STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, July 2026, pp 698–715.
Published online in International Academic Press (www.IAPress.org)

Restricted Bayesian Inference for the Misspecified Random Repeated Measurements Model

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Abstract This article presents a new technique for Bayesian inference in the random repeated measurements model. The fundamental idea underlying this work is the application of Bayesian inference to estimate the model parameters of interest. We then focus on theoretical results that allow the incorporation of linear constraints into the Bayesian estimator under model misspecification. It is essential to investigate the mathematical properties of the parameters; accordingly, this article also examines the asymptotic properties of the restricted Bayesian estimator for the misspecified repeated measurements model. It is shown that the Bayesian estimator of the second component of the parameter vector, under an underfitted model, is weakly consistent provided that certain conditions are satisfied. Moreover, in the presence of linear constraints, overfitting reduces the asymptotic efficiency of the Bayesian estimator of the second component of the parameter vector under certain conditions.

Keywords Bayesian inference, repeated measurements model, model misspecification, restricted Bayesian estimation, asymptotic properties, weak consistency

AMS 2010 subject classifications 62F15, 62J05, 62F30, 62F35.

DOI: 10.19139/soic-2310-5070-3842

1. Introduction

There are several reasons to employ the repeated measurements model, which is one of the most important and widely used models across various scientific domains, including engineering, agriculture, and medical sciences. The primary objective of a repeated measurements model is to investigate and analyze patterns in response variables over time. In addition, measuring the same subject more than once is often more informative than measuring several subjects only once. Numerous studies in the literature have addressed the repeated measurements framework; some of these are briefly mentioned here, including Al-Mouel and Wang [1], Naik and Rao [2], Schober and Vetter [3], Mohaisen and AL-Mouel [4], Kaifeng et al. [5], and Savitsky et al. [6].

There are also several reasons why a misspecified model may be intentionally employed. Typically, model selection is based on the interpretability of the parameters with respect to certain quantities within the context of the problem. Another reason for adopting simpler but misspecified models is mathematical convenience. In most situations, the estimation process becomes easier if the number of distribution functions is small. Furthermore, the need for computational simplicity might lead to adopting an overly simple but misspecified model. This will imply that there is always a balance between having a more detailed model which provides better estimation, and a more compact and restrictive model that is easy to interpret and compute [7, 8]. Bayesian inference is regarded as one of the most effective statistical approaches for integrating data in complex settings, a task that

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

has become increasingly important in modern applications. Nevertheless, Bayesian inference under a misspecified model may lead to suboptimal results. Several studies have addressed this issue, including those by Frazier, Robert, and Rousseau [9], Grünwald and Van Ommen [10], Stephens et al. [11], Nott, Drovandi, and Frazier [12], Blasi and Walker [13], Lanzani [14], Weerasinghe et al. [15], and Al-Isawi et al. [16, 17]. This article is, to the best of our knowledge, the first attempt to estimate the parameters of a repeated measurements model that incorporates a random effect in addition to random error using a novel approach. The article focuses on Bayesian inference for the model parameters under misspecification, with linear constraints imposed on the parameters, and establishes several mathematical properties of the Bayesian estimator in this setting. A practical experiment is also conducted using simulation methods, in which data are generated according to the mathematical model proposed in this study. The general least squares method, the Bayesian method, and the constrained Bayesian method are then applied by incorporating a set of linear constraints on the model parameters. Particular attention is given to the central research issue of model misspecification, considering both underfitting and overfitting scenarios.

Moreover, three prior probability distributions for the parameter vector were considered. The criteria used to compare the estimation methods are the average mean squared error (AMSE) and the average mean absolute error (AMAE).

The remainder of the article is organized as follows. Section 2 describes the materials, definitions, and preliminary results required for this work, and presents the basic concepts underlying the research. Section 3 addresses the main research problem, providing a detailed description of the model, including the mathematical formulation and an investigation of the generalized least squares and Bayesian estimators of the parameters. This section also considers Bayesian estimation under linear constraints and verifies the asymptotic properties of the restricted Bayesian estimator for the repeated measurements model under misspecification, as well as its finite-sample properties. Section 4 presents the results, including a practical application using simulation. The discussion and conclusion are provided in Section 5.

2. Materials

In this section, we outline the materials, definitions, and preliminary results required to support the mathematical development of this work. By presenting these auxiliary results and basic concepts, we establish the necessary theoretical framework to investigate Bayesian inference and model misspecification in the context of random repeated measurements. The following definitions and theorems are taken from [18] and [19].

Definition 1
Let A be $p \times p$ matrix. We say that A is a nonnegative definite and write $A \geq 0$ if $b^{'}\text{Ab} \geq 0$ for all $p-$ dimensional vectors b.

Definition 2
Let A be $p \times p$ matrix. We say that A is a positive definite and write $A > 0$ if $b^{'}\text{Ab} > 0$ for all nonzero $p-$ dimensional vectors b.

Theorem 1
If a and b are jointly multivariate normal with $\Sigma_{\text{ba}} \neq 0$, then the conditional distribution of b given a is multivariate normal with $E\left(b|a\right) = \mu_b + \Sigma_{\text{ba}}\Sigma_{\text{aa}}^{-1}\Sigma_{\text{ab}}$, and
$\text{cov}\left(b|a\right) = \Sigma_{\text{bb}} - \Sigma_{\text{ba}}\Sigma_{\text{aa}}^{-1}\Sigma_{\text{ab}}$.

Theorem 2
Consider the partitioned matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. *If A and A_{11} be nonsingular, then*
$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}A_{22.1}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22.1}^{-1} \\ -A_{22.1}^{-1}A_{21}A_{11}^{-1} & A_{22.1}^{-1} \end{pmatrix}$, *where $A_{22.1}^{-1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$, and if A and A_{22} be nonsingular, then*
$A^{-1} = \begin{pmatrix} A_{11.2}^{-1} & -A_{11.2}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{11.2}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}A_{11.2}^{-1}A_{12}A_{22}^{-1} \end{pmatrix}$, *where $A_{11.2}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.*

3. Model and Methodology

In this section, we investigate the main research problem by detailing the mathematical formulation of the random repeated measurements model and the resulting generalized least squares and Bayesian estimators. Furthermore, we examine the behavior of these estimators under linear constraints and establish both their asymptotic and finite-sample properties within the context of model misspecification.

3.1. Problem description

The model presented in this study is as follows:

$$Y = \alpha j_N + X\beta + u + \mathcal{E} = Z\varphi + \Lambda, \tag{1}$$

such that Y is $N \times 1$ response vector, where $N = n \times \tau$, j_N is an $N \times 1$ vector of ones, $Z = [j_N \;\; X]_{N\times(K+1)}$, and $\varphi = [\alpha \; \beta]^T$ is the parameter vector. The covariance matrix of Λ is given by

$$\begin{aligned}\Omega &= E(\Lambda\Lambda^T)\\ &= \sigma_\varepsilon^2(I_n \otimes (I_\tau - \tau^{-1}J_\tau)) + (\sigma_\varepsilon^2 + \tau\sigma_u^2)(I_n \otimes \tau^{-1}J_\tau)\\ &= \sigma_\varepsilon^2(I_n \otimes I_\tau) + \sigma_u^2(I_n \otimes J_\tau)\\ &= I_n \otimes (\sigma_\varepsilon^2 I_\tau + \sigma_u^2 J_\tau).\end{aligned} \tag{2}$$

where J_τ is a matrix of ones. From (2), we obtain the inverse covariance matrix (3) as

$$\Omega^{-1} = I_n \otimes (\sigma_\varepsilon^2 I_\tau + \sigma_u^2 J_\tau)^{-1}. \tag{3}$$

The general least squares estimator of φ is defined in (4) as

$$\widehat{\varphi}_{GL} = (Z^T\Omega^{-1}Z)^{-1}Z^T\Omega^{-1}Y. \tag{4}$$

The Bayesian estimator of φ, assuming a prior distribution $N(m, S)$, is given by (5) as

$$\widehat{\varphi}_B = (Z^T\Omega^{-1}Z + S^{-1})^{-1}(Z^T\Omega^{-1}Y + S^{-1}m). \tag{5}$$

Finally, under linear constraints $A\varphi = a$, the restricted Bayesian estimator is defined in (6) as

$$\widehat{\varphi}_{CB} = \widehat{\varphi}_B + (Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T\left[A(Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T\right]^{-1}(a - A\widehat{\varphi}_B). \tag{6}$$

3.2. Misspecification Repeated Measurements Model

Let the model in (1) be partitioned as in (7).

$$Y = Z_1\varphi_1 + Z_2\varphi_2 + \Lambda. \tag{7}$$

where Z_1 and Z_2 are of dimensions $(N \times K_1)$ and $(N \times K_2)$, respectively, and φ_1 and φ_2 are of dimensions $(K_1 \times 1)$ and $(K_2 \times 1)$, where $K_2 = K - K_1 + 1$ and $K_1 < K$.

Correspondingly, we partition $Z = (Z_1, Z_2)$, $\varphi = (\varphi_1^T, \varphi_2^T)^T$, and the matrix $Z^T\Omega^{-1}Z + S^{-1}$ conformably as

$$Z^T\Omega^{-1}Z + S^{-1} = \begin{pmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{pmatrix}, \tag{8}$$

where the block matrix in (8) defines $\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}$ are the corresponding block matrices.

Theorem 3
Suppose the model (1) under model misspecification with linear constraints $A_{h\times(K+1)}\varphi = a_{h\times 1}$. Let $A = (I \quad 0)$ and $a = 0$, then the conditional expected and the mode of φ_2 given $\varphi_1 = 0$ is $\widehat{\varphi}_{2CB}$.

Proof
Since

$$f(\varphi|Y) \propto \exp\left\{-\frac{1}{2}(\varphi - \widehat{\varphi}_B)^T \left(Z^T\Omega^{-1}Z + S^{-1}\right)(\varphi - \widehat{\varphi}_B)\right\},$$

we can write the joint posterior as

$$f(\varphi_1, \varphi_2|Y) \propto \exp\left\{-\frac{1}{2}\begin{bmatrix}\varphi_1 - \widehat{\varphi}_{1B}\\ \varphi_2 - \widehat{\varphi}_{2B}\end{bmatrix}^T \begin{pmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{pmatrix}\begin{bmatrix}\varphi_1 - \widehat{\varphi}_{1B}\\ \varphi_2 - \widehat{\varphi}_{2B}\end{bmatrix}\right\}.$$

Expanding the quadratic form gives

$$\propto \exp\left\{-\frac{1}{2}\Big[(\varphi_1 - \widehat{\varphi}_{1B})^T\psi_{11}(\varphi_1 - \widehat{\varphi}_{1B}) + 2(\varphi_1 - \widehat{\varphi}_{1B})^T\psi_{12}(\varphi_2 - \widehat{\varphi}_{2B}) + (\varphi_2 - \widehat{\varphi}_{2B})^T\psi_{22}(\varphi_2 - \widehat{\varphi}_{2B})\Big]\right\}.$$

Completing the square in φ_2, we obtain

$$\propto \exp\left\{-\frac{1}{2}\left(\varphi_2 - \widehat{\varphi}_{2B} + \psi_{22}^{-1}\psi_{21}(\varphi_1 - \widehat{\varphi}_{1B})\right)^T \psi_{22}\left(\varphi_2 - \widehat{\varphi}_{2B} + \psi_{22}^{-1}\psi_{21}(\varphi_1 - \widehat{\varphi}_{1B})\right)\right.$$

$$\left. -\frac{1}{2}(\varphi_1 - \widehat{\varphi}_{1B})^T(\psi_{11} - \psi_{12}\psi_{22}^{-1}\psi_{21})(\varphi_1 - \widehat{\varphi}_{1B})\right\}.$$

Hence, from the multivariate normal distribution, the conditional mean is

$$E(\varphi_2|\varphi_1) = \widehat{\varphi}_{2B} - \psi_{22}^{-1}\psi_{21}(\varphi_1 - \widehat{\varphi}_{1B}).$$

In particular, for $\varphi_1 = 0$,

$$E(\varphi_2|\varphi_1 = 0) = \widehat{\varphi}_{2B} + \psi_{22}^{-1}\psi_{21}\widehat{\varphi}_{1B},$$

and taking $a = 0$, we obtain the constrained estimator.

$$\widehat{\varphi}_{CB} = \widehat{\varphi}_B - \left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T\left[A(Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T\right]^{-1} A\widehat{\varphi}_B,$$

Using the partitioned inverse of ψ,

$$\begin{pmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{pmatrix}^{-1} = \begin{pmatrix}\psi^{11} & \psi^{12}\\ \psi^{21} & \psi^{22}\end{pmatrix},$$

standard algebra yields

$$\widehat{\varphi}_{2CB} = \widehat{\varphi}_{2B} + \psi_{22}^{-1}\psi_{21}\widehat{\varphi}_{1B}.$$

Moreover, from the conditional density of the multivariate normal distribution, the mode coincides with the conditional mean, hence

$$\text{mode}(\varphi_2|\varphi_1 = 0) = \widehat{\varphi}_{2CB}.$$

This completes the proof. □

3.3. Asymptotic properties

In this sub-section, we investigate the asymptotic properties of the restricted Bayesian estimator under the misspecification repeated measurements model.

Theorem 4
Suppose $Y = Z\varphi + \Lambda$ under model misspecification with $A_{h\times(K+1)}\varphi = a_{h\times 1}$. Let $A = (I \;\; 0)$ and $a = 0$. Then, $\widehat{\varphi}_{2CB}$ under the underfitting model is weakly consistent if $\phi_{12} = 0$, where

$$\phi = \lim_{N\to\infty} \frac{1}{N} Z^T \Omega^{-1} Z = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

Proof
The constrained Bayesian estimator is given by

$$\widehat{\varphi}_{CB} = \widehat{\varphi}_B + \left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T \left[A\left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T\right]^{-1} (a - A\varphi).$$

Taking probability limits as $N \to \infty$, we obtain

$$\text{plim}_{N\to\infty}\, \widehat{\varphi}_{CB} = \varphi + \phi^{-1}A^T \left(A\phi^{-1}A^T\right)^{-1} (a - A\varphi).$$

Now let $a = 0$ and $A = (I \;\; 0)$. Then

$$\text{plim}_{N\to\infty}\, \widehat{\varphi}_{CB} = \varphi - \phi^{-1}A^T \left(A\phi^{-1}A^T\right)^{-1} A\varphi.$$

Using the block inverse structure of ϕ^{-1},

$$\phi^{-1} = \begin{pmatrix} \phi_{11}^{-1} & 0 \\ -\phi_{22}^{-1}\phi_{21}\phi_{22}^{-1} & \phi_{22}^{-1} \end{pmatrix},$$

we substitute into the expression to obtain

$$\text{plim}_{N\to\infty} \begin{pmatrix} \widehat{\varphi}_{1CB} \\ \widehat{\varphi}_{2CB} \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} - \begin{pmatrix} I & 0 \\ -\phi_{22}^{-1}\phi_{21} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

Hence,

$$\text{plim}_{N\to\infty} \begin{pmatrix} \widehat{\varphi}_{1CB} \\ \widehat{\varphi}_{2CB} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_2 + \phi_{22}^{-1}\phi_{21}\varphi_1 \end{pmatrix}.$$

□

Theorem 5
Suppose we have a misspecified repeated measurements model with linear constraints $A_{h\times(K+1)}\varphi = a_{h\times 1}$. Let $A = (I \;\; 0)$ and $a = 0$. Then, overfitting reduces the asymptotic efficiency of the Bayes estimator of φ_2, and underfitting increases its asymptotic precision, unless $\phi_{12} = 0$, where

$$\phi = \lim_{N\to\infty} \left(\frac{1}{N} Z'\Omega^{-1}Z\right) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

Proof
Since

$$\text{var}\Big(\sqrt{N}(\widehat{\varphi}_B - \varphi)\Big) = \left(\frac{1}{N} Z^T\Omega^{-1}Z\right)^{-1},$$

we obtain the partitioned form

$$\phi^{-1} = \begin{pmatrix} \phi^{11} & \phi^{12} \\ \phi^{21} & \phi^{22} \end{pmatrix}.$$

Hence,

$$\operatorname{var}\left(\sqrt{N}(\widehat{\varphi}_{2B}-\varphi_2)\right)=\phi_{22}^{-1}+\phi_{22}^{-1}\phi_{21}\left(\phi_{11}-\phi_{12}\phi_{22}^{-1}\phi_{21}\right)^{-1}\phi_{12}\phi_{22}^{-1} \tag{9}$$

Also, for the constrained Bayes estimator,

$$\operatorname{var}\left(\sqrt{N}(\widehat{\varphi}_{CB}-\varphi)\right)=\left[I-\phi^{-1}A^T\left(A\phi^{-1}A^T\right)^{-1}A\right]\phi^{-1}\left[I-A^T\left(A\phi^{-1}A^T\right)^{-1}A\phi^{-1}\right].$$

For $A=(I\ 0)$, this simplifies to

$$\operatorname{var}\left(\sqrt{N}(\widehat{\varphi}_{2CB}-\varphi_2)\right)=\phi_{22}^{-1}. \tag{10}$$

Thus, if $\phi_{12}=0$, then from (9) and (10),

$$\operatorname{var}\left(\sqrt{N}(\widehat{\varphi}_{2B}-\varphi_2)\right)=\operatorname{var}\left(\sqrt{N}(\widehat{\varphi}_{2CB}-\varphi_2)\right),$$

whereas if $\phi_{12}\neq 0$, the difference of the variances above is positive definite, implying loss of efficiency under overfitting and gain under underfitting. □

3.4. The finite sample

In the following theorems, we investigate the finite sample theory to the restricted Bayesian estimator of the misspecification random repeated measurements model.

Theorem 6
Suppose overfitting repeated measurements model

$$Y=Z_1\varphi_1+Z_2\varphi_2+\Lambda$$

with

$$A_{h\times(K+1)}\varphi=a_{h\times 1}.$$

Let $A=(I\ \ 0)$ and $a=0$. If $\varphi_1=0$, then

$$\operatorname{Bias}(\widehat{\varphi}_{\mathrm{2CB}})=\operatorname{Bias}(\widehat{\varphi}_{2B})+\psi_{22}^{-1}\psi_{21}\operatorname{Bias}(\widehat{\varphi}_{\mathrm{1B}}).$$

Proof
Since

$$\widehat{\varphi}_B=\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}\left(Z^T\Omega^{-1}Y+S^{-1}m\right),$$

and using $Z^T\Omega^{-1}Z\widehat{\varphi}_{\mathrm{GL}}=Z^T\Omega^{-1}Y$, we obtain

$$\begin{aligned}\widehat{\varphi}_B&=\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}S^{-1}m\\&\quad+\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}\left(Z^T\Omega^{-1}Z\,\widehat{\varphi}_{\mathrm{GL}}\right).\end{aligned} \tag{11}$$

Using the matrix identity in (11), we can rewrite the estimator as

$$\begin{aligned}\widehat{\varphi}_B&=\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}S^{-1}m\\&\quad+\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}Z^T\Omega^{-1}Z\,\widehat{\varphi}_{\mathrm{GL}}.\end{aligned} \tag{12}$$

From (12), taking expectation yields

$$E(\widehat{\varphi}_B)=\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}S^{-1}m+\varphi-S^{-1}\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}\varphi,$$

thus

$$\operatorname{Bias}(\widehat{\varphi}_B)=\left(Z^T\Omega^{-1}Z+S^{-1}\right)^{-1}S^{-1}(m-\varphi).$$

Now, using (11)–(12) in the constrained estimator,

$$\widehat{\varphi}_{\mathrm{CB}} = \widehat{\varphi}_B + \left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T \left[A\left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T\right]^{-1} (a - A\widehat{\varphi}_B),$$

we obtain

$$\mathrm{Bias}(\widehat{\varphi}_{\mathrm{CB}}) = \ell \mathrm{Bias}(\widehat{\varphi}_B) + \left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T \left[A\left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T\right]^{-1} (a - A\varphi).$$

where $\ell = \left[I - (Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T[A(Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T]^{-1}A\right]$.

Since $a = 0$ and $A = (I \;\; 0)$, it follows that

$$\mathrm{Bias}(\widehat{\varphi}_{\mathrm{CB}}) = \ell \mathrm{Bias}(\widehat{\varphi}_B) - \left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T \left[A\left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T\right]^{-1} A\varphi.$$

Using the partitioned form implied by (11)–(12), we obtain

$$\mathrm{Bias}(\widehat{\varphi}_{2\mathrm{CB}}) = \mathrm{Bias}(\widehat{\varphi}_{2B}) + \psi_{22}^{-1}\psi_{21}\,\mathrm{Bias}(\widehat{\varphi}_{1\mathrm{B}}).$$

Finally, since $\varphi_1 = 0$, the result follows. □

Theorem 7

Suppose that the overfitting repeated measurements model

$$Y = Z_1\varphi_1 + Z_2\varphi_2 + \Lambda$$

with

$$A_{h\times(K+1)}\varphi = a_{h\times 1}.$$

Let $A = (I \;\; 0)$ and $a = 0$. If $\varphi_1 \neq 0$, then

$$\mathrm{Bias}(\widehat{\varphi}_{2\mathrm{CB}}) = \mathrm{Bias}(\widehat{\varphi}_{2B}) + \psi_{22}^{-1}\psi_{21}\left[\mathrm{Bias}(\widehat{\varphi}_{1B}) + \varphi_1\right].$$

Proof

Since

$$\begin{pmatrix}\mathrm{Bias}(\widehat{\varphi}_{1\mathrm{CB}})\\ \mathrm{Bias}(\widehat{\varphi}_{2\mathrm{CB}})\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \mathrm{Bias}(\widehat{\varphi}_{2B}) + \psi_{22}^{-1}\psi_{21}\,\mathrm{Bias}(\widehat{\varphi}_{1B}) + \psi_{22}^{-1}\psi_{21}\varphi_1\end{pmatrix},$$

then from Theorem 6, if $\widehat{\varphi}_1 \neq 0$, we obtain

$$\mathrm{Bias}(\widehat{\varphi}_{2\mathrm{CB}}) = \mathrm{Bias}(\widehat{\varphi}_{2B}) + \psi_{22}^{-1}\psi_{21}\,\mathrm{Bias}(\widehat{\varphi}_{1B}) + \psi_{22}^{-1}\psi_{21}\varphi_1.$$

Factoring the last two terms yields

$$\mathrm{Bias}(\widehat{\varphi}_{2\mathrm{CB}}) = \mathrm{Bias}(\widehat{\varphi}_{2B}) + \psi_{22}^{-1}\psi_{21}\left[\mathrm{Bias}(\widehat{\varphi}_{1B}) + \varphi_1\right].$$

□

Theorem 8

Suppose we have repeated measurements model with linear constraints

$$A_{h\times(K+1)}\varphi = a_{h\times 1}.$$

Let $A = (I \;\; 0)$ and $a = 0$. Then underfitting (reduced model) improves the matrix $\mathrm{MMSE}(\widehat{\varphi}_B)$ if and only if

$$\gamma = \varphi_1^T\left[\mathrm{MMSE}(\widehat{\varphi}_{11B})\right]^{-1}\varphi_1 \leq 1.$$

Proof
Since

$$\widehat{\varphi}_B = (Z^T\Omega^{-1}Z + S^{-1})^{-1}S^{-1}m + (Z^T\Omega^{-1}Z + S^{-1})^{-1}Z^T\Omega^{-1}Z\widehat{\varphi}_{GL},$$

then

$$E(\widehat{\varphi}_B) = (Z^T\Omega^{-1}Z + S^{-1})^{-1}S^{-1}m + \varphi - S^{-1}(Z^T\Omega^{-1}Z + S^{-1})^{-1}\varphi,$$

and

$$\text{Bias}(\widehat{\varphi}_B) = E(\widehat{\varphi}_B) - \varphi = (Z^T\Omega^{-1}Z + S^{-1})^{-1}S^{-1}(m - \varphi) = PS^{-1}F,$$

where

$$P = (Z^T\Omega^{-1}Z + S^{-1})^{-1}, \qquad F = (m - \varphi).$$

Since

$$\text{Bias}(\widehat{\varphi}_{CB}) = \ell\,\text{Bias}(\widehat{\varphi}_B) + (Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T\left[A(Z^T\Omega^{-1}Z + S^{-1})^{-1}A^T\right]^{-1}(a - A\varphi),$$

then

$$\text{Bias}(\widehat{\varphi}_{CB}) = \ell\,\text{Bias}(\widehat{\varphi}_B) + PA^T(APA^T)^{-1}(a - A\varphi),$$

where

$$\ell = \left[I - PA^T(APA^T)^{-1}A\right].$$

Then

$$\text{MMSE}(\widehat{\varphi}_B) = P(Z^T\Omega^{-1}Z)P + PS^{-1}FF^TS^{-1}P,$$

and

$$\text{MMSE}(\widehat{\varphi}_{CB}) = \ell\,P(Z^T\Omega^{-1}Z)P\,\ell^T + \ell\,PS^{-1}FF^TS^{-1}P\,\ell^T.$$

Let

$$C = \text{MMSE}(\widehat{\varphi}_B) - \text{MMSE}(\widehat{\varphi}_{CB}),$$

then

$$\begin{aligned} C = \text{MMSE}(\widehat{\varphi}_B) - \ell\,\text{MMSE}(\widehat{\varphi}_B)\,\ell^T - PA^T(APA^T)^{-1}(a - A\varphi)(a - A\varphi)^T(APA^T)^{-1}AP \\ -\ell\,PS^{-1}F(a - A\varphi)^T(APA^T)^{-1}AP \\ -PA^T(APA^T)^{-1}(a - A\varphi)F^TS^{-1}P\ell^T. \end{aligned}$$

Hence,

$$\begin{aligned} ACA^T = A\left\{\text{MMSE}(\widehat{\varphi}_B) - \ell\,[\text{MMSE}(\widehat{\varphi}_B)]\ell^T\right\}A^T \\ -APA^T(APA^T)^{-1}(a - A\varphi)(a - A\varphi)^T(APA^T)^{-1}APA^T \\ -A\ell PS^{-1}F(a - A\varphi)^T(APA^T)^{-1}APA^T \\ -APA^T(APA^T)^{-1}(a - A\varphi)F^TS^{-1}P\ell^TA^T. \end{aligned}$$

Then, as

$$A\ell = 0$$

we get

$$ACA^T = A\,\text{MMSE}(\widehat{\varphi}_B)A^T - (a - A\varphi)(a - A\varphi)^T.$$

And, $\widehat{\varphi}_{CB}$ is preferred to $\widehat{\varphi}_B$ if ACA^T is at least positive semidefinite, i.e. if and only if

$$\eta^T\left(A\,\text{MMSE}(\widehat{\varphi}_B)A^T - (a - A\varphi)(a - A\varphi)^T\right)\eta \geq 0, \qquad \forall\,\eta \neq 0.$$

This indicates that

$$-\eta^T\left[(a - A\varphi)(a - A\varphi)^T\right]\eta \geq -\eta^T\left(A\,\text{MMSE}(\widehat{\varphi}_B)A^T\right)\eta,$$

or if and only if

$$\zeta = \frac{\eta^T \left[(a - A\varphi)(a - A\varphi)^T\right] \eta}{\eta^T \left(A\,\mathrm{MMSE}(\widehat{\varphi}_B)A^T\right) \eta} \leq 1, \qquad \forall\, \eta \neq 0.$$

This inequality holds for all η if and only if

$$\gamma = \sup_\eta(\zeta) \leq 1,$$

which is the necessary and sufficient condition to provide

$$\gamma = (a - A\varphi)^T \left[A\,\mathrm{MMSE}(\widehat{\varphi}_B)A^T\right]^{-1} (a - A\varphi) \leq 1.$$

Now, if $a = 0$ and $A = (I \;\; 0)$, then

$$\gamma = (-A\varphi)^T \left[A\,\mathrm{MMSE}(\widehat{\varphi}_B)A^T\right]^{-1} (-A\varphi) \leq 1$$

$$= \varphi^T A^T \left[A\,\mathrm{MMSE}(\widehat{\varphi}_B)A^T\right]^{-1} A\varphi \leq 1$$

$$= \begin{pmatrix}\varphi_1^T & \varphi_2^T\end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} \left[(0 \;\; I) \begin{pmatrix} \mathrm{MMSE}(\widehat{\varphi}_{11B}) & \mathrm{MMSE}(\widehat{\varphi}_{12B}) \\ \mathrm{MMSE}(\widehat{\varphi}_{21B}) & \mathrm{MMSE}(\widehat{\varphi}_{22B}) \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}\right]^{-1} (I \;\; 0) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \leq 1$$

$$= \varphi_1^T \left[(0 \;\; I) \begin{pmatrix} \mathrm{MMSE}(\widehat{\varphi}_{11B}) \\ \mathrm{MMSE}(\widehat{\varphi}_{21B}) \end{pmatrix}\right]^{-1} \varphi_1 \leq 1$$

$$\Longrightarrow \quad \varphi_1^T \left[\mathrm{MMSE}(\widehat{\varphi}_{11B})\right]^{-1} \varphi_1 \leq 1.$$

□

4. Simulation Results

This section presents a practical application based on a simulation study. Statistical data were generated according to the random repeated measurements model and analyzed using the theoretical methods developed in this research. The simulation approach was employed to illustrate the applicability of the proposed estimators and to validate the theoretical findings.

The response vector was generated according to model (1). There were four distinct values of sample size, which were $75, 150, 300$, and 500. The number of sections was set at five possible values for each sample size, such that $2, 3, 5, 7$, and 9. For each sample size, the study was repeated 1000 times.

The model included seven explanatory variables. Furthermore, four linear restrictions on the model parameters were imposed as follows:

$$\begin{aligned} \beta_1 + \beta_2 - 4\beta_3 &= 2\beta_6 - \beta_7, \\ 3\beta_1 + \beta_3 - \beta_4 &= \beta_2 + 9\beta_5, \\ 5\beta_2 + \beta_3 + \beta_5 &= 3\beta_4 + 2\beta_7, \\ \beta_1 + 4\beta_2 - \beta_6 &= 2\beta_4 - 3\beta_5. \end{aligned}$$

In addition, the underfit model contained four variables. In connection with the distribution of the parameter vector prior to the Bayesian approach, three conjugate distributions were investigated. For each prior, the Bayes' estimator was estimated, followed by its constrained form for the case of misspecification.

The prior distributions are specified as follows:

(i) **Conjugate prior distribution:**

$$\varphi \sim N_8(m_1, S_1),$$

where

$$m_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad S_1 = \mathrm{diag}(9,\ 1,\ 3,\ 2.4,\ 6,\ 10,\ 0.75,\ 5).$$

(ii) **Conjugate prior distribution:**

$$\varphi \sim N_8(m_2, S_2),$$

where

$$m_2 = \begin{pmatrix} 6 \\ -3 \\ 12 \\ -5 \\ 4 \\ 10 \\ 0 \\ 8 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0.93 & -3 & 0.41 & 1.22 & 0.91 & -0.67 & 0.036 & 0.8 \\ -3 & 0.50 & 0.86 & 0.71 & 2.3 & 3.01 & 1.77 & 0.42 \\ 0.41 & 0.86 & 3 & 0.09 & 0.88 & 0.22 & -0.66 & 1.03 \\ 1.22 & 0.71 & 0.09 & 2.4 & 6 & 0.074 & 0.59 & -0.91 \\ 0.91 & 2.3 & 0.88 & 6 & 0.44 & 1.05 & 4.1 & 2.01 \\ -0.67 & 3.01 & 0.22 & 0.074 & 1.05 & 10 & 1.91 & 0.26 \\ 0.036 & 1.77 & -0.66 & 0.59 & 4.1 & 1.91 & 1.1 & 4.04 \\ 0.8 & 0.42 & 1.03 & -0.91 & 2.01 & 0.26 & 4.04 & 5 \end{pmatrix}.$$

(iii) **Conjugate prior distribution:**

$$\varphi \sim N_8(m_3, S_3),$$

where

$$m_3 = \begin{pmatrix} 0.6 \\ 0.13 \\ 0.25 \\ 1.09 \\ 0 \\ 7.3 \\ 2.8 \\ 0.43 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0.64 & 0.77 & 0 & -2.05 & 0.78 & 3.1 & 4.11 & 0.01 \\ 0.77 & 0.96 & 3.89 & 0.098 & 8.02 & 0 & 0.049 & 0.72 \\ 0 & 3.89 & 1.02 & 0 & 0 & 1.01 & 7.01 & -3.04 \\ -2.05 & 0.098 & 0 & 0.09 & -6.06 & 0 & 0.033 & -0.401 \\ 0.78 & 8.02 & 0 & -6.06 & 2.2 & 3.80 & -0.99 & 3.11 \\ 3.1 & 0 & 1.01 & 0 & 3.80 & 0.02 & 0 & 0.12 \\ 4.11 & 0.049 & 7.01 & 0.033 & -0.99 & 0 & 0.009 & 0 \\ 0.01 & 0.72 & -3.04 & -0.401 & 3.11 & 0.12 & 0 & 5 \end{pmatrix}.$$

The criteria used to compare the estimation methods were the average mean squared error (AMSE) and the average mean absolute error (AMAE). The results are summarized in Tables 1-6 and illustrated in Figures 1-6.

Table 1 contains the results of the AMSE using the first prior distribution, and Table 2 shows the corresponding results of the AMAE. Table 3 and Table 4 show the AMSE and AMAE, respectively, using the second prior distribution. Lastly, Table 5 and Table 6 show the results of the AMSE and AMAE using the third prior distribution.

Table 1. Results of the (AMSE) criterion (for the first prior distribution)

Sample Size	Sections	GLS	Bayesian	Restricted Bayes	
				Underfitting	Overfitting
75	2	3.84428	3.66125	2.54499	3.22559
	3	3.31991	2.99699	2.11923	3.11233
	5	3.42228	2.95015	2.10021	2.91166
	7	3.11655	2.83997	1.98969	2.81210
	9	2.98901	2.11169	1.44114	2.10921
150	2	3.06711	2.99890	2.36212	2.89119
	3	3.06121	2.90676	2.10141	2.80121
	5	3.01093	2.89598	2.02801	2.70014
	7	3.00441	2.59799	1.21002	2.10012
	9	3.00169	2.52218	1.12016	2.02011
300	2	2.99981	2.75552	2.21745	2.56668
	3	2.88929	2.64489	1.99767	2.33119
	5	2.84785	2.51165	1.59654	2.22112
	7	2.77997	2.33467	1.11032	1.99936
	9	2.71229	2.21114	1.10001	1.76996
500	2	2.81066	2.66779	1.43021	1.78963
	3	2.11668	1.99669	1.25665	1.64323
	5	2.00831	1.97771	1.12035	1.41124
	7	1.99127	1.66542	1.00764	1.11039
	9	1.97015	1.60446	0.99176	1.01971

Having enumerated and presented the values from Table 1 numerically, we continue by depicting the methods used in estimation graphically for better ease in comparing them based on the different sample sizes and models.

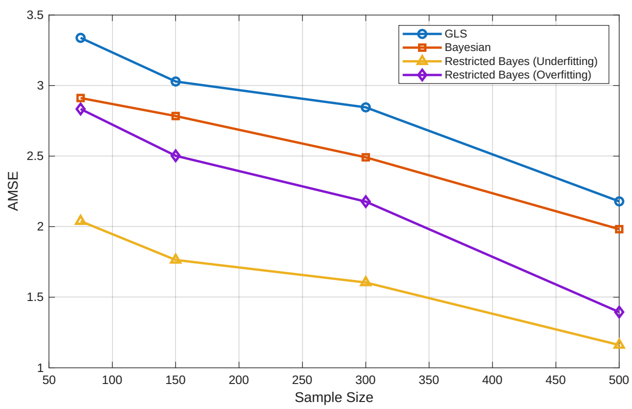

Figure 1. Graphical representation of the AMSE criterion for the first prior distribution.

As seen from the graphical representations presented in Figure 1, the numerical results presented in Table 1 are strongly supported. It can be clearly noted that the restricted Bayes estimator performs better than the other estimators using the AMSE criteria for all cases. In addition, the performance of this estimator improves with an increase in the sample size.

Table 2. Results of the (AMAE) criterion (for the first prior distribution)

Sample Size	Sections	GLS	Bayesian	Restricted Bayes	
				Underfitting	Overfitting
75	2	2.77333	2.56552	2.11116	2.21124
	3	2.61227	2.24778	2.01017	2.11771
	5	2.11575	2.10015	1.99979	2.00776
	7	2.10154	2.00331	1.98667	2.00083
	9	2.00651	1.99119	1.55451	1.97011
150	2	2.56701	2.35756	2.10010	2.11121
	3	2.47111	2.10197	2.00031	2.10111
	5	2.10176	2.01106	1.98776	2.00011
	7	2.00459	1.99699	1.96654	1.99102
	9	2.00331	1.96701	1.42222	1.95111
300	2	2.33331	2.22221	2.00115	2.10101
	3	2.21132	2.11066	1.99944	2.00661
	5	2.10001	2.00551	1.96666	1.99999
	7	1.99997	1.99551	1.91101	1.96674
	9	1.99876	1.98777	1.22201	1.77552
500	2	2.11111	2.00113	1.99119	2.00001
	3	2.10014	1.99989	1.98431	1.99135
	5	2.00036	1.99224	1.87641	1.89997
	7	1.97766	1.90010	1.66073	1.75879
	9	1.95668	1.71007	0.99999	1.00099

After observing the outcomes in Table 2, a visual comparison has been provided to further demonstrate the effect of the AMAE on various methods of estimation.

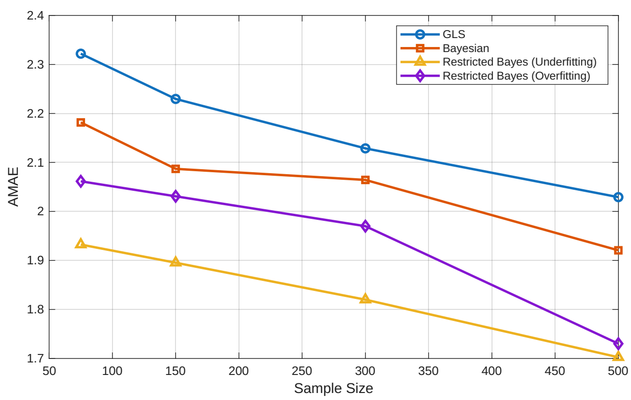

Figure 2. Graphical representation of the AMAE criterion for the first prior distribution.

Figure 2 validates the numerical outcomes in Table 2, where the restricted Bayes estimation in the underfitting scenario produces the minimum values of the AMAE compared to all other settings.

Table 3. Results of the (AMSE) criterion (for the second prior distribution)

Sample Size	Sections	GLS	Bayesian	Restricted Bayes	
				Underfitting	Overfitting
75	2	1.11319	1.10012	1.22775	1.78837
	3	2.20110	1.76664	1.92119	2.01031
	5	1.66207	1.41104	1.81143	1.95505
	7	1.12400	1.01003	1.66633	1.79973
	9	1.12110	1.04336	1.39993	1.54445
150	2	1.10560	1.10077	1.11997	1.44134
	3	1.05232	1.02099	1.10113	1.20109
	5	1.03363	1.01119	1.08888	1.11003
	7	1.02211	1.01001	1.05506	1.07973
	9	1.02055	1.00990	1.03136	1.05525
300	2	1.10333	1.10001	1.11016	1.24427
	3	1.10110	1.02001	1.10088	1.20010
	5	1.03001	1.01003	1.06661	1.10354
	7	1.02001	1.00977	1.04101	1.05955
	9	1.01613	1.00558	1.02200	1.04401
500	2	1.10101	1.09665	1.11002	1.21101
	3	1.10010	1.01011	1.03310	1.16424
	5	1.02102	1.00919	1.03011	1.10091
	7	1.01910	1.00545	1.02110	1.04884
	9	1.01102	1.00333	1.01859	1.01502

In order to support the numerical results presented in Table 3, the performance of the AMSE for the second prior probability distribution is depicted below.

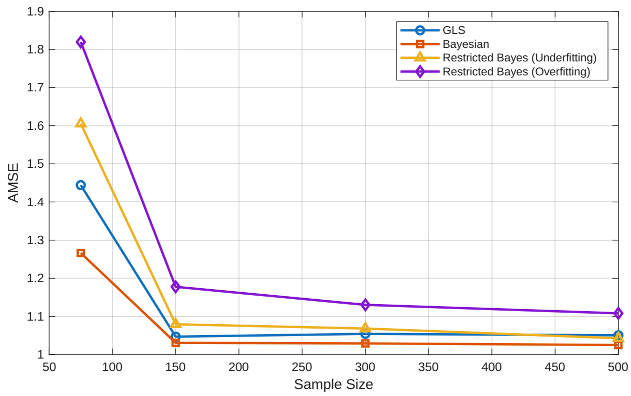

Figure 3. Graphical representation of the AMSE criterion for the second prior distribution.

The performance depicted in Figure 3 reveals that the Bayesian estimator yields the minimum AMSE value compared with other estimators, as expected from the numerical results presented in Table 3.

Table 4. Results of the (AMAE) criterion (for the second prior distribution)

Sample Size	Sections	GLS	Bayesian	Restricted Bayes	
				Underfitting	Overfitting
75	2	1.66222	1.45441	1.87778	2.01010
	3	1.50116	1.13667	1.77661	1.99669
	5	1.11663	1.10113	1.55444	1.77974
	7	1.11105	1.01128	1.34323	1.55543
	9	1.01976	1.01001	1.10047	1.38785
150	2	1.50944	1.39112	1.81121	2.00111
	3	1.42101	1.11200	1.75542	2.00010
	5	1.11157	1.10002	1.50111	1.70113
	7	1.11011	1.01009	1.30101	1.44404
	9	1.01900	1.00997	1.03012	1.32223
300	2	1.49169	1.28423	1.80101	1.91014
	3	1.32014	1.10124	1.71197	1.83327
	5	1.11012	1.01003	1.44311	1.69695
	7	1.10089	1.00968	1.27892	1.33455
	9	1.01611	1.00916	1.02601	1.20266
500	2	1.44114	1.22246	1.74333	1.80121
	3	1.29917	1.10077	1.61652	1.75572
	5	1.09055	1.00208	1.34722	1.59779
	7	1.03411	0.99998	1.11301	1.30016
	9	1.00564	0.94159	1.04515	1.10041

Graphical representation of the AMAE criterion based on Table 4 is shown below.

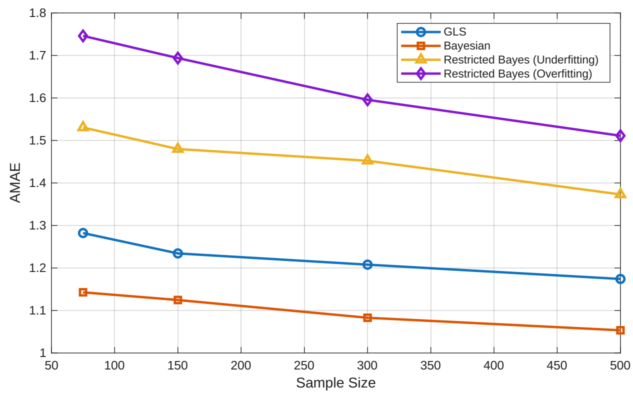

Figure 4. Graphical representation of the AMAE criterion for the second prior distribution.

Figure 4 validates the findings presented in Table 4, where Bayesian estimation provides a lower value of AMAE than GLS and restricted Bayes estimators.

Table 5. Results of the (AMSE) criterion (for the third prior distribution)

Sample Size	Sections	GLS	Bayesian	Restricted Bayes	
				Underfitting	Overfitting
75	2	1.62526	1.44132	1.52496	1.58855
	3	1.20771	1.16167	1.17955	1.18111
	5	1.20106	1.11013	1.13033	1.15004
	7	1.02043	1.00331	1.01206	1.01413
	9	1.00772	1.00132	1.00186	1.01006
150	2	1.61061	1.43113	1.51121	1.53461
	3	1.20113	1.16013	1.16663	1.17211
	5	1.16711	1.10101	1.11036	1.13101
	7	1.01967	1.00301	1.01101	1.01121
	9	1.00658	1.00101	1.00163	1.01001
300	2	1.60198	1.41113	1.47876	1.51151
	3	1.20101	1.13311	1.16333	1.16611
	5	1.14414	1.10010	1.11022	1.12101
	7	1.01133	1.00211	1.01013	1.01101
	9	1.00431	1.00011	1.00141	1.00168
500	2	1.57976	1.40104	1.38735	1.49945
	3	1.20062	1.13101	1.12791	1.16406
	5	1.13753	1.10001	1.10301	1.11112
	7	1.01010	1.00196	1.01001	1.01100
	9	1.00382	1.00009	1.00112	1.00143

To further interpret the AMSE results in Table 5, a graphical comparison is presented below.

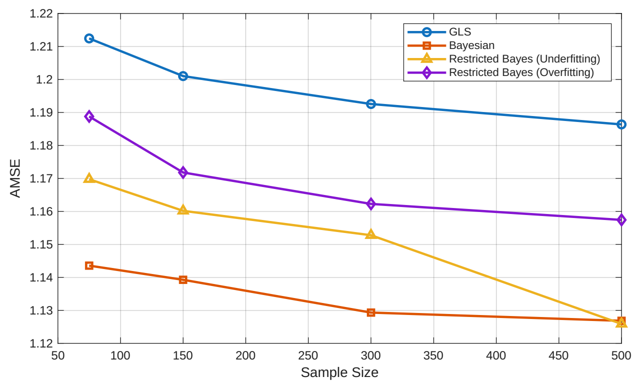

Figure 5. Graphical representation of the AMSE criterion for the third prior distribution.

As illustrated in Figure 5, the Bayesian and restricted Bayes estimators exhibit strong performance, with the Bayesian estimator generally achieving the lowest AMSE values, consistent with Table 5.

Table 6. Results of the (AMAE) criterion (for the third prior distribution)

Sample Size	Sections	GLS	Bayesian	Restricted Bayes	
				Underfitting	Overfitting
75	2	1.65335	1.59456	1.51145	1.53257
	3	1.63237	1.45975	1.43433	1.44124
	5	1.57895	1.43334	1.41015	1.42113
	7	1.33312	1.33321	1.31211	1.32201
	9	1.31211	1.31133	1.30111	1.30336
150	2	1.59665	1.55354	1.51124	1.56996
	3	1.55865	1.44346	1.49132	1.55535
	5	1.54556	1.41142	1.45605	1.55475
	7	1.31121	1.30010	1.39636	1.44113
	9	1.30013	1.26792	1.30011	1.33775
300	2	1.56675	1.53537	1.55455	1.55793
	3	1.54455	1.43456	1.54412	1.52554
	5	1.51176	1.39936	1.51777	1.51998
	7	1.31001	1.30002	1.36678	1.41446
	9	1.30001	1.25213	1.30231	1.33363
500	2	1.55541	1.52565	1.53656	1.55352
	3	1.53135	1.43111	1.52111	1.52323
	5	1.49857	1.34676	1.40321	1.50117
	7	1.29952	1.27625	1.29815	1.41011
	9	1.29864	1.24456	1.26112	1.31011

Finally, to visualize the AMAE results reported in Table 6, we present the corresponding graphical comparison.

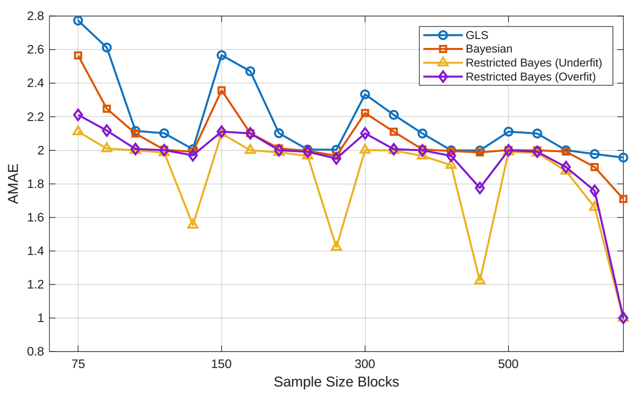

Figure 6. Graphical representation of the AMAE criterion for the third prior distribution.

The graphical results in Figure 6 confirm the numerical findings in Table 6, showing that the Bayesian estimator provides the most stable and lowest AMAE values across all configurations.

5. Discussion and Conclusion

Regarding the results obtained under the first prior distribution, Table 1 shows that the restricted Bayesian method provides the best performance for estimating the parameters of a misspecified random repeated measurements model under the underfitting scenario. In this case, the values of the average mean squared error (AMSE) are the lowest across all considered sample sizes and numbers of sections, compared with the other estimation methods. Moreover, these AMSE values are also lower than those obtained under the overfitting scenario.

From Table 1, the lowest AMSE value is 0.99176, which occurs at a sample size of 500 and nine sections when using the restricted Bayesian estimator under underfitting. In contrast, the highest AMSE value, 3.84428, is observed for the general least squares estimator at a sample size of 75 with two sections.

In addition, Table 2 presents the results for the AMAE. It can be observed that, once again, the restricted Bayesian approach exhibits the lowest AMAE values for all sample sizes and section numbers when the underfitting case is considered. The smallest value is 0.99999 with a sample size of 500 and 9 sections, whereas the largest one, which is 2.77333, pertains to the general least squares estimator with the original model.

According to Table 3, which gives the AMSE values using the second prior distribution, it is clear that the Bayesian estimation method provides the smallest AMSE values for all sample sizes and numbers of sections. Likewise, Table 4 indicates that the Bayesian estimator also attains the smallest AMAE values for all cases.

Based on these results, the minimum AMSE value for the Bayesian method is 1.00333, obtained at a sample size of 500 with nine sections. The corresponding minimum AMAE value is 0.94159, also at the same configuration.

Moreover, Tables 5 and 6 present the results for the third conjugate prior distribution. These show that the Bayesian estimator again provides the best performance in terms of both criteria, with AMSE = 1.00009 and AMAE = 1.24456.

A set of conclusions can be drawn from the theoretical and numerical results as follows.

First, regarding the theoretical results:

1. The constrained Bayesian estimator of the misspecified repeated measurements model satisfies

$$\widehat{\varphi}_{\text{CB}} = \widehat{\varphi}_B + \left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T \left[A\left(Z^T\Omega^{-1}Z + S^{-1}\right)^{-1} A^T\right]^{-1} (A - A\varphi),$$

as established in the main derivations.

2. From Theorems 6 and 7, the conditional structure of the estimator implies

$$E(\varphi_2 \mid \varphi_1 = 0) = \text{mode}(\varphi_2 \mid \varphi_1 = 0) = \widehat{\varphi}_{2\text{CB}} = \widehat{\varphi}_{2B} + \psi_{22}^{-1}\psi_{21}.$$

3. The restricted Bayes estimator of the misspecified repeated measurements model with $A = (I \;\; 0)$ and $a = 0$ is weakly consistent when $\phi_{12} = 0$, where

$$\phi = \lim_{N\to\infty}\left(\frac{1}{N}Z^T\Omega^{-1}Z\right) = \begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}.$$

4. Under overfitting, the Bayesian estimator of φ_2 loses asymptotic efficiency unless $\phi_{12} = 0$, while underfitting improves its asymptotic precision unless $\phi_{12} = 0$.
5. From Theorems 6–8, under misspecification with linear constraints, the bias behavior is characterized by

$$\text{Bias}(\widehat{\varphi}_{2\text{CB}}) = \text{Bias}(\widehat{\varphi}_{2B}) + \psi_{22}^{-1}\psi_{21}\big(\text{Bias}(\widehat{\varphi}_{1B}) + \varphi_1\big),$$

with the special case $\varphi_1 = 0$ reducing accordingly.

6. The underfitting model improves the MMSE matrix of $\widehat{\varphi}_B$ if and only if

$$\gamma = \varphi_1^T\left[\text{MMSE}(\widehat{\varphi}_B)\right]^{-1}\varphi_1 \le 1.$$

7. A natural extension of this work is the development of model selection and estimation procedures for nonlinear repeated measurements models.

Second, regarding the numerical results:

1. For the first prior distribution, the restricted Bayesian estimator under the underfitting scenario consistently outperforms all competing methods in terms of both AMSE and AMAE across all sample sizes and numbers of sections.
2. For the second prior distribution, the Bayesian estimator achieves the lowest AMSE and AMAE values for all configurations considered.
3. All posterior results obtained from different priors demonstrate that the Bayesian procedure is very robust against any form of model misspecification and point to promising avenues for further research, especially on nonlinear and high-dimensional repeated measurements models.

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