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Plithogenic Vertex Graph and Neutrosophic Vertex HyperGraph

Takaaki Fujita [1], Raed Hatamleh [2,*], and Ajoy Kanti Das [3]

[1] *Independent Researcher, Tokyo, Japan. Email: Takaaki.fujita060@gmail.com*
[2] *Department of Mathematics, Faculty of Science, Jadara University, Irbid-Jordan. Email: raed@jadara.edu.jo*
[3] *Department of Mathematics, Tripura University, Agartala-799022, Tripura, India. Email: ajoykantidas@gmail.com*

Abstract Plithogenic sets extend fuzzy, intuitionistic fuzzy, and neutrosophic frameworks by incorporating attribute-dependent appurtenance degrees together with contradiction functions. These ideas provide a flexible mathematical setting for representing vague, incomplete, inconsistent, and attribute-dependent information. When transferred to graph-theoretic structures, they allow uncertainty and contradiction to be modeled not only at the level of individual elements but also through relations, hyperedges, and higher-order interactions.
This paper investigates several vertex-based and edge-based uncertainty graph structures, including Plithogenic Graphs, Plithogenic Vertex Graphs, Plithogenic Edge Graphs, Intuitionistic Fuzzy Vertex Graphs, Intuitionistic Fuzzy Edge Graphs, Neutrosophic Vertex HyperGraphs, Neutrosophic Edge HyperGraphs, Neutrosophic Vertex SuperHyperGraphs, and Neutrosophic Edge SuperHyperGraphs. The proposed formulations clarify how membership, nonmembership, truth, indeterminacy, falsity, appurtenance, and contradiction-related information can be assigned to vertices, edges, hyperedges, and superhypergraph components. In particular, the framework emphasizes the distinction between vertex-induced and edge-induced uncertainty models and shows how graph, hypergraph, and SuperHyperGraph structures can be used to represent pairwise, multiway, and hierarchical relations under fuzzy, neutrosophic, and plithogenic uncertainty. The study provides a unified perspective for developing more expressive graph-based models of complex systems involving ambiguity, inconsistency, and higher-order relational information.

Keywords Plithogenic Set, Plithogenic Graph, Neutrosophic Graph, Neutrosophic Set

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1. Introduction

1.1. Neutrosophic Set

Over the past several decades, many mathematical frameworks have been proposed for representing and processing uncertainty, such as Fuzzy Sets[1], Intuitionistic Fuzzy Sets[2], HyperFuzzy Sets[3, 4, 5], Picture Fuzzy Sets[6], Hesitant Fuzzy Sets[7],
and Neutrosophic Sets[8]. In this paper, our focus is on the Neutrosophic Set paradigm. In a Neutrosophic Set, each element is described by three membership functions: a degree of truth T, a degree of indeterminacy I, and a degree of falsity F. These values are typically taken in $[0, 1]$ and satisfy

$$0 \leq T + I + F \leq 3,$$

*Correspondence to: Raed Hatamleh (Email: raed@jadara.edu.jo). Department of Mathematics, Faculty of Science, Jadara University, Irbid-Jordan.

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as introduced and further developed in [9]. Neutrosophic Sets have been studied from both foundational and applied perspectives in various fields [10, 11, 12].
A number of important refinements and extensions of the basic Neutrosophic Set have been investigated in the literature, including Bipolar Neutrosophic Sets [13], Hesitant Neutrosophic Sets [14], Double-valued Neutrosophic Sets[15, 16], Triple-valued Neutrosophic Sets[17, 18], and q-rung Orthopair Neutrosophic Sets [19]. These generalized models have recently attracted considerable research interest.
It is also well known that the Neutrosophic framework contains several earlier theories of uncertainty as particular cases, including Fuzzy Sets [1], Intuitionistic Fuzzy Sets [20], Vague Sets [21], and Hesitant Fuzzy Sets [22, 23]. In addition, Plithogenic Sets have been proposed as a further generalization of Neutrosophic Sets, providing a richer structure for modeling complex, possibly conflicting attributes [24, 25]. These families of models have been successfully employed in a wide range of areas, including control theory, applied sciences, and multi-criteria decision-making.
For convenience, Table 1 gives a concise comparison of Classical, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets in terms of their membership components and basic characteristics. In view of these developments and applications, the systematic study of Neutrosophic Sets remains an important and active research topic.

Table 1. Compact comparison of Classical, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Sets

Model	Membership components	Short description
Classical Set	$\chi_A(x) \in \{0,1\}$	Membership is given by the characteristic function $\chi_A(x)$; each element x either definitely belongs to A ($\chi_A(x) = 1$) or definitely does not belong ($\chi_A(x) = 0$), without intermediate grades.
Fuzzy Set	$\mu_A(x) \in [0,1]$	Assigns to each element x a membership grade $\mu_A(x)$ in the unit interval, allowing a gradual transition from 0 (completely not in A) to 1 (fully in A).
Intuitionistic Fuzzy Set	$\mu_A(x), \nu_A(x) \in [0,1]$ with $0 \leq \mu_A(x) + \nu_A(x) \leq 1$	Uses a membership degree $\mu_A(x)$ and a nonmembership degree $\nu_A(x)$; the hesitation degree is $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, modeling incomplete or missing information about x.
Neutrosophic Set	$T_A(x), I_A(x), F_A(x) \in [0,1]$	Associates with each element x three components: truth $T_A(x)$, indeterminacy $I_A(x)$, and falsity $F_A(x)$, subject to $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, thereby accommodating uncertainty, inconsistency, and indeterminacy in a flexible way.
Plithogenic Set	$v(x) \in V$, $\mathrm{DAF}(x, v(x)) \in [0,1]^s$, $\mathrm{DCF}(v(x), v_0) \in [0,1]^t$	Each element x is associated with attribute value(s) $v(x)$. Its degree of appurtenance is described by the DAF, while the degree of contradiction between an attribute value $v(x)$ and a dominant value v_0 is described by the DCF. These two functions enable the model to handle attribute-dependent membership and contradiction, thereby generalizing fuzzy, intuitionistic fuzzy, and neutrosophic models.

The above ideas have also been explored in the field of Graph Theory. In particular, the notions of Fuzzy Graphs[26], Intuitionistic Fuzzy Graphs[27], Neutrosophic Graphs[28, 29], and Plithogenic Graphs[30, 31] have been actively studied. Furthermore, to represent increasingly complex real-world networks in a clearer and more expressive manner, these concepts have been extended using higher-order structures such as HyperGraphs [32, 33] and SuperHyperGraphs [34]. An overview of Graphs, HyperGraphs, and SuperHyperGraphs is presented in Table 2.

Table 2. Compact overview of Graphs, HyperGraphs, and SuperHyperGraphs

Structure	Vertex / edge form	Short overview
Graph	$G = (V, E)$ with a finite vertex set V and $E \subseteq \binom{V}{2}$ (each edge is an unordered pair $\{u, v\}$).	Models pairwise relationships; every edge connects exactly two vertices, suitable for standard network structures (social networks, transport links, etc.).
HyperGraph	$H = (V, E)$ with a finite vertex set V and $E \subseteq \mathcal{P}^*(V)$ (each hyperedge is a nonempty subset of V).	Generalizes graphs by allowing a single hyperedge to join any number of vertices, enabling direct modeling of multi-way/group interactions in one layer.
SuperHyperGraph	$$\mathrm{SHG}^{(n)} = (V, E)$$ with a finite base set V_0, supervertex set $$V \subseteq \mathcal{P}^n(V_0)$$, and superedge set $E \subseteq \mathcal{P}^*(V)$.	Extends hypergraphs by iterating the powerset construction n times: vertices are higher–order objects (sets of sets, etc.), and superedges connect these supervertices, capturing hierarchical, multi-level "groups of groups" and higher–order dependencies.

1.2. Our Contributions

Some of the basic notions used in this paper are adapted from established neutrosophic graph, neutrosophic hypergraph, plithogenic graph, and SuperHyperGraph frameworks. Therefore, the purpose of this paper is not merely to restate these definitions. The main goal is to isolate vertex-based and edge-based versions of these models and to develop structural operations and invariants that are specific to the proposed setting.
The main contributions of this paper are as follows.

(i) We reorganize plithogenic, intuitionistic fuzzy, and neutrosophic vertex/edge graph structures in a unified finite graph-theoretic framework.

(ii) We distinguish the proposed Plithogenic Vertex Graph and Plithogenic Edge Graph from earlier plithogenic graph models by introducing a signed attribute map

$$\sigma : ML \to +1, -1,$$

which separates favorable and unfavorable attributes and changes the induced edge aggregation rule from a single monotone rule to a signed min/max rule.

(iii) We show that the signed plithogenic construction induces threshold-core filtrations of crisp graphs. These threshold cores provide graph invariants of the proposed Plithogenic Vertex Graph and are not present in the earlier plithogenic graph formulations.

(iv) For Neutrosophic Vertex/Edge HyperGraphs, we explicitly separate the background definitions adapted from the literature from the new operations introduced in this paper, including neutrosophic transversals, dual neutrosophic hypergraphs, and threshold colorability.

(v) For Neutrosophic Vertex/Edge SuperHyperGraphs, we extend these operations to the higher-order SuperHyperGraph setting, where vertices are themselves higher-order objects generated by iterated powerset constructions.

Table 3. Difference between existing models and the revised contribution of this paper

Model / literature	Main content in earlier works	Added role in this paper
Neutrosophic HyperGraphs [35, 36, 37]	Hyperedges are equipped with neutrosophic membership information, usually through truth, indeterminacy, and falsity grades.	The basic definition is treated as background. The revised paper adds threshold transversals, dual neutrosophic hypergraphs, and threshold colorability as hypergraph-specific operations.
SuperHyperGraphs [34, 38]	Supervertices and superedges are constructed through iterated powerset structures, allowing higher-order and hierarchical connectivity.	The basic SuperHyperGraph construction is used as the underlying carrier. The revised paper adds neutrosophic vertex/edge labeling together with threshold operations, dualization, and colorability in the superhypergraph setting.
Plithogenic Graphs [39, 40]	Vertices and/or edges are equipped with plithogenic appurtenance and contradiction functions.	The revised paper introduces a signed attribute map $\sigma : ML \to \{+1, -1\}$, separating favorable and unfavorable attributes. This sign map determines whether edge appurtenance is induced by a minimum or a maximum rule and produces threshold-core graph filtrations.
This paper	–	The contribution is not the repetition of the carrier definitions, but the combination of signed plithogenic aggregation, threshold-core invariants, neutrosophic transversals, duals, and coloring operations for vertex/edge hypergraph and SuperHyperGraph models.

Table 3 presents the differences between existing models and the revised contributions of this paper.
Compared with the existing literature, the novelty of this paper lies in the introduction of signed plithogenic aggregation, threshold-core graph filtrations, and threshold-based operations for neutrosophic hypergraphs and SuperHyperGraphs. These constructions provide additional graph-theoretic invariants, such as threshold cores, transversals, dual structures, and colorability, rather than merely repeating known neutrosophic, plithogenic, or SuperHyperGraph definitions.

2. Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Throughout this paper, all structures and sets are assumed to be finite. Moreover, the graph is assumed to be undirected and simple.

2.1. Intuitionistic Fuzzy Graph

An intuitionistic fuzzy graph assigns to each vertex and edge a pair of degrees (membership and nonmembership), together with a hesitation margin, allowing one to model relational uncertainty more expressively [41]. It is well known that this framework generalizes the classical fuzzy graph [42, 43].

Definition 2.1 (Intuitionistic Fuzzy Graph)**.** [44] Let $G = (V, E)$ be a finite, simple, undirected (crisp) graph with $E \subseteq \binom{V}{2}$. An *intuitionistic fuzzy graph* (IFG) on G is a pair

$$(A, B),$$

where

- $A = (\mu_A, \nu_A)$ assigns to each vertex $v \in V$ two numbers $\mu_A(v), \nu_A(v) \in [0, 1]$ (membership and nonmembership) with

$$0 \;\leq\; \mu_A(v) + \nu_A(v) \;\leq\; 1, \qquad \pi_A(v) := 1 - \mu_A(v) - \nu_A(v) \in [0, 1].$$

- $B = (\mu_B, \nu_B)$ assigns to each edge $e = \{u, v\} \in E$ two numbers $\mu_B(e), \nu_B(e) \in [0, 1]$ with

$$0 \;\leq\; \mu_B(e) + \nu_B(e) \;\leq\; 1, \qquad \pi_B(e) := 1 - \mu_B(e) - \nu_B(e) \in [0, 1].$$

These assignments satisfy the standard consistency (compatibility) constraints for every $e = \{u, v\} \in E$:

$$\mu_B(\{u, v\}) \;\leq\; \min\{\mu_A(u), \mu_A(v)\}, \qquad \nu_B(\{u, v\}) \;\geq\; \max\{\nu_A(u), \nu_A(v)\}.$$

2.2. Neutrosophic Set and Graph

Neutrosophic sets model elements via independent truth, indeterminacy, and falsity degrees in $[0, 1]$, accommodating incomplete, inconsistent, and ambiguous information effectively [9]. A Neutrosophic Set generalizes the notions of Fuzzy Sets [1] and Intuitionistic Fuzzy Sets [45]. Neutrosophic graphs represent vertices and edges with truth, indeterminacy, and falsity memberships, enabling uncertain, contradictory, or incomplete modeling in networks[46]. Related notions include the Quadripartitioned Neutrosophic Graph [47, 48], the Pentapartitioned Neutrosophic Graph [49], the Interval-valued Neutrosophic Graph [50, 51], the Bipolar Neutrosophic Graph [52, 53, 54], and the Neutrosophic Directed Graph [55, 56, 57], among others.

Definition 2.2 (Neutrosophic Graph (single–valued))**.** [28] Let V be a finite vertex set and $E \subseteq \binom{V}{2}$ an undirected edge set. A *(single–valued) neutrosophic graph* on (V, E) is a pair

$$G = (A, B),$$

where

- $A = (T_A, I_A, F_A)$ assigns to each $v \in V$ its truth-, indeterminacy-, and falsity-memberships $T_A(v), I_A(v), F_A(v) \in [0, 1]$ with

$$0 \leq T_A(v) + I_A(v) + F_A(v) \leq 3 \quad (\forall v \in V),$$

- $B = (T_B, I_B, F_B)$ assigns to each $e = \{u, v\} \in E$ its truth-, indeterminacy-, and falsity-memberships $T_B(e), I_B(e), F_B(e) \in [0, 1]$ with

$$0 \leq T_B(e) + I_B(e) + F_B(e) \leq 3 \quad (\forall e \in E),$$

and the vertex–edge compatibility constraints hold for every edge $e = \{u, v\} \in E$:

$$T_B(\{u, v\}) \leq \min\{T_A(u), T_A(v)\},$$

$$I_B(\{u, v\}) \geq \max\{I_A(u), I_A(v)\},$$

$$F_B(\{u, v\}) \geq \max\{F_A(u), F_A(v)\}.$$

A neutrosophic vertex graph assigns to each vertex degrees of truth, indeterminacy, and falsity, thereby inducing compatible edge evaluations to effectively represent uncertainty [4]. A neutrosophic edge graph assigns truth, indeterminacy, and falsity degrees directly to each edge, with corresponding constraints on vertices to capture relational uncertainty. These structures are known to generalize fuzzy vertex[57]/edge graphs[58].

Definition 2.3 (Neutrosophic Vertex Graph)**.** [59] Let V be a finite vertex set and $E \subseteq \binom{V}{2}$ be (crisp) edges. A *neutrosophic vertex graph* on (V, E) consists of vertex assignments

$$A = (T_A, I_A, F_A) : V \to [0, 1]^3, \qquad 0 \le T_A(v) + I_A(v) + F_A(v) \le 3 \ \ (\forall v \in V).$$

When one wishes to induce edge grades from the vertex information, the *canonical* edge relation $B_A = (T_{B_A}, I_{B_A}, F_{B_A})$ on E is defined by

$$T_{B_A}(\{u, v\}) := \min\{T_A(u), T_A(v)\},$$
$$I_{B_A}(\{u, v\}) := \max\{I_A(u), I_A(v)\},$$
$$F_{B_A}(\{u, v\}) := \max\{F_A(u), F_A(v)\}.$$

This choice is consistent with the vertex–edge bounds in the neutrosophic graph definition above.

Definition 2.4 (Neutrosophic Edge Graph)**.** Let V be a finite vertex set and $E \subseteq \binom{V}{2}$ be (crisp) edges. A *neutrosophic edge graph* on (V, E) assigns to each edge $e \in E$ the triple

$$B = (T_B, I_B, F_B) : E \to [0, 1]^3, \qquad 0 \le T_B(e) + I_B(e) + F_B(e) \le 3 \ \ (\forall e \in E).$$

If vertex grades $A = (T_A, I_A, F_A)$ are also specified, they must be compatible with the edges in the sense that, for every $e = \{u, v\} \in E$,

$$T_B(\{u, v\}) \le \min\{T_A(u), T_A(v)\}, \qquad I_B(\{u, v\}) \ge \max\{I_A(u), I_A(v)\}, \qquad F_B(\{u, v\}) \ge \max\{F_A(u), F_A(v)\}.$$

2.3. Plithogenic Set and Graph

A Plithogenic Set [34] models elements with attribute-based membership and contradiction functions, extending fuzzy[58], intuitionistic [60], and neutrosophic sets[59]. A plithogenic graph assigns attribute-based membership vectors and contradiction functions to vertices or edges, generalizing fuzzy, intuitionistic, and neutrosophic models [4].

Definition 2.5 (Plithogenic Set)**.** [61] Let S be a universal set and $P \subseteq S$ a nonempty subset. A *Plithogenic Set* is a quintuple

$$PS = (P, \ v, \ Pv, \ pdf, \ pCF),$$

where

- v is an attribute,
- Pv is the set of possible values of the attribute v,
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)*,†
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

The DCF satisfies, for all $a, b \in Pv$,

$$\text{Reflexivity: } pCF(a, a) = 0, \qquad \text{Symmetry: } pCF(a, b) = pCF(b, a).$$

Here $s \in \mathbb{N}$ is the appurtenance dimension and $t \in \mathbb{N}$ the contradiction dimension.

†In the literature, DAF is defined in slightly different ways: some variants use powerset–valued constructions, others the simple cube $[0, 1]^s$. We adopt the latter (classical) form here; cf. [62].

Definition 2.6 (Plithogenic Graph)**.** (cf. [31, 63]) Let $G = (V, E)$ be a crisp (simple, undirected) graph with $E \subseteq \{\{x, y\} : x, y \in V,\ x \neq y\}$. A *plithogenic graph* is a pair

$$PG = (PM, PN),$$

where the vertex and edge components are specified as follows.

Vertex component.

$$PM = (M,\ \ell,\ ML,\ \mathrm{adf},\ \mathrm{acf}),$$

with

- $M \subseteq V$ a chosen vertex subset;
- ℓ an attribute attached to vertices;
- ML the set of possible values of ℓ;
- $\mathrm{adf} : M \times ML \to [0, 1]^s$ the vertex DAF;
- $\mathrm{acf} : ML \times ML \to [0, 1]^t$ the vertex DCF.

Edge component.

$$PN = (N,\ m,\ ML',\ \mathrm{bdf},\ \mathrm{bCf}),$$

with

- $N \subseteq E$ a chosen edge subset;
- m an attribute attached to edges;
- ML' the set of possible values of m;
- $\mathrm{bdf} : N \times ML' \to [0, 1]^s$ the edge DAF;
- $\mathrm{bCf} : ML' \times ML' \to [0, 1]^t$ the edge DCF.

All inequalities in $[0, 1]^k$ are interpreted *componentwise*. Fix $s, t \in \mathbb{N}$. The following axioms are required.

(A1) **Edge–vertex compatibility (appurtenance bound).** For all $\{x, y\} \in N$ and $a, b \in ML$,

$$\mathrm{bdf}\big(\{x, y\}, (a, b)\big) \ \leq\ \min\big\{\mathrm{adf}(x, a),\ \mathrm{adf}(y, b)\big\}. \tag{1}$$

(A2) **Contradiction consistency (edge vs. vertices).** For all $(a, b), (c, d) \in ML'$,

$$\mathrm{bCf}\big((a, b), (c, d)\big) \ \leq\ \min\big\{\mathrm{acf}(a, c),\ \mathrm{acf}(b, d)\big\}. \tag{2}$$

(A3) **Reflexivity and symmetry of DCF.**

$$\mathrm{acf}(u, u) = 0, \quad \mathrm{acf}(u, v) = \mathrm{acf}(v, u) \qquad (\forall\, u, v \in ML),$$

$$\mathrm{bCf}(u, u) = 0, \quad \mathrm{bCf}(u, v) = \mathrm{bCf}(v, u) \qquad (\forall\, u, v \in ML').$$

When $s = t = 1$, all maps are scalar-valued in $[0, 1]$ and (1)-(2) are scalar inequalities.

2.4. Hypergraphs and Superhypergraphs

A *hypergraph* extends an ordinary graph by permitting each edge to join an arbitrary nonempty subset of vertices; this allows multiway (higher-order) relationships to be modeled within a single framework [42, 64]. A *SuperHyperGraph* captures hierarchical interaction patterns by iterating the powerset construction to a prescribed depth n, thereby organizing vertices and edges across multiple levels [65, 34].

Definition 2.7 (Hypergraph [66, 67]). A *hypergraph* is a pair $H = (V(H), E(H))$ with $V(H) \neq \varnothing$ and $E(H) \subseteq \mathcal{P}^*(V(H))$. Throughout, both $V(H)$ and $E(H)$ are finite.

Example 2.8 (Hypergraph: Project Teams and Tasks). Consider a company with employees $V = \{a, b, c, d\}$. Each task may require any number of employees. Model the assignment by the hypergraph

$$H = (V, E), \qquad E = \left\{ e_1 = \{a, b\},\ e_2 = \{b, c, d\},\ e_3 = \{a, d\} \right\}.$$

Here, each hyperedge e_i is the (nonempty) set of employees assigned to task i: task 1 needs $\{a, b\}$, task 2 needs $\{b, c, d\}$, and task 3 needs $\{a, d\}$. Because hyperedges may contain more than two vertices, this representation naturally captures multi-person collaborations that a simple graph (pairwise edges only) cannot express.

Definition 2.9 (n-th powerset). [68] For a set X, define $\mathcal{P}_1(X) = \mathcal{P}(X)$ and, for $n \geq 1$,

$$\mathcal{P}_{n+1}(X) \ = \ \mathcal{P}\big(\mathcal{P}_n(X)\big).$$

When excluding the empty set, write $\mathcal{P}^*_n(X) = \mathcal{P}_n(X) \setminus \{\emptyset\}$.

Definition 2.10 (n-SuperHyperGraph). [34] Fix a finite base set V_0 and $n \in \mathbb{N}_0$. Define $\mathcal{P}^0(V_0) = V_0$ and $\mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big)$ for $k \geq 0$. An n-*SuperHyperGraph* is a pair

$$\mathrm{SHG}^{(n)} = (V, E), \qquad V \subseteq \mathcal{P}^{\,n}(V_0), \quad E \subseteq \mathcal{P}^*(V),$$

whose elements of V are the n-*supervertices* and whose members of E are nonempty sets of supervertices (the n-*superedges*).

Example 2.11 (n-SuperHyperGraph: Consortia of Project Teams ($n = 2$)). Let the base set of individual researchers be $V_0 = \{A, B, C\}$. Then $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ is the set of all *teams*, and $\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big)$ is the set of *consortia of teams*. Define the supervertex set

$$V = \Big\{ v_A = \{\{A\}\},\ \ v_B = \{\{B\}\},\ \ v_{AB} = \big\{\{A\}, \{B\}\big\} \Big\} \subseteq \mathcal{P}^2(V_0),$$

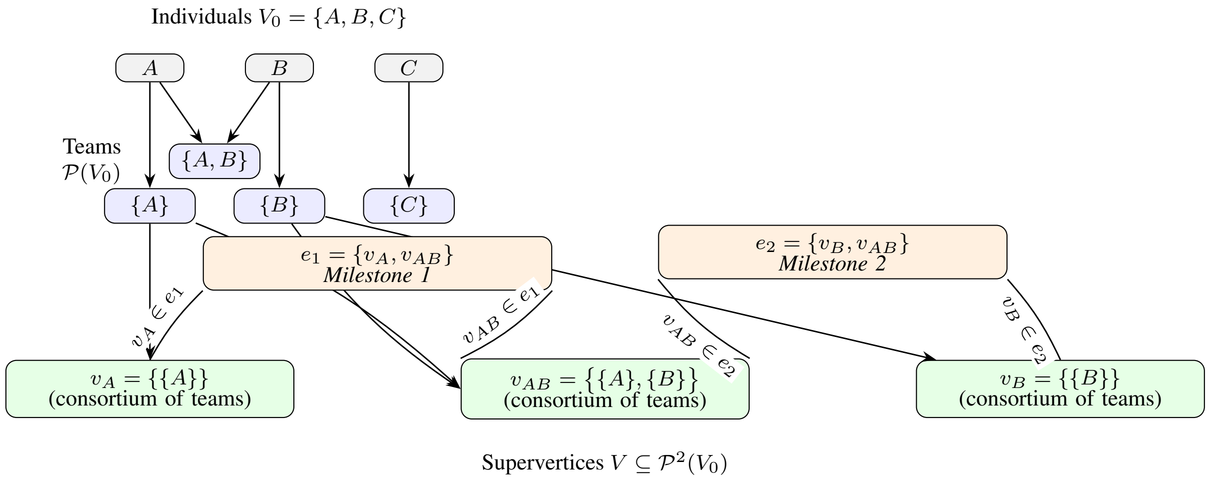
and superedges (joint milestones that require multiple consortia)

$$E = \big\{ e_1 = \{v_A, v_{AB}\},\ \ e_2 = \{v_B, v_{AB}\} \big\} \subseteq \mathcal{P}^*(V).$$

The pair $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph: elements of V are *supervertices* (consortia of teams), and each $e_i \in E$ is a nonempty set of such supervertices (a *superedge*) that must coordinate to deliver a milestone. This models hierarchical collaboration: individuals $\to$ teams $\to$ consortia, with superedges capturing multi-consortia interactions typical in large research programs.
The diagram for this example is shown in Fig. 1.

2.5. Neutrosophic n-Superhypergraph

A single-valued neutrosophic hypergraph is a hypergraph constructed from a single-valued neutrosophic graph [35, 69], and a single-valued neutrosophic superhypergraph is a superhypergraph constructed from a single-valued neutrosophic graph. A single-valued neutrosophic hypergraph and a neutrosophic n-superhypergraph are given as follows [70, 34].



Hierarchy. Individuals $\to$ teams $\to$ consortia (supervertices). **Superedges.** Each e_i collects multiple consortia that must coordinate to deliver a joint milestone.

Figure 1. 2-SuperHyperGraph modeling consortia of project teams: individuals $\to$ teams $\to$ consortia, with superedges as joint milestones.

Definition 2.12 (Single-Valued Neutrosophic Hypergraph)**.** (cf.[36]) Let $V = \{v_1, \ldots, v_n\}$ be a finite vertex set and let $E = \{E_i\}_{i=1}^m$ be a family of nontrivial single-valued neutrosophic subsets of V such that

$$V = \bigcup_{i=1}^{m} \mathrm{supp}(E_i).$$

Each neutrosophic hyperedge E_i is given by

$$E_i = \big\{(v, T_{E_i}(v), I_{E_i}(v), F_{E_i}(v)) : v \in V\big\},$$

where

$$T_{E_i}, I_{E_i}, F_{E_i} : V \longrightarrow [0,1] \quad \text{satisfy} \quad 0 \le T_{E_i}(v) + I_{E_i}(v) + F_{E_i}(v) \le 3 \;\; \forall v \in V.$$

Then $H = (V, E)$ is called a *single-valued neutrosophic hypergraph*.

Definition 2.13 (Neutrosophic n-Superhypergraph)**.** (cf. [34]) Let V_0 be a finite base set and define recursively

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \ge 0).$$

An *n-Superhypergraph* is a pair $\mathrm{SHT}^{(n)} = (V, E)$ with

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V).$$

A *Neutrosophic n-Superhypergraph* on (V, E) is a tuple

$$\big(V, E,\ T_V, I_V, F_V,\ T_E, I_E, F_E\big),$$

where

- $T_V, I_V, F_V : V \to [0,1]$ assign to each n-supervertex $v \in V$ its truth, indeterminacy, and falsity degrees, with

$$0 \le T_V(v) + I_V(v) + F_V(v) \le 3 \quad (\forall v \in V).$$

- $T_E, I_E, F_E : E \times V \to [0,1]$ encode a neutrosophic *incidence* (membership) of v into a superedge $e \in E$, subject to
$$0 \leq T_E(e,v) + I_E(e,v) + F_E(e,v) \leq 3 \quad (\forall e \in E,\ v \in V),$$
the *vertex–domination* (componentwise) constraints
$$T_E(e,v) \leq T_V(v), \quad I_E(e,v) \leq I_V(v), \quad F_E(e,v) \leq F_V(v),$$
and the *support* condition
$$v \notin e \implies T_E(e,v) = I_E(e,v) = F_E(e,v) = 0.$$

Thus T_V, I_V, F_V describe the intrinsic neutrosophic status of supervertices, while T_E, I_E, F_E specify how those supervertices contribute neutrosophically to each superedge.

Example 2.14 (Neutrosophic 2-Superhypergraph)**.** Let the base set be $V_0 = \{a, b\}$. Then $\mathcal{P}^1(V_0) = \big\{\{a\}, \{b\}, \{a,b\}\big\}$ and $\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big)$. Choose the supervertex set
$$V = \Big\{ v_a = \{\{a\}\}, \quad v_b = \{\{b\}\}, \quad v_{ab} = \big\{\{a\}, \{b\}\big\} \Big\} \subseteq \mathcal{P}^2(V_0),$$
and superedges
$$E = \big\{\ e_1 = \{v_a, v_{ab}\},\ \ e_2 = \{v_b, v_{ab}\}\ \big\} \subseteq \mathcal{P}(V).$$
Assign neutrosophic *vertex* degrees (each component in $[0,1]$, sums ≤ 3):
$$\langle T_V(v_a), I_V(v_a), F_V(v_a)\rangle = (0.70, 0.20, 0.10),$$
$$\langle T_V(v_b), I_V(v_b), F_V(v_b)\rangle = (0.60, 0.20, 0.20),$$
$$\langle T_V(v_{ab}), I_V(v_{ab}), F_V(v_{ab})\rangle = (0.90, 0.05, 0.05).$$
Define *edge incidences* only on vertices that belong to each superedge (support condition), and make them componentwise dominated by the corresponding vertex degrees:
$$\begin{aligned}
\text{on } e_1 = \{v_a, v_{ab}\}: \quad & \langle T_E(e_1,v_a), I_E(e_1,v_a), F_E(e_1,v_a)\rangle = (0.60, 0.15, 0.05), \\
& \langle T_E(e_1,v_{ab}), I_E(e_1,v_{ab}), F_E(e_1,v_{ab})\rangle = (0.85, 0.05, 0.05); \\
\text{on } e_2 = \{v_b, v_{ab}\}: \quad & \langle T_E(e_2,v_b), I_E(e_2,v_b), F_E(e_2,v_b)\rangle = (0.55, 0.15, 0.15), \\
& \langle T_E(e_2,v_{ab}), I_E(e_2,v_{ab}), F_E(e_2,v_{ab})\rangle = (0.80, 0.05, 0.05).
\end{aligned}$$
For any $e \in E$ and $v \in V \setminus e$, set $T_E(e,v) = I_E(e,v) = F_E(e,v) = 0$. By construction,
$$T_E(e,v) \leq T_V(v), \quad I_E(e,v) \leq I_V(v), \quad F_E(e,v) \leq F_V(v),$$
and each incidence triple sums to ≤ 3. Therefore $\big(V, E; T_V, I_V, F_V; T_E, I_E, F_E\big)$ is a neutrosophic 2-superhypergraph. (One may view v_a, v_b, v_{ab} as single/team units, and e_1, e_2 as coalitions with graded participation.) The example is illustrated in Fig. 2.

3. Review and Main Results

We now present the results established in this paper.

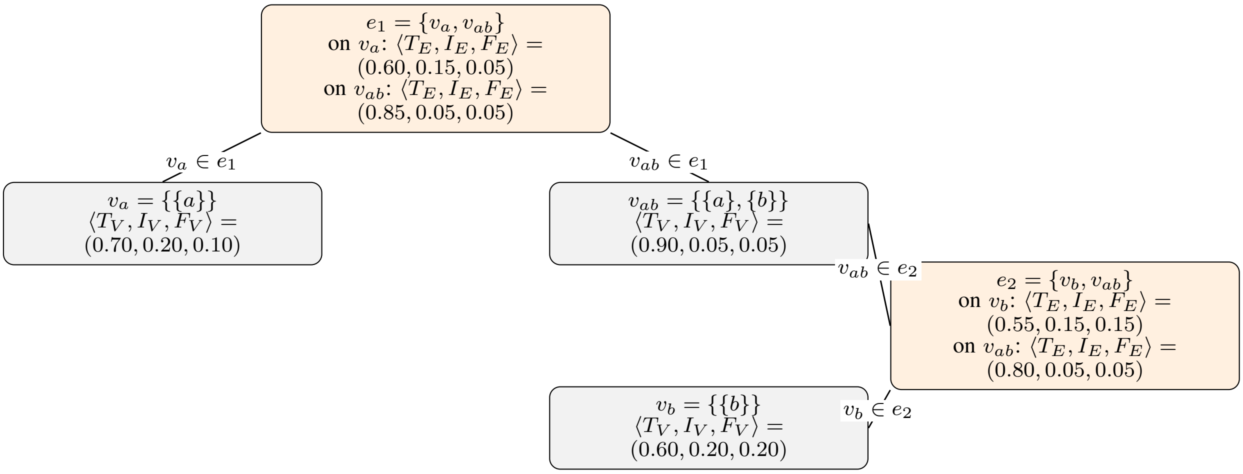


Figure 2. A neutrosophic 2-superhypergraph with supervertices v_a, v_b, v_{ab} and superedges e_1, e_2.

3.1. Intuitionistic Fuzzy Vertex Graph

An intuitionistic fuzzy vertex graph assigns to each vertex membership and nonmembership degrees with hesitation, inducing compatible edge assessments automatically.

Definition 3.1 (Intuitionistic Fuzzy Vertex Graph)**.** Let $G = (V, E)$ be as above. An *intuitionistic fuzzy vertex graph* (IFVG) on G specifies only vertex grades

$$A = (\mu_A, \nu_A) : V \to [0,1]^2, \qquad 0 \le \mu_A(v) + \nu_A(v) \le 1 \ \ (\forall v \in V),$$

with hesitation $\pi_A(v) = 1 - \mu_A(v) - \nu_A(v)$.
When one wishes to derive edge grades from the vertex information, the *canonical induced* edge assignment $B_A = (\mu_{B_A}, \nu_{B_A})$ on E is defined by

$$\mu_{B_A}(\{u,v\}) := \min\{\mu_A(u), \mu_A(v)\}, \qquad \nu_{B_A}(\{u,v\}) := \max\{\nu_A(u), \nu_A(v)\},$$

which automatically satisfies $0 \le \mu_{B_A} + \nu_{B_A} \le 1$ and the IFG compatibility inequalities.

Example 3.2 (Intuitionistic Fuzzy Vertex Graph (IFVG))**.** Let $G = (V, E)$ with $V = \{a, b, c\}$ and $E = \big\{\{a, b\}, \{b, c\}\big\}$. Assign vertex grades

$$\begin{aligned}
\mu_A(a) = 0.80,\ \nu_A(a) = 0.10 \quad (\pi_A(a) = 0.10),\\
\mu_A(b) = 0.60,\ \nu_A(b) = 0.20 \quad (\pi_A(b) = 0.20),\\
\mu_A(c) = 0.30,\ \nu_A(c) = 0.50 \quad (\pi_A(c) = 0.20),
\end{aligned}$$

so $0 \le \mu_A(v) + \nu_A(v) \le 1$ for all $v \in V$. The canonical induced edge grades are

$$\mu_{B_A}(\{u,v\}) = \min\{\mu_A(u), \mu_A(v)\}, \qquad \nu_{B_A}(\{u,v\}) = \max\{\nu_A(u), \nu_A(v)\}.$$

Hence

$$\begin{aligned}
\{a,b\}: &\quad \mu_{B_A} = \min(0.80, 0.60) = 0.60, \quad \nu_{B_A} = \max(0.10, 0.20) = 0.20, \quad \pi_{B_A} = 0.20;\\
\{b,c\}: &\quad \mu_{B_A} = \min(0.60, 0.30) = 0.30, \quad \nu_{B_A} = \max(0.20, 0.50) = 0.50, \quad \pi_{B_A} = 0.20,
\end{aligned}$$

and in each case $0 \le \mu_{B_A} + \nu_{B_A} \le 1$. Thus (A, B_A) defines an IFVG on G.

3.2. Intuitionistic Fuzzy Edge Graph

An intuitionistic fuzzy edge graph labels every edge with membership and nonmembership degrees under hesitation, constraining vertex grades for consistency.

Definition 3.3 (Intuitionistic Fuzzy Edge Graph)**.** Let $G = (V, E)$ be as above. An *intuitionistic fuzzy edge graph* (IFEG) on G specifies only edge grades

$$B = (\mu_B, \nu_B) : E \to [0,1]^2, \qquad 0 \le \mu_B(e) + \nu_B(e) \le 1 \ \ (\forall e \in E),$$

with hesitation $\pi_B(e) = 1 - \mu_B(e) - \nu_B(e)$.
If vertex grades $A = (\mu_A, \nu_A)$ are also provided, then (A, B) must satisfy, for all $e = \{u, v\} \in E$,

$$\mu_B(\{u,v\}) \ \le \ \min\{\mu_A(u), \mu_A(v)\}, \qquad \nu_B(\{u,v\}) \ \ge \ \max\{\nu_A(u), \nu_A(v)\}.$$

Example 3.4 (Intuitionistic Fuzzy Edge Graph (IFEG))**.** Let $G = (V, E)$ with $V = \{a, b, c\}$ and $E = \big\{\{a,b\}, \{b,c\}\big\}$. Specify edge grades

$$\begin{aligned} \{a,b\}: \ \mu_B = 0.55, \ \nu_B = 0.25 \quad (\pi_B = 0.20), \\ \{b,c\}: \ \mu_B = 0.35, \ \nu_B = 0.45 \quad (\pi_B = 0.20), \end{aligned}$$

so $0 \le \mu_B(e) + \nu_B(e) \le 1$ for all $e \in E$. Provide vertex grades

$$\mu_A(a) = 0.70, \ \nu_A(a) = 0.20; \qquad \mu_A(b) = 0.60, \ \nu_A(b) = 0.25; \qquad \mu_A(c) = 0.40, \ \nu_A(c) = 0.45,$$

each satisfying $0 \le \mu_A(v) + \nu_A(v) \le 1$. Then the IF edge–vertex compatibility holds:

$$\begin{aligned} \{a,b\}: \ \mu_B = 0.55 \le \min(0.70, 0.60) = 0.60, \quad \nu_B = 0.25 \ge \max(0.20, 0.25) = 0.25; \\ \{b,c\}: \ \mu_B = 0.35 \le \min(0.60, 0.40) = 0.40, \quad \nu_B = 0.45 \ge \max(0.25, 0.45) = 0.45. \end{aligned}$$

Hence (A, B) defines a consistent IFEG on G.

3.3. Plithogenic Vertex Graph

A plithogenic vertex graph attaches attribute based memberships to vertices with contradiction function, generating edge memberships via signed aggregation rules.

Definition 3.5 (Plithogenic Vertex Graph (PVG))**.** Let $G = (V, E)$ be a finite simple undirected graph. Fix

- an *attribute alphabet* ML for vertices;
- a *sign map* $\sigma : ML \to \{+1, -1\}$ indicating, for each attribute value $a \in ML$, whether higher membership is *favorable* $(+1)$ or *unfavorable* (-1);
- a *degree of appurtenance function (DAF)*

$$\mathrm{adf} : V \times ML \longrightarrow [0,1]^s,$$

  assigning to each vertex $x \in V$ and attribute value $a \in ML$ an s-tuple of membership degrees (componentwise order on $[0,1]^s$);
- a *degree of contradiction function (DCF)*

$$\mathrm{acf} : ML \times ML \longrightarrow [0,1]^t,$$

  with $\mathrm{acf}(u,u) = 0$ and $\mathrm{acf}(u,v) = \mathrm{acf}(v,u)$ (componentwise).

The *canonical induced edge DAF* is the map

$$\mathrm{bdf} : E \times ML \to [0,1]^s, \qquad \mathrm{bdf}(\{x,y\},a) := \begin{cases} \min\bigl\{\mathrm{adf}(x,a),\ \mathrm{adf}(y,a)\bigr\}, & \sigma(a) = +1, \\ \max\bigl\{\mathrm{adf}(x,a),\ \mathrm{adf}(y,a)\bigr\}, & \sigma(a) = -1, \end{cases}$$

where $\min / \max$ act componentwise on $[0,1]^s$. The structure

$$PG_V \ = \ \bigl(V, E;\ ML, \sigma;\ \mathrm{adf}, \mathrm{acf};\ \mathrm{bdf}\bigr)$$

is called a *Plithogenic Vertex Graph (PVG)*. (Other t-norm / t-conorm pairs may replace $\min / \max$; we fix these to connect with neutrosophic graphs.)

Remark 3.6 (Role of the sign map)**.** The sign map

$$\sigma : ML \to \{+1, -1\}$$

is not merely a notational addition. It changes the edge-induction mechanism. For a favorable attribute a with $\sigma(a) = +1$, the induced edge appurtenance is governed by a minimum rule:

$$\mathrm{bdf}(\{x,y\},a) = \min\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\}.$$

This expresses the principle that a favorable relation is limited by the weaker endpoint.
For an unfavorable attribute a with $\sigma(a) = -1$, the induced edge risk or opposition is governed by a maximum rule:

$$\mathrm{bdf}(\{x,y\},a) = \max\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\}.$$

This expresses the principle that an unfavorable relation is dominated by the more obstructive endpoint.
Therefore, the sign map changes the semantics and the induced graph structure. It also leads to different threshold-core filtrations for favorable and unfavorable attributes.

Example 3.7 (Plithogenic Vertex Graph (PVG): Team formation)**.** Let $G = (V, E)$ with $V = \{x, y, z\}$ and $E = \bigl\{\{x,y\}, \{y,z\}\bigr\}$. Take a vertex attribute alphabet

$$ML = \{\texttt{rel}, \texttt{risk}\}, \qquad \sigma(\texttt{rel}) = +1 \text{ (higher is favorable)}, \quad \sigma(\texttt{risk}) = -1 \text{ (higher is unfavorable)}.$$

Work with $s = t = 1$. Set a symmetric vertex DCF $\mathrm{acf}(\texttt{rel},\texttt{rel}) = \mathrm{acf}(\texttt{risk},\texttt{risk}) = 0$, $\mathrm{acf}(\texttt{rel},\texttt{risk}) = \mathrm{acf}(\texttt{risk},\texttt{rel}) = 0.7$. Give vertex DAF values (componentwise scalars in $[0,1]$):

$$\begin{aligned} \mathrm{adf}(x,\texttt{rel}) = 0.90,\ \ \mathrm{adf}(x,\texttt{risk}) = 0.20, \\ \mathrm{adf}(y,\texttt{rel}) = 0.70,\ \ \mathrm{adf}(y,\texttt{risk}) = 0.40, \\ \mathrm{adf}(z,\texttt{rel}) = 0.50,\ \ \mathrm{adf}(z,\texttt{risk}) = 0.60. \end{aligned}$$

The canonical induced edge DAF is

$$\mathrm{bdf}(\{u,v\},a) = \begin{cases} \min\{\mathrm{adf}(u,a), \mathrm{adf}(v,a)\}, & a = \texttt{rel}, \\ \max\{\mathrm{adf}(u,a), \mathrm{adf}(v,a)\}, & a = \texttt{risk}. \end{cases}$$

Hence

$$\begin{aligned} \{x,y\} : \mathrm{bdf}(\cdot,\texttt{rel}) = \min(0.90, 0.70) = 0.70, \quad \mathrm{bdf}(\cdot,\texttt{risk}) = \max(0.20, 0.40) = 0.40; \\ \{y,z\} : \mathrm{bdf}(\cdot,\texttt{rel}) = \min(0.70, 0.50) = 0.50, \quad \mathrm{bdf}(\cdot,\texttt{risk}) = \max(0.40, 0.60) = 0.60. \end{aligned}$$

Thus $PG_V = (V, E; ML, \sigma; \mathrm{adf}, \mathrm{acf}; \mathrm{bdf})$ is a PVG that aggregates vertex “reliability” and “risk” into edge memberships via the signed rule.

Theorem 3.8 (PVG generalizes the Neutrosophic Vertex Graph)
Every single-valued Neutrosophic Vertex Graph $A = (T_A, I_A, F_A) : V \to [0,1]^3$ (with canonical induced edges $T_{B_A}(\{x,y\}) = \min\{T_A(x), T_A(y)\},\ I_{B_A}(\{x,y\}) = \max\{I_A(x), I_A(y)\},\ F_{B_A}(\{x,y\}) = \max\{F_A(x), F_A(y)\}$) is a special case of a PVG.

Proof
Given (V, E) and (T_A, I_A, F_A), define a PVG by

$$ML = \{\mathtt{T}, \mathtt{I}, \mathtt{F}\}, \qquad \sigma(\mathtt{T}) = +1,\ \ \sigma(\mathtt{I}) = \sigma(\mathtt{F}) = -1,$$

take $s = t = 1$, set the vertex DAF

$$\mathrm{adf}(x, \mathtt{T}) = T_A(x), \quad \mathrm{adf}(x, \mathtt{I}) = I_A(x), \quad \mathrm{adf}(x, \mathtt{F}) = F_A(x),$$

and choose any symmetric DCF with $\mathrm{acf}(u,u) = 0$ (its values are immaterial for this embedding). By the PVG definition,

$$\mathrm{bdf}(\{x,y\}, \mathtt{T}) = \min\{T_A(x), T_A(y)\} = T_{B_A}(\{x,y\}),$$

$$\mathrm{bdf}(\{x,y\}, \mathtt{I}) = \max\{I_A(x), I_A(y)\} = I_{B_A}(\{x,y\}), \qquad \mathrm{bdf}(\{x,y\}, \mathtt{F}) = \max\{F_A(x), F_A(y)\} = F_{B_A}(\{x,y\}).$$

Thus the PVG reproduces exactly the neutrosophic vertex graph (both vertex and canonically induced edge degrees), proving that NVGs are PVGs with the above specialization. □

Theorem 3.9 (Symmetry, bounds, and monotonicity of the induced edge DAF)
Let

$$PG_V = (V, E;\ ML, \sigma;\ \mathrm{adf}, \mathrm{acf};\ \mathrm{bdf})$$

be a Plithogenic Vertex Graph. Then for every edge $e = \{x,y\} \in E$ and every attribute $a \in ML$ the following statements hold componentwise in $[0,1]^s$. Write

$$u := \mathrm{adf}(x,a), \qquad v := \mathrm{adf}(y,a).$$

(i) **Symmetry.**

$$\mathrm{bdf}(\{x,y\}, a) = \mathrm{bdf}(\{y,x\}, a).$$

(ii) **Bounds.** One has

$$\min\{u,v\} \ \le\ \mathrm{bdf}(\{x,y\}, a) \ \le\ \max\{u,v\},$$

where $\min$ and $\max$ are taken componentwise. In particular,

- if $\sigma(a) = +1$, then $\mathrm{bdf}(\{x,y\}, a) = \min\{u,v\}$,
- if $\sigma(a) = -1$, then $\mathrm{bdf}(\{x,y\}, a) = \max\{u,v\}$.

(iii) **Monotonicity.** Fix $a \in ML$ and $\{x,y\} \in E$. Let $\mathrm{adf}' : V \times ML \to [0,1]^s$ be another vertex DAF on the same ML and σ such that

$$\mathrm{adf}(x,a) \le \mathrm{adf}'(x,a), \qquad \mathrm{adf}(y,a) \le \mathrm{adf}'(y,a)$$

componentwise. Let bdf' be the induced edge DAF obtained from adf' by the same rule as in the definition of PG_V. Then

$$\mathrm{bdf}(\{x,y\}, a) \ \le\ \mathrm{bdf}'(\{x,y\}, a)$$

componentwise in $[0,1]^s$.

Proof
Since $e = \{x, y\}$ is an unordered pair, by definition of bdf we have

$$\mathrm{bdf}(\{x,y\},a) = \begin{cases} \min\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\}, & \sigma(a) = +1, \\ \max\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\}, & \sigma(a) = -1, \end{cases}$$

and the same expression is obtained if one writes $\{y, x\}$ instead of $\{x, y\}$. Thus (i) holds.
For (ii), work component wise. Fix a coordinate $j \in \{1, \dots, s\}$ and let $u_j, v_j \in [0,1]$ be the j-th coordinates of u, v. Then

$$\min\{u_j, v_j\} \le \max\{u_j, v_j\}, \qquad \min\{u_j, v_j\} \le u_j, v_j \le \max\{u_j, v_j\}.$$

If $\sigma(a) = +1$, then

$$\big(\mathrm{bdf}(\{x,y\},a)\big)_j = \min\{u_j, v_j\},$$

so $\min\{u_j, v_j\} \le (\mathrm{bdf})_j \le \max\{u_j, v_j\}$ is immediate. If $\sigma(a) = -1$, then

$$\big(\mathrm{bdf}(\{x,y\},a)\big)_j = \max\{u_j, v_j\},$$

and again $\min\{u_j, v_j\} \le (\mathrm{bdf})_j \le \max\{u_j, v_j\}$ holds. Since this argument applies to every coordinate j, the stated component wise inequalities follow, together with the two particular cases in the theorem.
For (iii), fix a and $e = \{x, y\}$. Put

$$u_j = (\mathrm{adf}(x,a))_j, \quad v_j = (\mathrm{adf}(y,a))_j, \quad u'_j = (\mathrm{adf}'(x,a))_j, \quad v'_j = (\mathrm{adf}'(y,a))_j.$$

By assumption $u_j \le u'_j$ and $v_j \le v'_j$ for each j.
If $\sigma(a) = +1$, then

$$\big(\mathrm{bdf}(\{x,y\},a)\big)_j = \min\{u_j, v_j\}, \qquad \big(\mathrm{bdf}'(\{x,y\},a)\big)_j = \min\{u'_j, v'_j\}.$$

The ordinary minimum on $\mathbb{R}$ is monotone in each argument: whenever $u_j \le u'_j$ and $v_j \le v'_j$ one has

$$\min\{u_j, v_j\} \le \min\{u'_j, v'_j\}.$$

Hence $(\mathrm{bdf})_j \le (\mathrm{bdf}')_j$ for all j.
If $\sigma(a) = -1$, then

$$\big(\mathrm{bdf}(\{x,y\},a)\big)_j = \max\{u_j, v_j\}, \qquad \big(\mathrm{bdf}'(\{x,y\},a)\big)_j = \max\{u'_j, v'_j\},$$

and the ordinary maximum is also monotone in each argument, so

$$\max\{u_j, v_j\} \le \max\{u'_j, v'_j\},$$

giving $(\mathrm{bdf})_j \le (\mathrm{bdf}')_j$ again. Thus in all cases $\mathrm{bdf}(\{x,y\},a) \le \mathrm{bdf}'(\{x,y\},a)$ component wise, as required. □

Theorem 3.10 (Homogeneous scaling of vertex memberships in a PVG)
Let

$$PG_V = (V, E;\ ML, \sigma;\ \mathrm{adf}, \mathrm{acf};\ \mathrm{bdf})$$

be a Plithogenic Vertex Graph and let $\lambda \in [0,1]$ be fixed. Define a new vertex DAF

$$\mathrm{adf}_\lambda : V \times ML \longrightarrow [0,1]^s, \qquad \mathrm{adf}_\lambda(x,a) := \lambda\, \mathrm{adf}(x,a),$$

and let bdf_λ be the induced edge DAF built from adf_λ by the same rule and sign map σ as in the definition of PG_V. Then:

(i) $(V, E;\ ML, \sigma;\ \mathrm{adf}_\lambda, \mathrm{acf};\ \mathrm{bdf}_\lambda)$ is again a Plithogenic Vertex Graph.

(ii) For every edge $e = \{x, y\} \in E$ and every attribute $a \in ML$,

$$\mathrm{bdf}_\lambda(\{x, y\}, a) \;=\; \lambda\, \mathrm{bdf}(\{x, y\}, a)$$

componentwise in $[0, 1]^s$.

Proof
(i) Fix $(x, a) \in V \times ML$. Each coordinate of $\mathrm{adf}(x, a)$ lies in $[0, 1]$, and $\lambda \in [0, 1]$, so each coordinate of $\mathrm{adf}_\lambda(x, a) = \lambda\, \mathrm{adf}(x, a)$ also lies in $[0, 1]$. Hence adf_λ maps $V \times ML$ to $[0, 1]^s$. The DCF acf is left unchanged, so it remains reflexive and symmetric as in the original PVG. The induced edge map bdf_λ is obtained from adf_λ using the same min/max rule, therefore each coordinate of $\mathrm{bdf}_\lambda(e, a)$ still belongs to $[0, 1]$ for all (e, a). Consequently the PVG axioms are satisfied, and $(V, E; ML, \sigma; \mathrm{adf}_\lambda, \mathrm{acf}; \mathrm{bdf}_\lambda)$ is a Plithogenic Vertex Graph.
(ii) Fix $e = \{x, y\} \in E$ and $a \in ML$. Write

$$u := \mathrm{adf}(x, a), \quad v := \mathrm{adf}(y, a), \quad u_\lambda := \mathrm{adf}_\lambda(x, a) = \lambda u, \quad v_\lambda := \mathrm{adf}_\lambda(y, a) = \lambda v.$$

We work componentwise. Let $j \in \{1, \ldots, s\}$ and denote the j-th coordinates by $u_j, v_j, u_{\lambda,j}, v_{\lambda,j} \in [0, 1]$.
If $\sigma(a) = +1$, then by definition

$$\big(\mathrm{bdf}(\{x, y\}, a)\big)_j = \min\{u_j, v_j\}, \qquad \big(\mathrm{bdf}_\lambda(\{x, y\}, a)\big)_j = \min\{u_{\lambda,j}, v_{\lambda,j}\} = \min\{\lambda u_j, \lambda v_j\}.$$

Since $\lambda \geq 0$, the usual minimum on $\mathbb{R}$ satisfies

$$\min\{\lambda u_j, \lambda v_j\} = \lambda\, \min\{u_j, v_j\}.$$

Therefore

$$\big(\mathrm{bdf}_\lambda(\{x, y\}, a)\big)_j = \lambda \big(\mathrm{bdf}(\{x, y\}, a)\big)_j.$$

If $\sigma(a) = -1$, then

$$\big(\mathrm{bdf}(\{x, y\}, a)\big)_j = \max\{u_j, v_j\}, \qquad \big(\mathrm{bdf}_\lambda(\{x, y\}, a)\big)_j = \max\{\lambda u_j, \lambda v_j\}.$$

Again, for $\lambda \geq 0$ the usual maximum satisfies

$$\max\{\lambda u_j, \lambda v_j\} = \lambda\, \max\{u_j, v_j\},$$

so

$$\big(\mathrm{bdf}_\lambda(\{x, y\}, a)\big)_j = \lambda \big(\mathrm{bdf}(\{x, y\}, a)\big)_j.$$

In both cases, and for every coordinate j, we obtain

$$\mathrm{bdf}_\lambda(\{x, y\}, a) = \lambda\, \mathrm{bdf}(\{x, y\}, a)$$

component wise in $[0, 1]^s$. This proves (ii). □

The previous definitions assign graded appurtenance values to vertices and induce edge values by signed aggregation. We now show that this construction has a concrete combinatorial consequence: each attribute and each threshold determine a crisp core graph. Hence a Plithogenic Vertex Graph is not only a graded object but also a finite filtration of crisp graphs.

Definition 3.11 (Plithogenic threshold core)**.** Let

$$PG_V = (V, E;\ ML, \sigma; \mathrm{adf}, \mathrm{acf}; \mathrm{bdf})$$

be a Plithogenic Vertex Graph. Fix an attribute value $a \in ML$, a coordinate $j \in \{1, \ldots, s\}$, and a threshold $\alpha \in [0, 1]$.

If $\sigma(a) = +1$, define

$$V_\alpha^{a,j,+} = \{x \in V : (\mathrm{adf}(x,a))_j \geq \alpha\},$$

and

$$E_\alpha^{a,j,+} = \{\{x,y\} \in E : (\mathrm{bdf}(\{x,y\},a))_j \geq \alpha\}.$$

If $\sigma(a) = -1$, define

$$V_\alpha^{a,j,-} = \{x \in V : (\mathrm{adf}(x,a))_j \leq \alpha\},$$

and

$$E_\alpha^{a,j,-} = \{\{x,y\} \in E : (\mathrm{bdf}(\{x,y\},a))_j \leq \alpha\}.$$

The corresponding crisp graph is called the *threshold core* of PG_V at (a,j,α).

Theorem 3.12 (Threshold-core characterization of PVGs)
Let

$$PG_V = (V, E;\ ML, \sigma; \mathrm{adf}, \mathrm{acf}; \mathrm{bdf})$$

be a Plithogenic Vertex Graph, and fix $a \in ML$, $j \in \{1, \ldots, s\}$, and $\alpha \in [0,1]$.

(i) If $\sigma(a) = +1$, then

$$E_\alpha^{a,j,+} = \{\{x,y\} \in E : x, y \in V_\alpha^{a,j,+}\}.$$

Therefore the positive threshold core is precisely the subgraph induced by $V_\alpha^{a,j,+}$.

(ii) If $\sigma(a) = -1$, then

$$E_\alpha^{a,j,-} = \{\{x,y\} \in E : x, y \in V_\alpha^{a,j,-}\}.$$

Therefore the negative threshold core is precisely the subgraph induced by $V_\alpha^{a,j,-}$.

Proof
We prove the two cases separately.
First assume $\sigma(a) = +1$. By the definition of the induced edge DAF,

$$(\mathrm{bdf}(\{x,y\},a))_j = \min\{(\mathrm{adf}(x,a))_j, (\mathrm{adf}(y,a))_j\}.$$

Hence

$$(\mathrm{bdf}(\{x,y\},a))_j \geq \alpha$$

holds if and only if both

$$(\mathrm{adf}(x,a))_j \geq \alpha \quad \text{and} \quad (\mathrm{adf}(y,a))_j \geq \alpha$$

hold. This is equivalent to $x, y \in V_\alpha^{a,j,+}$. Therefore

$$E_\alpha^{a,j,+} = \{\{x,y\} \in E : x, y \in V_\alpha^{a,j,+}\}.$$

Next assume $\sigma(a) = -1$. Then

$$(\mathrm{bdf}(\{x,y\},a))_j = \max\{(\mathrm{adf}(x,a))_j, (\mathrm{adf}(y,a))_j\}.$$

Thus

$$(\mathrm{bdf}(\{x,y\},a))_j \leq \alpha$$

holds if and only if both

$$(\mathrm{adf}(x,a))_j \leq \alpha \quad \text{and} \quad (\mathrm{adf}(y,a))_j \leq \alpha$$

hold. This is equivalent to $x, y \in V_\alpha^{a,j,-}$. Therefore

$$E_\alpha^{a,j,-} = \{\{x,y\} \in E : x, y \in V_\alpha^{a,j,-}\}.$$

The theorem follows. □

Table 4 presents a comparison between Neutrosophic Vertex Graphs, existing Plithogenic Graphs, and the proposed Plithogenic Vertex Graph.

Table 4. Comparison between Neutrosophic Vertex Graphs, existing Plithogenic Graphs, and the proposed Plithogenic Vertex Graph

Aspect	Neutrosophic Vertex Graph	Existing Plithogenic Graph	Proposed Plithogenic Vertex Graph
Vertex information	Uses truth, indeterminacy, and falsity degrees.	Uses attribute-based appurtenance and contradiction values.	Uses attribute-dependent appurtenance values with signed attribute interpretation.
Attribute treatment	Does not explicitly distinguish attribute values.	Allows attribute values through plithogenic functions.	Treats multiple attribute values separately and distinguishes favorable and unfavorable effects.
Contradiction	No explicit contradiction function between attribute values.	May include contradiction functions.	Combines contradiction functions with vertex-based graph structure and signed aggregation.
Induced edge behavior	Edges are constrained by values endpoint T, I, and F values.	Edges may carry plithogenic information.	Edges are induced by a minimum rule for favorable attributes and a maximum rule for unfavorable attributes.
Expressive power	Models uncertain vertex information.	Models attribute-dependent uncertainty and contradiction.	Models attribute-dependent, contradiction-aware, and direction-sensitive uncertainty on vertices and induced edges.

3.4. Plithogenic Edge Graph

A plithogenic edge graph assigns attribute based memberships to edges with contradiction function, optionally bounded by vertex memberships for coherence (cf.[71, 72]).

Definition 3.13 (Plithogenic Edge Graph (PEG))**.** (cf.[73]) Let $G = (V, E)$ be a finite simple undirected graph. Fix

- an *attribute alphabet* ML' for edges;
- a *sign map* $\sigma' : ML' \to \{+1, -1\}$;
- an *edge DAF*
$$\mathrm{bdf} : E \times ML' \longrightarrow [0, 1]^s;$$
- an *edge DCF*
$$\mathrm{bCf} : ML' \times ML' \longrightarrow [0, 1]^t,$$
with $\mathrm{bCf}(u, u) = 0$ and $\mathrm{bCf}(u, v) = \mathrm{bCf}(v, u)$ (componentwise).

Optionally, if vertex degrees $\mathrm{adf} : V \times ML' \to [0, 1]^s$ are given on the same alphabet, we require the *compatibility bounds* for every $e = \{x, y\} \in E$ and $a \in ML'$:

$$\sigma'(a) = +1 : \quad \mathrm{bdf}(e, a) \ \le\ \min\big\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\big\},$$

$$\sigma'(a) = -1 : \quad \mathrm{bdf}(e, a) \ \ge\ \max\big\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\big\},$$

all componentwise in $[0,1]^s$. The structure

$$PG_E \;=\; \bigl(V, E;\; ML', \sigma';\; \mathrm{bdf}, \mathrm{bCf}\bigr)$$

is called a *Plithogenic Edge Graph (PEG)*.

Example 3.14 (Plithogenic Edge Graph (PEG): Network links)**.** Let $G = (V, E)$ with $V = \{u, v, w\}$ and $E = \bigl\{\{u, v\}, \{v, w\}\bigr\}$. Use the edge attribute alphabet

$$ML' = \{\texttt{bw}, \texttt{lat}\}, \qquad \sigma'(\texttt{bw}) = +1 \text{ (more bandwidth is better)}, \quad \sigma'(\texttt{lat}) = -1 \text{ (more latency is worse)}.$$

Take $s = t = 1$ and set a symmetric edge DCF with $\mathrm{bCf}(\texttt{bw}, \texttt{bw}) = \mathrm{bCf}(\texttt{lat}, \texttt{lat}) = 0$, $\mathrm{bCf}(\texttt{bw}, \texttt{lat}) = \mathrm{bCf}(\texttt{lat}, \texttt{bw}) = 0.6$. Assign edge DAF values

$$\begin{aligned} \{u, v\}:\ & \mathrm{bdf}(\cdot, \texttt{bw}) = 0.80, \quad \mathrm{bdf}(\cdot, \texttt{lat}) = 0.40, \\ \{v, w\}:\ & \mathrm{bdf}(\cdot, \texttt{bw}) = 0.60, \quad \mathrm{bdf}(\cdot, \texttt{lat}) = 0.50. \end{aligned}$$

Optionally provide vertex DAF on the same alphabet to check compatibility:

$$\begin{aligned} &\mathrm{adf}(u, \texttt{bw}) = 0.90,\ \mathrm{adf}(v, \texttt{bw}) = 0.80,\ \mathrm{adf}(w, \texttt{bw}) = 0.60, \\ &\mathrm{adf}(u, \texttt{lat}) = 0.30,\ \mathrm{adf}(v, \texttt{lat}) = 0.40,\ \mathrm{adf}(w, \texttt{lat}) = 0.50. \end{aligned}$$

Then the PEG bounds hold componentwise:

$$\begin{aligned} \{u, v\}:\ & \mathrm{bdf}(\texttt{bw}) = 0.80 \le \min(0.90, 0.80) = 0.80, \quad \mathrm{bdf}(\texttt{lat}) = 0.40 \ge \max(0.30, 0.40) = 0.40; \\ \{v, w\}:\ & \mathrm{bdf}(\texttt{bw}) = 0.60 \le \min(0.80, 0.60) = 0.60, \quad \mathrm{bdf}(\texttt{lat}) = 0.50 \ge \max(0.40, 0.50) = 0.50. \end{aligned}$$

Thus $PG_E = (V, E; ML', \sigma'; \mathrm{bdf}, \mathrm{bCf})$ is a coherent PEG describing link quality using bandwidth (positive) and latency (negative).

Theorem 3.15 (PEG generalizes the Neutrosophic Edge Graph)
Every single-valued Neutrosophic Edge Graph $B = (T_B, I_B, F_B) : E \to [0,1]^3$ (with optional vertex grades $A = (T_A, I_A, F_A)$ satisfying the usual compatibility $T_B(\{x, y\}) \le \min\{T_A(x), T_A(y)\}$, $I_B(\{x, y\}) \ge \max\{I_A(x), I_A(y)\}$, $F_B(\{x, y\}) \ge \max\{F_A(x), F_A(y)\}$) is a special case of a PEG.

Proof
Fix $ML' = \{\texttt{T}, \texttt{I}, \texttt{F}\}$ and $s = t = 1$, and define

$$\sigma'(\texttt{T}) = +1, \qquad \sigma'(\texttt{I}) = \sigma'(\texttt{F}) = -1.$$

Set the edge DAF by identification:

$$\mathrm{bdf}(e, \texttt{T}) = T_B(e), \quad \mathrm{bdf}(e, \texttt{I}) = I_B(e), \quad \mathrm{bdf}(e, \texttt{F}) = F_B(e), \qquad (e \in E),$$

and choose any symmetric DCF bCf with $\mathrm{bCf}(u, u) = 0$. If vertex grades A are given, define

$$\mathrm{adf}(x, \texttt{T}) = T_A(x),\ \ \mathrm{adf}(x, \texttt{I}) = I_A(x),\ \ \mathrm{adf}(x, \texttt{F}) = F_A(x)$$

(on the same alphabet), so that the PEG compatibility bounds become

$$\mathrm{bdf}(e, \texttt{T}) \le \min\bigl\{\mathrm{adf}(x, \texttt{T}), \mathrm{adf}(y, \texttt{T})\bigr\}, \quad \mathrm{bdf}(e, \texttt{I}) \ge \max\bigl\{\mathrm{adf}(x, \texttt{I}), \mathrm{adf}(y, \texttt{I})\bigr\},$$

$$\mathrm{bdf}(e, \texttt{F}) \ge \max\bigl\{\mathrm{adf}(x, \texttt{F}), \mathrm{adf}(y, \texttt{F})\bigr\}, \qquad (e = \{x, y\}),$$

which are exactly the neutrosophic edge–vertex constraints. Hence every neutrosophic edge graph is realized as a PEG under this specialization. $\square$

Theorem 3.16 (Stability under edge-induced subgraphs for PEGs)
Let

$$PG_E = \big(V, E;\ ML', \sigma';\ \mathrm{bdf}, \mathrm{bCf}\big)$$

be a Plithogenic Edge Graph on a finite simple undirected graph $G = (V, E)$. Let $E' \subseteq E$ and let

$$V' \ :=\ \{x \in V :\ \exists e \in E' \text{ with } x \in e\}.$$

Define the edge-induced subgraph $G' = (V', E')$ and the restricted structure

$$PG'_E \ :=\ \big(V', E';\ ML', \sigma';\ \mathrm{bdf}', \mathrm{bCf}'\big),$$

where

$$\mathrm{bdf}' \ :=\ \mathrm{bdf}\big|_{E' \times ML'} \quad \text{and} \quad \mathrm{bCf}' \ :=\ \mathrm{bCf}.$$

If a vertex DAF $\mathrm{adf} : V \times ML' \to [0,1]^s$ is given and satisfies the compatibility bounds of the PEG definition, then we also restrict it to

$$\mathrm{adf}' \ :=\ \mathrm{adf}\big|_{V' \times ML'}.$$

Then PG'_E (together with adf' when present) is again a Plithogenic Edge Graph.

Proof
By construction, $G' = (V', E')$ is a finite simple undirected graph, because $E' \subseteq E$ and we only remove edges and possibly isolated vertices.
The attribute alphabet ML' and the sign map $\sigma' : ML' \to \{+1, -1\}$ are not changed, so all their properties are trivially preserved. The edge DCF is also unchanged: $\mathrm{bCf}' = \mathrm{bCf}$, hence for all $u, v \in ML'$ we still have

$$\mathrm{bCf}'(u,u) = \mathrm{bCf}(u,u) = 0, \qquad \mathrm{bCf}'(u,v) = \mathrm{bCf}(u,v) = \mathrm{bCf}(v,u) = \mathrm{bCf}'(v,u),$$

component wise in $[0,1]^t$.
It remains to verify the compatibility bounds. Assume a vertex DAF adf is given on $V \times ML'$ and satisfies the PEG inequalities for PG_E. For any edge $e = \{x, y\} \in E'$ and any attribute $a \in ML'$, we have in PG_E:

$$\sigma'(a) = +1 : \quad \mathrm{bdf}(e,a) \ \le\ \min\big\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\big\},$$

$$\sigma'(a) = -1 : \quad \mathrm{bdf}(e,a) \ \ge\ \max\big\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\big\},$$

all componentwise in $[0,1]^s$.
By definition of the restrictions,

$$\mathrm{bdf}'(e,a) = \mathrm{bdf}(e,a), \qquad \mathrm{adf}'(x,a) = \mathrm{adf}(x,a), \quad \mathrm{adf}'(y,a) = \mathrm{adf}(y,a).$$

Therefore, the same inequalities hold in PG'_E:

$$\sigma'(a) = +1 : \quad \mathrm{bdf}'(e,a) \ =\ \mathrm{bdf}(e,a) \ \le\ \min\big\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\big\} \ =\ \min\big\{\mathrm{adf}'(x,a), \mathrm{adf}'(y,a)\big\},$$

$$\sigma'(a) = -1 : \quad \mathrm{bdf}'(e,a) \ =\ \mathrm{bdf}(e,a) \ \ge\ \max\big\{\mathrm{adf}(x,a), \mathrm{adf}(y,a)\big\} \ =\ \max\big\{\mathrm{adf}'(x,a), \mathrm{adf}'(y,a)\big\},$$

again componentwise in $[0,1]^s$.
Thus all axioms in the definition of a Plithogenic Edge Graph are satisfied by PG'_E, so PG'_E is a PEG. $\square$

Theorem 3.17 (Closure of PEGs under edgewise min/max aggregation)
Let $G = (V, E)$ be a finite simple undirected graph and fix ML', $\sigma' : ML' \to \{+1, -1\}$, an edge DCF $\mathrm{bCf} : ML' \times ML' \to [0,1]^t$ with $\mathrm{bCf}(u,u) = 0$ and $\mathrm{bCf}(u,v) = \mathrm{bCf}(v,u)$, and a vertex DAF $\mathrm{adf} : V \times ML' \to [0,1]^s$.
Suppose $\mathrm{bdf}_1, \mathrm{bdf}_2 : E \times ML' \to [0,1]^s$ define two Plithogenic Edge Graphs on the same data:

$$PG_E^{(k)} = \big(V, E; ML', \sigma';\ \mathrm{bdf}_k, \mathrm{bCf}\big), \quad k = 1, 2,$$

so for every $e = \{x, y\} \in E$, $a \in ML'$ we have

$$\sigma'(a) = +1 : \quad \mathrm{bdf}_k(e, a) \ \leq \ \min\big\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\big\},$$

$$\sigma'(a) = -1 : \quad \mathrm{bdf}_k(e, a) \ \geq \ \max\big\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\big\},$$

component wise in $[0, 1]^s$ for $k = 1, 2$.
Define two new edge DAFs $\mathrm{bdf}_\wedge, \mathrm{bdf}_\vee : E \times ML' \to [0, 1]^s$ by

$$\mathrm{bdf}_\wedge(e, a) \ := \ \min\big\{\mathrm{bdf}_1(e, a), \mathrm{bdf}_2(e, a)\big\},$$

$$\mathrm{bdf}_\vee(e, a) \ := \ \max\big\{\mathrm{bdf}_1(e, a), \mathrm{bdf}_2(e, a)\big\},$$

where the min and max are taken component wise in $[0, 1]^s$. Then each of

$$\big(V, E; ML', \sigma';\ \mathrm{bdf}_\wedge, \mathrm{bCf}\big), \qquad \big(V, E; ML', \sigma';\ \mathrm{bdf}_\vee, \mathrm{bCf}\big)$$

is again a Plithogenic Edge Graph (with the same $G, ML', \sigma', \mathrm{bCf}, \mathrm{adf}$).

Proof
The underlying graph G, the attribute alphabet ML', the sign map σ' and the edge DCF bCf remain unchanged, so the only property to verify is the system of compatibility bounds with the vertex DAF adf.
Fix an edge $e = \{x, y\} \in E$ and an attribute $a \in ML'$. For each coordinate index $j \in \{1, \dots, s\}$, write

$$\alpha_j \ := \ \big(\mathrm{bdf}_1(e, a)\big)_j, \qquad \beta_j \ := \ \big(\mathrm{bdf}_2(e, a)\big)_j,$$

and

$$c_j^- \ := \ \big(\max\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\}\big)_j, \qquad c_j^+ \ := \ \big(\min\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\}\big)_j.$$

By the PEG assumptions on bdf_1 and bdf_2, we have, for every j,

$$\sigma'(a) = +1 \implies \alpha_j \leq c_j^+, \quad \beta_j \leq c_j^+,$$

$$\sigma'(a) = -1 \implies \alpha_j \geq c_j^-, \quad \beta_j \geq c_j^-.$$

First, consider $\mathrm{bdf}_\wedge$. For each (e, a) and coordinate j we have

$$\big(\mathrm{bdf}_\wedge(e, a)\big)_j = \min\{\alpha_j, \beta_j\}.$$

If $\sigma'(a) = +1$, then $\alpha_j \leq c_j^+$ and $\beta_j \leq c_j^+$, hence

$$\min\{\alpha_j, \beta_j\} \ \leq \ c_j^+.$$

If $\sigma'(a) = -1$, then $\alpha_j \geq c_j^-$ and $\beta_j \geq c_j^-$, so

$$\min\{\alpha_j, \beta_j\} \ \geq \ c_j^-.$$

Translating back to vector notation, this shows that for every $e = \{x, y\}$, $a \in ML'$,

$$\sigma'(a) = +1 : \quad \mathrm{bdf}_\wedge(e, a) \ \leq \ \min\big\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\big\},$$

$$\sigma'(a) = -1 : \quad \mathrm{bdf}_\wedge(e, a) \ \geq \ \max\big\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\big\},$$

component wise in $[0, 1]^s$. Thus $\big(V, E; ML', \sigma'; \mathrm{bdf}_\wedge, \mathrm{bCf}\big)$ is a PEG.
Next, consider $\mathrm{bdf}_\vee$. For each (e, a) and coordinate j,

$$\big(\mathrm{bdf}_\vee(e, a)\big)_j = \max\{\alpha_j, \beta_j\}.$$

If $\sigma'(a) = +1$, then again $\alpha_j \le c_j^+$ and $\beta_j \le c_j^+$, hence

$$\max\{\alpha_j, \beta_j\} \;\le\; c_j^+.$$

If $\sigma'(a) = -1$, then $\alpha_j \ge c_j^-$ and $\beta_j \ge c_j^-$, so

$$\max\{\alpha_j, \beta_j\} \;\ge\; c_j^-.$$

Equivalently, for every $e = \{x, y\}$, $a \in ML'$,

$$\sigma'(a) = +1: \quad \mathrm{bdf}_\vee(e, a) \;\le\; \min\bigl\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\bigr\},$$

$$\sigma'(a) = -1: \quad \mathrm{bdf}_\vee(e, a) \;\ge\; \max\bigl\{\mathrm{adf}(x, a), \mathrm{adf}(y, a)\bigr\},$$

component wise in $[0, 1]^s$.
Therefore, both $\mathrm{bdf}_\wedge$ and $\mathrm{bdf}_\vee$ satisfy the PEG compatibility bounds, and each defines a Plithogenic Edge Graph on the same underlying data. This proves the closure of PEGs under edgewise componentwise min and max aggregation of the edge DAF. □

3.5. Neutrosophic Vertex HyperGraph

A Neutrosophic Vertex HyperGraph assigns each vertex neutrosophic truth, indeterminacy, and falsity degrees, while hyperedges connect subsets, and incidences respect bounds derived from vertices componentwise.

Definition 3.18 (Neutrosophic Vertex HyperGraph)**.** Let $H = (V, E)$ be a hypergraph. A *Neutrosophic Vertex HyperGraph* on H is a tuple

$$\mathrm{NVH}(H) \;=\; \bigl(V, E;\; T_V, I_V, F_V;\; T_{inc}, I_{inc}, F_{inc}\bigr),$$

where

- $T_V, I_V, F_V : V \to [0, 1]$ are vertex neutrosophic grades satisfying

$$0 \;\le\; T_V(v) + I_V(v) + F_V(v) \;\le\; 3 \qquad (\forall v \in V),$$

- $T_{inc}, I_{inc}, F_{inc} : E \times V \to [0, 1]$ are *incidence* grades with

$$v \notin e \implies T_{inc}(e, v) = I_{inc}(e, v) = F_{inc}(e, v) = 0,$$

  and the (componentwise) vertex–domination constraints

$$T_{inc}(e, v) \le T_V(v), \qquad I_{inc}(e, v) \le I_V(v), \qquad F_{inc}(e, v) \le F_V(v) \quad (\forall e \in E,\ \forall v \in V).$$

A common canonical choice is $T_{inc}(e, v) = T_V(v)\mathbf{1}_{\{v\in e\}}$, $I_{inc}(e, v) = I_V(v)\mathbf{1}_{\{v\in e\}}$, $F_{inc}(e, v) = F_V(v)\mathbf{1}_{\{v\in e\}}$.

Example 3.19 (Task teams as a vertex hypergraph)**.** Let $V = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ be employees and

$$E = \bigl\{\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{B}, \mathrm{C}\}, \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}\bigr\}$$

be feasible coalitions. Set

$$\langle T_V, I_V, F_V\rangle(\mathrm{A}) = (0.9, 0.05, 0.05),$$
$$\langle T_V, I_V, F_V\rangle(\mathrm{B}) = (0.7, 0.2, 0.1),$$
$$\langle T_V, I_V, F_V\rangle(\mathrm{C}) = (0.6, 0.3, 0.1),$$

and take the canonical incidence $T_{inc}(e, v) = T_V(v)\mathbf{1}_{\{v\in e\}}$ (similarly for I, F). This produces an $\mathrm{NVH}(H)$: each coalition (hyperedge) inherits the vertex grades of its members, while absent members contribute zero.

Theorem 3.20 (The vertex model generalizes hypergraphs and neutrosophic vertex graphs)
Let $\mathrm{NVH}(H)$ be as in Definition 3.18.

(a) (*To hypergraphs*) Every hypergraph $H = (V, E)$ is recovered as a crisp special case of a Neutrosophic Vertex HyperGraph by setting, for all $v \in V$ and $(e, v) \in E \times V$,

$$T_V(v) = 1,\ I_V(v) = 0,\ F_V(v) = 0, \qquad T_{inc}(e, v) = \mathbf{1}_{\{v \in e\}},\ I_{inc}(e, v) = F_{inc}(e, v) = 0.$$

(b) (*To neutrosophic vertex graphs*) If $H = (V, E)$ is 2-uniform (i.e. every $e \in E$ has $|e| = 2$), then $\mathrm{NVH}(H)$ reduces to a *neutrosophic vertex graph* by keeping the same vertex triples (T_V, I_V, F_V) and viewing E as the usual edge set of a simple graph.

Proof
(a) The stated assignments satisfy $T_V + I_V + F_V = 1 \le 3$ and, for all (e, v), $v \notin e \Rightarrow T_{inc} = I_{inc} = F_{inc} = 0$ by definition. If $v \in e$, then $T_{inc}(e, v) = 1 \le T_V(v) = 1$ and $I_{inc}, F_{inc} = 0 \le I_V(v), F_V(v) = 0$, so the vertex–domination constraints hold. Forgetting the neutrosophic labels recovers exactly the original (V, E), hence the embedding is faithful.
(b) When every hyperedge has cardinality 2, we may regard H as a simple undirected graph on V with edge set $E \subseteq \binom{V}{2}$. Keeping the vertex assignments (T_V, I_V, F_V) gives a *neutrosophic vertex graph*. The incidence maps $T_{inc}, I_{inc}, F_{inc}$ are consistent with (and bounded by) the vertex labels, so they do not alter the usual neutrosophic vertex graph structure; one may take the canonical choice $T_{inc}(e, v) = T_V(v)$, etc., for $v \in e$. □

We next introduce a finite combinatorial invariant associated with a Neutrosophic Vertex HyperGraph. The invariant records how the underlying hypergraph changes when one observes the structure at different truth thresholds.
Let

$$\mathrm{NVH}(H) = (V, E; T_V, I_V, F_V; T_{inc}, I_{inc}, F_{inc})$$

be a Neutrosophic Vertex HyperGraph on $H = (V, E)$. Denote the incidence set by

$$\mathcal{I}(H) = \{(e, v) \in E \times V : v \in e\}.$$

Definition 3.21 (Truth-threshold hypergraph)**.** For $\alpha \in [0, 1]$, define

$$V_\alpha^T = \{v \in V : T_V(v) \ge \alpha\}.$$

For each hyperedge $e \in E$, define its α-truth core by

$$e_\alpha^T = \{v \in e : T_{inc}(e, v) \ge \alpha\}.$$

The α-truth threshold hypergraph is

$$H_\alpha^T = (V_\alpha^T, E_\alpha^T), \qquad E_\alpha^T = \{e_\alpha^T : e \in E,\ e_\alpha^T \ne \varnothing\}.$$

The family

$$\mathcal{S}_T(\mathrm{NVH}) = \{H_\alpha^T : \alpha \in [0, 1]\}$$

is called the *truth-threshold spectrum* of the Neutrosophic Vertex HyperGraph.

Theorem 3.22 (Finite threshold-spectrum theorem)
Let $\mathrm{NVH}(H)$ be a finite Neutrosophic Vertex HyperGraph. Then:

(i) If $0 \le \alpha \le \beta \le 1$, then

$$V_\beta^T \subseteq V_\alpha^T$$

and, for every $e \in E$,

$$e_\beta^T \subseteq e_\alpha^T.$$

Hence the truth-threshold hypergraphs form a nested family.

(ii) The family H_α^T can change only when α crosses a value belonging to the finite set

$$\Lambda_T = \{T_V(v) : v \in V\} \cup \{T_{inc}(e, v) : (e, v) \in \mathcal{I}(H)\}.$$

(iii) Consequently, the number of distinct truth-threshold hypergraphs is at most

$$|\Lambda_T| + 1 \le |V| + |\mathcal{I}(H)| + 1.$$

Proof
(i) Let $0 \le \alpha \le \beta \le 1$. If $v \in V_\beta^T$, then $T_V(v) \ge \beta \ge \alpha$, and therefore $v \in V_\alpha^T$. Hence $V_\beta^T \subseteq V_\alpha^T$. Similarly, if $v \in e_\beta^T$, then $T_{inc}(e, v) \ge \beta \ge \alpha$, so $v \in e_\alpha^T$. Therefore $e_\beta^T \subseteq e_\alpha^T$.
(ii) Suppose that $\alpha, \beta \in [0, 1]$ lie in the same connected component of $[0, 1] \setminus \Lambda_T$. Then no vertex value $T_V(v)$ and no incidence value $T_{inc}(e, v)$ lies strictly between α and β. Hence the truth comparisons

$$T_V(v) \ge \alpha, \qquad T_{inc}(e, v) \ge \alpha$$

have the same truth values as

$$T_V(v) \ge \beta, \qquad T_{inc}(e, v) \ge \beta$$

for all $v \in V$ and all $(e, v) \in \mathcal{I}(H)$. Therefore $V_\alpha^T = V_\beta^T$ and $e_\alpha^T = e_\beta^T$ for every $e \in E$. Thus the threshold hypergraph changes only at values of Λ_T.
(iii) Since Λ_T is finite and has cardinality at most $|V| + |\mathcal{I}(H)|$, the interval $[0, 1]$ is divided into at most $|\Lambda_T| + 1$ regions on which H_α^T is constant. Therefore the number of distinct truth-threshold hypergraphs is at most $|\Lambda_T| + 1 \le |V| + |\mathcal{I}(H)| + 1$. □

The preceding definition of a Neutrosophic Vertex HyperGraph is close to existing neutrosophic hypergraph models. We therefore now introduce operations that are specific to the present vertex-incidence formulation.
Let

$$\mathrm{NVH}(H) = (V, E; T_V, I_V, F_V; T_{inc}, I_{inc}, F_{inc})$$

be a Neutrosophic Vertex HyperGraph, where $H = (V, E)$.

Definition 3.23 (Truth-threshold support)**.** For $\alpha \in [0, 1]$, the α-truth support of a hyperedge $e \in E$ is defined by

$$e_\alpha^T = \{v \in e : T_{inc}(e, v) \ge \alpha\}.$$

The associated truth-threshold hypergraph is

$$H_\alpha^T = (V_\alpha^T, E_\alpha^T),$$

where

$$V_\alpha^T = \{v \in V : T_V(v) \ge \alpha\}, \qquad E_\alpha^T = \{e_\alpha^T : e \in E,\ e_\alpha^T \ne \varnothing\}.$$

Definition 3.24 (Neutrosophic truth transversal)**.** Let $\alpha \in [0, 1]$. A subset $S \subseteq V$ is called an α-truth transversal of $\mathrm{NVH}(H)$ if

$$S \cap e_\alpha^T \ne \varnothing \qquad (\forall e \in E \text{ with } e_\alpha^T \ne \varnothing).$$

The minimum cardinality of such a set is denoted by

$$\tau_T^\alpha(\mathrm{NVH}(H)).$$

Definition 3.25 (Truth-threshold dual hypergraph)**.** For $\alpha \in [0, 1]$, the truth-threshold dual hypergraph

$$(H_\alpha^T)^*$$

is defined as the hypergraph whose vertex set is E_α^T, and whose hyperedges are indexed by vertices $v \in V_\alpha^T$:

$$E_v^* = \{e_\alpha^T \in E_\alpha^T : v \in e_\alpha^T\}.$$

Empty dual hyperedges are omitted.

Theorem 3.26 (Transversal-dual correspondence)
Let $\mathrm{NVH}(H)$ be a Neutrosophic Vertex HyperGraph and let $\alpha \in [0,1]$. A subset $S \subseteq V$ is an α-truth transversal of $\mathrm{NVH}(H)$ if and only if the family

$$\{E_v^* : v \in S\}$$

covers the vertex set of the truth-threshold dual hypergraph $(H_\alpha^T)^*$.

Proof
By definition, S is an α-truth transversal if and only if every nonempty threshold hyperedge $e_\alpha^T \in E_\alpha^T$ contains at least one vertex $v \in S$. This is equivalent to saying that, for every $e_\alpha^T \in E_\alpha^T$, there exists $v \in S$ such that $e_\alpha^T \in E_v^*$. Hence the dual hyperedges indexed by vertices of S cover all vertices of the dual hypergraph. This proves the assertion. □

Definition 3.27 (Truth-threshold proper coloring)**.** Let $\alpha \in [0,1]$. A map

$$c : V_\alpha^T \to \{1, 2, \ldots, k\}$$

is called a proper k-coloring of the truth-threshold hypergraph H_α^T if every hyperedge $e_\alpha^T \in E_\alpha^T$ with $|e_\alpha^T| \geq 2$ contains two vertices with different colors. Equivalently, no threshold hyperedge of cardinality at least two is monochromatic.
The least such k is denoted by

$$\chi_T^\alpha(\mathrm{NVH}(H)).$$

Proposition 3.28 (Monotonicity of threshold chromatic number)
If $0 \leq \alpha \leq \beta \leq 1$, then

$$\chi_T^\beta(\mathrm{NVH}(H)) \leq \chi_T^\alpha(\mathrm{NVH}(H)).$$

Proof
For $0 \leq \alpha \leq \beta \leq 1$, every β-truth support is contained in the corresponding α-truth support:

$$e_\beta^T \subseteq e_\alpha^T.$$

Thus H_β^T is obtained from H_α^T by deleting vertices from threshold hyperedges and possibly deleting empty hyperedges. Any proper coloring of H_α^T, restricted to V_β^T, remains proper for H_β^T. Therefore the minimum number of colors required at threshold β cannot exceed the minimum number required at threshold α. □

3.6. Neutrosophic Edge HyperGraph

A Neutrosophic Edge HyperGraph assigns to each hyperedge neutrosophic truth, indeterminacy, and falsity degrees, leaving vertices crisp, modeling uncertain multiway relations and connectivity across subsets.

Definition 3.29 (Neutrosophic Edge HyperGraph)**.** Let $H = (V, E)$ be a hypergraph. A *Neutrosophic Edge HyperGraph* on H is a tuple

$$\mathrm{NEH}(H) \;=\; \big(V, E;\; T_E, I_E, F_E\big),$$

where $T_E, I_E, F_E : E \to [0,1]$ assign to each hyperedge $e \in E$ its neutrosophic truth-, indeterminacy-, and falsity-degrees, with

$$0 \;\leq\; T_E(e) + I_E(e) + F_E(e) \;\leq\; 3 \qquad (\forall e \in E).$$

(Optionally, if vertex neutrosophic labels T_V, I_V, F_V are also given, one may impose the compatibility bounds $T_E(e) \leq \min_{v \in e} T_V(v)$, $I_E(e) \geq \max_{v \in e} I_V(v)$, $F_E(e) \geq \max_{v \in e} F_V(v)$, but these are not required in the basic definition.)

Example 3.30 (Project difficulty as an edge hypergraph)**.** Let the same (V, E) as above be given. Define edge grades as perceived *project feasibility*:

$$\langle T_E, I_E, F_E \rangle \big(\{\mathrm{A}, \mathrm{B}\}\big) = (0.8, 0.1, 0.1),$$

$$\langle T_E, I_E, F_E \rangle \big(\{\mathrm{B}, \mathrm{C}\}\big) = (0.6, 0.25, 0.15),$$

$$\langle T_E, I_E, F_E \rangle \big(\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}\big) = (0.7, 0.2, 0.1).$$

Then $\mathrm{NEH}(H)$ records, for each coalition, its truth/uncertainty/falsity degree as a single triple attached to the hyperedge.

Theorem 3.31 (The edge model generalizes hypergraphs and neutrosophic edge graphs)
Let $\mathrm{NEH}(H)$ be as in Definition 3.29.

(a) (*To hypergraphs*) Every hypergraph $H = (V, E)$ embeds into a Neutrosophic Edge HyperGraph by setting, for all $e \in E$,

$$T_E(e) = 1, \qquad I_E(e) = 0, \qquad F_E(e) = 0.$$

(b) (*To neutrosophic edge graphs*) If $H = (V, E)$ is 2-uniform, then giving edge labels $(T_E, I_E, F_E) : E \to [0, 1]^3$ yields exactly a *neutrosophic edge graph* on the simple graph (V, E).

Proof
(a) The assignments satisfy $T_E + I_E + F_E = 1 \leq 3$ for each e. Discarding the neutrosophic information recovers the original hypergraph (V, E), showing a faithful embedding.
(b) When $|e| = 2$ for all $e \in E$, the pair (V, E) is a simple undirected graph. Triples $(T_E(e), I_E(e), F_E(e))$ attached to each $e \in E$ are precisely the edge grades of a neutrosophic edge graph, with the same admissibility condition $0 \leq T_E + I_E + F_E \leq 3$. □

3.7. Neutrosophic Vertex SuperHyperGraph

A Neutrosophic Vertex SuperHyperGraph equips each n-level supervertex with neutrosophic truth, indeterminacy, and falsity degrees, while superedges connect supervertices across layers, capturing hierarchical uncertainty patterns.

Definition 3.32 (Neutrosophic Vertex SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *Neutrosophic Vertex SuperHyperGraph* on (V, E) is a tuple

$$\mathrm{NVSHG}^{(n)} \;=\; \big(V, E;\; T_V, I_V, F_V;\; T_{inc}, I_{inc}, F_{inc}\big),$$

where

- $T_V, I_V, F_V : V \to [0, 1]$ assign to each supervertex $v \in V$ its truth-, indeterminacy-, and falsity-memberships, with
$$0 \;\leq\; T_V(v) + I_V(v) + F_V(v) \;\leq\; 3 \qquad (\forall v \in V).$$
- $T_{inc}, I_{inc}, F_{inc} : E \times V \to [0, 1]$ are neutrosophic *incidence* maps (membership of v in e) such that, for all $e \in E$ and $v \in V$,
$$v \notin e \implies T_{inc}(e, v) = I_{inc}(e, v) = F_{inc}(e, v) = 0,$$
and the (componentwise) vertex–domination constraints hold:
$$T_{inc}(e, v) \leq T_V(v), \quad I_{inc}(e, v) \leq I_V(v), \quad F_{inc}(e, v) \leq F_V(v).$$

A canonical choice is $T_{inc}(e, v) = T_V(v)\mathbf{1}_{\{v \in e\}}$ and similarly for I and F.

Example 3.33 (Neutrosophic Vertex SuperHyperGraph (real-life, emergency response, $n = 2$)). *Scenario.* Three agencies coordinate emergency response: EMS (E), Fire (F), and Police (P). Take the base set $V_0 = \{E, F, P\}$ and form $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Choose the supervertex set

$$V = \Big\{v_E = \{\{E\}\},\ v_F = \{\{F\}\},\ v_{EF} = \big\{\{E\}, \{F\}\big\}\Big\} \subseteq \mathcal{P}^2(V_0),$$

and superedges (multi-team "operations")

$$e_{\text{med}} = \{v_E, v_{EF}\}, \qquad e_{\text{fire}} = \{v_F, v_{EF}\}, \qquad E = \{e_{\text{med}}, e_{\text{fire}}\} \subseteq \mathcal{P}^*(V).$$

Vertex neutrosophic grades (T: suitability, I: uncertainty, F: unsuitability) are

$$\begin{aligned}
\langle T_V(v_E), I_V(v_E), F_V(v_E)\rangle &= (0.75,\ 0.15,\ 0.10),\\
\langle T_V(v_F), I_V(v_F), F_V(v_F)\rangle &= (0.65,\ 0.20,\ 0.15),\\
\langle T_V(v_{EF}), I_V(v_{EF}), F_V(v_{EF})\rangle &= (0.90,\ 0.06,\ 0.04),
\end{aligned}$$

(all sums ≤ 3). Incidence triples for each operation (only listed pairs are nonzero; all others vanish by support) are

$$\begin{aligned}
\langle T_{inc}(e_{\text{med}}, v_E), I_{inc}(e_{\text{med}}, v_E), F_{inc}(e_{\text{med}}, v_E)\rangle &= (0.70,\ 0.15,\ 0.10),\\
\langle T_{inc}(e_{\text{med}}, v_{EF}), I_{inc}(e_{\text{med}}, v_{EF}), F_{inc}(e_{\text{med}}, v_{EF})\rangle &= (0.85,\ 0.05,\ 0.04),\\
\langle T_{inc}(e_{\text{fire}}, v_F), I_{inc}(e_{\text{fire}}, v_F), F_{inc}(e_{\text{fire}}, v_F)\rangle &= (0.60,\ 0.20,\ 0.15),\\
\langle T_{inc}(e_{\text{fire}}, v_{EF}), I_{inc}(e_{\text{fire}}, v_{EF}), F_{inc}(e_{\text{fire}}, v_{EF})\rangle &= (0.80,\ 0.06,\ 0.04).
\end{aligned}$$

These satisfy the vertex–domination constraints $T_{inc}(e, v) \leq T_V(v)$, $I_{inc}(e, v) \leq I_V(v)$, $F_{inc}(e, v) \leq F_V(v)$ componentwise. Interpretation: the joint team v_{EF} has high suitability for both operations (large T_{inc}, small I_{inc}, F_{inc}), while the single-agency teams contribute with their own (more uncertain) profiles.

Theorem 3.34 (Vertex model generalizes both NV-hypergraphs and superhypergraphs)
Let $\mathrm{NVSHG}^{(n)}$ be as in Definition 3.32.

(a) (*Reduction to Neutrosophic Vertex HyperGraph*) For $n = 1$, any $\mathrm{NVSHG}^{(1)}$ is exactly a *Neutrosophic Vertex HyperGraph* on the hypergraph (V, E): supervertices are ordinary vertices and superedges are ordinary hyperedges. Conversely, any Neutrosophic Vertex HyperGraph arises as an $\mathrm{NVSHG}^{(1)}$.

(b) (*Reduction to crisp n-SuperHyperGraph*) Every n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is recovered as a crisp special case of $\mathrm{NVSHG}^{(n)}$ by setting, for all $v \in V$, $e \in E$,

$$T_V(v) = 1,\ I_V(v) = 0,\ F_V(v) = 0, \qquad T_{inc}(e, v) = \mathbf{1}_{\{v \in e\}},\ I_{inc}(e, v) = 0,\ F_{inc}(e, v) = 0.$$

Forgetting the neutrosophic labels recovers (V, E).

Proof
(a) When $n = 1$ we have $V \subseteq \mathcal{P}(V_0)$ and $E \subseteq \mathcal{P}^*(V)$, i.e. (V, E) is an ordinary hypergraph. The data (T_V, I_V, F_V) (on vertices) and $(T_{inc}, I_{inc}, F_{inc})$ (on incidences) together with the support and domination constraints are precisely the neutrosophic vertex hypergraph structure; hence the notions coincide.
(b) The assigned triples satisfy $T_V + I_V + F_V = 1 \leq 3$ and, for each (e, v), $v \notin e \Rightarrow (T_{inc}, I_{inc}, F_{inc}) = (0, 0, 0)$, while $v \in e$ gives $T_{inc}(e, v) = 1 \leq T_V(v) = 1$ and $I_{inc} = F_{inc} = 0 \leq I_V(v), F_V(v)$. Thus all constraints in Definition 3.32 hold. Discarding the labels returns exactly the underlying $\mathrm{SHG}^{(n)}$, proving a faithful embedding. □

Theorem 3.35 (Stability under neutrosophic vertex-induced sub-SuperHyperGraphs)
Let

$$\mathrm{NVSHG}^{(n)} = \big(V, E;\ T_V, I_V, F_V;\ T_{inc}, I_{inc}, F_{inc}\big)$$

be a Neutrosophic Vertex SuperHyperGraph on the n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$.
Let $V' \subseteq V$ be nonempty and define

$$E' := \{ e \in E : e \subseteq V' \}, \qquad \mathrm{SHG}'^{(n)} := (V', E').$$

Define restricted neutrosophic maps by

$$T'_V := T_V\big|_{V'}, \quad I'_V := I_V\big|_{V'}, \quad F'_V := F_V\big|_{V'},$$

and

$$T'_{inc} := T_{inc}\big|_{E' \times V'}, \quad I'_{inc} := I_{inc}\big|_{E' \times V'}, \quad F'_{inc} := F_{inc}\big|_{E' \times V'}.$$

Then

$$\mathrm{NVSHG}'^{(n)} := \big(V', E';\ T'_V, I'_V, F'_V;\ T'_{inc}, I'_{inc}, F'_{inc} \big)$$

is again a Neutrosophic Vertex SuperHyperGraph on $\mathrm{SHG}'^{(n)}$.

Proof
First, (V', E') is an n-SuperHyperGraph because $E' \subseteq E$ and each $e \in E'$ is a nonempty subset of V' by definition; thus $\mathrm{SHG}'^{(n)}$ is well-defined.
We verify the axioms of Definition 3.32.
(1) Vertex neutrosophic bounds. For any $v \in V'$ we have

$$T'_V(v) = T_V(v), \quad I'_V(v) = I_V(v), \quad F'_V(v) = F_V(v).$$

Since $\mathrm{NVSHG}^{(n)}$ is neutrosophic,

$$0 \le T_V(v) + I_V(v) + F_V(v) \le 3 \qquad (\forall v \in V).$$

Hence, for all $v \in V'$,

$$0 \le T'_V(v) + I'_V(v) + F'_V(v) = T_V(v) + I_V(v) + F_V(v) \le 3.$$

Thus the vertex-level neutrosophic constraint is preserved on V'.
(2) Support of incidence maps. Take any $e \in E'$ and $v \in V'$. If $v \notin e$ in $\mathrm{SHG}'^{(n)}$, then also $v \notin e$ in the original (V, E). By Definition 3.32 for $\mathrm{NVSHG}^{(n)}$,

$$v \notin e \implies T_{inc}(e, v) = I_{inc}(e, v) = F_{inc}(e, v) = 0.$$

By restriction,

$$T'_{inc}(e, v) = T_{inc}(e, v) = 0, \quad I'_{inc}(e, v) = I_{inc}(e, v) = 0, \quad F'_{inc}(e, v) = F_{inc}(e, v) = 0.$$

Thus the support condition is satisfied in $\mathrm{NVSHG}'^{(n)}$.
(3) Vertex–domination constraints. Let $e \in E'$ and $v \in e \subseteq V'$. In $\mathrm{NVSHG}^{(n)}$ we have

$$T_{inc}(e, v) \le T_V(v), \quad I_{inc}(e, v) \le I_V(v), \quad F_{inc}(e, v) \le F_V(v).$$

Since

$$T'_{inc}(e, v) = T_{inc}(e, v), \quad I'_{inc}(e, v) = I_{inc}(e, v), \quad F'_{inc}(e, v) = F_{inc}(e, v),$$

and

$$T'_V(v) = T_V(v), \quad I'_V(v) = I_V(v), \quad F'_V(v) = F_V(v),$$

we obtain the same inequalities in the restricted structure:

$$T'_{inc}(e, v) \le T'_V(v), \quad I'_{inc}(e, v) \le I'_V(v), \quad F'_{inc}(e, v) \le F'_V(v).$$

Hence all conditions of Definition 3.32 hold for $\mathrm{NVSHG}'^{(n)}$, so it is a Neutrosophic Vertex SuperHyperGraph. □

Theorem 3.36 (Convex combination of Neutrosophic Vertex SuperHyperGraphs)
Let $\mathrm{SHG}^{(n)} = (V, E)$ be a fixed n-SuperHyperGraph. Suppose

$$\mathrm{NVSHG}_1^{(n)} = \big(V, E;\ T_V^{(1)}, I_V^{(1)}, F_V^{(1)};\ T_{inc}^{(1)}, I_{inc}^{(1)}, F_{inc}^{(1)}\big),$$

$$\mathrm{NVSHG}_2^{(n)} = \big(V, E;\ T_V^{(2)}, I_V^{(2)}, F_V^{(2)};\ T_{inc}^{(2)}, I_{inc}^{(2)}, F_{inc}^{(2)}\big)$$

are two Neutrosophic Vertex SuperHyperGraphs on the same (V, E).
Fix $\lambda \in [0, 1]$ and define new maps for every $v \in V$ and $(e, v) \in E \times V$ by the componentwise convex combinations

$$T_V^{(\lambda)}(v) := \lambda\, T_V^{(1)}(v) + (1-\lambda)\, T_V^{(2)}(v),$$

$$I_V^{(\lambda)}(v) := \lambda\, I_V^{(1)}(v) + (1-\lambda)\, I_V^{(2)}(v),$$

$$F_V^{(\lambda)}(v) := \lambda\, F_V^{(1)}(v) + (1-\lambda)\, F_V^{(2)}(v),$$

and

$$T_{inc}^{(\lambda)}(e, v) := \lambda\, T_{inc}^{(1)}(e, v) + (1-\lambda)\, T_{inc}^{(2)}(e, v),$$

$$I_{inc}^{(\lambda)}(e, v) := \lambda\, I_{inc}^{(1)}(e, v) + (1-\lambda)\, I_{inc}^{(2)}(e, v),$$

$$F_{inc}^{(\lambda)}(e, v) := \lambda\, F_{inc}^{(1)}(e, v) + (1-\lambda)\, F_{inc}^{(2)}(e, v).$$

Then

$$\mathrm{NVSHG}_\lambda^{(n)} := \big(V, E;\ T_V^{(\lambda)}, I_V^{(\lambda)}, F_V^{(\lambda)}; T_{inc}^{(\lambda)}, I_{inc}^{(\lambda)}, F_{inc}^{(\lambda)}\big)$$

is a Neutrosophic Vertex SuperHyperGraph on $\mathrm{SHG}^{(n)}$.

Proof
We check each axiom of Definition 3.32.
(1) Vertex neutrosophic bounds. For any $v \in V$, since $\mathrm{NVSHG}_k^{(n)}$ $(k = 1, 2)$ are neutrosophic,

$$0 \ \le\ T_V^{(k)}(v) + I_V^{(k)}(v) + F_V^{(k)}(v) \ \le\ 3.$$

Using $0 \le \lambda \le 1$, we have

$$\begin{aligned} T_V^{(\lambda)}(v) + I_V^{(\lambda)}(v) + F_V^{(\lambda)}(v) = \lambda\big(&T_V^{(1)}(v) + I_V^{(1)}(v) + F_V^{(1)}(v)\big) \\ &+ (1-\lambda)\big(T_V^{(2)}(v) + I_V^{(2)}(v) + F_V^{(2)}(v)\big). \end{aligned}$$

Because each bracketed sum lies in $[0, 3]$,

$$0 \ \le\ T_V^{(\lambda)}(v) + I_V^{(\lambda)}(v) + F_V^{(\lambda)}(v) \ \le\ \lambda \cdot 3 + (1-\lambda) \cdot 3 \ =\ 3.$$

Thus the vertex-level neutrosophic constraint holds for all $v \in V$.
(2) Support of incidence maps. Let $e \in E$ and $v \in V$ with $v \notin e$. For $k = 1, 2$, $\mathrm{NVSHG}_k^{(n)}$ satisfies

$$v \notin e \implies T_{inc}^{(k)}(e, v) = I_{inc}^{(k)}(e, v) = F_{inc}^{(k)}(e, v) = 0.$$

Therefore

$$\begin{aligned} &T_{inc}^{(\lambda)}(e, v) = \lambda\, T_{inc}^{(1)}(e, v) + (1-\lambda)\, T_{inc}^{(2)}(e, v) = \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0, \\ &I_{inc}^{(\lambda)}(e, v) = \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0, \\ &F_{inc}^{(\lambda)}(e, v) = \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0. \end{aligned}$$

Hence the support condition $v \notin e \Rightarrow$ incidence triple $= 0$ is preserved.

(3) Vertex–domination constraints. Fix $e \in E$ and $v \in V$ (the case $v \notin e$ is already handled). For $k = 1, 2$ we have

$$T^{(k)}_{inc}(e, v) \le T^{(k)}_V(v), \quad I^{(k)}_{inc}(e, v) \le I^{(k)}_V(v), \quad F^{(k)}_{inc}(e, v) \le F^{(k)}_V(v),$$

with all inequalities in $[0, 1]$.
Using the fact that λ and $1 - \lambda$ are nonnegative and sum to 1, we obtain

$$\begin{aligned}
T^{(\lambda)}_{inc}(e, v) &= \lambda\, T^{(1)}_{inc}(e, v) + (1 - \lambda)\, T^{(2)}_{inc}(e, v) \\
&\le \lambda\, T^{(1)}_V(v) + (1 - \lambda)\, T^{(2)}_V(v) = T^{(\lambda)}_V(v), \\
I^{(\lambda)}_{inc}(e, v) &= \lambda\, I^{(1)}_{inc}(e, v) + (1 - \lambda)\, I^{(2)}_{inc}(e, v) \\
&\le \lambda\, I^{(1)}_V(v) + (1 - \lambda)\, I^{(2)}_V(v) = I^{(\lambda)}_V(v), \\
F^{(\lambda)}_{inc}(e, v) &= \lambda\, F^{(1)}_{inc}(e, v) + (1 - \lambda)\, F^{(2)}_{inc}(e, v) \\
&\le \lambda\, F^{(1)}_V(v) + (1 - \lambda)\, F^{(2)}_V(v) = F^{(\lambda)}_V(v).
\end{aligned}$$

Thus, for all (e, v),

$$T^{(\lambda)}_{inc}(e, v) \le T^{(\lambda)}_V(v), \quad I^{(\lambda)}_{inc}(e, v) \le I^{(\lambda)}_V(v), \quad F^{(\lambda)}_{inc}(e, v) \le F^{(\lambda)}_V(v).$$

Therefore all axioms of Definition 3.32 are satisfied by $\mathrm{NVSHG}^{(n)}_\lambda$, so it is a Neutrosophic Vertex SuperHyperGraph. This shows that the class of NVSHGs on a fixed $\mathrm{SHG}^{(n)}$ is closed under convex combinations of their neutrosophic vertex and incidence degrees. □

3.8. Neutrosophic Edge SuperHyperGraph

A Neutrosophic Edge SuperHyperGraph assigns neutrosophic truth, indeterminacy, and falsity degrees to n-level superedges, with crisp supervertices, capturing uncertain multilayer connectivity across powerset hierarchies effectively.

Definition 3.37 (Neutrosophic Edge SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *Neutrosophic Edge SuperHyperGraph* on (V, E) is a tuple

$$\mathrm{NESHG}^{(n)} \;=\; \big(V, E;\; T_E, I_E, F_E\big),$$

where $T_E, I_E, F_E : E \to [0, 1]$ attach to each superedge $e \in E$ its neutrosophic truth-, indeterminacy-, and falsity-memberships, with

$$0 \;\le\; T_E(e) + I_E(e) + F_E(e) \;\le\; 3 \qquad (\forall e \in E).$$

(If vertex labels T_V, I_V, F_V are also given, one may optionally impose compatibility bounds $T_E(e) \le \min_{v \in e} T_V(v)$, $I_E(e) \ge \max_{v \in e} I_V(v)$, $F_E(e) \ge \max_{v \in e} F_V(v)$; these are not required in the basic definition.)

Example 3.38 (Neutrosophic Edge SuperHyperGraph (real-life, urban logistics, $n = 2$))**.** *Scenario.* A city coordinates freight transfers across hub-of-hubs. Let $V_0 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ denote physical hubs. In $\mathcal{P}^2(V_0)$ take the supervertices

$$V = \Big\{ v_{\mathrm{A}} = \{\{\mathrm{A}\}\},\ v_{\mathrm{AB}} = \big\{\{\mathrm{A}\}, \{\mathrm{B}\}\big\},\ v_{\mathrm{BC}} = \big\{\{\mathrm{B}\}, \{\mathrm{C}\}\big\} \Big\} \subseteq \mathcal{P}^2(V_0).$$

Define three superedges (multi-hub corridors)

$$e_1 = \{v_{\mathrm{A}}, v_{\mathrm{AB}}\}, \qquad e_2 = \{v_{\mathrm{AB}}, v_{\mathrm{BC}}\}, \qquad e_3 = \{v_{\mathrm{A}}, v_{\mathrm{BC}}\}, \qquad E = \{e_1, e_2, e_3\}.$$

Assign neutrosophic *edge* grades (T: on-time reliability, I: operational uncertainty, F: disruption risk):

$$\begin{aligned}
\langle T_E(e_1), I_E(e_1), F_E(e_1)\rangle &= (0.88,\ 0.08,\ 0.04), \\
\langle T_E(e_2), I_E(e_2), F_E(e_2)\rangle &= (0.62,\ 0.25,\ 0.13), \\
\langle T_E(e_3), I_E(e_3), F_E(e_3)\rangle &= (0.55,\ 0.30,\ 0.15),
\end{aligned}$$

each summing to ≤ 3. Here vertices need not carry neutrosophic labels—the uncertainty is encoded at the corridor (superedge) level. Interpretation: the corridor e_1 is highly reliable (large T_E), e_2 is moderately reliable with congestion-driven uncertainty (larger I_E), and e_3 faces higher disruption risk (larger F_E), e.g., due to roadworks or weather.

Theorem 3.39 (Edge model generalizes both NE-hypergraphs and superhypergraphs)
Let $\mathrm{NESHG}^{(n)}$ be as in Definition 3.37.

(a) (*Reduction to Neutrosophic Edge HyperGraph*) For $n = 1$, any $\mathrm{NESHG}^{(1)}$ is exactly a *Neutrosophic Edge HyperGraph* on the hypergraph (V, E): superedges become ordinary hyperedges with neutrosophic edge labels (T_E, I_E, F_E). Conversely, any Neutrosophic Edge HyperGraph arises as an $\mathrm{NESHG}^{(1)}$.

(b) (*Reduction to crisp n-SuperHyperGraph*) Every n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is recovered as a crisp special case of $\mathrm{NESHG}^{(n)}$ by setting, for all $e \in E$,

$$T_E(e) = 1, \qquad I_E(e) = 0, \qquad F_E(e) = 0.$$

Forgetting the neutrosophic labels recovers (V, E).

Proof
(a) For $n = 1$ we again have an ordinary hypergraph (V, E). Attaching to each hyperedge $e \in E$ the triple $(T_E(e), I_E(e), F_E(e)) \in [0, 1]^3$ with $T_E + I_E + F_E \leq 3$ is precisely the definition of a neutrosophic edge hypergraph; hence the notions coincide.
(b) The constant assignments satisfy $T_E + I_E + F_E = 1 \leq 3$ for every $e \in E$, and they do not alter the incidence structure. Discarding the labels recovers the original $\mathrm{SHG}^{(n)}$, establishing a faithful embedding. □

Theorem 3.40 (Edge-induced Neutrosophic sub-SuperHyperGraph)
Let

$$\mathrm{NESHG}^{(n)} \;=\; \bigl(V, E;\; T_E, I_E, F_E\bigr)$$

be a Neutrosophic Edge SuperHyperGraph on the n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ as in Definition 3.37.
Let $E' \subseteq E$ be nonempty and define the induced vertex set

$$V' \;:=\; \bigl\{\, v \in V : \; \exists e \in E' \text{ with } v \in e \,\bigr\}.$$

Set

$$\mathrm{SHG}'^{(n)} \;:=\; (V', E').$$

Define restricted neutrosophic edge maps

$$T'_E := T_E\big|_{E'}, \qquad I'_E := I_E\big|_{E'}, \qquad F'_E := F_E\big|_{E'}.$$

Then

$$\mathrm{NESHG}'^{(n)} \;:=\; \bigl(V', E';\; T'_E, I'_E, F'_E\bigr)$$

is again a Neutrosophic Edge SuperHyperGraph on $\mathrm{SHG}'^{(n)}$.

Proof
First we check that $\mathrm{SHG}'^{(n)} = (V', E')$ is an n-SuperHyperGraph. By construction $E' \subseteq E$, and each $e \in E'$ is a nonempty subset of V because (V, E) is an n-SuperHyperGraph. Moreover, V' is exactly the union of all superedges in E', hence each $e \in E'$ satisfies

$$e \;\subseteq\; V'.$$

Therefore $E' \subseteq \mathcal{P}^*(V')$ and $\mathrm{SHG}'^{(n)}$ is a well-defined n-SuperHyperGraph.
Next we verify the axioms of Definition 3.37 for the restricted structure $\mathrm{NESHG}'^{(n)}$.

Take any $e \in E'$. By definition of the restrictions,

$$T'_E(e) = T_E(e), \qquad I'_E(e) = I_E(e), \qquad F'_E(e) = F_E(e).$$

Since $\mathrm{NESHG}^{(n)}$ is neutrosophic edge-labeled, each $e \in E$ satisfies

$$0 \le T_E(e) + I_E(e) + F_E(e) \le 3.$$

In particular this holds for every $e \in E'$, so

$$0 \le T'_E(e) + I'_E(e) + F'_E(e) = T_E(e) + I_E(e) + F_E(e) \le 3,$$

for all $e \in E'$.
Furthermore, each of T'_E, I'_E, F'_E takes values in $[0,1]$, because they coincide with T_E, I_E, F_E on E', and by Definition 3.37 we know $T_E, I_E, F_E : E \to [0,1]$.
Thus $(V', E'; T'_E, I'_E, F'_E)$ satisfies exactly the same pointwise constraints as in Definition 3.37, now over E'. Therefore $\mathrm{NESHG}'^{(n)}$ is a Neutrosophic Edge SuperHyperGraph on $\mathrm{SHG}'^{(n)}$. □

Theorem 3.41 (Convex combination of Neutrosophic Edge SuperHyperGraphs)
Let $\mathrm{SHG}^{(n)} = (V, E)$ be a fixed n-SuperHyperGraph and let

$$\mathrm{NESHG}_1^{(n)} = \big(V, E;\ T_E^{(1)}, I_E^{(1)}, F_E^{(1)}\big), \qquad \mathrm{NESHG}_2^{(n)} = \big(V, E;\ T_E^{(2)}, I_E^{(2)}, F_E^{(2)}\big)$$

be two Neutrosophic Edge SuperHyperGraphs on (V, E).
Fix $\lambda \in [0,1]$ and define, for each $e \in E$,

$$T_E^{(\lambda)}(e) := \lambda\, T_E^{(1)}(e) + (1-\lambda)\, T_E^{(2)}(e),$$

$$I_E^{(\lambda)}(e) := \lambda\, I_E^{(1)}(e) + (1-\lambda)\, I_E^{(2)}(e),$$

$$F_E^{(\lambda)}(e) := \lambda\, F_E^{(1)}(e) + (1-\lambda)\, F_E^{(2)}(e).$$

Then

$$\mathrm{NESHG}_\lambda^{(n)} := \big(V, E;\ T_E^{(\lambda)}, I_E^{(\lambda)}, F_E^{(\lambda)}\big)$$

is a Neutrosophic Edge SuperHyperGraph on $\mathrm{SHG}^{(n)}$.

Proof
We must show that for each $e \in E$,

$$T_E^{(\lambda)}(e),\ I_E^{(\lambda)}(e),\ F_E^{(\lambda)}(e) \in [0,1] \quad \text{and} \quad 0 \le T_E^{(\lambda)}(e) + I_E^{(\lambda)}(e) + F_E^{(\lambda)}(e) \le 3.$$

(1) Componentwise bounds in $[0,1]$. Fix $e \in E$ and consider, for example, $T_E^{(\lambda)}(e)$. Since $\mathrm{NESHG}_k^{(n)}$ $(k = 1, 2)$ are neutrosophic, we have

$$0 \le T_E^{(k)}(e) \le 1 \qquad (k = 1, 2).$$

Because $0 \le \lambda \le 1$ and $0 \le 1-\lambda \le 1$, we obtain

$$T_E^{(\lambda)}(e) = \lambda\, T_E^{(1)}(e) + (1-\lambda)\, T_E^{(2)}(e) \ge 0 \cdot T_E^{(1)}(e) + 0 \cdot T_E^{(2)}(e) = 0,$$

and

$$T_E^{(\lambda)}(e) = \lambda\, T_E^{(1)}(e) + (1-\lambda)\, T_E^{(2)}(e) \le \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.$$

Hence $T_E^{(\lambda)}(e) \in [0,1]$. The same argument, applied to $I_E^{(k)}(e)$ and $F_E^{(k)}(e)$, gives

$$I_E^{(\lambda)}(e) \in [0,1], \qquad F_E^{(\lambda)}(e) \in [0,1].$$

(2) Neutrosophic sum constraint. Again fix $e \in E$. Since $\mathrm{NESHG}_k^{(n)}$ are neutrosophic,

$$0 \le T_E^{(k)}(e) + I_E^{(k)}(e) + F_E^{(k)}(e) \le 3 \qquad (k = 1, 2).$$

Compute the sum for the convex combination:

$$\begin{aligned} & T_E^{(\lambda)}(e) + I_E^{(\lambda)}(e) + F_E^{(\lambda)}(e) \\ &= \lambda\, T_E^{(1)}(e) + (1-\lambda)\, T_E^{(2)}(e) + \lambda\, I_E^{(1)}(e) + (1-\lambda)\, I_E^{(2)}(e) \\ &\qquad + \lambda\, F_E^{(1)}(e) + (1-\lambda)\, F_E^{(2)}(e) \\ &= \lambda\big(T_E^{(1)}(e) + I_E^{(1)}(e) + F_E^{(1)}(e)\big) + (1-\lambda)\big(T_E^{(2)}(e) + I_E^{(2)}(e) + F_E^{(2)}(e)\big). \end{aligned}$$

Because each bracketed term lies in $[0, 3]$, we obtain the lower bound

$$T_E^{(\lambda)}(e) + I_E^{(\lambda)}(e) + F_E^{(\lambda)}(e) \ge \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0,$$

and the upper bound

$$T_E^{(\lambda)}(e) + I_E^{(\lambda)}(e) + F_E^{(\lambda)}(e) \le \lambda \cdot 3 + (1-\lambda) \cdot 3 = 3.$$

Thus, for every $e \in E$,

$$0 \le T_E^{(\lambda)}(e) + I_E^{(\lambda)}(e) + F_E^{(\lambda)}(e) \le 3,$$

and each component is in $[0, 1]$. Consequently, $\mathrm{NESHG}_\lambda^{(n)}$ satisfies all conditions of Definition 3.37, so it is a Neutrosophic Edge SuperHyperGraph on $\mathrm{SHG}^{(n)}$. □

3.9. SuperHypergraph-Level Threshold Operations

Let

$$\mathrm{NVSHG}^{(n)} = (V, E; T_V, I_V, F_V; T_{inc}, I_{inc}, F_{inc})$$

be a Neutrosophic Vertex SuperHyperGraph.

Definition 3.42 (Truth-threshold SuperHyperGraph core)**.** For $\alpha \in [0, 1]$, define

$$V_\alpha^T = \{v \in V : T_V(v) \ge \alpha\}$$

and, for each superedge $e \in E$,

$$e_\alpha^T = \{v \in e : T_{inc}(e, v) \ge \alpha\}.$$

The truth-threshold SuperHyperGraph core is

$$\mathrm{SHG}_\alpha^T = (V_\alpha^T, E_\alpha^T), \qquad E_\alpha^T = \{e_\alpha^T : e \in E,\ e_\alpha^T \ne \varnothing\}.$$

Definition 3.43 (SuperHyperGraph threshold transversal)**.** A subset $S \subseteq V$ is called an α-truth SuperHyperGraph transversal if

$$S \cap e_\alpha^T \ne \varnothing$$

for every nonempty threshold superedge $e_\alpha^T \in E_\alpha^T$. The minimum cardinality of such a set is denoted by

$$\tau_T^\alpha(\mathrm{NVSHG}^{(n)}).$$

Proposition 3.44 (Reduction to the hypergraph case)
If $n = 1$, then the truth-threshold SuperHyperGraph core and the SuperHyperGraph threshold transversal reduce to the corresponding truth-threshold hypergraph core and neutrosophic truth transversal.

Proof
When $n = 1$, the supervertices are ordinary subsets generated at the first powerset level, and the superedges are ordinary hyperedges on this vertex family. Hence the definitions of V_α^T, e_α^T, and τ_T^α coincide exactly with the corresponding hypergraph-level definitions. □

For convenience, Table 5 summarizes the relationships among the main structures considered in this paper, including their underlying carriers and their roles as generalizations or special cases.

Table 5. Relationships among the main structures considered in this paper

Structure	Underlying carrier	Main information	Relationship to other structures
Intuitionistic Fuzzy Vertex Graph	Graph	Vertex membership and nonmembership degrees	Vertex-based intuitionistic fuzzy extension of a graph
Intuitionistic Fuzzy Edge Graph	Graph	Edge membership and nonmembership degrees	Edge-based intuitionistic fuzzy extension of a graph
Neutrosophic Vertex Graph	Graph	Vertex truth, indeterminacy, and falsity degrees	Vertex-based neutrosophic extension of a graph
Plithogenic Vertex Graph	Graph	Vertex appurtenance, contradiction, and signed attribute effects	Generalizes the Neutrosophic Vertex Graph and extends existing plithogenic graph models
Plithogenic Edge Graph	Graph	Edge appurtenance, contradiction, and signed attribute effects	Edge-based plithogenic extension of a graph
Neutrosophic Vertex HyperGraph	HyperGraph	Vertex and incidence neutrosophic information	Neutrosophic extension of a hypergraph; contains the crisp hypergraph as a special case
Neutrosophic Edge HyperGraph	HyperGraph	Edge-based neutrosophic information	Edge-based neutrosophic extension of a hypergraph
Neutrosophic Vertex SuperHyperGraph	(n)-SuperHyperGraph	Vertex and incidence neutrosophic information on higher-order carriers	Higher-order extension of the Neutrosophic Vertex HyperGraph
Neutrosophic Edge SuperHyperGraph	(n)-SuperHyperGraph	Edge-based neutrosophic information on higher-order carriers	Higher-order extension of the Neutrosophic Edge HyperGraph

4. Conclusion

This paper investigated Plithogenic Graphs, Plithogenic Vertex Graphs, and Plithogenic Edge Graphs. It also examined Intuitionistic Fuzzy Vertex Graphs and Edge Graphs, Neutrosophic Vertex HyperGraphs and Edge HyperGraphs, and Neutrosophic Vertex SuperHyperGraphs and Edge SuperHyperGraphs.

The study clarified how vertex-based and edge-based uncertainty information can be represented in graph, hypergraph, and SuperHyperGraph settings. In particular, the proposed formulations show how membership, nonmembership, indeterminacy, falsity, appurtenance, and contradiction-related information can be assigned to vertices, edges, hyperedges, and higher-order supervertices. These constructions provide a unified viewpoint for modeling pairwise, multiway, and hierarchical relations under fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic uncertainty.

Moreover, the paper emphasized the distinction between vertex-induced and edge-induced models. Vertex-based models are useful when uncertainty is primarily attached to individual objects, whereas edge-based models are useful when uncertainty is primarily attached to relations or interactions. Hypergraph and SuperHyperGraph versions further allow the representation of group interactions, higher-order dependencies, and hierarchical structures that cannot be fully described by ordinary graphs alone.

In future work, we plan to explore extensions based on Directed Graphs [74], Bidirected Graphs [75], and Directed HyperGraphs [76, 77], aiming to further generalize the proposed framework. These extensions are non-trivial because directionality changes the nature of relational uncertainty. For example, a directed plithogenic edge from x

to y may have a different appurtenance or contradiction value from the reverse edge from (y) to (x). Hence, directed plithogenic models may require asymmetric contradiction functions, ordered endpoint constraints, and separate in-flow and out-flow uncertainty measures. Possible future directions include the study of directed plithogenic vertex and edge graphs, neutrosophic directed hypergraphs, threshold-based connectivity, transversals, coloring, dual structures, and algorithmic applications to decision-making, network analysis, and complex systems with uncertain or contradictory information.
As this paper is primarily theoretical, no computational experiment or empirical validation is included at this stage. In future work, we plan to develop decision-making algorithms, simulations, and case studies based on the proposed graph, hypergraph, and SuperHyperGraph models. We also aim to compare these structures with standard Neutrosophic Graphs and related models in order to evaluate their practical advantages in handling incomplete, inconsistent, and ambiguous information.

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Data Availability

This research is purely theoretical and does not involve the collection, generation, or analysis of empirical data. Therefore, no datasets are associated with this study. Future researchers are encouraged to conduct empirical investigations to further develop, apply, and validate the concepts introduced in this paper.

On the Use of AI

AI-assisted tools, including LLM-based software, were used solely for editorial support (e.g., grammar and readability checks), format verification, and limited help in identifying potential LaTeX compilation issues. These tools were not used to generate scientific results, proofs, data, or conclusions, and their use did not affect the integrity of the research.

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