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A Refined Study of T-C-Compactness within T-topological Structures

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Abstract This research aimed at defining C-compact spaces with three topologies and studying their properties and relationships with other T-topological spaces. The theoretical outcomes are spotlighted and evidence is provided on their generalizability to various well-known theorems related to C-compact spaces. The findings are backed up with illustrative examples.

Keywords T-Compactness, T-C-Separation Axioms, T-extremely disconnected space.

AMS 2010 subject classifications 54E55, 54A10, 54A20, 54D20, 54D30, 54A05

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1. introduction

1.1. Bi-topological Spaces

In this study, we research into the T-C-compact spaces and tripartite locally C-compact spaces in T-topological spaces and discuss the findings. However, we start the discussion by shedding light on the concept of bi-topological spaces and overviewing relevant previous research and studies on them, with particular emphasis on triple topological spaces. In essence, the theory and definitions of the T-topological spaces emerged as a generalization of the results of studies of binary topological spaces and numerous related studies clarified and discussed the relations between these types of spaces.

The notion of C-compactness in topological space (X, τ) was introduced by Arthur (1945). Since then, this concept attracted attention of topology researchers who started studying the various features and characteristics of these spaces and their relations with the different types of topologies and closed sets. Examples encompass the studies of Willard (1970), Engleking (1989), and Matveev (1994). Explanations of the expandability, feebly expandability, and almost expandability of the bi-topological space were provided by Oudetallah (2021a, 2021b).

The basic concepts of bi-topological spaces were established by Kelly [1]. In 1963, he defined the bi-topological space as a non-empty set ξ on which two topologies are defined and he assigned it the symbol $(\xi, \tau_\xi^1, \tau_\xi^2)$. In this study [1], many spaces were defined like the pairwise regular spaces and pairwise normal spaces. Furthermore, he generalized several topological concepts, including the Hausdörff and pairwise Hausdörff concepts. This approach

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attracted attention of topology researchers, which, eventually, led to formulation of the definition of the pairwise compact spaces and inspired studies of their properties.

Since that time, many of the researchers interested in this research theme have used these concepts to define varying bi-topological concepts such as the separation axioms and covering properties, in addition to investigating the relationships between them. For example, Kim [2] coined a definition for the pairwise compact bi-topological spaces. He addressed the problem of defining pairwise compactness suitably for a bitopological space (X, τ_1, τ_2) under the condition that the space can be both pairwise compact and pairwise uniform without τ_2 being equal to τ_1. Fletcher et al. [29] studied the pairwise uniform spaces, with special attention to the case of a set with two topologies, and analyzed the conditions under which the two topologies are equal. They also addressed the problem of defining 'compactness' and 'para-compactness' in the bi-topological spaces. These studies focused on defining the compact binary spaces, studying their properties, and providing and discussing examples on each case.

In 1969, Fletscher et al. [29] defined a number of types of covers in the bi-topological spaces like the p-open and $\tau_1\tau_2$-open covers. A cover $\tilde{U} = \{u_\alpha : \alpha \in \Delta\}$ of a bitopological space $(\xi, \tau_\xi^1, \tau_\xi^2)$ is called $\tau_1\tau_2$-open cover if $\tilde{U} \subseteq \tau_1 \cup \tau_2$. However, if $\tilde{U}$ contains one member or more of both τ_1 and τ_2, then $\tilde{U}$ is called a p-open cover. Considering this, several types of spaces were defined, including the pairwise compact spaces and the pairwise countably compact spaces. Today, this study [29] is considered as one of the best studies that contributed to establishment of research works in the binary topological spaces.

In 1972, semi-compactness of bi-topological spaces was defined by Datta [30], who also investigated the properties of these spaces. Birsan [4] too introduced and discussed concepts relating to compactness in the bi-topological spaces. He defined and used concepts such as P-compactness and inquired into its relation with other properties like being closed subset in ambient space. By so doing, he contributed to the study of differing compactness definitions and properties.

In 1975, Cook and Reilly [2] researched into the relations between three topological concepts: B-compact spaces, p-compact spaces, and s-compact spaces. Building on the distinguished scientific approach to this subject, many important studies have emerged in this field. These encircle the study of Fora and Hdieb [9] in 1983, which made satisfactory generalization of the findings of most of the related previous studies, through which the following concepts were presented: pairwise Lindel$\ddot{O}$f spaces, semi Lindel$\ddot{O}$f spaces, and B-Lindel$\ddot{O}$f spaces. The findings of this study [2] contributed to the development and validation of numerous topological concepts. Further, the relationships between those concepts and other well-known topological concepts were described. It is worth underlining that this study shed light on definitions of important types of functions on bi-topological spaces. For instance, it articulated that a function $h : (\xi, \tau_1, \tau_2) \to (\xi, \tau_1', \tau_2')$ is described as P-open (P-closed, P- continuous, P-homeomorphism if the functions $h : (\xi, \tau_1) \to (\xi, \tau_1',)$ and $h : (\xi, \tau_2) \to (\xi, \tau_2')$, both, are open, closed, continuous, or homeomorphism functions, respectively.

In 2000, Dochviri [27], generalized the concepts of S-closed and semi-compact topological spaces to bi-topological spaces. Moreover, he defined the characteristic properties of (i, j)-S closed bi-topological spaces with respect to the (i, j)-quasi-H-compact, (i, j)-semi-compact, and p-extremally disconnected spaces, where $i, j \in \{1, 2\}, i \neq j$.

In 2007, Al-Hawary and Al-Omari [28] researched into quasi-b-open sets, which was relatively new notion by that time. They introduced this type of sets, investigated their features, defined the concept of quasi-b-continuity between bi-topological spaces, and provided a decomposition of which. Furthermore, they investigated group of quasi-b-homeomorphisms and defined a number of new bi-topological spaces. Atoom et al. (2021) introduced the concepts and notions of the perfect functions in the bi-topological spaces, which give two kinds of functions, that is, the S-Lindel$\ddot{o}$f perfect function and p-Lindel$\ddot{o}$f perfect function. These researchers, additionally, studied images and inverse images of particular bi-topological characteristics of these functions. During this process, they established theorems of products of these functions.

1.2. T-topological Spaces

In this paper, we study the T-C-compact spaces and tripartite locally C-compact spaces in T-topological spaces and discuss the findings. The T-topological spaces attracted the attention of many scientists and most of the relevant previous works have focused on generalizing the already known concepts and ideas in topological spaces to the

T-topological spaces, which are, in effect, a generalization of the bi-topological spaces, which themselves are generalization of the case of single topological spaces. The present work focuses on a non-empty set ξ having three topologies τ_1, τ_2, τ_3 that is described as a T-topological space and denoted by $(\xi, \tau_1, \tau_2, \tau_3)$. Explorations of T-topological spaces began in 2000 with the work of Kovár (2000), who defined important separation axioms and clarified them through various distinctive examples. These studies inspired numerous researchers to delve deeper into this field. As an example, Hameed and Abid (2011) introduced and investigated the concept of $123b\tau$-closeness in T-topological spaces. They used the main concept in the work of Kovár (2000) to introduce new types of separation axioms in the T-topological spaces and to explore their characteristics. These researchers denoted these axioms as $123b\tau$-T_k spaces, where $k = 0, 1, 2$.

These concepts paved the way for definition of new separation axioms in the T-topological spaces based on sorts of open sets that are specific to these spaces rather than based on the traditional concept of open sets. Three new separation axioms were introduced, which Hameed and Abid (2011) denoted as $123b\tau$-T_0, $123b\tau$-T_1, and $123b\tau$-T_2 spaces. In other respects, Sweedy and Hassan (2011) introduced the concept of α-continuous functions, detailing their characteristics in the T-topological spaces. He also looked into the α-continuous functions in the T-topological spaces and examined their properties and effects on these spaces. In this respect, Palaniammal (2011) performed in-depth study of T-topological spaces in his doctoral dissertation and obtained important results that helped him to develop definitions for the T-α-continuous and T-β-continuous functions. Later to this, Tapi et al. (2016) proposed two open sets in the T-topological spaces, i.e., the pre-open sets and semi-open sets, and inquired into their fundamental features. In addition to this, they examined two important types of the continuous functions; the T-semi-continuous and T-pri-continuous functions, and analyzed their properties. Further, they established the concept of α-T-open sets and inspected their properties. They employed this concept to define one of the important types of functions, which is the α-T-functions. Qoqazeh et al. (2023) defined new covering properties in the T-topological spaces, that is, T- Lindelöf space, and studied its characteristics and relations with other sorts of T-topological spaces. In addition, they investigated effects of certain sorts of functions on the T-Lindelöf spaces. Moreover, they investigated the conditions under which the T-topological space reduces to single topological space. As well, definitions of a variety of concepts in singular and bi-topological spaces were studied and generalized to the T-topological spaces. Many illustrative examples were given and well-known facts and theorems related to the Lindelöf spaces were generalized.

Qawaqneh et al. (2026), discussed new concepts of generalized $\varphi - S$ contractions of integral types (A) and (B) and proved related fixed-point theorems within the context of a supra-metric space. Pandiselvi & Jeyaraman (2025) discussed and introduced the concept of "neutrosophic metric-like spaces"examining their properties which represents a development and extension of both intuitionistic fuzzy metric spaces and classical metric-like spaces. To see more [15].

2. Motivations and Main Contributions

Motivations:

1. **Addressing structural gaps:** Research centered on the traditional concept of "compactness" often fails to capture the true behavior of T-type topological structures, necessitating the development of a new, improved approach that accounts for and enhances the quality of specific topological constraints.
2. **Enhancing analytical precision:** There is a critical need to refine and scrutinize the definition of "compactness" to better address the inherent complexities of T-type topological spaces that are fundamental to constructing realistic and robust theoretical models.
3. **An introductory pathway to applied sciences:** Amid the rapid evolution of computing and decision-making environments, this study offers a deeper understanding of the concept of "T-C-Compactness," providing a theoretical foundation for analyzing stability and convergence in data-driven models.
4. **Theoretical Generalization:** This study stems from the potential to generalize certain established topological properties, thereby offering the opportunity to provide a more flexible and practically applicable framework within broader mathematical fields.

The main contributions of this work are put out as follows:

1. **A clear, formal presentation of the concept of "T-C-compactness":** We formally discuss and introduce this concept within the context of T-type topological structures, thereby providing a topological foundation that has not previously been sufficiently utilized or developed.
2. **Characterization Theorems:** We establish and prove a number of fundamental facts and characterization theorems that describe and elucidate the behavior of the "T-C-compactness" property under various mathematical operations, thereby providing future researchers with the tools to assess the applicability of this property in different contexts.
3. **Comparative Analysis:** This study presents a comprehensive and clear comparison between "T-C-compactness" and traditional compactness, clearly outlining the structural improvements made and the new conditions required for the former concept to hold.
4. **Real-world applications regarding stability and convergence:** We demonstrate the practical utility of the "T-C-compactness" concept by applying stability conditions within T-type topological spaces. This effectively contributes to a better understanding of the behavior of these structures in iterative models and models based on development and optimization.

In the next section, we study the T-C-compactness concept in the T-topological spaces, pinpoint some of its major properties, and relate this concept to other types of spaces. We start with a well-known definition that will be used in the sequel.

3. Some Definition in T-Topological Space

3.1. On T-C-compact space

In the current study, we define the T-C-compact concept in a triple topological space.

Definition 1
1. A T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is S-compact if for any $\tau_\xi^1\tau_\xi^2\tau_\xi^3$- open cover of the space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ has a finite subcover.
2. A T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is T-compact if any $\tau_\xi^{1,2,3}$- open cover of the space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ has a finite subcover.
3. A T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is C-compact if any τ_ξ^i -open cover of $\xi, i = 1, 2, 3$ has τ_ξ^j-finite subcover, $j = 1, 2, 3$ and τ_ξ^k-finite subcover, where $i \neq j \neq k$.

Definition 2
Let $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ be a T-topological space and $z \subset \xi$, then $\forall i = 1, 2, 3$ and

(i) z is T-regular open set if $z = Int_{\tau_\xi^i}(CL(z))$,
(ii) z is T-regular closed set if $z = CL_{\tau_\xi^i}(Int(z))$, and
(iii) z is T-semi-open if $\exists q_{\tau_\xi^i}$ an open set : $q_{\tau_\xi^i} \subseteq z \subseteq CL_{\tau_\xi^i}(q)$.

Remark 3
If $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-topological space and $z \subset \xi$, then: $(\xi - z)$ is T-regular open set iff z is T-regular closed set.

Theorem 4
If $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-topological space, then each T-open set is T-semi-open.

Proof. If z is a T-regular open, we have $Int_{\tau_\xi^i}(z) \subseteq z \subseteq CL_{\tau_\xi^i}(z) \quad \forall i = 1, 2, 3$. So, $q_{\tau_\xi^i} \subseteq z \subseteq CL_{\tau_\xi^i}(q)$. Therefore, z is a T-semi open set.

Definition 5
[17] When each open cover of $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ has a finite subcollection where its closure covers ξ, then $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is quasi H-closed space.

Definition 6
[17] If all open cover of $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ has a finite subcollection where its closure covers ξ, then $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a p-quasi-H-closed space.

Definition 7
When any τ_ξ^i open cover of a T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ has a finite subcollection with property that its closure's interior covers ξ, then the topological space $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is T-nearly compact, $\forall i = 1, 2, 3$.

Definition 8
Let $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ be a T- topological space, then if any τ_ξ^i-S-open cover of ξ has a finite subcollection, such that its closure covers ξ, so $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-S-closed space $\forall i = 1, 2, 3$.

Definition 9
A T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is called a T-C-compact if for any closed subset A of ξ and if any τ_ξ^i -open cover $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$, there exists a τ_ξ^j - open sets $\{w_{\alpha_1}, w_{\alpha_2}, ..., w_{\alpha_n}\}$ for which $A \subset \bigcup_{i=1}^{n} \overline{w_{\alpha_i}}$, for each $i, j = 1, 2, 3$, $i \neq j$.

Definition 10
A T- Hausdörff space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is called T-H-closed if for any τ_ξ^i- open cover $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ has a finite τ_ξ^j- collection $\{w_{\alpha_1}, w_{\alpha_2}, ..., w_{\alpha_n}\}$ with $w_{\alpha_n} \subset \bigcup_{i=1}^{n} \overline{w_{\alpha_i}}$, $i, j = 1, 2, 3$, $i \neq j$.

Definition 11
A subset A of a T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is called a T-regular open, if $Int_{\tau_\xi^i}(CL_{\tau_\xi^i}(A)) = A$ in each topological space τ_ξ^i, $i = 1, 2, 3$.

Theorem 12
The T-topological space $(\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is defined as C-compact space if and only if, for any subset A of ξ and for any τ_ξ^i-open cover $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ of A, where w_α is T-regular open for all $\alpha \in \lambda$, $\exists$ a τ_ξ^i-collection $\{w_{\alpha_k}\}_{k=1}^n$ of $\tilde{w}$ to the extent that $w_{\alpha_k} \subset \bigcup_{i=1}^{n} \overline{w_{\alpha_i}}$,$i = 1, 2, 3$.

Proof. $\Rightarrow$ Suppose that ξ is a T-C-compact and $A \subset \xi$. Let $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ be a τ_ξ^i open cover of A for which w_α is T-regular open $\forall i = 1, 2, 3$, $\forall \alpha \in \lambda$. So, we have $[Int(\overline{w_{\alpha_i}}) = w_{\alpha_i}]$, $i = 1, 2, 3$. Because $Int(\overline{w_{\alpha_i}})$ is τ_ξ^i-open set $\forall \alpha \in \lambda$, since ξ is T-C-compact, hence the proof.
$\Leftarrow$ Consider the τ_ξ^i-regular open cover, $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ of A, $i = 1, 2, 3$. Then,

$$A \subset \bigcup_{\alpha \in \lambda} w_\alpha \subset \bigcup_{\alpha \in \lambda} Int(\overline{w_\alpha}) \subset \bigcup_{\alpha \in \lambda} \overline{w_\alpha}.$$

Since A has an open cover such, $\{Int(\overline{w_\alpha}), \alpha \in \lambda\}$, then let $Int(\overline{w_\alpha}) = r_\alpha \quad \forall \alpha \in \lambda$. Accordingly, we obtain

$$A \subset \bigcup_{\alpha \in \lambda} r_\alpha$$

Thus, consequent to the conditions, $A \subset \bigcup_{\alpha \in \lambda} r_{\alpha_k}$ and, thus, ξ is T-C-compact space.

3.2. Relation between C-compactness and C-separation axioms in a T-topological space

We continue to derive results theoretically in this section, but this time we show how the C-compactness in a T-topological space and the C-separation axioms relate to each other.

Definition 13
Consider that $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-topological space, then $\forall\, i, j \in 1, 2, 3$ and $i \neq j$ we have:

$i.$ ξ is a T-c-T_0-space, if $\forall\, g \neq h$ in ξ, $\exists$ a τ_ξ^i-open set W_g in ξ with $g \in \overline{W_g}$ and $h \notin \overline{W_g}$, or $\exists$ a τ_ξ^j-open set V_h in ξ with $h \in \overline{V_h}$ and $g \notin \overline{V_h}$.
$ii.$ ξ is a T-c-T_1-space if $\forall\, g \neq h$ in ξ, $\exists$ a τ_ξ^i-open set W_g in ξ with $g \in \overline{W_g}$, $h \notin \overline{W_g}$, and $\exists$ a τ_ξ^j-open set V_h in ξ with $h \in \overline{V_h}$, $g \notin \overline{V_h}$.
$iii.$ ξ is a T-c-T_2-space if $\forall\, g \neq h$ in ξ, $\exists$ a τ_ξ^i-open set W_g and a τ_ξ^j-open set V_h such that $g \in \overline{W_g}$, $h \in \overline{V_h}$, and $\overline{W_g} \cap \overline{V_h} = \emptyset$.
$iv.$ ξ is T-C-regular if $\forall\, g \notin A$ where A is a τ_ξ^i-closed set in ξ, $\exists$ a τ_ξ^i-open set W_g and a τ_ξ^j-open set V_A such that $g \in W_g$, $A \subseteq V_A$, and $\overline{W_g} \cap \overline{V_A} = \emptyset$.
$v.$ ξ is a T-C-T_3-space if it is a T-C-T_1 and T-C-regular space.
$vi.$ ξ is T-c-normal, if for any two disjoint $\tau_\xi^{i}{*}$-closed sets $A,\ B$ in ξ, $\exists$ a τ_ξ^i-open set W_A in ξ and a τ_ξ^j-open set V_B in ξ such that $A \subseteq \overline{W}_A$, $B \subseteq \overline{V}_B$, and $\overline{W}_A \cap \overline{V}_B = \emptyset$.

Theorem 14
Given the C-compact subset M of a C-T_2-space that contains h, then $\forall g \neq h, \exists w_g, v_M$ open sets such that $g \in \overline{w_g}, M \subset \overline{v_M}$ and $\overline{w_g} \cap \overline{v_M} = \emptyset$.

Proof. Assume that $m \in M$ and $g \neq a, g \notin M$ and ξ is T-C-T_2-space, then $\exists$ two τ_ξ^i-open sets $w_g(m), v(m)$ in which $w_g(m)$ and $m \in \overline{v}(m)$ with $\overline{w_g}(m) \cap \overline{v}(m) = \emptyset$ for all $j \neq i, \quad i, j = 1, 2, 3$. Thereupon, we have the open cover $\tilde{V} = \{v(m) : a \in M\}$ of M and $\exists\{v_{\alpha_k}\}_{k=1}^n$ in which $\overline{M} \subset \bigcup_{k=1}^n \overline{v_{\alpha_k}(m)}$. Consequently, we get $g \in \overline{v_\alpha(m)}$ and $M \subset \overline{v}$, where $\overline{v} = \bigcup_{k=1}^n \overline{v_{\alpha_k}(m)}$, and so,

$$\overline{w(m)} \cap \overline{v} = \overline{w(m)} \cap \bigcup_{k=1}^{n} \overline{v_{\alpha_k}(m)} = \bigcup_{k=1}^{n} \overline{w(m)} \cap \overline{v_{\alpha_k}(m)} = \bigcup_{k=1}^{n} \emptyset = \emptyset.$$

Theorem 15
Provided that, N, M are disjoint τ_ξ^i-T-C-compact contained in a T-T_2-space $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$, then they can be separated by distinct τ_ξ^j-open sets, v_M and w_N, that is $N \subset w_N$ and $M \subset v_M \quad \forall j \neq i, j, i = 1, 2, 3$.

Proof. Let N and M be two disjoint τ_ξ^i-T-C-compact sets. Assume that $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is T-T_2-space. Then $\forall a \in N$ we have $a \notin M$, since M is a τ_ξ^i-T-C-compact in ξ. Now, based on a previous theorem, $\exists B_j$-open collections, $w(a), v(M)$ satisfy $a \in w(a)$, $M \subset v(M)$.

Thus $\tilde{W} = \{w(a) : a \in N\}$ is a τ_ξ^j-open cover of N. Hence $N = \bigcup_{a \in N}^n (w(a))$. Since that N is a τ_ξ^j-T-C-compact, $\exists\{w_k(a) : 1 \leq k \leq n\}$ a T-regular open sets in which $N \subset \bigcup_{k=1}^n Int(\overline{w_k(a)})$, (say $\bigcup_{k=1}^n Int(\overline{w_k(a)}) = w_N$) for each $i = 1, 2$ and 3.

As mentioned, $N \subset w_N$ and $M \subset v_M$. It is enough to show that $w_N \cap v_M = \emptyset$.

$$w_N \cap v_M = \left(\bigcup_{k=1}^{n} Int(\overline{w_k(a)}) \right) \cap v_M = \bigcup_{k=1}^{n} Int(\overline{w_k(a)}) \cap v_M,$$

though $\underline{w_k(a)} \cap v_M = \emptyset$ along with $Int(\overline{w_k(a)}) \subset w_k(a)$. Thereupon, $Int(\overline{w_k(a)}) \cap v_M = \emptyset$ as well as $\bigcup_{k=1}^n Int(\overline{w_k(a)}) \cap v_M = \bigcup_{k=1}^n \emptyset = \emptyset$.

Definition 16
If every open set is τ_ξ^i-clopen, then $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is T-extremely disconnected $\forall i = 1, 2, 3$.

Theorem 17
The space $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is T-C-compact iff it is T-extremely disconnected compact.

Proof. Consider the open cover $\{w_\alpha : \alpha \in \lambda\}$ of N, then $\forall\ i = 1, 2, 3,\ \xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-C-compact. Since $N \subseteq \xi$, then $\bigcup_{\alpha\in\lambda} w_\alpha$. Now, let $\xi_e = (\xi_e - N) \cup N$.

Hence, $\xi_e = (\xi_e - N) \cup (\bigcup_{\alpha\in\lambda} w_\alpha)$. Since $\{\xi_e - N\} \cup \{w_\alpha : \alpha \in \lambda\}$ covers ξ because ξ is a T-C-compact space.

As a result, ξ has a τ_ξ^i-finite subcover say, $\{\xi_e - N\} \cup \{w_{\alpha_k} : \alpha \in \lambda,\ k = 1, 2, ..., n\}$

Consequently, $\xi_e = (\xi_e - N) \cup \bigcup_{k=1}^n w_{\alpha_k}$. Thus, $N = \bigcup_{k=1}^n w_{\alpha_k}$, since ξ is a T-C-compact. So $N = \bigcup_{k=1}^n \overline{w}_{\alpha_k}$, so, $w_{\alpha_k} = \overline{w}_{\alpha_k}$, $\forall\ \alpha \in \lambda$, therefore ξ is T-extremely disconnected space.

Consequently, suppose that $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-extremely disconnected compact space. Let $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ be a τ_ξ^i-open cover of ξ and $N \subset \xi$, then $\tilde{W}$ is a τ_ξ^i-open cover of N.

Since ξ is compact and $N \subseteq \xi$ then there exists a finite subsets $\{w_{\alpha_1}, w_{\alpha_2}, ..., w_{\alpha_n}\}$ of $\tilde{W}$ such that $N \subseteq \bigcup_{k=1}^n w_{\alpha_n}$.

Now, $\xi = N \cup (\xi - N) = \bigcup_{k=1}^n w_{\alpha_k} \cup (\xi - N)$, so, $(\xi - N) \cup \{w_{\alpha_k} : \alpha \in \lambda,\quad k = 1, 2, ..., n\}$ is a finite subcover of $\tilde{W}$ for ξ. Hence ξ is T-C-compact.

Theorem 18
Each T-compact space is a T-C-compact.

Proof. Suppose that $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ be a τ_ξ^i - open cover of N. Then, $\{\xi - N\} \cup \{w_\alpha : \alpha \in \lambda\}$ constitutes a τ_ξ^i-open cover of ξ, $\forall i = 1, 2, 3$. With respect to ξ, $\xi \subset \bigcup_{k=1}^n w_{\alpha_k} \cup (\xi - N)$ and, $N \subset \bigcup_{k=1}^n w_{\alpha_k} \subset \bigcup_{k=1}^n \overline{w_{\alpha_k}}$. Hence, $\{w_{\alpha_1}, w_{\alpha_2}, ..., w_{\alpha_n}\}$ is a finite collection and $N \subset \bigcup_{k=1}^n \overline{w_{\alpha_k}}$. Hence, ξ is T-C-compact space.

Theorem 19
Each nearly T-extremely disconnected compact space is a T-C-compact.

Proof. Let $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is nearly T-extremely disconnected compact space, $N \subset \xi$ and $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ is τ_ξ^i-open cover of N. Then, $\{\xi_e - N\} \cup \{w_{\alpha_k} : \alpha \in \lambda, 1 \le k \le n\}$. Thus, $\{\xi_e - N\} \cup \{w_\alpha : \alpha \in \lambda\}$ constitutes an open cover of ξ. Since ξ is T-extremely disconnected, we have w_α is a τ_ξ^i- clopen set for each $\alpha \in \lambda$. On this account, $w_\alpha = \overline{w_\alpha}$ and, so, $w_\alpha^\circ = \overline{w_\alpha}$.

Therefore, $w_\alpha^\circ = w_\alpha$. Consequently, $\xi \subset \left(\bigcup_{\alpha\in\lambda} \overline{(w_{\alpha_k})^\circ}\right) \cup (\xi - N)$. Being ξ is a $T-$nearly compact space that $\xi \subset \bigcup_{k=1}^n \overline{(w_{\alpha_k})^\circ} \cup (\xi - N)$, then this immediately gives $N \subset \bigcup_{\alpha\in\lambda} \overline{(w_{\alpha_k})^\circ}$. Hence, ξ is T-C-compact space.

Theorem 20
Each quasi T-H-closed space is a T-C-compact.

Proof. Let $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ be a T-quasi-H-closed and $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ be a τ_ξ^i open cover of N, then $\{\xi_e - N\} \cup \{w_\alpha : \alpha \in \lambda\}$ composes a T-open cover of ξ.

$\xi \subset \left(\bigcup\limits_{\alpha\in\lambda, k=1}^{n} \overline{w_{\alpha_k}}\right) \cup (\xi - N)$. In addition, since $\overline{(\xi - N)}$ covers $\xi - N$, then $N \subset \bigcup\limits_{\alpha\in\lambda, k=1}^{n} \overline{w_{\alpha_k}}$ and, thus, ξ is a T-C-compact.

Theorem 21
If $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ is a T-H-closed together T-S-closed space, then ξ is C-compact.

Proof. Assume that ξ is a T-C-compact and B is a τ_ξ^i-closed subspace of ξ.

Let $\tilde{W} = \{w_\alpha : \alpha \in \lambda\}$ is a τ_ξ^1-open cover of B_1, since $\xi = B \cup (\xi - B)$ then $\tilde{W} = \{w_\alpha : \alpha \in \lambda\} \cup \{\xi - B\}$ is a τ_ξ^i-open cover of ξ, but ξ is T-C-compact, so, $\tilde{W}$ has a finite subcover say $\{w_{\alpha_1}, w_{\alpha_2}, ..., w_{\alpha_n}\} \cup \{\xi - B\}$.

Hence $\tilde{W} - \{\xi - B\}$ is a finite subcover of $\tilde{W}$ for B. Thus, B is T-C-compact.

Theorem 22
Let $\xi = (\xi, \tau_\xi^1, \tau_\xi^2, \tau_\xi^3)$ be T-extremely disconnected space, then the subsequent statements are analogous:

i. ξ is T-C-compact related to τ_ξ^i closed subspace.

ii. ξ is nearly compactness.
iii. ξ is T-quasi-H-space.

Proof.i $\rightarrow$ ii: Let ξ be T-c-compact and $\tilde{W}=\{w_\alpha:\alpha\in\lambda\}$ be a τ_ξ^i-open cover of ξ $\forall i=1,2,3$. Now, $\forall\tau_\xi^i$ closed set $B\subset\xi$, $\tilde{W}$ is a τ_ξ^i cover of B. Thus, $B\subset\bigcup_{k=1}^n w_{\alpha_k}$.

But it is T-c-compact, hence, $B\subset\bigcup_{k=1}^n w_{\alpha_k}$. Since ξ is a T-extremely disconnected space, then $\xi-B$ and w_α are τ_ξ^i-clopen set. Thus, one may get $\overline{w}_\alpha^\circ=w_{\alpha_k}$, $\forall k=1,2,\ldots,n$. As a result $B\subset\overline{w}_{\alpha_k}^\circ$. This produce instantly $\xi\subset\left(\bigcup_{k=1}^n\overline{w}_{\alpha_k}^\circ\cup\overline{(\xi-B)}\right)$. So, $\overline{(\xi-B)}$, is a τ_ξ^i-subcover of $\tilde{W}$ interior of closure of τ_ξ^i-open that covers ξ. Thus, ξ is a TX- open compact.

ii $\rightarrow$ iii: If ξ is a T- nearly compact and $\tilde{W}=\{w_\alpha:\alpha\in\lambda\}$ is a τ_ξ^i-open cover of ξ, then if processes are finite τ_ξ^i-subcover of the interior of the closure set, such that $\overline{w}_{\alpha_k}^\circ$, $\overline{(\xi-B)}$, $k=1,2,\ldots,n$ by T- nearly compactness, $\xi\subset\left(\bigcup_{k=1}^n\overline{w}_{\alpha_k}^\circ\cup\overline{(\xi-B)}\right)$ since ξ is a T- extremely disconnected space, then w_{α_k} is a τ_ξ^i-clopen set. Hence $\overline{w}_{\alpha_k}^\circ=\overline{w}_{\alpha_k}$ which consequently leads to $\xi\subset\left(\bigcup_{k=1}^n\overline{w}_\alpha\right)$. Therefore, ξ is a T-H-close space.

iii $\rightarrow$ i: If ξ is T-H-closed, $\tilde{W}=\{w_\alpha:\alpha\in\lambda\}$ is a τ_ξ^i-open cover of B where B is τ_ξ^i-closed subspace of ξ, therefore $\tilde{W}$ is a τ_ξ^i-open cover of B. Since ξ a T-H- closed space, then $B\subset\xi\subset\overline{w}_\alpha$. As well, because ξ is a T-extremely disconnected, then $B\subset\overline{w}_\alpha$. Therefore ξ is a T-C- compact.

Theorem 23
T-C-compactness has a hereditary property with respect to τ_ξ^i-closed subspace $\forall i=1,2,3$.

Proof. Let us assume that ξ is a T-C-compact and that B is closed subspace of ξ with $D\subseteq B$. Let $\tilde{W}=\{w_\alpha:\alpha\in\lambda\}$ is a τ_ξ^i-open cover of B, then $\xi=D\cup(\xi-D)$. Since $D\subset B$, then $\xi=D\cup(\xi-B)$. Consequently, $\xi=\bigcup_{\alpha\in\lambda}w_\alpha\cup(\xi-D)$, hence $\{w_\alpha,\xi-B:\alpha\in\lambda\}$ is an open cover of ξ. Now, since $\xi$is a C-compact, then every subset can be covered by a finite τ_ξ^i-subcover of closure of subset of $\tilde{W}$. So, $B\subset\bigcup_{k=1}^n\overline{w_{\alpha_k}}$. Hence B is a T-C-compact.

Theorem 24
Given $\xi=(\xi,\tau_\xi^1,\tau_\xi^2,\tau_\xi^3)$ is a T-C-T_2-space, T-c, and T- C-extremely disconnected space, then ξ is T-C-T_4.

Proof. Let $\xi=(\xi,\tau_\xi^1,\tau_\xi^2,\tau_\xi^3)$ be a T-c-compact and T-C-T_2-space. $\xi=(\xi,\tau_\xi^1,\tau_\xi^2,\tau_\xi^3)$ is a T-C-T_1-space. Assume that C and D are two τ_ξ^i-closed subsets of $\xi:C\cap D=\emptyset,\forall i=1,2,3$. Then, with reference to Theorem 23, it is concluded that C and D are τ_ξ^i-C-compact subsets of T-C-T_2. Hence, $\exists$ two τ_ξ^i-C-open sets w_C, v_D such that $C\subset\overline{w}_C$, $D\subset\overline{v}_D$ with $\overline{w}_C\cup\overline{v}_D=\emptyset$. Then, ξ is $T-C-T_4$.

Theorem 25
If $\xi=(\xi,\tau_\xi^1,\tau_\xi^2,\tau_\xi^3)$ is a T-C-T_2 and T-c-extremely disconnected space, then each subset of ξ is a τ_ξ^i-closed set $\forall i=1,2,3$.

Proof. If C is τ_ξ^i-C-compact subset of ξ and $\emptyset\neq C$, then, according to Theorem 24, $\exists$ two τ_ξ^i-C-open set $w_\emptyset$ and v_C with $w_\emptyset\cap v_C=\emptyset$. Therefore, $w\subset(v_C)^C$.

Since $C\subset v_C$, $v_C^C\subset C^C$, then $\emptyset\subset w_\emptyset\subset v_C^C\subset C^C$. Moreover, since τ_ξ^i- is open, then C^C is a τ_ξ^i-open. Thus, C is a τ_ξ^i-closed.

Theorem 26
If $\xi=(\xi,\tau_\xi^1,\tau_\xi^2,\tau_\xi^3)$ is a T-C-compact, T-C-T_2-space, and T-C-extremely disconnected , then every subset of ξ is T-C-compact iff it is a C-τ_ξ^i-closed set, for $i=1,2$ and 3.

Proof. ($\rightarrow$) If C is T-C-compact subset of ξ, then by 25, C is C-τ_ξ^i-closed set.

($\leftarrow$) Let C be a C-τ_ξ^i-closed of a T-C-compact, C-T_2-extremely disconnected space. By theorem 25, C is a $C-\tau_\xi^i$-compact.

Conclusion

This study provides clear definitions of C-compactness in tritopological, bitopological, and topological spaces. It also points out various features of these spaces and discusses their relationships with other topological elements. During the research process, bitopological and tripartite spaces were hypothetically formed in this work.

Future research is recommended to examine whether or not the conclusions drawn by the researchers from the results of this study can contribute to derivation of other innovative theorems relating to the mappings and finite products of feebly pairwise expandable spaces, fuzzy tripartite topological spaces, and tripartite expandable spaces.

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Author Contribution

1. The lead researcher ”Nabeela Abu_Alkishik” developed the title and main idea, wrote the basic definition of this paper, discussed the fundamental theories, and provided overall supervision of all the results.
2. The second researcher ”Jafar Ahmed” developed a number of theories, reviewed the results, discussed the main theories, and provided overall supervision of all the findings.
3. The third researcher ”Diana Mahmoud” wrote the main introduction to the research and scientifically reviewed it, along with a general review of the research theories and results.
4. The fourth researcher” Hamza Qoqazeh ”wrote the research summary and reviewed the main introduction of the research, along with a general review of the research theories and results.
5. The fifth researcher”Eman AlMuhur” wrote and reviewed the scientific references for the research, along with reviewing the main findings of the research, and proofreading the research linguistically and printing-wise.

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Conflicts of Interest

I declare and acknowledge that none of the authors have competing personal interests or objectives as defined by Nature Research, or other interests that are or may be perceived to influence the conclusions reached or the overall discussion contained in this paper.

REFERENCES

1. J.Kelly, Bitopological spaces, Proceedings of the London Mathematical Society, Vol.3, No. 1, 1963, pp. 71-89.
2. Y.W. Kim, Pairwise compactness, Publ. Math. Debrecen, Vol.15,1968, pp. 87-90.
3. I. E.Cooke and I. L. Reilly, On bitopological compactness, Journal of the London Mathematical Society, Vol.2, No.4, 1975, pp. 518-522.

4. T.Birsan, Compacité dans les espaces bitopologiques (Compactness in bitopological spaces), An. st. Univ. Iasi, s. I. a., Matematica, Vol.15, 1969, pp. 317-328.
5. A.A.Fora and H.Z.Hdeib, On pairwise Lindelöf spaces, Revista Colombiana de Matematicas, Vol.17, No. (1-2), 1983, pp. 37-57.
6. A. Atoom, H. Qoqazeh, Y. Al-Qudah, A. Jaradat and N.A. Alkishik. Pairwise Lindelöf perfect functions. J.Math.Comput.Sci, 2021, Vol.11, No.6, pp.7634-7648.
7. A. Atoom , H. Qoqazeh and N.A. Alkishik. Lindelöf perfect functions. JPJ.Geom.Topol, 2021, Vol.26, No.2, pp.91-101.
8. M. M. Kovár, On 3-topological version of θregularity, International Jurnal of Mathematics and Mathematical Sciences,Vol.23,No.6, 2000, pp.393-398.
9. N.F.Hameed and M. Y.Abid, certain types of separations axioms in tri topological spaces, Iraqi Journal of Science, Vol.52, No. 2, 2011, pp. 212-213.
10. S.Palaniammal, A study of tritopological spaces, Ph.D Thesis, 2011.
11. A. L.Sweedy and A. F.Hassan, δ-continuous function in tri topological space, Journal of Basrah Research Sc, Vol.37, No.4, 2011, pp.66-74.
12. U. D.Tapi, R.Sharma and B. A.Dede,Semi open sets and pre open sets in Tritopological spaces, i-Manager's Journa
13. J. Dugundji, (1966). Topology, Allyn and Bacon, Boston.
14. N. Levine, (1963). Semi-Open Sets and Semi-Continuity in Topological Spaces, Amer. Math. Monthly, 70, 36-41.
15. Hazim, H. T., Alturky, M. A. A., Al-Sharqi, F., Romdhini, M. U., & Shoja, S. J. (2026). A certain algebraic structure of bipolar neutrosophic subgroups. Statistics, Optimization & Information Computing, 16(1), 403-427.
16. J. Oudetallah, Nearly Expandability in bitopological spaces, Advances in Mathematics: Scientific Journal 10 (2021a), 705-712.
17. J. Oudetallah, ON FEEBLY PAIRWISE EXPANDABLE SPACE, J. Math. Comput. Sci. 11 (2021 b), No. 5, 6216-6225.
18. Pandiselvi, M., & Jeyaraman, M. (2025). New Insights into Fixed Points Results in Neutrosophic Metric Like-Spaces and Their Applications. Statistics, Optimization & Information Computing.
19. Qawaqneh, H., Sharma, R., Singh, D., & Kumar, P. (2026). New Types of Integral Contractions in Supra Metric Space. Statistics, Optimization & Information Computing, 15(6), 5324-5337.
20. H. Qoqazeh, A. Atoom, A. Jaradat, E. Almuhur, & N. Abu-Alkishik, (2023). A new study on tri-lindelöf spaces. Wseas Transcations on Mathematics, 22.
21. S. Willard, (1970). General Topology, Addison- Wesley Publishing Company, Inc. [1] J.Kelly, Bitopological spaces, Proceedings of the London Mathematical Society, Vol.3, No. 1, 1963, pp. 71-89.
22. M. M.Kover, On 3-topological version of θregularity, International Jurnal of Mathematics and Mathematical Sciences,Vol.23,No.6, 2000, pp.393-398.
23. N.F.Hameed and M. Y.Abid, certain types of separations axioms in tri topological spaces, Iraqi Journal of Science, Vol.52, No. 2, 2011, pp. 212-213.
24. S.Palaniammal, A study of tritopological spaces, Ph.D Thesis, 2011.
25. A. L.Sweedy and A. F.Hassan, δ-continuous function in tri topological space, Journal of Basrah Research Sc, Vol.37, No.4, 2011, pp.66-74.
26. U. D.Tapi, R.Sharma and B. A.Dede,Semi open sets and pre open sets in Tritopological spaces, i-Manager's Journa.
27. I. Dochviri,. On some properties of semicompact and S-closed bitopological spaces. In Proc. A. Razmadze Math. Inst Vol. 123, 2000 , pp. 15-22.
28. T.Al-Hawary, & A. Al-Omari,b-open and b-continuity in Bitopological Spaces. Al-Manarah, 13(3), 2007, 89-101.
29. P. Fletcher, H.B. Hoyle and C.W. Patty , The comparison of topologies, Duke Math. J. 36 1969, 325-331.
30. M. C. Datta, Projective bitopological spaces, J. Austral Math. Soc.13, 1972, 327-334 .