at P_{12}, the accumulated waiting times at stations P_{11} and P_{12} increase progressively with each operational cycle.

3.5.4. Baseline FIFO Solution via Max-Plus Algebra

The Baseline FIFO solution using Max-Plus algebra is executed sequentially and deterministically without an optimization (re-planning) process. In this scenario, the input release time (U) is fixed based on a constant minimum interval $\tau_{\text{FIFO}} = 4.00$ minutes, and the state (X) and output (Y) values are recursively computed using Max-Plus operations ($\oplus \equiv \max$ and $\otimes \equiv +$).

1) Formulation of Recursive FIFO Equations

   The time-dependency equation governing machine start times under the FIFO baseline adheres to the canonical form:

$$X_i(k+1) = \max(X_i(k) + d_i),\ \max_{j \in \text{inflow}} \left(X_j(k+1) + d_j + t_{(j \to i)}\right),\ \max_m \left(U_m(k+1) + t_{(U_m \to i)}\right) \quad (13)$$

   Because the sugarcane feeding time U_1 is scheduled at constant intervals:

$$U_1(k+1) = U_1(k) + \tau_{\text{FIFO}} = U_1(k) + 4.00$$

2) Results of Recursive Numerical Computation ($k = 1$ and $k = 2$)

   First Cycle ($k = 1$) Raw Juice Output ($Y(1)$):

$$Y(1) = C \otimes X(1) = 2.50 + X_{12}(1) = 2.50 + 13.35 = 15.85 \text{ minutes}$$

   Second Cycle ($k = 2$) Raw Juice Output ($Y(2)$):

$$Y(2) = 2.50 + X_{12}(2) = 2.50 + 17.35 = 19.85 \text{ minutes}$$

In the FIFO strategy, constant feeding of $U_1(2)$ at minute 4.00 causes queue accumulation of juice at the screen (X_{11}) and tank (X_{12}), thereby delaying the release time of batch 2 to 19.85 minutes. Conversely, MPC utilizes explicit system matrices to dynamically adjust input trajectories, achieving a faster total completion time of 18.85 minutes.

Table 3. Summary Evaluation of Comparative Results

Event Metric	**FIFO Baseline**	**Max-Plus MPC** ($N_p = 2$)	**Difference (Efficiency)**
Output Batch 1 ($Y(1)$)	15.85 minutes	15.85 minutes	0.00 minutes
Output Batch 2 ($Y(2)$)	19.85 minutes	18.85 minutes	-1.00 minutes
Total Completion Time	19.85 minutes	18.85 minutes	5.0% faster

3.6. Consistency Check of the Max-Plus Model

The system in Subsection 3.2 can be represented as a timed event graph with 12 stations. A path from station j to station i carries a weight of $d_j + t_{(j\to i)}$, while each station contains a self-loop with weight d_i holding a single token. These token-holding paths enforce the operational constraint that batch $k+1$ can only be processed after batch k is completed at the same station. Tokenless paths construct the implicit matrix A_0 (representing interdependencies within the same event cycle), whereas token-holding paths form matrix A_1 (representing interdependencies with the previous cycle). The explicit form is derived via $A_{\text{exp}} = A_0^* \otimes A_1$ and $B_{\text{exp}} = A_0^* \otimes B_0$, where $A_0^* = E \oplus A_0 \oplus A_0^2 \oplus \cdots$. This Kleene star series converges if and only if all circuits in the A_0 graph have non-positive weights [1, 2].

Subsection 3.2 incorporates four imbibition juice feedback flows ($P_{10} \to P_8$, $P_9 \to P_7$, $P_8 \to P_6$, and $P_{12} \to P_6$) into the same cycle $k+1$ matrix alongside the feedforward paths. This placement yields five tokenless circuits with positive weights (Table 4). The circuit with the largest weight, $P_6 \to P_7 \to P_{11} \to P_{12} \to P_6$, has a total weight of 6.90 minutes; consequently, each loop traversal adds to the station start times, causing the series A_0^* to diverge to $+\infty$. Thus, the A^* and A_{exp} matrices presented in Subsection 3.4 cannot be rigorously derived from the formal definition of the Kleene star operator. The entries in both matrices correspond only to path summations truncated up to a bounded path length.

Table 4. Tokenless Circuits with Positive Weights in the Matrix Structure of Subsection 3.2

No.	**Circuit**	**Stations Traversed**	**Weight (minutes)**
1	$P_6 \to P_7 \to P_{11} \to P_{12} \to P_6$	Mill 1, Mill 2, Juice Screen, Juice Tank	6.90
2	$P_6 \to P_{11} \to P_{12} \to P_6$	Mill 1, Juice Screen, Juice Tank	6.00
3	$P_6 \to P_7 \to P_8 \to P_6$	Mill 1, Mill 2, Mill 3	2.90
4	$P_7 \to P_8 \to P_9 \to P_7$	Mill 2, Mill 3, Mill 4	2.90
5	$P_8 \to P_9 \to P_{10} \to P_8$	Mill 3, Mill 4, Mill 5	2.90

This study evaluates two mathematically well-defined alternative structures. Model M0 eliminates all four feedback flows, rendering A_0 acyclic (devoid of cycles). Model M1 retains the feedback flows but treats them as dependencies on the preceding batch, placing a single token along each path. Model M1 is selected as the primary formulation because the juice extracted from downstream mills and sprayed onto upstream mills originates from a batch that has already passed through those mills, thereby exhibiting inter-cycle dependency. Imbibition water addition follows a fixed operational rule $U_2(k) = U_1(k) + \delta$, where $\delta = 10.95$ minutes, representing the travel time of sugarcane to the inlet of Mill 3 (corresponding to element $B_{\text{exp}}(8,1)$ in Subsection 3.3). Under this rule, both input columns can be consolidated into $b_{\text{eff}} = B_{\text{exp}}(\cdot,1) \oplus [\delta \otimes B_{\text{exp}}(\cdot,2)]$, effectively reducing the decision variable of the MPC to solely the sugarcane entry time $U_1(k)$.

The U_1 column of matrix B_{exp} in both M0 and M1 is identical to that in Subsection 3.4. The travel time of the first batch from sugarcane entry to the Juice Tank is 13.35 minutes, thereby confirming $Y(1) = 15.85$ minutes. Differences arise in the elements involving feedback flows. Out of the 144 elements in A_{exp}, 23 elements in M0 and 30 elements in M1 differ from the original values in the draft, whereas 6 out of the 12 elements in column U_2 differ across both models. The element $A_{\text{exp}}(12,12)$ in M1 equals 6.90 minutes, significantly exceeding $d_{12} = 2.00$

minutes, because juice from the Juice Tank in the preceding batch must loop back to Mill 1 prior to crushing the subsequent batch. In M0, this element takes a value of 2.00 minutes.

3.7. System Cycle Time

For the system $x(k) = A_{\text{exp}} \otimes x(k-1) \oplus b_{\text{eff}} \otimes u(k)$, the long-term output rate is bounded by the max-plus eigenvalue λ^*, which corresponds to the maximum cycle mean of A_{exp}. This value is computed using Karp's algorithm (1978) [16]:

$$\lambda^* = \max_{1 \le i \le n} \left(\min_{0 \le m \le n-1} \left(\frac{D_n(i) - D_m(i)}{n-m} \right) \right), \qquad D_m = A_{\text{exp}} \otimes D_{m-1}, \qquad D_0 = 0 \tag{14}$$

where $n = 12$ and $D_m(i)$ represents the maximum path weight of length m ending at station i. The result from Karp's algorithm is cross-checked via the ratio of total circuit weight to token count for every simple circuit in the event graph. Both approaches yield $\lambda^* = 6.90$ minutes per batch for Model M1, equivalent to 8.70 batches per hour, with the critical circuit identified as $P_6 \to P_7 \to P_{11} \to P_{12} \to P_6$ (Table 5). Power iteration of $x(k) = A_{\text{exp}} \otimes x(k-1)$ initialized from $x(0) = 0$ yields $x_{12}(400)/400 = 6.9161$ minutes per batch; the discrepancy relative to 6.90 arises from transient terms decaying asymptotically as $1/k$. For Model M0, λ^* is strictly 2.00 minutes per batch, governed by the maximum processing durations (d_1, d_{11}, and d_{12}).

Table 5. Five Circuits with the Largest Weight-per-Token Ratios in Model M1

No.	Circuit	Weight (minutes)	Tokens	Ratio (minutes/batch)
1	$P_6 \to P_7 \to P_{11} \to P_{12} \to P_6$	6.90	1	6.90
2	$P_6 \to P_{11} \to P_{12} \to P_6$	6.00	1	6.00
3	$P_8 \to P_9 \to P_{10} \to P_8$	2.90	1	2.90
4	$P_6 \to P_7 \to P_8 \to P_6$	2.90	1	2.90
5	$P_7 \to P_8 \to P_9 \to P_7$	2.90	1	2.90

This finding has a direct impact on the feeding strategy. In Model M1, both the FIFO interval $\tau_{\text{FIFO}} = 4,00$ minutes and the minimum MPC input spacing $\tau_{\text{in}} = 3,00$ minutes are tighter than the system cycle time. Sugarcane fed every 4,00 minutes cannot be processed faster than 6,90 minutes per batch, causing a discrepancy of 2,90 minutes per batch to accumulate as queue delay. Conversely, in Model M0, the 4,00-minute feeding interval is more relaxed than the 2,00-minute cycle time, preventing FIFO from inducing continuous queue accumulation.

3.8. Validasi terhadap Contoh Numerik $N_p = 2$

For M0, $U_1 = (0;3)$ yields $Y(2) = 18.85$ minutes and $J = 31,91$, whereas $U_1 = (0;4)$ results in $Y(2) = 19,85$ minutes. Both values are identical to those in Subsection 3.5, and the Linear Programming (LP) solution under M0 similarly selects $U_1(2) = 3,00$ minutes with $J^* = 31,91$ (Table 6). The 1,00-minute (5.0%) discrepancy between MPC and FIFO in this example stems entirely from the difference in input spacing—specifically 3,00 versus 4,00 minutes—because input trajectories do not interact with queue delays under M0.

Conversely, for M1, both scenarios yield $Y(2) = 22,75$ minutes because the second batch is held back by the feedback juice flow from the first batch. The LP optimization under M1 selects $U_1(2) = 6,90$ minutes, representing a postponed sugarcane release until the system is ready without shifting the output completion time, yielding $J^* = 35,03$. This behavior highlights the inherent *Just-In-Time* (JIT) property of Max-Plus MPC [7].

Table 6. Reproduction of Numerical Example $N_p = 2$ for Models M0 and M1 ($\lambda = 0.10$; $\delta = 10.95$ minutes)

Model	Scenario	$U_1(2)$	$Y(1)$	$Y(2)$	ΣY	J
M0	Manuscript MPC trajectory	3.00	15.85	18.85	34.70	31.91
	Manuscript FIFO trajectory	4.00	15.85	19.85	35.70	32.71
	LP Optimum	3.00	15.85	18.85	34.70	31.91
M1	Manuscript MPC trajectory	3.00	15.85	22.75	38.60	35.81
	Manuscript FIFO trajectory	4.00	15.85	22.75	38.60	35.61
	LP Optimum	6.90	15.85	22.75	38.60	35.03
Manuscript	Manuscript MPC values	3.00	15.85	18.85	34.70	31.91
	Manuscript FIFO values	4.00	15.85	19.85	–	–

3.9. Simulation Design for MPC and Baseline FIFO

At each event k, the output prediction over a prediction horizon N_p is expressed as:

$$\tilde{y}(k+j) = h_j \oplus \bigoplus_{l=0}^{j} \left[g_{j-l} \otimes u(k+l) \right], \qquad j = 0, 1, \ldots, N_p - 1 \tag{15}$$

Where $h_j = C \otimes A_{\text{exp}}^{\otimes(j+1)} \otimes x(k-1)$ and $g_m = C \otimes A_{\text{exp}}^{\otimes m} \otimes b_{\text{eff}}$. The maximum operator in Equation (15) is handled via the epigraph form. An auxiliary variable $\hat{y}_j$ satisfies $\hat{y}_j \geq h_j$ and $\hat{y}_j \geq g_{j-l} + v_l$ for all $l \leq j$, with $v_l = u(k+l)$. Because the coefficient of $\hat{y}_j$ in the objective function is positive, at the optimal point $\hat{y}_j$ equals the maximum of its right-hand side, casting the MPC problem into a Linear Programming (LP) formulation:

$$\min J = \sum_j \hat{y}_j - 2\lambda \sum_j v_j + \rho \sum_j s_j \tag{16}$$

Subject to:

$$\hat{y}_j \leq r(k+j) + s_j, \qquad s_j \geq 0, \qquad j = 1, 2, \ldots, N_p$$

$$v_0 - u(k-1) \geq \tau_{\text{in}} \tag{17}$$

$$v_j - v_{j-1} \geq \tau_{\text{in}}, \qquad j = 1, 2, \ldots, N_c - 1 \tag{18}$$

$$v_j - v_{j-1} = v_{(N_c-1)} - v_{(N_c-2)}, \qquad j = N_c, \ldots, N_P - 1 \tag{19}$$

The 2λ factor in Equation (16) arises because $U_2 = U_1 + \delta$ is summed within the Just-In-Time (JIT) term of the objective function in Subsection 3.4.2. Constraint (19) enforces a constant input interval beyond the control horizon N_c, following De Schutter and van den Boom (2001) [7]. Output due dates are treated as soft constraints with tardiness variables s_j weighted by $\rho = 1000$, maintaining Linear Programming (LP) feasibility even when the plant (controlled physical system) runs slower than planned. A receding horizon strategy is implemented by executing v_0 as $u(k)$, measuring the plant state $x(k)$, and re-solving the LP at the subsequent event cycle. The LP optimization is solved using the HiGHS solver [17] via the SciPy library [18].

Because the original draft does not specify output due date values, reference timestamps are set as $r(k) = R_0 + (k-1)T_{\text{ref}}$, where $R_0 = Y(1) + 2.00 = 17.85$ minutes and $T_{\text{ref}} = 1.05\lambda^* = 7.245$ minutes per batch. Other parameters adhere to Subsection 3.4, specifically $\tau_{\text{in}} = 3.00$ minutes, $\tau_{\text{FIFO}} = 4.00$ minutes, and $\lambda = 0.10$. Simulations are executed for $K = 40$ batches initialized from empty system conditions ($x(0) = \varepsilon$), across five prediction/control horizon pairs $(N_p, N_c) = (9, 6), (25, 20), (55, 50), (75, 70)$, and $(100, 90)$. The FIFO baseline releases sugarcane at $U_1(k) = (k-1)\tau_{\text{FIFO}}$ under identical imbibition water addition rules.

In addition to nominal MPC, an MPC-R variant is constructed at the $(N_p, N_c) = (25, 20)$ configuration. This variant adds a travel time margin $\mu = 0.10 \times Y(1) = 1.585$ minutes to each input path term ($g_m \leftarrow g_m + \mu$) to

accommodate processing delays of up to 10% not captured by the nominal model. In robustness testing, all MPC variants retain nominal models and reference trajectories while operating against a perturbed plant.

Performance is measured via lead time $L(k) = Y(k) - U_1(k)$; average Work-In-Process (WIP, batch count in system) based on Little's Law [19] as $\sum L(k)/T_{\text{op}}$, where T_{op} denotes the operational span from initial release to final output; maximum WIP; total queue wait time per batch (start time minus material arrival timestamp across all stations); *makespan* $Y(K)$; output rate; percentage due-date compliance (proportion of batches with $Y(k) \leq r(k)$); and average tardiness delay.

3.10. Simulation Results under Nominal Conditions

3.10.1. MPC with Five Horizon Pairs

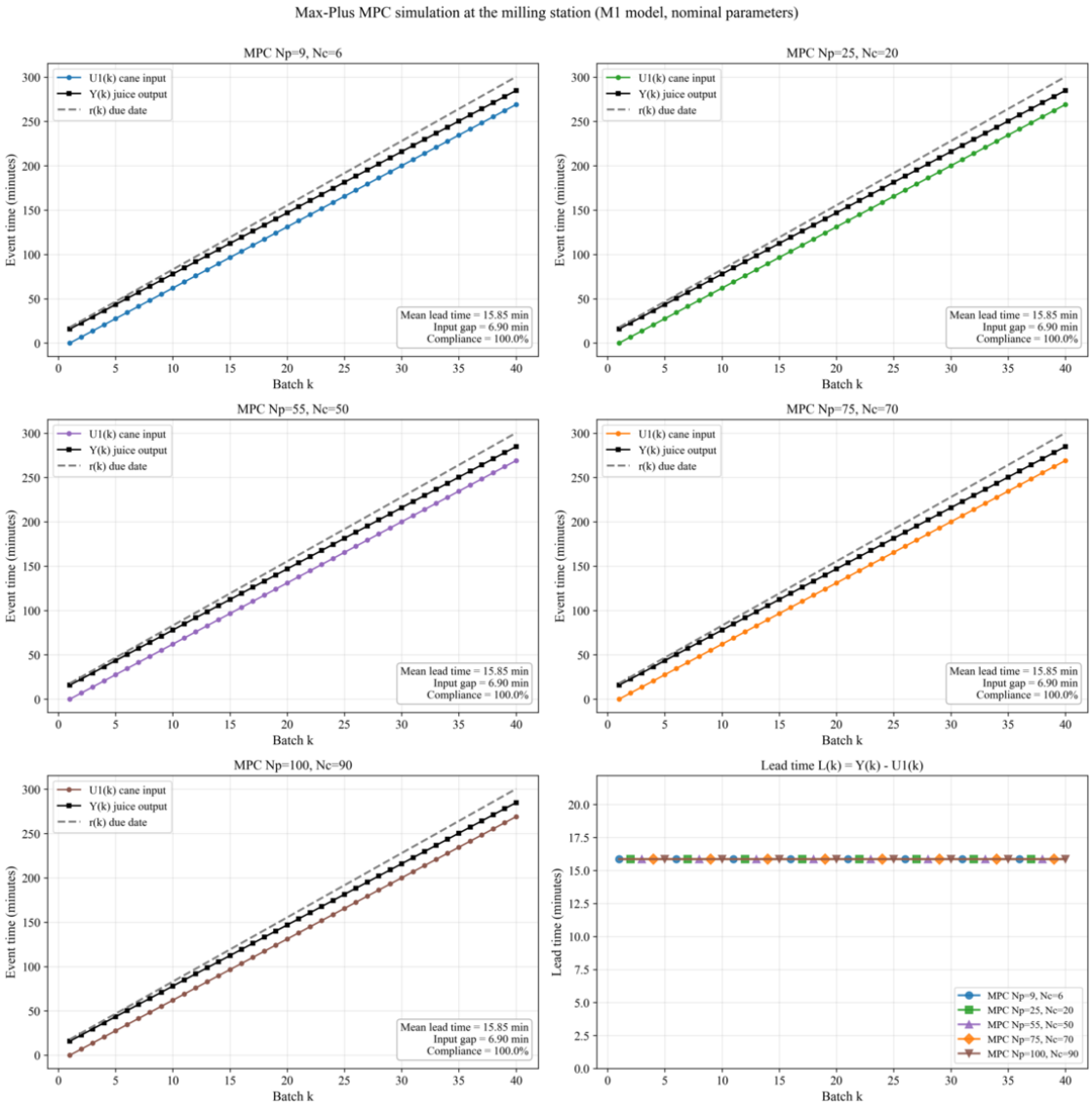


Figure 2. Sugarcane entry time $U_1(k)$, juice completion time $Y(k)$, and output due date $r(k)$ across five MPC horizon pairs; the bottom-right panel displays the lead time for each configuration (Model M1, nominal parameters)

Figure 2 demonstrates that all five MPC configurations release sugarcane at a 6.90-minute interval, matching the system cycle time, even though the constraint only specifies a minimum spacing of 3.00 minutes. The raw juice output completion time follows $Y(k) = 15.85 + 6.90(k-1)$ minutes and consistently remains below the due date, resulting in a constant lead time of 15.85 minutes across all batches. Discrepancies in performance metrics among horizon configurations are smaller than 0.001 (Table 7). In this deterministic system governed by a single critical circuit, the optimal release delay can be adequately determined using a prediction horizon of $N_p = 9$; longer horizons merely expand the Linear Programming (LP) problem size. The average LP computation time per event cycle is 4.49 ms and 4.03 ms for $N_p = 9$ and 25, respectively, and increases to 9.11 ms, 13.98 ms, and 17.12 ms for $N_p = 55.75$, and 100. One-step prediction errors under nominal conditions are zero (Figure 3) because the model perfectly mirrors the physical plant.

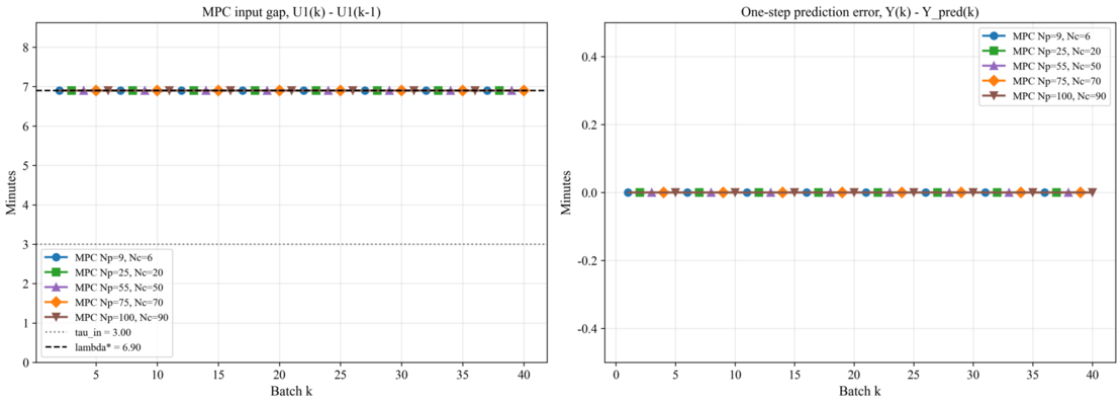


Figure 3. MPC input spacing $U_1(k) - U_1(k-1)$ and one-step prediction error under nominal conditions

3.10.2. FIFO Baseline

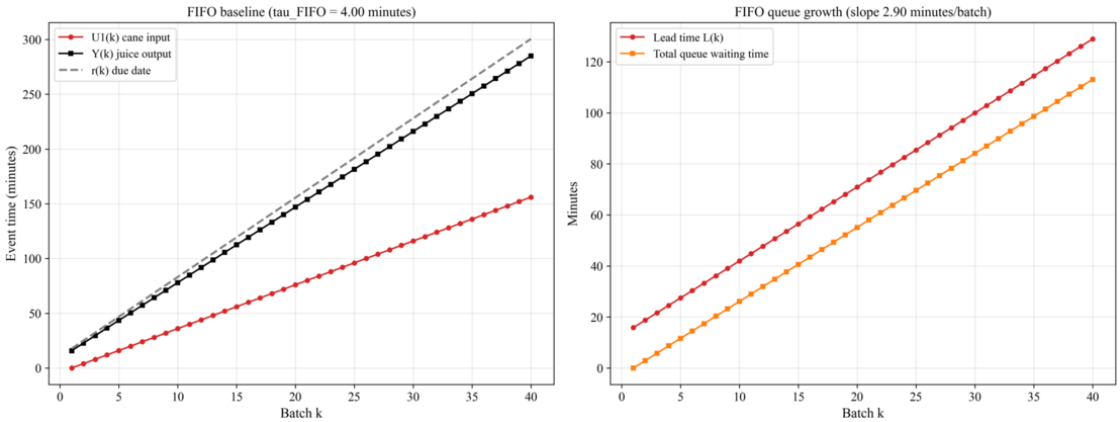


Figure 4. Entry time, completion time, and due date for the FIFO baseline (left), alongside lead time growth and queue wait time (right)

FIFO releases sugarcane every 4.00 minutes, but the juice completion time remains bound by the system cycle time of 6.90 minutes per batch (Figure 4). The FIFO lead time increases linearly with a slope of 2.90 minutes per batch, escalating from 15.85 minutes for the first batch to 128.95 minutes for the 40^{th} batch, averaging 72.40 minutes. The entire queue wait time accumulates at Mill 1 (Figure 5), which serves as the confluence point between incoming prepared cane and the juice feedback flows from Mill 3 and the Juice Tank. The average queue wait time

reaches 56.55 minutes per batch, with an average WIP of 10.16 batches and a maximum WIP of 19 batches. The description in Subsection 3.5 projected that FIFO queue accumulation would occur at the Juice Screen and Juice Tank. Under Model M1, however, neither station experiences queueing because the inflow entering both units is already bottlenecked by Mill 1.

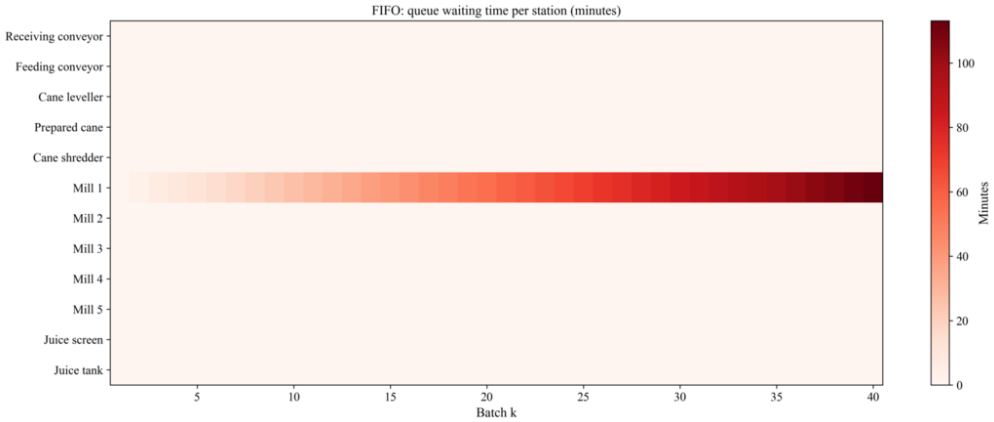


Figure 5. Distribution of FIFO queue waiting times by station and by batch

3.10.3. Comparison of MPC and FIFO

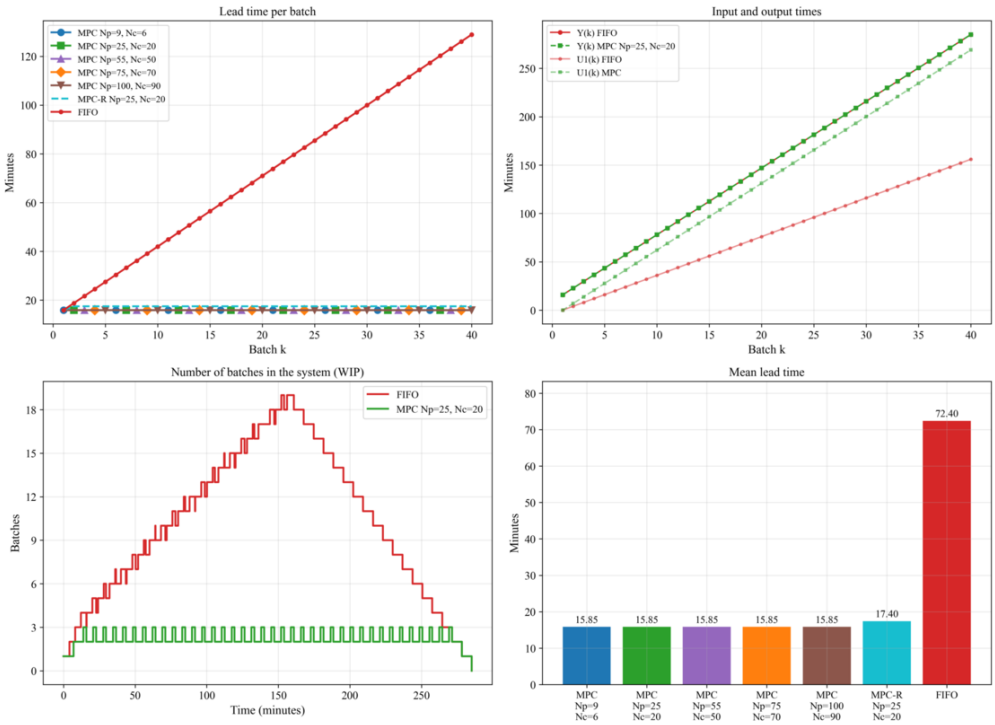


Figure 6. Comparison of MPC and FIFO: per-batch lead time, entry and completion times, number of batches in the system (WIP), and average lead time

Table 7. Performance Metrics under Nominal Conditions ($K = 40$ batches)

No.	Strategy	Makespan (min)	Rate (batch/h)	Avg. lead time (min)	Max. lead time (min)	Avg. WIP (batch)	Max. WIP (batch)	Waiting time/batch (min)	Due-date compliance (%)	LP time (ms)
1	MPC $N_p = 9$ $N_c = 6$	284.95	8.42	15.85	15.85	2.22	3	0.00	100.0	4.49
2	MPC $N_p = 25$ $N_c = 20$	284.95	8.42	15.85	15.85	2.22	3	0.00	100.0	4.03
3	MPC $N_p = 55$ $N_c = 50$	284.95	8.42	15.85	15.85	2.22	3	0.00	100.0	9.11
4	MPC $N_p = 75$ $N_c = 70$	284.95	8.42	15.85	15.85	2.22	3	0.00	100.0	13.98
5	MPC $N_p = 100$ $N_c = 90$	284.95	8.42	15.85	15.85	2.22	3	0.00	100.0	17.12
6	MPC-R $N_p = 25$ $N_c = 20$	284.95	8.42	17.40	17.44	2.44	3	1.55	100.0	2.42
7	FIFO	284.95	8.42	72.40	128.95	10.16	19	56.55	100.0	–

Table 7 and Figure 6 summarize the comparison under nominal conditions. MPC and FIFO yield the same makespan (284.95 minutes), the same output rate (8.42 batches per hour), and 100% deadline compliance. The difference between the two strategies lies in the amount of material held within the system. MPC reduces the average lead time from 72.40 to 15.85 minutes and the average WIP from 10.16 to 2.22 batches—reductions of 78.1% in each case—and lowers the maximum WIP from 19 to 3 batches (Figure 7). The buffer capacity requirement upstream of Mill 1 decreases by the same magnitude.

The Gantt chart for the first four batches (Figure 8) illustrates the mechanism underlying this discrepancy. Baseline FIFO releases sugarcane at 0, 4, 8, and 12 minutes, whereas MPC releases sugarcane at 0, 6.90, 13.80, and 20.70 minutes. The output completion times for both strategies are identical—namely, 15.85, 22.75, 29.65, and 36.55 minutes—demonstrating that the earlier releases under FIFO merely prolong the waiting time of sugarcane in front of Mill 1. Meanwhile, the robust variant (MPC-R) yields the same makespan, with an average lead time of 17.40 minutes and an average WIP of 2.44 batches. The additional lead time of approximately 1.55 minutes compared to nominal MPC is a direct consequence of incorporating the travel time margin.

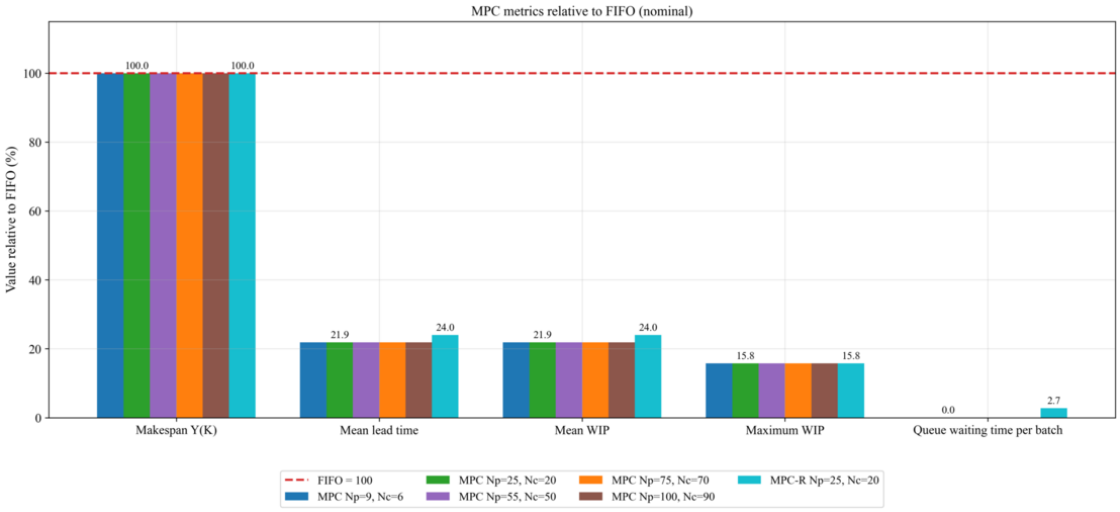


Figure 7. MPC and MPC-R performance metrics relative to FIFO (FIFO = 100) under nominal conditions

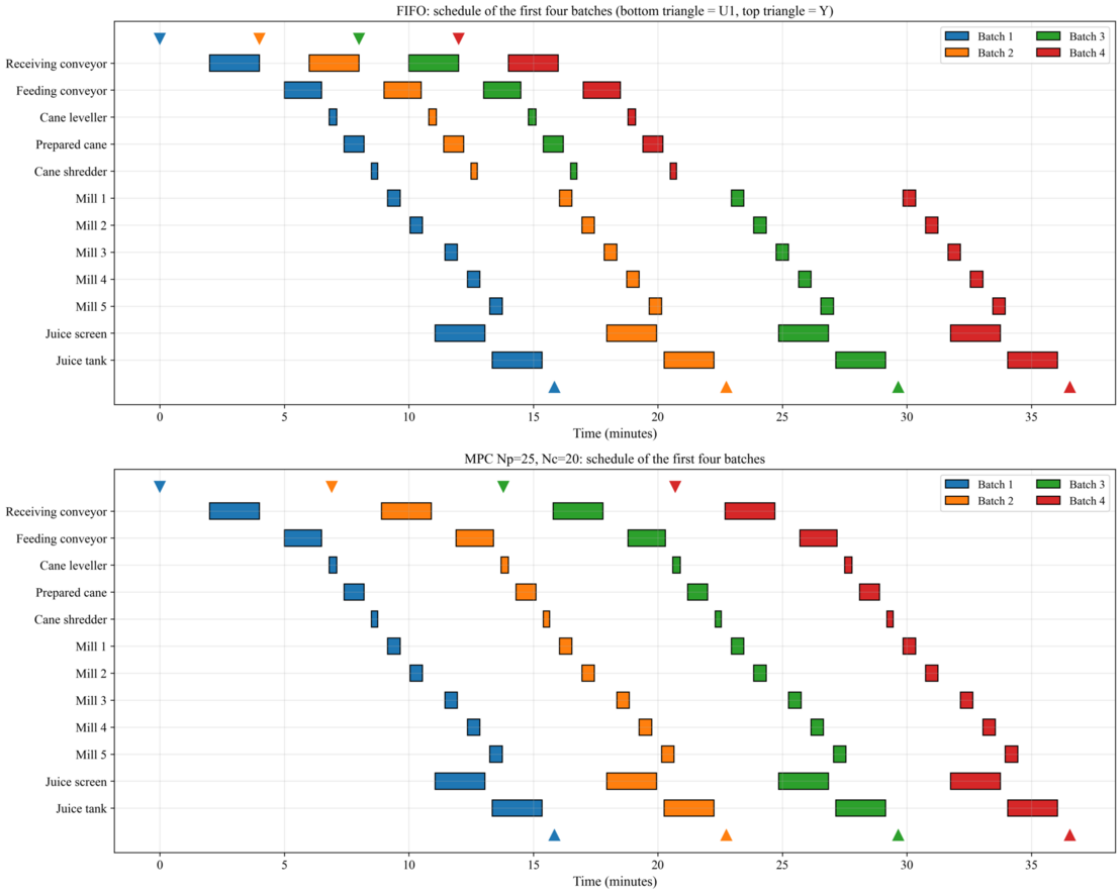


Figure 8. Gantt chart of the first four batches under FIFO (top) and MPC with $N_p = 2$ (bottom)

3.11. Sensitivity Analysis

A one-at-a-time (OAT) sensitivity analysis was conducted by perturbing each parameter to the minimum and maximum boundaries of the optimization range specified in Table 1, while holding all other parameters constant at their nominal values. Out of 33 parameters, only 8 parameters alter the system cycle time (Figure 9 and Table 8). The processing times of the Juice Screen (d_11) and the Juice Tank (d_12) exhibit the largest impact range, each contributing 2.00 minutes (λ^* ranges from 5.90 to 7.90 minutes per batch). The next largest impacts stem from the juice transport time from the Juice Tank to Mill 1 (t_{17}; ranging from 6.60 to 7.40 minutes per batch) and from Mill 2 to the Juice Screen (t_{15}; ranging from 6.70 to 7.20 minutes per batch). All four of these parameters reside on the critical circuit.

The parameters associated with the conveyor, cane leveler, and cane cutter do not alter the cycle time, but they directly affect the first-batch lead time $Y(1)$. Among the 19 parameters that influence $Y(1)$, the largest ranges originate from d_{11} and d_{12} (2.00 minutes), as well as d_1, d_2, and t_1 (1.50 minutes). Consequently, the capacity of the milling station in this model is governed more strongly by the juice circulation loop than by the sugarcane conveyor speed. Because the OAT approach does not capture parameter interactions [20], these results reflect the ranking of local parameter sensitivity around the nominal operating point.

Table 8. OAT Sensitivity Analysis Results across the Optimization Range in Table 1 (Parameters with the Largest Impact)

No.	Parameter	λ^* at Min. Bound	λ^* at Max. Bound	λ^* Range	$Y(1)$ at Min. Bound	$Y(1)$ at Max. Bound	$Y(1)$ Range
1	d_{12} Juice Tank	5.90	7.90	2.00	14.85	16.85	2.00
2	d_{11} Juice Screen	5.90	7.90	2.00	14.85	16.85	2.00
3	t_{12} Juice Tank → Mill 1	6.60	7.40	0.80	15.85	15.85	0.00
4	t_{15} Mill 2 → Juice Screen	6.70	7.20	0.50	15.65	16.15	0.50
5	d_6 Mill 1	6.75	7.10	0.35	15.70	16.05	0.35
6	d_7 Mill 2	6.75	7.10	0.35	15.70	16.05	0.35
7	t_7 Mill 1 → Mill 2	6.75	7.10	0.35	15.70	16.05	0.35
8	t_{16} Juice Screen → Juice Tank	6.80	7.10	0.30	15.75	16.05	0.30
9	d_1 Receiving Conveyor	6.90	6.90	0.00	15.35	16.85	1.50
10	d_2 Feeding Conveyor	6.90	6.90	0.00	15.35	16.85	1.50
12	t_1 Sugarcane → Receiving Conveyor	6.90	6.90	0.00	15.35	16.85	1.50

Note: All values in minutes; λ^* in minutes per batch.

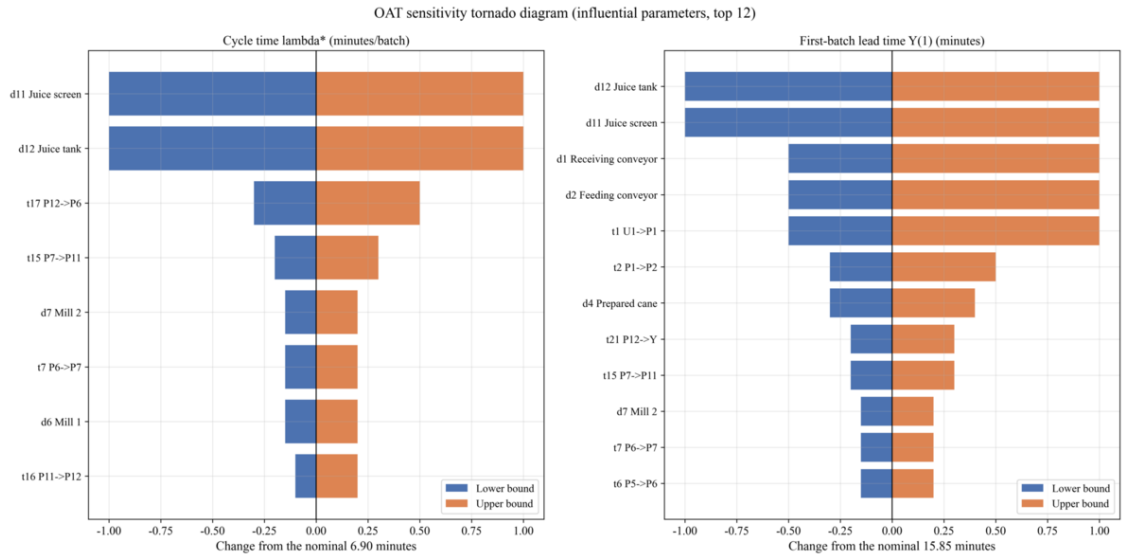


Figure 9. Tornado diagram of OAT sensitivity with respect to cycle time λ^* (left) and first-batch lead time $Y(1)$ (right)

Figure 10 illustrates the sensitivity to controller parameters. For λ values between 0.05 and 0.40, the MPC consistently selects an input interval of 6.90 minutes with a lead time of 15.85 minutes. At $\lambda = 0$, the objective function provides no incentive to delay batch releases, resulting in multiple equivalent optimal solutions for the LP formulation. The solver selects the earliest release schedule (3.00-minute interval), yielding an average lead time of 91.90 minutes and an average WIP of 12.90 batches, while the makespan remains unchanged. Consequently, a small positive λ value is sufficient to induce Just-In-Time (JIT) behavior.

Under the FIFO strategy, the average lead time decreases from 111.40 minutes ($\tau_{\mathrm{FIFO}} = 2$ minutes) to 33.40 minutes ($\tau_{\mathrm{FIFO}} = 6$ minutes), and becomes equal to that of the MPC (15.85 minutes) when $\tau_{\mathrm{FIFO}} = 7$ or 8 minutes—that is, when the feeding interval is no tighter than the system cycle time. However, feeding intervals looser than the cycle time reduce the FIFO output rate to 8.31 batches per hour ($\tau_{\mathrm{FIFO}} = 7$ minutes) and 7.32 batches per hour ($\tau_{\mathrm{FIFO}} = 8$ minutes). A similar degradation occurs in MPC if τ_{in} is fixed at 7 or 8 minutes, as the input spacing constraint forces releases to be less frequent than the system capacity allows. FIFO can only match MPC performance if its interval is tuned precisely to the system cycle time, a value that can only be determined through max-plus model analysis.

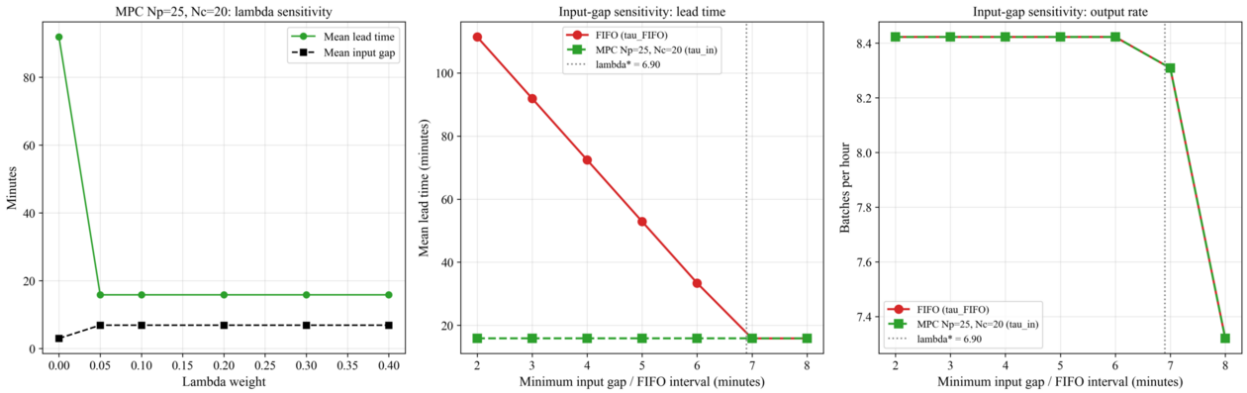


Figure 10. MPC sensitivity to penalty weight λ (left), along with MPC and FIFO sensitivity to input interval with respect to lead time (middle) and output rate (right)

3.12. Robustness Analysis

3.12.1. Deterministic Perturbations of $\pm 5\%$ *and* $\pm 10\%$

In this test, all processing times and transport times of the plant were multiplied by scaling factors of 0.90; 0.95; 1.00; 1.05; and 1.10, while the MPC retained its nominal model and nominal due dates. Consequently, the plant cycle time shifted to 6.210, 6.555, 6.900, 7.245, and 7.590 minutes per batch, respectively. All five prediction horizons of the nominal MPC yielded identical results for each perturbation factor (Figure 11); thus, Table 9 presents only the configuration (25.20).

For faster plant dynamics (factors of 0.90 and 0.95), all three strategies satisfy all due dates and yield identical makespans. The nominal MPC successfully tracks the actual plant cycle time because the measured state $x(k)$ corrects predictions at every event, resulting in average lead times of 14.89 and 15.37 minutes, compared to 57.36 and 64.88 minutes for FIFO.

For slower plant dynamics, the performance of the nominal MPC degrades. At a factor of 1.05, due-date compliance for the nominal MPC drops to 10.0% with an average tardiness of 5.15 minutes, whereas FIFO maintains 100% compliance. The makespan of the nominal MPC increases to 311.78 minutes, which is longer than that of FIFO (299.20 minutes), because its average input interval (7.57 minutes) exceeds the plant cycle time (7.245 minutes). The nominal model underestimates the lead time of the input path by 0.79 minutes (15.85 versus 16.64 minutes), which is larger than the 0.345-minute underestimation of the cycle time. Consequently, JIT batch releases are delayed, and this delay propagates to subsequent batches through the juice recirculation path. At a

Table 9. Deterministic Robustness Test Results (MPC Using the Nominal Model)

No.	Factor	Plant λ^* (min/batch)	Strategy	Makespan (min)	Average Lead Time (min)	Average WIP (batch)	Due-Date Compliance (%)	Average Tardiness (min)	Average Input Interval (min)
1	$\times 0.90$	6.210	MPC $N_p = 25$, $N_c = 20$ MPC-R $N_p = 25$, $N_c = 20$ FIFO	256.46	14.89	2.32	100.0	0.00	6.19
2	$\times 0.95$	6.555	MPC $N_p = 25$, $N_c = 20$ MPC-R $N_p = 25$, $N_c = 20$ FIFO	270.70	15.37	2.27	100.0	0.00	6.55
3	$\times 1.00$	6.900	MPC $N_p = 25$, $N_c = 20$ MPC-R $N_p = 25$, $N_c = 20$ FIFO	284.95	15.85	2.22	100.0	0.00	6.90
4	$\times 1.05$	7.245	MPC $N_p = 25$, $N_c = 20$ MPC-R $N_p = 25$, $N_c = 20$ FIFO	311.78	16.64	2.14	10.0	5.15	7.57
5	$\times 1.10$	7.590	MPC $N_p = 25$, $N_c = 20$ MPC-R $N_p = 25$, $N_c = 20$ FIFO	338.60	17.44	2.06	2.5	18.90	8.24

factor of 1.10, nominal MPC compliance drops to only 2.5% with an average tardiness of 18.90 minutes (maximum 38.20 minutes), while FIFO achieves 5.0% compliance with an average tardiness of 6.32 minutes (maximum 13.04 minutes).

MPC-R effectively overcomes these limitations. Across all perturbation factors, the makespan, due-date compliance, and tardiness of MPC-R match those of FIFO, including 100% compliance at a factor of 1.05, while

maintaining significantly shorter average lead times of only 16.44 to 18.35 minutes compared to 57.36 to 87.44 minutes for FIFO. At a factor of 1.10, the plant cycle time (7.590 minutes per batch) exceeds T_{ref} (7.245 minutes per batch); thus, no input timing can fulfill all due dates. Under this condition, the completion times of FIFO are completely governed by plant capacity due to persistent queue accumulation, meaning earlier batch releases fail to accelerate output. MPC-R achieves the same tardiness as FIFO while maintaining an average WIP of only 2.34 batches, compared to 11.16 batches for FIFO (Figure 12).

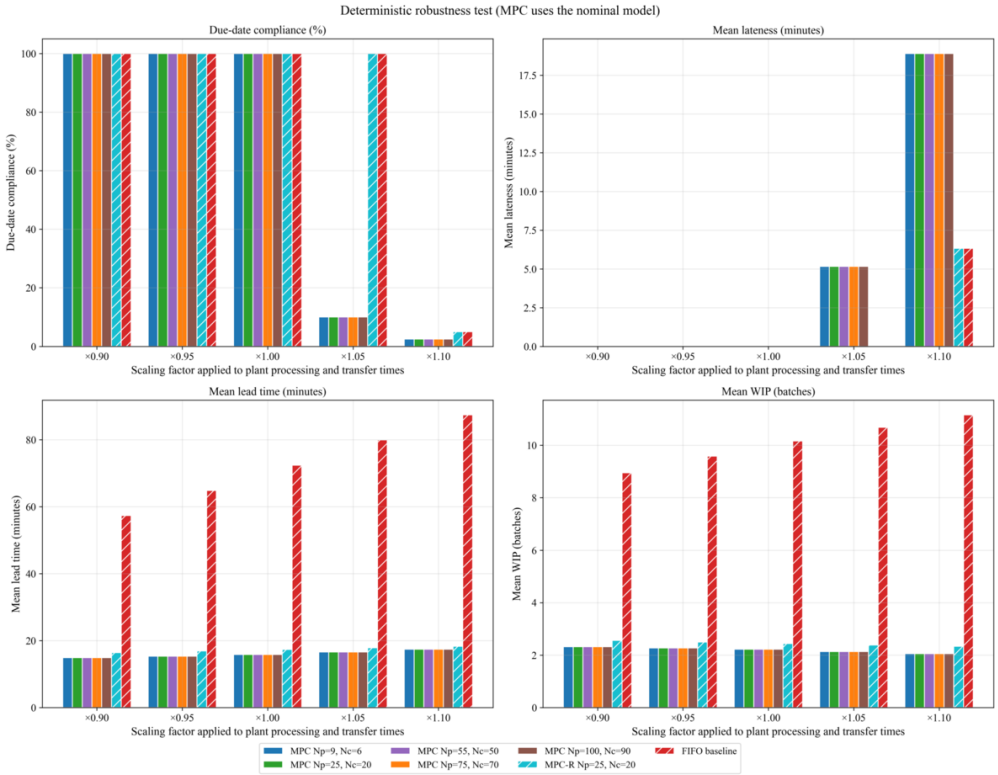


Figure 11. Due-date compliance, average tardiness, average lead time, and average WIP under deterministic robustness testing

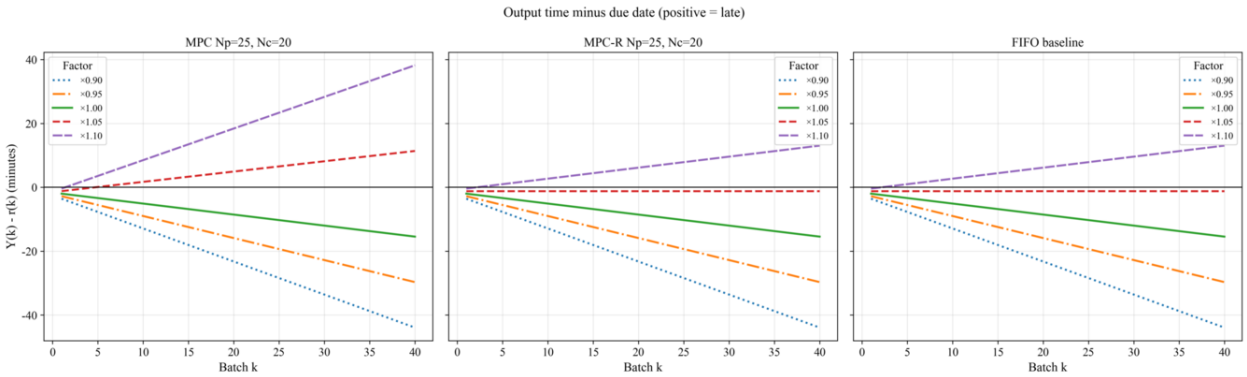


Figure 12. Difference between exit time and due date, $Y(k) - r(k)$, for nominal MPC, MPC-R, and FIFO across five perturbation factors (positive values indicate tardiness)

3.12.2. *Stochastic Perturbations of* $\pm 5\%$ *and* $\pm 10\%$

In the stochastic test, each processing time and transport time for every batch was independently multiplied by a random factor following a uniform distribution $U(1-p, 1+p)$, with $p = 0.05$ and 0.10, using 20 replications for each disturbance level. All strategies evaluated the same sequence of disturbances across corresponding replications (common random numbers); thus, pairwise differences between strategies were statistically tested using the Wilcoxon signed-rank test [21]. The 5% margin on T_{ref} proved sufficient to absorb these random disturbances, with all three strategies achieving full due-date compliance across all replications (Table 10).

Table 10. Stochastic Robustness Test Results (Mean ± Standard Deviation across 20 Replications)

Disturbance	Strategy	Average Lead Time (min)	Average WIP (batch)	Makespan (min)	Due-Date Compliance (%)
$\pm 5\%$	MPC $N_p = 25$, $N_c = 20$	15.90 ± 0.03	2.216 ± 0.002	287.01 ± 0.70	100.0
	MPC-R $N_p = 25$, $N_c = 20$	17.40 ± 0.03	2.441 ± 0.002	285.11 ± 0.66	100.0
	FIFO	72.50 ± 0.42	10.172 ± 0.039	285.11 ± 0.66	100.0
$\pm 10\%$	MPC $N_p = 25$, $N_c = 20$	15.97 ± 0.05	2.206 ± 0.006	289.64 ± 1.49	100.0
	MPC-R $N_p = 25$, $N_c = 20$	17.42 ± 0.04	2.440 ± 0.004	285.53 ± 1.06	100.0
	FIFO	72.69 ± 0.58	10.183 ± 0.050	285.53 ± 1.06	100.0

The average lead time of nominal MPC was 15.90 ± 0.03 minutes under $\pm 5\%$ disturbances and 15.97 ± 0.05 minutes under $\pm 10\%$ disturbances, whereas FIFO yielded 72.50 ± 0.42 minutes and 72.69 ± 0.58 minutes, respectively. The MPC lead time was lower across all 20 replications at both disturbance levels, with median differences of -56.64 and -56.67 minutes (Wilcoxon statistic $W = 0$; $p = 1.9 \times 10^{-6}$). The same pattern applies to the average WIP, with median differences of -7.96 and -7.97 batches.

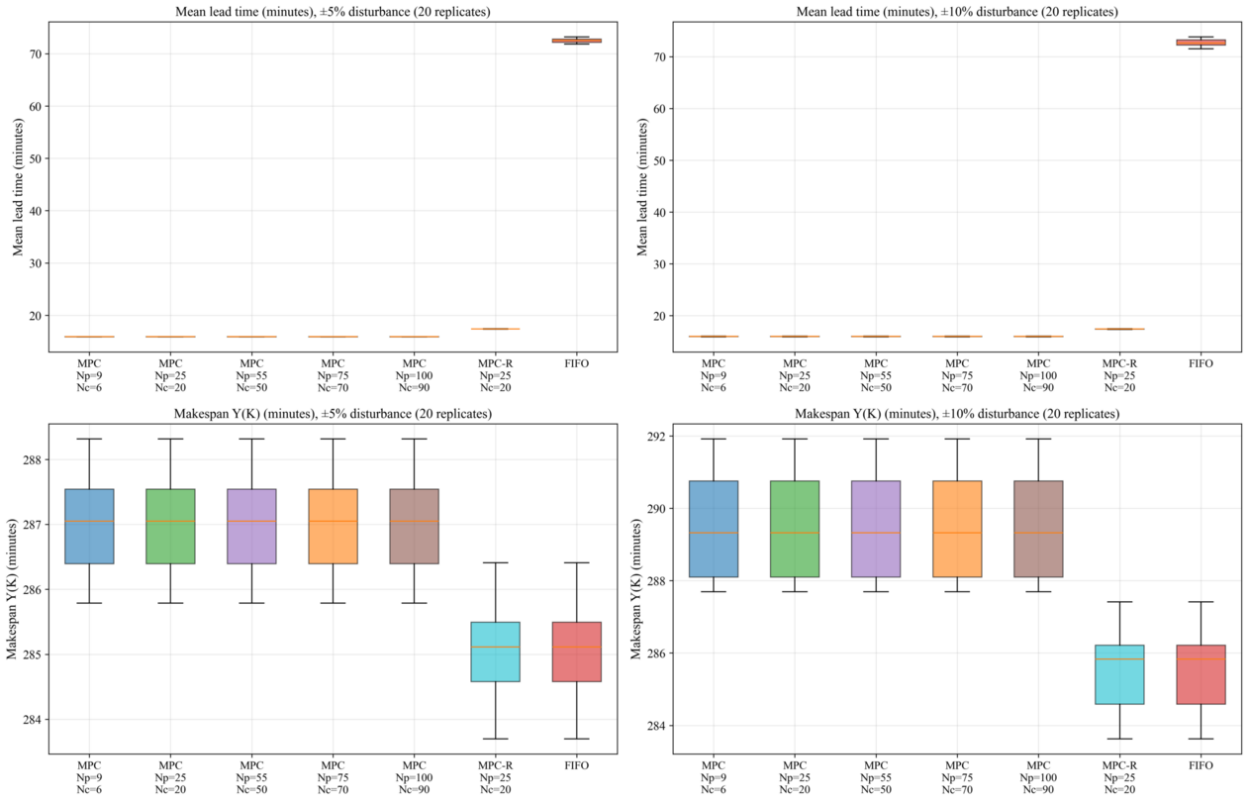


Figure 13. Distribution of average lead time (top) and makespan (bottom) under stochastic perturbations of $\pm 5\%$ (left) and $\pm 10\%$ (right)

The makespan of nominal MPC was longer than that of FIFO across all 20 replications, namely 287.01 ± 0.70 minutes compared to 285.11 ± 0.66 minutes under $\pm 5\%$ disturbances (a difference of 1.41 to 2.80 minutes) and 289.64 ± 1.49 minutes compared to 285.53 ± 1.06 minutes under $\pm 10\%$ disturbances (a difference of 2.23 to 6.19 minutes), with $p = 1.9 \times 10^{-6}$ at both disturbance levels. These average differences correspond to 0.67% and 1.44% of the FIFO makespan, respectively, and stem from the same mechanism as in the deterministic test: batches with longer-than-nominal processing times are released late by nominal MPC. MPC-R produces a makespan identical to FIFO in every replication, with an average lead time of 17.40 ± 0.03 minutes and 17.42 ± 0.04 minutes, and an average WIP of 2.441 and 2.440 batches. All five prediction horizons of nominal MPC again yield identical results (Figure 13). The Wilcoxon test in this section compares simulation replications and is not intended as an inference regarding factory operational data.

3.13. Discussion

Simulation results on Model M1 demonstrate that the primary benefit of max-plus MPC at the milling station lies in the control of material in process (work-in-process). The completion times of raw juice are strictly bounded by the critical circuit formed by juice recirculation, and input timing adjustments cannot force output faster than this underlying cycle time. A FIFO policy that feeds cane at intervals tighter than the cycle time merely shifts the cane queue upstream to Milling Unit 1. The reduction in lead time from 72.40 to 15.85 minutes and the reduction in buffer requirements from 19 to 3 batches remain operationally meaningful, even though the makespan remains unchanged.

Robustness testing reveals that MPC utilizing a nominal model is sensitive to process slowing down. The release delays that are optimal under nominal conditions become excessively delayed when the plant runs slower. Juice recirculation couples consecutive batches, causing minor delays in one batch to accumulate in subsequent batches. The 10% travel time margin in MPC-R addresses this vulnerability at the expense of an additional lead time of approximately 1.5 minutes per batch. The magnitude of this margin represents a trade-off between robustness and WIP; a smaller margin shortens lead time but is expected to protect only against smaller process slowdowns. Online parameter estimation (adaptive MPC) or min-max formulations over uncertain parameter sets warrant further study as alternative approaches.

This comprehensive simulation on Model M1 revises the preliminary analytical estimates from M0, which previously suggested a makespan reduction of approximately 7.6%, a WIP reduction exceeding 70%, 100% MPC due-date compliance under $\pm 5\%$ and $\pm 10\%$ perturbations, and 68% FIFO compliance under $+10\%$ perturbations, without providing underlying calculations. The present simulation yields a 0.00% change in makespan under nominal conditions, an average WIP reduction of 78.1%, nominal MPC due-date compliance of 10.0% and 2.5% at factors of 1.05 and 1.10, and FIFO compliance of 100% and 5.0% at those same factors, respectively. The claim of WIP reduction exceeding 70% is consistent with the simulation findings, whereas the other claims are not supported by the simulation results. The simulation proves that input timing adjustments cannot achieve a makespan faster than the lower bound imposed by the critical circuit cycle time (6.90 minutes). The true primary advantage of MPC lies in preventing material accumulation (WIP) rather than accelerating makespan.

Several limitations should be noted. The timing parameters were obtained from Table 1 and have not been validated against historical factory operational data. Output due dates were established as design assumptions because they were not specified in the manuscript, meaning compliance percentages depend on the chosen values of R_0 and T_{ref}. Imbibition water addition rules were kept fixed, and Model M1 assumes that each recirculation path carries exactly one token. The OAT analysis does not capture parameter interactions, and the identical performance observed across prediction horizons applies specifically to systems characterized by a single dominant critical circuit and due dates that are relaxed relative to system capacity.

4. Conclusion

This study modeled a 12-station sugar factory milling tandem as a Max-Plus Linear (MPL) system and compared constrained Model Predictive Control (MPC) with a baseline First-In, First-Out (FIFO) strategy through

simulations, sensitivity analyses, and robustness tests. Placing imbibition juice recirculation within the same event cycle resulted in positive-weight circuits, causing the Kleene star operator to be undefined. A consistent system structure was obtained by modeling recirculation as a dependence on the preceding batch. This structure yields a minimum cycle time of 6,90 minutes per batch (8.70 batches per hour), governed by the circuit encompassing Milling Unit 1, Milling Unit 2, Juice Screen, and Juice Tank. The numerical example for $N_p = 2$ in Subsection 3.5 applies strictly to the model without recirculation, where the 1.00-minute advantage of MPC stems solely from differences in input spacing.

Under nominal conditions across 40 batches, MPC releases sugarcane in alignment with the system cycle time, reducing average lead time from 72.40 to 15.85 minutes and average Work-In-Process (WIP) from 10.16 to 2.22 batches (both representing a 78.1% reduction), while achieving identical makespan and due-date compliance to FIFO. All five prediction/control horizon pairs (N_p, N_c) generated identical control decisions; thus, a horizon configuration of $(9, 6)$ is sufficient, requiring a computation time of approximately 4 ms per event. Cycle time exhibited the highest sensitivity to the processing times of the Juice Screen and Juice Tank (a range of 2.00 minutes) and the transport time of juice from the Juice Tank to Milling Unit 1 (a range of 0.80 minutes).

Nominal MPC is robust to process speedups; however, under process slowdowns of 5% and 10%, its due-date compliance degrades to 10.0% and 2.5%, respectively—lower than that of FIFO (100% and 5.0%). Under stochastic perturbations of $\pm 5\%$ and $\pm 10\%$, nominal MPC maintains an average lead time of approximately 16 minutes and 100% due-date compliance, but its makespan is 0.67% and 1.44% longer than FIFO across all replications. MPC augmented with a 10% travel time margin (MPC-R) achieves identical makespan and due-date compliance to FIFO across all tested scenarios, while maintaining an average lead time of only 16.44 to 18.35 minutes compared to 57.36 to 87.44 minutes for FIFO. Within the evaluated range of processing time uncertainty (up to 10%), MPC-R represents the most suitable configuration for implementation in the milling station.

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