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On Boundedness, Continuity, and Hyers–Ulam Stability of Interval-Valued Stochastic Processes in a Probabilistic Framework

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Abstract In this paper, we investigate interval-valued stochastic processes within the framework of pseudo-order relations and analyze their fundamental analytical properties. We give a probabilistically meaningful definition of boundedness for interval-valued processes and use it to establish a single, consolidated local-to-global boundedness theorem, together with a corrected treatment of continuity in probability and of Hyers–Ulam stability for interval-valued stochastic mappings. Every construction that relies on interval subtraction, interval absolute value, or the ordering of negative intervals is given an explicit. The proposed approach provides a consistent structure for handling uncertainty arising from both randomness and interval-valued data, and we illustrate it with a toy financial example in which a return is simultaneously random and only known up to a measurement interval. We show precisely how the developed results relate to, and in the component-wise pseudo-order case reduce to, existing real-valued results, and we discuss open problems for the genuinely set-valued (inclusion-order) setting.

Keywords Interval-valued stochastic processes; pseudo-order relations; boundedness; continuity; Hyers–Ulam stability; random uncertainty; probabilistic analysis

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1. Introduction

The analysis of stochastic processes plays a fundamental role in probability theory and its wide range of applications in science and engineering. In many real-world situations, uncertainty arises not only from randomness but also from imprecision in data, measurement errors, or incomplete information. To address such dual uncertainty, interval-valued stochastic processes have emerged as an effective mathematical framework, where the outcomes are represented by intervals instead of precise real values [1, 6, 7, 28]. This approach allows simultaneous modeling of randomness and vagueness, making it particularly suitable for complex systems encountered in finance, engineering, and biological sciences [2, 4, 31, 34].

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To make the dual-uncertainty motivation concrete, consider a daily asset return that is (a) random, because it depends on market conditions, and (b) only observable up to a bid–ask style measurement interval, because the reported closing quote is itself a range rather than a single tick. A purely stochastic model discards the measurement interval, while a purely interval (deterministic) model discards the randomness. Neither a fuzzy stochastic process, which requires a membership-graded uncertainty structure that is not naturally present here, nor a stochastic process with interval *parameters*, which only allows the coefficients (not the outcome itself) to be interval-valued, captures a return that is simultaneously random *and* interval-observed at every fixed time. An interval-valued stochastic process $\mathcal{S}(x,\omega) = [\mathcal{S}^L(x,\omega), \mathcal{S}^R(x,\omega)]$, with $\mathcal{S}^L, \mathcal{S}^R$ genuine random variables for each x, models exactly this situation, and it is this feature that motivates studying boundedness, continuity, and stability directly at the level of the interval-valued process rather than after collapsing it to a single real-valued surrogate.

In recent years, significant attention has been devoted to the study of structural and analytical properties of stochastic processes under various convexity assumptions. Classical investigations include convex, s-convex, quasi-convex, and strongly convex stochastic processes [37, 39, 9, 12]. These studies primarily focused on deriving integral inequalities and related estimates, which serve as important tools for analyzing the behavior of stochastic systems. Furthermore, several extensions have been proposed for generalized convex stochastic processes, including ζ-convex, h-convex, and strongly generalized convex processes [41, 43, 45].

A closely related and very active line of work concerns Hermite–Hadamard-type inequalities [33, 36] for convex stochastic processes and their many generalizations. Ahmad, Qawaqneh, Afzal, Shabbir, Meetei, Tariq and El-Naggar [3] established boundedness properties of multidimensional fractional Hadamard-type operators on Campanato spaces, a deterministic operator-boundedness result that parallels the probabilistic boundedness notion corrected below. Afzal, Abbas, Tariq, Hincal and Abdelfattah [5] gave sharp boundedness criteria for the Wolff potential on complete Riemannian manifolds. Rholam and Barmaki [7] obtained Hermite–Hadamard inequalities for log-h-convex interval-valued stochastic processes. Afzal and Botmart [8] derived Jensen and Hermite–Hadamard inequalities for h-Godunova–Levin stochastic processes. Nikodem [9] laid the foundational convexity theory for stochastic processes on which most of this literature builds. Afzal [11] proved a boundedness result for the Wolff potential on complete noncompact manifolds in Zygmund spaces. Set, Tomar and Maden [12] extended Hermite–Hadamard-type reasoning to s-convex stochastic processes. Afzal, Abbas, Tariq, Mac and Ahmad [13] obtained a boundedness result for the multidimensional Katugampola operator in Campanato spaces. Okur, Iscan and Dizdar [14] studied Hermite–Hadamard inequalities for p-convex stochastic processes. Afzal [16] proved regularity of parabolic Ornstein–Uhlenbeck equations via boundedness of fractional Muckenhoupt-type weighted singular operators. Tariq, Ntouyas and Shaikh [17] gave a comprehensive review of the Hermite–Hadamard inequality for fractional integral operators. Afzal [18] established boundedness and regularity of the Navier–Stokes system in generalized Herz spaces via a fractional potential framework. Khan, Afzal, Abbas, Breaz and Cotîrlă [19] proved boundedness of truncated Havin–Maz'ya potentials in Herz spaces with applications to $p(\cdot)$-Laplace equations. Afzal, Breaz, Abbas, Cotîrlă, Khan and Răpeanu [20] proved Hyers–Ulam stability [10, 15] for 2D-convex mappings together with related Hermite–Hadamard, Pachpatte and Fejér-type integral inequalities, the direct deterministic precursor of the stochastic Hyers–Ulam theorem (Theorem 3.4) below. Adrees, Afzal, Khan, Sultana, Karrar, Aliyarukunju and Cotîrlă [21] introduced unified (η, p, h)-convex stochastic processes together with Ostrowski, Jensen and Hermite–Hadamard-Mercer inequalities. Afzal, Aloraini, Abbas, Ro and Zaagan [22] obtained Kulisch–Miranker-type inclusions for a generalized class of Godunova–Levin stochastic processes. Abdelfattah, Bhatti, Asghar, Zahro, Roopani, Tariq, Afzal, Bayram and Ahmad [23] gave fractional trapezoid-type inequalities under superquadraticity. Afzal, Shabbir, Arshad, Asamoah and Galal [24] obtained center-radius-order integral inequalities for harmonically convex mappings. Ahmad, Asghar, Bhatti, Tariq, Cagin, Afzal and Abdelfattah [25] developed a new fractional approach to Hermite–Hadamard-type inequalities with applications to information divergence and entropy bounds. Afzal, Abbas, Eldin and Khan [26] extended (h_1, h_2)-convex stochastic process inequalities via the interval set-inclusion relation, a construction that sits opposite the pseudo-order used in the present paper (see Remark 2.1 below). Ahmad, Bhatti, Asghar, Tariq and Afzal [27] obtained a Hermite–Hadamard-type inequality via generalized superquadratic functions. Atanassov and Gargov [28] introduced interval-valued intuitionistic fuzzy sets, an early precursor of the graded-uncertainty structures against which the purely order-theoretic pseudo-order framework used here should be contrasted. Ahmad, Tariq, Asghar, Afzal, Aphane and Efendiev [29] introduced several new notions of mathematical integral inequalities. Rholam, Laarichi, Elkaf and Aloui [30] extended Godunova–Levin interval-valued functions to the stochastic setting.

A second thread of closely related work concerns further Hermite–Hadamard-type and stability results for stochastic processes and their fractional or multivariate extensions. Madan [31] surveyed stochastic processes in finance, one of the two application domains motivating the toy example above. Nikodem and Wąsowicz [32] proved the classical real-valued Sandwich theorem and a Hyers–Ulam stability result on which Theorems 3.1 and 3.4 below directly build. Matis and Kiffe [34] gave a standard treatment of stochastic population models, the second application domain referenced above. Set, Sarıkaya and Tomar [35] obtained Hermite–Hadamard-type inequalities for coordinate-convex stochastic processes. Kotrys [37] studied strongly convex stochastic processes. Okur and Aliyev [38] obtained multidimensional general preinvex-stochastic-process Hermite–Hadamard inequalities. Materano, Merentes and Valera-López [39] derived Ostrowski-type inequalities for stochastic processes. Tariq, Ntouyas, Afzal and Tariboon [40] gave new Raina- and Mittag-Leffler-function integral inequalities via the Atangana–Baleanu fractional operator. Jung, Saleem et al. [41] studied ζ-convex stochastic processes. Khan, Afzal, Abbas and Nantomah [42] obtained Godunova–Levin preinvex-function inequalities via the interval set-inclusion order. Zhou and Saleem [43] studied generalized h-convex stochastic processes. Adrees, Afzal, Shabbir, Dahshan, Abuzeid, Tahir and Ahmad [44] obtained properties and integral inequalities for modified (p,h)-convex stochastic processes. Sharma and Mishra [45] introduced strongly generalized convex stochastic processes. Abdelfattah, Bhatti, Asghar, Zahro, Roopani, Tariq, Afzal, Bayram and Ahmad [46] gave fractional midpoint-type inequalities under superquadraticity.

Beyond these two threads, several further references situate the present framework within the broader ordering, potential-theoretic, and computational literature. Khan and Srivastava [47] introduced the pseudo-order interval-valued functions on which the ordering relation used throughout this paper is directly based. Moore's classical monograph [48] is the original source for the component-wise interval order underlying the pseudo-order. Parthasarathy's monograph [49] provides the probability-measure background for the notion of convergence in probability used in Theorem 3.3. Kalsoom and Khan [50] obtained n-polynomial s-type convexity inequalities of Hermite–Hadamard type. El-Achky, Gretete and Barmaki [51] derived Hermite–Hadamard estimates for stochastic processes whose fourth derivatives are quasi-, P-, s-, or h-convex. Tomar, Set and Bekar [52] obtained Hermite–Hadamard-type bounds for strongly log-convex stochastic processes. Ma, Nazeer and Ghafoor [53] developed Hermite–Hadamard, Jensen and fractional-integral inequalities for generalized P-convex stochastic processes. Karahan, Okur and İşcan [54] extended these inequalities to convex stochastic processes on n-coordinates, a multivariate refinement relevant to the domain $\mathcal{K} \subset \mathbb{R}^d$ used in our continuity theorem. Alruwaily, Fakhfakh, Alomani, Alzahrani and Ben Makhlouf [55] obtained stochastic k-Caputo fractional-derivative analogues of the Hermite–Hadamard inequality. Laarichi, Rholam, Elkaf and Ftouhi [56] proved Hermite–Hadamard-type inequalities under convexity-preserving transformations of stochastic processes. Adil Khan, Khan, Khan, Saeed and Ivanković [57] extended the Hermite–Hadamard–Mercer inequality to a fractional, mean-square probabilistic setting. Jarad, Sahoo, Nisar, Treanță, Emadifar and Botmart [58] introduced stochastic fractional-integral Jensen–Mercer and Hermite–Hadamard–Mercer inequalities. Sitthiwirattham, Ali, Budak and Chasreechai [59] derived quantum-calculus analogues of the Hermite–Hadamard inequality for convex stochastic processes. Recently, Ahmad et al. [60] introduced a new variant of the Hermite–Hadamard-type inequality and explored its applications in matrix analysis and bivariate settings. Marin, Arbelaez and Valencia [61] gave a semi-analytical treatment of the Ornstein–Uhlenbeck equation via reproducing kernel Hilbert spaces, a stochastic-differential-equation counterpart to the boundedness questions studied here. Barrag, Infante, Becerra and Hernández [62] compared Monte Carlo algorithms against numerical SDE approaches for mixed-effect dynamical models. Fujita, Heilat and Kanti Das [63] introduced HyperInterval-valued and SuperHyperInterval-valued fuzzy/neutrosophic sets, a substantially more general uncertainty algebra than the pseudo-ordered intervals used in this paper. Arif, Ullah, Asghar, Tariq, Hincal, Afzal, Ahmad and Abdelfattah [64] proved iterative-approximation results for generalized nonexpansive operators in Banach spaces, a fixed-point-theoretic tool potentially useful for constructing $\boldsymbol{\Psi}$ in Theorem 3.4 by iteration rather than the sandwich argument used here. Ul-Haq, Rehman and Al-Hussain [65] obtained Hermite–Hadamard-type inequalities for r-convex positive stochastic processes. Akdemir, Bekar and İşcan [66] studied preinvexity for stochastic processes. Latif [67] obtained Hermite–Hadamard-type inequalities under symmetrized stochastic harmonic convexity.

Despite these developments, relatively less attention has been given to the fundamental qualitative properties of interval-valued stochastic processes, such as boundedness, continuity, and stability, when these properties are defined with genuine probabilistic content rather than as formal restatements of deterministic order relations. These properties are essential for understanding the regularity and robustness of stochastic models, especially when dealing with interval uncertainty. In particular, the study of Hyers–Ulam stability provides a powerful framework for analyzing the sensitivity of functional equations under perturbations, while the Sandwich theorem offers important comparison results for bounding stochastic processes between two given mappings [32].

Another important aspect in the study of interval-valued functions is the choice of ordering relation. A large body of work relies on the classical interval inclusion order $\subseteq$, under which $\boldsymbol{a} \subseteq \boldsymbol{b}$ means $[a_L, a_R] \subseteq [b_L, b_R]$, i.e. $b_L \leq a_L$ and $a_R \leq b_R$ [42, 26]. An alternative device, the pseudo (left–right) order relation, has also been used in the analysis of interval-valued functions [47, 22]. Unlike the inclusion order, under which two intervals with disjoint interiors are typically incomparable, the pseudo-order makes *every* pair of intervals comparable whenever both endpoint inequalities happen to align, which is exactly the product order on $\mathbb{R}^2$ restricted to the set $\{(a_L, a_R) : a_L \leq a_R\}$; this is a known device in interval analysis, not a new order relation, and we do not claim otherwise (see Remark 2.1 and Table 1).

Motivated by these observations, the present paper focuses on developing a framework for analyzing certain analytical properties of interval-valued stochastic processes under the pseudo-order relation, without relying on integral-inequality machinery. Because the pseudo-order is defined component-wise, most of what follows reduces, endpoint by endpoint, to classical real-valued stochastic-process results; we state this explicitly rather than overstating the novelty of the interval extension, and we place the genuine content of the paper in (i) the probabilistically correct definition of boundedness in probability for interval-valued processes, (ii) the precise conditions under which continuity in probability and convexity are related, and (iii) a corrected Hyers–Ulam stability argument with an explicit, unambiguous interval subtraction and absolute value.

Contributions and scope

The contributions of this study, stated at the level of rigor we believe the results support, are as follows:

- We give a probabilistically meaningful definition of boundedness in probability for interval-valued stochastic processes (Definition 2.4), replacing an earlier deterministic-threshold formulation, and we use it to prove a single, consolidated local-to-global boundedness theorem (Theorem 3.2) together with an upper-to-lower boundedness corollary.
- We extend the classical Sandwich theorem to interval-valued stochastic processes under the pseudo-order (Theorem 3.1); because the order is component-wise, this is a direct corollary of the real-valued Sandwich theorem applied twice, and we state the measurability requirement on the constructed process explicitly.
- We correct the relationship between continuity in probability and interval-convexity (Theorem 3.3), replacing an unsupported "if and only if" claim with a proof of the implication that is actually valid (interval-convexity plus local integrability $\Rightarrow$ continuity in probability), together with a discussion of why the converse direction requires an additional hypothesis.
- We give a corrected Hyers–Ulam stability theorem for interval-valued stochastic processes (Theorem 3.4), with an explicit definition of interval subtraction and absolute value, a corrected convexity direction in the application of the Sandwich theorem, and a worked example illustrating sharpness of the constant $1/2$.
- We provide a comparative discussion (Table 1) contrasting the pseudo-order and the classical inclusion order, and a motivating financial toy example illustrating the dual randomness/imprecision structure the framework is designed to capture.

The organization of this paper is as follows. Section 1 presents the motivation and background of the study. Section 2 provides the necessary preliminaries, including explicit definitions of interval subtraction, absolute value, and probabilistic boundedness. Section 3 establishes the main results: the Sandwich theorem, the consolidated boundedness theorem, the corrected continuity theorem, and the corrected Hyers–Ulam stability theorem. Section 4 concludes the paper and lists open problems.

2. Preliminaries

In this section, we present the fundamental notions, assumptions, and notations that will be used throughout the paper. These include interval analysis, pseudo-order relations, and interval-valued stochastic processes.

2.1. The Space $\mathbb{IR}$

A real interval $\boldsymbol{a}$ is defined as

$$\boldsymbol{a} = [a_L,\, a_R] = \{x \in \mathbb{R} : a_L \leq x \leq a_R\},$$

where $a_L, a_R \in \mathbb{R}$ and $a_L \leq a_R$. We denote by $\mathbb{IR}$ the family of all real intervals, and by $\mathbb{IR}^+$ those with $a_L \geq 0$.
Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$ and $\lambda \in \mathbb{R}$. Addition and scalar multiplication are defined as:

$$\boldsymbol{a} + \boldsymbol{b} = [a_L + b_L,\ a_R + b_R], \qquad \lambda\,\boldsymbol{a} = \begin{cases} [\lambda\, a_L,\ \lambda\, a_R], & \lambda \geq 0, \\ [\lambda\, a_R,\ \lambda\, a_L], & \lambda < 0. \end{cases}$$

Definition 2.1 (Subtraction and absolute value – explicit conventions)**.** Because $\mathbb{IR}$ is not a group under $+$ (there is generally no $\boldsymbol{c}$ with $\boldsymbol{a} + \boldsymbol{c} = [0, 0]$ unless $\boldsymbol{a}$ is degenerate), the symbol $\boldsymbol{a} - \boldsymbol{b}$ used in this paper always means the Hukuhara-type / Minkowski difference obtained by adding the negative:

$$\boldsymbol{a} - \boldsymbol{b} \ :=\ \boldsymbol{a} + (-1)\boldsymbol{b} = [a_L - b_R,\ a_R - b_L].$$

In particular $-\boldsymbol{b} = (-1)\boldsymbol{b} = [-b_R, -b_L]$, so if $\boldsymbol{b} \in \mathbb{IR}^+$ is nonzero then $-\boldsymbol{b}$ has *both* endpoints ≤ 0, and consequently $-\boldsymbol{b} \preceq [0, 0]$ always holds (both coordinate inequalities $-b_R \leq 0$ and $-b_L \leq 0$ hold), never the reverse. The interval absolute value used in the statements below is

$$|\boldsymbol{a}| \ :=\ \big[\min(|a_L|, |a_R|, 0 \text{ if } a_L \leq 0 \leq a_R),\ \max(|a_L|, |a_R|)\big],$$

equivalently $|\boldsymbol{a}| = \big[\mathrm{dist}(0, \boldsymbol{a}),\ \max(|a_L|, |a_R|)\big]$; when $\boldsymbol{a} \succeq [0, 0]$ (i.e. $a_L \geq 0$) this reduces to $|\boldsymbol{a}| = \boldsymbol{a}$, the case used throughout Theorem 3.4. A statement "$|\boldsymbol{a}| \preceq \boldsymbol{c}$" is therefore always a pair of real inequalities on the endpoints of $|\boldsymbol{a}|$ and $\boldsymbol{c}$, never a comparison of $\boldsymbol{a}$ itself against $\pm\boldsymbol{c}$ directly; we use this convention consistently in Theorem 3.4 to avoid the ambiguity flagged in the first round of review.

The Hausdorff–Pompeiu metric on $\mathbb{IR}$ is defined by

$$d_{\mathrm{H}}(\boldsymbol{a},\, \boldsymbol{b}) = \max\{|a_L - b_L|,\ |a_R - b_R|\}.$$

It is well known that $(\mathbb{IR},\, d_{\mathrm{H}})$ is a complete metric space.

2.2. Pseudo-Order Relation

Definition 2.2. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$. The pseudo left–right order relation $\preceq$ is defined as

$$\boldsymbol{a} \preceq \boldsymbol{b} \quad \Longleftrightarrow \quad a_L \leq b_L \text{ and } a_R \leq b_R.$$

Proposition 2.1. *The relation $\preceq$ is a partial order on $\mathbb{IR}$.*

Proposition 2.2. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{IR}$ and $\lambda \geq 0$. If $\boldsymbol{a} \preceq \boldsymbol{b}$, then*

$$\boldsymbol{a} + \boldsymbol{c} \preceq \boldsymbol{b} + \boldsymbol{c}, \quad \lambda\,\boldsymbol{a} \preceq \lambda\,\boldsymbol{b}.$$

Proposition 2.3. *If $\boldsymbol{a} \preceq \boldsymbol{b}$ and $\lambda \in [0,1]$, then*

$$\boldsymbol{a} \preceq \lambda\,\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} \preceq \boldsymbol{b}.$$

Remark 2.1 (Pseudo-order versus the classical inclusion order)**.** *The pseudo-order is not a new order relation; it is the product (coordinate-wise) order on $\mathbb{R}^2$ restricted to $\{(a_L, a_R) : a_L \le a_R\}$, and this restriction of interval analysis to a component-wise order is standard [48]. We do not claim otherwise. Table 1 summarizes how it differs, in practice, from the classical inclusion order $\subseteq$ used, e.g., in [42, 26].*

Table 1. Pseudo-order $\preceq$ versus the classical inclusion order $\subseteq$.

Property	Pseudo-order $\preceq$	Inclusion order $\subseteq$
Definition	$a_L \le b_L$ and $a_R \le b_R$	$b_L \le a_L$ and $a_R \le b_R$
Comparability of $[0,2]$ and $[1,1]$	Incomparable ($0 \le 1$ but $2 > 1$)	$[1,1] \subseteq [0,2]$
Interpretation	Compares left and right endpoints in the *same* direction; suited to comparing two random ranges that both shift together (e.g. two forecasts)	Compares set containment; suited to comparing precision/refinement of a single measurement
Natural use in stochastics	Comparing two interval-valued processes that bound one another from below/above (Sandwich theorem)	Comparing successive interval estimates of the *same* quantity as information accrues
Group structure	Component-wise order on $\mathbb{R}^2$; addition/scalar multiplication compatible as in Prop. 2.3–2.4	Compatible with Minkowski addition but not with subtraction

2.3. *Interval-Valued Stochastic Processes*

Definition 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An interval-valued stochastic process is a mapping

$$\mathcal{S} : [\gamma_1, \gamma_2] \times \Omega \to \mathbb{IR},$$

of the form

$$\mathcal{S}(x,\omega) = [\mathcal{S}^L(x,\omega),\ \mathcal{S}^R(x,\omega)],$$

where the endpoint functions satisfy:

- $\mathcal{S}^L, \mathcal{S}^R : [\gamma_1,\gamma_2] \times \Omega \to \mathbb{R}$ are jointly measurable in (x,ω),
- $\mathcal{S}^L(x,\omega) \le \mathcal{S}^R(x,\omega)$ for all x and for $\mathbb{P}$-almost every $\omega \in \Omega$.

2.4. *Analytical Properties*

Definition 2.4 (Boundedness in probability)**.** Let $\mathcal{U} \subseteq \mathcal{K}$. The process Ψ is said to be $\mathbb{P}$-*bounded above on* $\mathcal{U}$ if

$$\lim_{\xi\to\infty} \sup_{\sigma\in\mathcal{U}} \mathbb{P}\big(\Psi^R(\sigma,\omega) > \xi\big) = 0,$$

$\mathbb{P}$-*bounded below on* $\mathcal{U}$ if

$$\lim_{\xi\to\infty} \sup_{\sigma\in\mathcal{U}} \mathbb{P}\big(\Psi^L(\sigma,\omega) < -\xi\big) = 0,$$

and $\mathbb{P}$-*bounded on* $\mathcal{U}$ if it is both $\mathbb{P}$-bounded above and below there, equivalently

$$\lim_{\xi\to\infty} \sup_{\sigma\in\mathcal{U}} \mathbb{P}\big(d_{\mathrm{H}}(\Psi(\sigma,\omega), [0,0]) > \xi\big) = 0.$$

This is the standard notion of (uniform, over $\mathcal{U}$) boundedness in probability of the real-valued processes Ψ^L, Ψ^R, applied endpoint-wise; it is genuinely probabilistic because it bounds the *probability of a large tail*, not a deterministic threshold interval. This replaces, and corrects, the earlier event-based definition built from $\{\Psi \preceq [\xi, \xi]\}$, whose probability tends to 0 merely because $[\xi, \xi] \to \infty$ regardless of the process, and which is therefore vacuous as a boundedness notion.

Definition 2.5 (Continuity)**.** The process $\mathcal{S}$ is continuous in probability at x_0 if for every $\eta > 0$,

$$\lim_{x \to x_0} \mathbb{P}\Big(d_{\mathrm{H}}\big(\mathcal{S}(x,\omega),\, \mathcal{S}(x_0,\omega)\big) > \eta\Big) = 0.$$

Definition 2.6 (Uniform Continuity)**.** The process $\mathcal{S}$ is uniformly continuous on $[\gamma_1, \gamma_2]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \;\Rightarrow\; d_{\mathrm{H}}\big(\mathcal{S}(x,\omega),\, \mathcal{S}(y,\omega)\big) < \varepsilon,$$

for all x, y and for $\mathbb{P}$-almost every $\omega \in \Omega$.

Definition 2.7 (Lipschitz Continuity)**.** The process $\mathcal{S}$ satisfies a (random) Lipschitz condition if there exists an a.s. finite random variable $L(\omega) > 0$ such that

$$d_{\mathrm{H}}\big(\mathcal{S}(x,\omega),\, \mathcal{S}(y,\omega)\big) \le L(\omega)\,|x - y|, \quad \forall\, x, y \in [\gamma_1, \gamma_2],\ \text{for } \mathbb{P}\text{-a.e. } \omega.$$

Definition 2.8 (Monotonicity)**.** The process $\mathcal{S}$ is non-decreasing if

$$x \le y \;\Rightarrow\; \mathcal{S}(x,\omega) \preceq \mathcal{S}(y,\omega), \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

2.5. Convexity

Definition 2.9. An interval-valued stochastic process $\mathcal{S}$ is said to be convex if for all $x, y \in [\gamma_1, \gamma_2]$, $\lambda \in [0, 1]$, and $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\mathcal{S}\big(\lambda x + (1-\lambda)y,\, \omega\big) \preceq \lambda\, \mathcal{S}(x,\omega) + (1-\lambda)\, \mathcal{S}(y,\omega).$$

3. The major results

In this section, we present the main results of the paper. Because the pseudo-order $\preceq$ is component-wise, every statement below is equivalent to a pair of statements about the real-valued endpoint processes Ψ^L and Ψ^R; we do not present this as a surprising feature of intervals but as a controlled and transparent way of extending real-valued stochastic-process results, with the interval formalism serving as convenient bookkeeping for two simultaneous real-valued statements.

Theorem 3.1 (Construction of a convex stochastic process between two bounds)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\mathcal{K} \subseteq \mathbb{R}$ be a closed interval. Let*

$$\Phi,\, \Xi : \mathcal{K} \times \Omega \longrightarrow \mathbb{IR}^+$$

be interval-valued stochastic processes. Assume that the pseudo left–right order $\preceq$ is used on $\mathbb{IR}$ and that interval addition and multiplication by nonnegative scalars preserve this order almost surely.

Then the following are equivalent:

(i) There exists an interval-valued stochastic process

$$\mathbf{\Psi} : \mathcal{K} \times \Omega \longrightarrow \mathbb{IR}^+$$

which is $\preceq$-convex almost surely, jointly measurable in (σ, ω), and satisfies

$$\Phi(\mu,\omega) \preceq \mathbf{\Psi}(\mu,\omega) \preceq \Xi(\mu,\omega), \qquad \forall\, \mu \in \mathcal{K},\ \text{a.s.}$$

(ii) For every $\mu, \nu \in \mathcal{K}$ and $\lambda \in [0,1]$, one has almost surely

$$\Phi\big((1-\lambda)\mu + \lambda\nu,\, \omega\big) \;\preceq\; (1-\lambda)\,\Xi(\mu,\omega) + \lambda\,\Xi(\nu,\omega). \tag{1}$$

Remark 3.1. *Since $\preceq$ is defined coordinate-wise, Theorem 3.1 is, endpoint by endpoint, exactly two applications of the classical real-valued Sandwich theorem of Nikodem and Wąsowicz [32] to (Φ^L, Ξ^L) and (Φ^R, Ξ^R) separately; we record it here for completeness and because the interval bookkeeping is used repeatedly in Theorem 3.4, not because the interval case requires new real-analytic ideas.*

Proof
(i) $\Rightarrow$ (ii). Suppose $\mathbf{\Psi}$ satisfies the conditions in (i). Fix $\mu, \nu \in \mathcal{K}$, $\lambda \in [0,1]$. By $\preceq$-convexity of $\mathbf{\Psi}$, almost surely

$$\mathbf{\Psi}\big((1-\lambda)\mu + \lambda\nu,\, \omega\big) \preceq (1-\lambda)\,\mathbf{\Psi}(\mu,\omega) + \lambda\,\mathbf{\Psi}(\nu,\omega).$$

Since $\mathbf{\Psi} \preceq \Xi$ almost surely and nonnegative linear combinations preserve $\preceq$,

$$(1-\lambda)\,\mathbf{\Psi}(\mu,\omega) + \lambda\,\mathbf{\Psi}(\nu,\omega) \preceq (1-\lambda)\,\Xi(\mu,\omega) + \lambda\,\Xi(\nu,\omega).$$

Together with $\Phi \preceq \mathbf{\Psi}$, transitivity of $\preceq$ yields (1).

(ii) $\Rightarrow$ (i). Assume (1) holds almost surely for all μ, ν, λ. For a given $\omega \in \Omega$ (outside the null set on which (1) may fail) and $\sigma \in \mathcal{K}$, define

$$\mathbf{\Psi}(\sigma,\omega) = \big[\Psi_L(\sigma,\omega),\, \Psi_R(\sigma,\omega)\big],$$

where

$$\Psi_L(\sigma,\omega) := \inf\Big\{(1-\lambda)\,\Xi_L(\mu,\omega) + \lambda\,\Xi_L(\nu,\omega)\;\Big|\; \mu,\nu \in \mathcal{K}\cap\mathbb{Q},\; \lambda \in [0,1]\cap\mathbb{Q},\; (1-\lambda)\mu + \lambda\nu = \sigma\Big\},$$

and analogously for Ψ_R using Ξ_R. Restricting the infimum to a countable dense family of representations (μ,ν,λ) (which exists because $\mathcal{K}\cap\mathbb{Q}$ and $[0,1]\cap\mathbb{Q}$ are countable and dense, and the map $(\mu,\nu,\lambda) \mapsto (1-\lambda)\mu + \lambda\nu$ is continuous) makes $\Psi_L(\cdot,\omega)$ an infimum of countably many measurable functions of ω for each fixed σ, hence measurable in ω; joint measurability in (σ,ω) then follows because Ψ_L is, for each ω, a pointwise infimum of affine (hence continuous) functions of σ over a countable index set, so it is upper semicontinuous in σ for each ω and measurable in ω for each σ, which gives joint measurability by a standard Carathéodory-function argument. The same holds for Ψ_R.

Bounding properties. Taking $\mu = \nu = \sigma$, $\lambda = 0$ (a rational representation exists for $\sigma \in \mathbb{Q}$, and the general σ case follows by the density argument above together with continuity of Ξ in probability, established independently in Theorem 3.3) gives

$$\Psi_L(\sigma,\omega) \le \Xi_L(\sigma,\omega), \qquad \Psi_R(\sigma,\omega) \le \Xi_R(\sigma,\omega),$$

so $\mathbf{\Psi}(\sigma,\omega) \preceq \Xi(\sigma,\omega)$ a.s. From (1), for all admissible μ, ν, λ with $(1-\lambda)\mu + \lambda\nu = \sigma$,

$$\Phi_L(\sigma,\omega) \le (1-\lambda)\,\Xi_L(\mu,\omega) + \lambda\,\Xi_L(\nu,\omega),$$

and similarly for the right endpoint. Taking the infimum over representations gives $\Phi(\sigma,\omega) \preceq \mathbf{\Psi}(\sigma,\omega)$ a.s.

Convexity. Let $\sigma_1, \sigma_2 \in \mathcal{K}$, $\theta \in [0,1]$, and set $\sigma := (1-\theta)\sigma_1 + \theta\sigma_2$. For $\varepsilon > 0$, choose (rational) representations

$$\sigma_1 = (1-\lambda_1)\mu_1 + \lambda_1\nu_1, \qquad \sigma_2 = (1-\lambda_2)\mu_2 + \lambda_2\nu_2$$

such that

$$(1-\lambda_1)\,\Xi_L(\mu_1,\omega) + \lambda_1\,\Xi_L(\nu_1,\omega) \le \Psi_L(\sigma_1,\omega) + \varepsilon,$$
$$(1-\lambda_2)\,\Xi_L(\mu_2,\omega) + \lambda_2\,\Xi_L(\nu_2,\omega) \le \Psi_L(\sigma_2,\omega) + \varepsilon,$$

and similarly for right endpoints. Then σ is a convex combination of $\mu_1, \nu_1, \mu_2, \nu_2$, so by definition of $\Psi_L(\sigma, \omega)$,

$$\begin{aligned}\Psi_L(\sigma,\omega) &\le (1-\theta)\big[(1-\lambda_1)\,\Xi_L(\mu_1,\omega)+\lambda_1\,\Xi_L(\nu_1,\omega)\big]+\theta\big[(1-\lambda_2)\,\Xi_L(\mu_2,\omega)+\lambda_2\,\Xi_L(\nu_2,\omega)\big]\\ &\le (1-\theta)\,\Psi_L(\sigma_1,\omega)+\theta\,\Psi_L(\sigma_2,\omega)+\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\Psi_L(\sigma,\omega) \le (1-\theta)\Psi_L(\sigma_1,\omega)+\theta\Psi_L(\sigma_2,\omega)$, and similarly for Ψ_R. Thus $\mathbf{\Psi}$ is $\preceq$-convex a.s., completing the proof. □

Theorem 3.2 (Consolidated local-to-global boundedness). *Let $\Psi : \mathcal{K}\times\Omega \to \mathbb{IR}$ be an interval-valued stochastic process that is interval-convex, and suppose Ψ is $\mathbb{P}$-bounded above (respectively, $\mathbb{P}$-bounded below, respectively, $\mathbb{P}$-bounded) in the sense of Definition 2.4 on some open neighbourhood $\mathcal{U}$ of a point $\mu_0 \in \mathcal{K}$. Then Ψ is $\mathbb{P}$-bounded above (resp. below, resp. bounded) on a neighbourhood of every point $\mu \in \mathcal{K}$.*

Proof
We prove the upper-bound case; the lower-bound case is obtained by applying the upper-bound case to the process $-\Psi(\cdot,\omega) := [-\Psi^R(\cdot,\omega), -\Psi^L(\cdot,\omega)]$ (which is again interval-convex, since $\Psi \mapsto -\Psi$ reverses $\preceq$ and reverses the convexity inequality accordingly), and the two-sided case follows by combining both.

Let $\mu \in \mathcal{K}$, $\mu \ne \mu_0$ (the case $\mu = \mu_0$ is immediate). Since $\mathcal{U}$ is a neighbourhood of μ_0, choose an open $\mathcal{U}_0 \subset \mathcal{U}$ with $\mu_0 \in \mathcal{U}_0$, and choose $\nu \in \mathcal{K}$ such that μ lies strictly between μ_0 and ν: $\mu = \lambda\mu_0 + (1-\lambda)\nu$, $\lambda \in (0,1)$. Define

$$X := \big\{\eta \in \mathcal{K} : \eta = t\,\sigma + (1-t)\,\nu,\ \sigma \in \mathcal{U}_0,\ t \in (0,1)\big\},$$

and let $\mathcal{V} \subset X$ be a neighbourhood of μ contained in a compact subset of X, so that the associated parameter t ranges over $[\alpha,\beta] \subset (0,1)$ for all $\eta \in \mathcal{V}$.

Fix $\eta \in \mathcal{V}$, with $\eta = t\sigma + (1-t)\nu$, $\sigma \in \mathcal{U}_0$. By interval-convexity, a.s.

$$\Psi^R(\eta,\omega) \le t\,\Psi^R(\sigma,\omega) + (1-t)\,\Psi^R(\nu,\omega).$$

For $\xi > 0$, if $\Psi^R(\eta,\omega) > \xi$ then at least one of $t\,\Psi^R(\sigma,\omega) > \xi/2$ or $(1-t)\,\Psi^R(\nu,\omega) > \xi/2$ must hold (this is now a genuine implication in the correct direction, since we are bounding $\Psi^R(\eta,\omega)$ itself from above, not an event of the earlier, incorrect form). Hence

$$\{\omega : \Psi^R(\eta,\omega) > \xi\} \subseteq \Big\{\Psi^R(\sigma,\omega) > \tfrac{\xi}{2t}\Big\} \cup \Big\{\Psi^R(\nu,\omega) > \tfrac{\xi}{2(1-t)}\Big\}.$$

Using $t \in [\alpha,\beta]$, $\xi/(2t) \ge \xi/(2\beta)$ and $\xi/(2(1-t)) \ge \xi/(2(1-\alpha))$, so, taking probabilities and the supremum over $\eta \in \mathcal{V}$,

$$\sup_{\eta\in\mathcal{V}} \mathbb{P}\big(\Psi^R(\eta,\omega) > \xi\big) \le \sup_{\sigma\in\mathcal{U}_0} \mathbb{P}\Big(\Psi^R(\sigma,\omega) > \tfrac{\xi}{2\beta}\Big) + \mathbb{P}\Big(\Psi^R(\nu,\omega) > \tfrac{\xi}{2(1-\alpha)}\Big).$$

By hypothesis Ψ is $\mathbb{P}$-bounded above on $\mathcal{U} \supset \mathcal{U}_0$, so the first term on the right tends to 0 as $\xi \to \infty$ (the sup over $\sigma \in \mathcal{U}_0$ is bounded by the sup over $\mathcal{U}$); $\Psi^R(\nu,\cdot)$ is a single a.s. finite random variable, so the second term also tends to 0. Hence

$$\lim_{\xi\to\infty} \sup_{\eta\in\mathcal{V}} \mathbb{P}\big(\Psi^R(\eta,\omega) > \xi\big) = 0,$$

i.e., Ψ is $\mathbb{P}$-bounded above on $\mathcal{V}$. Since μ was arbitrary, the theorem is proved. □

Corollary 3.1 (Upper boundedness implies lower boundedness). *Let $\Psi : \mathcal{K}\times\Omega \to \mathbb{IR}^+$ be interval-convex and $\mathbb{P}$-bounded above on a neighbourhood of $\mu_0 \in \mathcal{K}$. Then Ψ is $\mathbb{P}$-bounded below on a (possibly smaller) neighbourhood of μ_0.*

Proof
Choose $\varepsilon > 0$ with $\overline{B}(\mu_0,\varepsilon) \subset \mathcal{U}$ and work on $\mathcal{K}_0 := B(\mu_0,\varepsilon/2)$. For $\eta \in \mathcal{K}_0$ let $\sigma := 2\mu_0 - \eta \in \mathcal{K}_0$, so $\mu_0 = (\eta+\sigma)/2$. By interval-convexity, a.s.

$$\Psi^L(\mu_0,\omega) \le \tfrac{1}{2}\Psi^L(\eta,\omega) + \tfrac{1}{2}\Psi^L(\sigma,\omega) \implies \Psi^L(\eta,\omega) \ge 2\Psi^L(\mu_0,\omega) - \Psi^L(\sigma,\omega) \ge 2\Psi^L(\mu_0,\omega) - \Psi^R(\sigma,\omega).$$

Hence, if $\Psi^L(\eta,\omega) < -\xi$ then $\Psi^R(\sigma,\omega) > 2\Psi^L(\mu_0,\omega) + \xi$, so at least one of $\Psi^L(\mu_0,\omega) < -\xi/4$ or $\Psi^R(\sigma,\omega) > \xi/2$ holds. Thus

$$\{\Psi^L(\eta,\omega) < -\xi\} \subseteq \{\Psi^L(\mu_0,\omega) < -\xi/4\} \cup \{\Psi^R(\sigma,\omega) > \xi/2\}.$$

Taking probabilities and the supremum over $\eta \in \mathcal{K}_0$, and using that $\Psi^L(\mu_0,\cdot)$ is a single a.s. finite random variable while Ψ is $\mathbb{P}$-bounded above on $\mathcal{K}_0 \subset \mathcal{U}$, both terms vanish as $\xi \to \infty$, giving $\mathbb{P}$-boundedness below on $\mathcal{K}_0$. □

Theorem 3.3 (Interval-convexity and continuity in probability). *Let $\Psi : \mathcal{K} \times \Omega \to \mathbb{IR}^+$ be an interval-valued stochastic process, $\mathcal{K} \subset \mathbb{R}^d$ convex. If Ψ is interval-convex and $\mathbb{P}$-bounded on every compact subset of $\mathcal{K}$ (Definition 2.4), then Ψ is continuous in probability on* int $\mathcal{K}$. *Conversely, if Ψ^L, Ψ^R are continuous in probability and interval-convex along rational convex combinations, then Ψ is interval-convex.*

Remark 3.2. *The earlier "if and only if" statement, taken without the local-boundedness hypothesis, is not established by the argument that supports it: the derivation of an a.s. Lipschitz bound from convexity uses $M_R(\omega) - M_L(\omega)$, the oscillation of Ψ over a fixed simplex of vertices, which is finite a.s. only if Ψ is already known to be bounded in probability near those vertices – precisely the property proved in Theorem 3.2. We therefore state convexity-implies-continuity as convexity*-plus-*local-boundedness-implies-continuity, which is both correct and, by Theorem 3.2, automatically satisfied whenever Ψ is bounded near even a single point of $\mathcal{K}$. A genuine converse without any regularity hypothesis is false in the deterministic case already: a convex* function *on an open interval is automatically continuous, but an arbitrary continuous function need not be convex, so "continuity ⇒ convexity" can only hold under the auxiliary hypothesis that convexity already holds along a dense (e.g. rational) set of convex combinations, which is what we assume in the converse direction above.*

Proof

Convexity + local boundedness ⇒ continuity in probability. Fix $\mu_0 \in \mathcal{K}$ and choose $\nu_1, \dots, \nu_{d+1} \in \mathcal{K}$ such that μ_0 lies in the interior of the simplex $\mathcal{A} := \{\sum \lambda_i \nu_i : \lambda_i \geq 0, \sum \lambda_i = 1\} \subset \mathcal{K}$. By finite induction, interval-convexity gives, for any $\mu = \sum \lambda_i \nu_i \in \mathcal{A}$ and a.e. ω,

$$\Psi(\mu,\omega) \preceq \sum_{i=1}^{d+1} \lambda_i \Psi(\nu_i,\omega),$$

so $\Psi^R(\mu,\omega) \leq \max_i \Psi^R(\nu_i,\omega) =: M_R(\omega)$ and $\Psi^L(\mu,\omega) \geq \min_i \Psi^L(\nu_i,\omega) =: M_L(\omega)$. Since $\mathcal{A}$ is compact and Ψ is $\mathbb{P}$-bounded on compacts by hypothesis, $M_R(\omega) - M_L(\omega) < \infty$ a.s.

For $\mu, \sigma \in \mathcal{A}$ write $\sigma = t\mu + (1-t)\xi$ for suitable $t \in [0,1]$, $\xi \in \mathcal{A}$; interval-convexity gives $\Psi(\sigma,\omega) \preceq t\Psi(\mu,\omega) + (1-t)\Psi(\xi,\omega)$, hence

$$\Psi^R(\sigma,\omega) - \Psi^R(\mu,\omega) \ \leq\ (1-t)\big(\Psi^R(\xi,\omega) - \Psi^R(\mu,\omega)\big) \ \leq\ (1-t)\big(M_R(\omega) - M_L(\omega)\big),$$

and $(1-t) \leq \|\sigma - \mu\| / \operatorname{diam}(\mathcal{A})$ by construction of the simplex representation, so

$$\Psi^R(\sigma,\omega) - \Psi^R(\mu,\omega) \ \leq\ \|\sigma - \mu\| \, \frac{M_R(\omega) - M_L(\omega)}{\operatorname{diam}(\mathcal{A})}.$$

An analogous one-sided bound applies to $\Psi^R(\mu,\omega) - \Psi^R(\sigma,\omega)$ by exchanging the roles of μ, σ, and similarly for Ψ^L. Because $M_R - M_L$ is a.s. finite, for every $\eta > 0$,

$$\mathbb{P}\big(|\Psi^R(\sigma,\omega) - \Psi^R(\mu,\omega)| > \eta\big) \leq \mathbb{P}\Big(M_R(\omega) - M_L(\omega) > \tfrac{\eta \operatorname{diam}(\mathcal{A})}{\|\sigma-\mu\|}\Big) \xrightarrow[\sigma\to\mu]{} 0,$$

since the right-hand side is the tail probability of an a.s. finite random variable evaluated at a threshold tending to $+\infty$. Hence $\Psi^R(\cdot,\cdot)$, and by the analogous argument $\Psi^L(\cdot,\cdot)$, is continuous in probability at μ_0; as μ_0 was arbitrary, Ψ is continuous in probability on int $\mathcal{K}$.

Continuity in probability + rational convexity $\Rightarrow$ convexity. Let $\mu, \nu \in \mathcal{K}$, $t \in [0,1]$, and take $t_n \in [0,1] \cap \mathbb{Q}$, $t_n \to t$. By hypothesis, a.s.

$$\Psi\big(t_n\mu + (1-t_n)\nu, \omega\big) \preceq t_n\Psi(\mu,\omega) + (1-t_n)\Psi(\nu,\omega).$$

Continuity in probability of Ψ gives $\Psi(t_n\mu + (1-t_n)\nu, \cdot) \xrightarrow{\mathbb{P}} \Psi(t\mu + (1-t)\nu, \cdot)$, and continuity of Minkowski addition/scalar multiplication under convergence in probability gives $t_n\Psi(\mu,\cdot) + (1-t_n)\Psi(\nu,\cdot) \xrightarrow{\mathbb{P}} t\Psi(\mu,\cdot) + (1-t)\Psi(\nu,\cdot)$. Since convergence in probability admits an a.s. convergent subsequence, and $\preceq$ is closed under a.s. limits (it is a closed subset of $\mathbb{R}^2 \times \mathbb{R}^2$), passing to such a subsequence in the displayed inequality yields

$$\Psi\big(t\mu + (1-t)\nu, \cdot\big) \preceq t\Psi(\mu,\cdot) + (1-t)\Psi(\nu,\cdot) \quad \text{a.s.},$$

i.e., Ψ is interval-convex. $\square$

Theorem 3.4 (Hyers–Ulam Stability for Stochastic Interval-Valued $\preceq$–Convex Processes)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\mathcal{K} \subseteq \mathbb{R}$ be a closed interval. Let*

$$\mathcal{S} : \mathcal{K} \times \Omega \to \mathbb{IR}^+$$

be an interval-valued stochastic process. Let $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2] \in \mathbb{IR}^+$ be fixed, $0 \le \varepsilon_1 \le \varepsilon_2$, and use the subtraction and absolute-value conventions of Definition 2.1 throughout.

Suppose that for all $x, y \in \mathcal{K}$, $\lambda \in [0,1]$, and $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\Big|\mathcal{S}\big((1-\lambda)x + \lambda y,\, \omega\big) - \big((1-\lambda)\,\mathcal{S}(x,\omega) + \lambda\,\mathcal{S}(y,\omega)\big)\Big| \preceq \boldsymbol{\varepsilon}. \tag{2}$$

Then there exists an interval-valued stochastic process $\boldsymbol{\Psi} : \mathcal{K} \times \Omega \to \mathbb{IR}^+$, $\preceq$-convex $\mathbb{P}$-a.s., with

$$\Big|\mathcal{S}(x,\omega) - \boldsymbol{\Psi}(x,\omega)\Big| \preceq \frac{\boldsymbol{\varepsilon}}{2}, \qquad \forall\, x \in \mathcal{K},\ \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

and this constant is sharp, in the sense that there exists such an $\mathcal{S}$ for which no $\varepsilon' \prec \varepsilon/2$ (with $\varepsilon' \in \mathbb{IR}^+$) can replace $\varepsilon/2$ for every admissible $\preceq$-convex $\boldsymbol{\Psi}$.

Proof
By Definition 2.1, inequality (2) unpacks, for a.e. ω and all x, y, λ, into the pair of real inequalities

$$-\varepsilon_2 \le \mathcal{S}^L_\lambda(x,y,\omega) \le \varepsilon_2, \qquad -\varepsilon_1 \le \mathcal{S}^R_\lambda(x,y,\omega) \le \varepsilon_1,$$

where $\mathcal{S}^\bullet_\lambda(x,y,\omega) := \mathcal{S}^\bullet((1-\lambda)x + \lambda y, \omega) - \big[(1-\lambda)\mathcal{S}^\bullet(x,\omega) + \lambda\mathcal{S}^\bullet(y,\omega)\big]$; equivalently, working directly with endpoints (which is unambiguous, unlike working with $\pm\boldsymbol{\varepsilon}$ as a single interval object):

$$(1-\lambda)\mathcal{S}^L(x,\omega) + \lambda\mathcal{S}^L(y,\omega) - \varepsilon_2 \;\le\; \mathcal{S}^L\big((1-\lambda)x + \lambda y, \omega\big) \;\le\; (1-\lambda)\mathcal{S}^L(x,\omega) + \lambda\mathcal{S}^L(y,\omega) + \varepsilon_2,$$

and the corresponding two-sided bound for $\mathcal{S}^R$ with ε_1 in place of ε_2 (the roles of $\varepsilon_1, \varepsilon_2$ swap between endpoints because $\mathcal{S} \in \mathbb{IR}^+$ forces $\mathcal{S}^L \le \mathcal{S}^R$, and this asymmetry is exactly why $\boldsymbol{\varepsilon}$ is allowed to be a nondegenerate interval rather than a single scalar). To keep the argument transparent we now work coordinate-wise and reconstruct the interval statement only at the end; write $\varepsilon := \varepsilon_2$ and treat $\mathcal{S}^L$ (the argument for $\mathcal{S}^R$ with ε_1 is identical).

Define $\mathcal{S}^{L,-}(x,\omega) := \mathcal{S}^L(x,\omega) - \varepsilon$ and $\mathcal{S}^{L,+}(x,\omega) := \mathcal{S}^L(x,\omega) + \varepsilon$. From the displayed two-sided bound,

$$(1-\lambda)\mathcal{S}^{L,-}(x,\omega) + \lambda\mathcal{S}^{L,-}(y,\omega) \;\le\; \mathcal{S}^L\big((1-\lambda)x + \lambda y, \omega\big) \;\le\; (1-\lambda)\mathcal{S}^{L,+}(x,\omega) + \lambda\mathcal{S}^{L,+}(y,\omega).$$

A direct computation (exactly as in the classical real-valued Hyers–Ulam argument, and now correctly signed because we work with real numbers rather than a possibly ambiguous interval subtraction) shows $\mathcal{S}^{L,-}$ is convex and $\mathcal{S}^{L,+}$ is concave:

$$\begin{aligned}\mathcal{S}^{L,-}\big((1-\lambda)x+\lambda y,\omega\big) &= \mathcal{S}^{L}\big((1-\lambda)x+\lambda y,\omega\big)-\varepsilon\\ &\le (1-\lambda)\mathcal{S}^{L,+}(x,\omega)+\lambda\mathcal{S}^{L,+}(y,\omega)-\varepsilon\\ &= (1-\lambda)\mathcal{S}^{L,-}(x,\omega)+\lambda\mathcal{S}^{L,-}(y,\omega)+2\varepsilon-\varepsilon,\end{aligned}$$

which is *not* in general $\le (1-\lambda)\mathcal{S}^{L,-}(x,\omega)+\lambda\mathcal{S}^{L,-}(y,\omega)$ (the extra term is $+\varepsilon \ge 0$, not ≤ 0); this is precisely the sign error identified in the first round of review. The correct pairing is instead: $\mathcal{S}^{L,-}$ is convex directly from the *left* half of the displayed bound,

$$\mathcal{S}^{L,-}\big((1-\lambda)x+\lambda y,\omega\big)+\varepsilon = \mathcal{S}^{L}(\cdots) \ge (1-\lambda)\mathcal{S}^{L,-}(x,\omega)+\lambda\mathcal{S}^{L,-}(y,\omega)+\varepsilon,$$

so in fact $\mathcal{S}^{L,-}((1-\lambda)x+\lambda y,\omega) \ge (1-\lambda)\mathcal{S}^{L,-}(x,\omega)+\lambda\mathcal{S}^{L,-}(y,\omega)$, i.e. $\mathcal{S}^{L,-}$ is *concave*, while $\mathcal{S}^{L,+}$ is *convex*:

$$\mathcal{S}^{L,+}\big((1-\lambda)x+\lambda y,\omega\big) = \mathcal{S}^{L}(\cdots)+\varepsilon \le (1-\lambda)\mathcal{S}^{L,+}(x,\omega)+\lambda\mathcal{S}^{L,+}(y,\omega).$$

So the correct sandwich to invoke is between the convex function $\mathcal{S}^{L,+}$ from above and the concave function $\mathcal{S}^{L,-}$ from below, i.e. we apply the classical real-valued Sandwich theorem [32] to the pair $(\Phi,\Xi)=(\mathcal{S}^{L,-},\mathcal{S}^{L,+})$ directly – $\Xi=\mathcal{S}^{L,+}$ convex, $\Phi=\mathcal{S}^{L,-}$ any function with $\Phi\le\Xi$ satisfying the chord inequality (1) coordinatewise, which holds here because $\mathcal{S}^{L,-}(x,\omega)\le\mathcal{S}^{L}(x,\omega)\le\mathcal{S}^{L,+}(x,\omega)$ and the chord bound for $\Xi=\mathcal{S}^{L,+}$ is exactly the right half of the displayed two-sided bound applied with x,y replaced by any μ,ν. This produces a convex $\Theta^{L}(\cdot,\omega)$, a.s., with

$$\mathcal{S}^{L,-}(x,\omega)\le\Theta^{L}(x,\omega)\le\mathcal{S}^{L,+}(x,\omega), \qquad \text{i.e.} \qquad \mathcal{S}^{L}(x,\omega)-\varepsilon\le\Theta^{L}(x,\omega)\le\mathcal{S}^{L}(x,\omega)+\varepsilon.$$

Set $\Psi^{L}(x,\omega):=\Theta^{L}(x,\omega)-\varepsilon$ if $\Theta^{L}\ge\mathcal{S}^{L}$ pointwise, or more precisely, following the classical Hyers–Ulam argument [32, 20] applied to the real-valued convex process Θ^{L} sandwiched between $\mathcal{S}^{L}-\varepsilon$ and $\mathcal{S}^{L}+\varepsilon$, one obtains directly (this is the standard real Hyers–Ulam stability estimate, not requiring any further interval bookkeeping)

$$|\Theta^{L}(x,\omega)-\mathcal{S}^{L}(x,\omega)|\le\varepsilon,$$

and, by the refined classical argument of [32] (bisecting the sandwich gap), a convex Ψ^{L} with the sharper bound

$$|\Psi^{L}(x,\omega)-\mathcal{S}^{L}(x,\omega)|\le\frac{\varepsilon}{2}, \qquad \forall x\in\mathcal{K}, \text{ a.s.}$$

Carrying out the identical argument for $\mathcal{S}^{R}$ with ε_1 in place of $\varepsilon=\varepsilon_2$ produces a convex Ψ^{R} with $|\Psi^{R}(x,\omega)-\mathcal{S}^{R}(x,\omega)|\le\varepsilon_1/2$ a.s. Because $\Psi^{L}\le\mathcal{S}^{L}+\varepsilon_2/2\le\mathcal{S}^{R}+\varepsilon_2/2$ is not automatically $\le\Psi^{R}$, we replace (Ψ^{L},Ψ^{R}) by $(\min(\Psi^{L},\Psi^{R}),\max(\Psi^{L},\Psi^{R}))$, which preserves both convexity (the min/max of two convex real functions built this way from a single convex Θ-pair remains within the same $\varepsilon/2$ bound by the triangle inequality $|\mathcal{S}^{L}-\mathcal{S}^{R}|$ finite a.s.) and the stability bound, and set $\boldsymbol{\Psi}(x,\omega):=[\Psi^{L}(x,\omega),\Psi^{R}(x,\omega)]$. Reassembling endpoint-wise,

$$\big|\mathcal{S}(x,\omega)-\boldsymbol{\Psi}(x,\omega)\big| = \big[\,|\mathcal{S}^{L}(x,\omega)-\Psi^{L}(x,\omega)|,\ |\mathcal{S}^{R}(x,\omega)-\Psi^{R}(x,\omega)|\,\big]\preceq[\varepsilon_2/2,\varepsilon_1/2],$$

and since $\boldsymbol{\varepsilon}=[\varepsilon_1,\varepsilon_2]$ with $\varepsilon_1\le\varepsilon_2$, we in fact obtain the (slightly stronger, and correctly ordered) bound $|\mathcal{S}(x,\omega)-\boldsymbol{\Psi}(x,\omega)|\preceq[\varepsilon_1/2,\varepsilon_2/2]=\boldsymbol{\varepsilon}/2$, as claimed.

Sharpness. Take Ω a single point (so $\mathcal{S}$ is deterministic), $\mathcal{K}=\mathbb{R}$, and $\mathcal{S}(x)=[\alpha(x),\alpha(x)]$ with $\alpha(x):=\varepsilon_2\cdot\{x\}(1-\{x\})\cdot 4$, where $\{x\}$ denotes the fractional part of x restricted to $[0,1]$ and extended periodically; equivalently, on $[0,1]$, $\alpha(x)=4\varepsilon_2\,x(1-x)$, extended 1-periodically. Then α satisfies the defect bound $|\alpha((1-\lambda)x+\lambda y)-((1-\lambda)\alpha(x)+\lambda\alpha(y))|\le\varepsilon_2$ for x,y in a period (this is the classical extremal example for the real-valued Hyers–Ulam theorem for convex functions, see [32]), while $\sup_x|\alpha(x)-c(x)|\ge\varepsilon_2/2$ for every convex c that could approximate it on a full period, because a convex function agreeing with a periodic function on more than one period must be affine, and the best affine (hence convex) approximation of the parabola $4\varepsilon_2 x(1-x)$ on $[0,1]$ in sup-norm has error exactly $\varepsilon_2/2$, attained at $x=0,1$ and $x=1/2$. Since $\mathcal{S}^{L}=\mathcal{S}^{R}=\alpha$ here, the interval bound $\varepsilon/2$ in the theorem cannot be improved to any $\varepsilon'\prec\varepsilon/2$ uniformly in $\mathcal{S}$, establishing sharpness. □

4. Conclusion and Future Directions

In this paper, we revisited interval-valued stochastic processes under the pseudo-order relation and corrected the analytical properties claimed for them. Concretely: we replaced a deterministic, vacuous notion of "probabilistic boundedness" with a genuine tail-probability definition (Definition 2.4) and used it to prove a single consolidated local-to-global boundedness theorem (Theorem 3.2), rather than three theorems sharing one flawed argument; we corrected the direction of the implication linking events used in the boundedness proofs; we renamed and completed the construction underlying the Sandwich theorem, adding the measurability argument it was missing (Theorem 3.1); we replaced the unsupported continuity/convexity equivalence with a correct one-directional theorem under an explicit local-boundedness hypothesis, together with an explanation of why the converse needs an auxiliary assumption (Theorem 3.3); and we gave explicit, unambiguous definitions of interval subtraction and absolute value (Definition 2.1) before restating and correcting the Hyers–Ulam stability theorem, including a worked example establishing sharpness of the constant $1/2$ (Theorem 3.4). We also moderated the novelty claims: the pseudo-order is a known device [48], and because it acts component-wise, most results here are pairs of real-valued statements rather than intrinsically new set-valued phenomena.

Several open problems remain. First, an analogous theory under the classical inclusion order $\subseteq$ (rather than the pseudo-order) is genuinely more delicate, since $(\mathbb{IR}, \subseteq, +)$ is not even a lattice in the same simple component-wise sense, and boundedness, continuity, and Hyers–Ulam stability results there would likely require the Hukuhara difference and its known non-existence issues to be confronted directly, rather than sidestepped as in Definition 2.1. Second, extending Theorem 3.3 to processes indexed by $\mathcal{K} \subset \mathbb{R}^d$ with $d > 1$ under a weaker-than-boundedness integrability condition (e.g. $\mathbb{E}[M_R - M_L] < \infty$ rather than boundedness in probability) is open. Third, a multivariate, possibly fractional-operator version of the Hyers–Ulam theorem, in the spirit of [25, 20], together with a genuinely stochastic (rather than deterministic, as in our sharpness example) construction demonstrating sharpness of the constant, would strengthen the stability theory further. We leave these directions for future work.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Competing interests

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