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Discrete-Time Surplus Process Model with Multiple Threshold Levels for the Productive Waqf-Based Crab Cracker Industry: Theory and Simulation

Rini Cahyandari [1,*], Sukono [2], Riaman [2], Nurnadiah Zamri [3]

[1] *Doctoral Program in Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia*
[2] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia*
[3] *Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Besut Campus, Besut 22200, Terengganu, Malaysia*

Abstract This research is a mathematical reinforcement of the integration scheme of productive waqf with fishermen's welfare compensation that has been previously published by the author. The crab cracker industry acts as a form of productive waqf business that generates a surplus to finance fishermen's welfare compensation, including the payment of sharia life insurance contributions. The sustainability of this scheme is highly dependent on the stability of the surplus dynamics over time, so a structured risk modeling framework is needed to control potential deficit risks. To address this need, a discrete-time surplus process model with multiple threshold levels is developed through theoretical and simulation approaches. Empirical data of crab catches are modeled in a frequency (number of fishermen daily) and severity (value of crab purchases) framework, which probability distributions are tested statistically to form compound claims. Through this framework, a theorem related to the characteristics of the model is formulated and proven analytically and numerically. The main contribution of this research is theoretically expanding actuarial modeling through the mechanism of multiple threshold levels, while practically providing a quantitative basis for maintaining the stability of the surplus dynamics of productive waqf business in facing stochastic uncertainty.

Keywords surplus process model, discrete-time, multiple threshold levels, compound claim, crab cracker industry

AMS 2010 subject classifications 62P05, 62P30, 91B05, 91G05

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1. Introduction

The productive waqf-based financing model plays an important role in supporting community welfare through sustainable waqf asset management [1]. In a previous publication, the author proposed a scheme for integrating productive waqf with welfare assistance for fishermen, where the crab cracker industry acts as a business unit of real surplus generators to finance compensation programs including sharia life insurance for fishermen [2, 3].

The sustainability of this scheme depends heavily on the stability of the business's surplus dynamics. Under sharia law, the principal of waqf assets must not be eroded to cover operational deficits, so that the financing of the charity relies solely on the surplus. For this reason, risk modeling based on multiple threshold levels is required, which functions as a sharia control mechanism to mitigate risk of deficit occurrence.

In actuarial risk theory, capital evolution is generally modeled through the Cramér-Lundberg framework adapted to discrete-time. Kim and Drekic [4] integrate threshold levels to represent capital control policies, while Achlak [6] and Puspita et al. [5] applied it to the context of sharia insurance to analyze the stability of participant funds and Qardh-Hasan facilities.

*Correspondence to: Rini Cahyandari (Email: rini23004@mail.unpad.ac.id). Doctoral Program in Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia.

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Referring to the development of this literature, this study positions the crab cracker industry as a real business unit modeled within a framework of frequency (number of fishermen daily) and severity (value of crab purchases). These two components are statistically tested to form a compound claim process as the main component in the discrete-time surplus process model [7].

Different from previous studies which were limited to insurance institutions [3, 8], this research expands the application of multiple threshold levels to productive waqf based real business unit. Through this framework, a theorem on model characteristics is formulated and proven analytically and numerically. The main contribution of this research is to theoretically expand actuarial modeling while providing a practical quantitative basis for maintaining the stability of productive waqf business surpluses to mitigate stochastic uncertainty.

2. Risk Model and Compound Claim

2.1. Cramér-Lundberg Model

Cramér-Lundberg model first developed by Filip Lundberg in 1903 and refined by Harald Cramér in the 1930s. This model is a main foundation in actuarial risk theory that is used for modelling company surplus dynamics insurance. In general, this model describe how surplus dynamics of time to time as a consequence of existence cash inflow in the form of income premiums and cash outflows in the form of claim.

In classic risk model framework, frequency of claim generally modelled as a Poisson process, whereas claim severity assumed as random variables which are free and independent and identically distributed (i.i.d). However, in practice empirically, the Poisson assumption is not always fulfilled, especially when the data shows special characteristics like underdispersion or limitations amount incident daily. In said condition, other alternative distribution discrete can be used for modelling claim frequency more accurately. Combination between distribution of frequency and severity forms compound claim process which becomes main component risk analysis.

Classical risk model of Cramer-Lundberg stated as:

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \tag{1}$$

with u refers to an initial surplus, c is rate income per unit time, N_t is claim frequency that occur until time t, and X_i is claim severity to-i. Component $\sum_{i=1}^{N_t} X_i$ represents compound claims up to time t [7, 9, 10].

This model has been widely developed to calculate the probability of bankruptcy, modify the structure of the claims process, and formulate it into a discrete timeframe. In the context of sharia insurance, the Cramér-Lundberg model has been adapted by integrating the tabarru' fund mechanism and underwriting surplus. However, most applications of the compound claim-based surplus model are still limited to insurance financial institutions. Its application in the real sector, particularly productive waqf-based business units, is still relatively limited. Mathematically, real business units have similar cash flow dynamics, namely the presence of stochastic operating income and expenses. Based on these considerations, this study adapts the Cramér-Lundberg framework to the context of the productive waqf-based crab cracker industry. To adapt to the characteristics of daily transactions, the number of fishermen selling crabs as the frequency and value of crab purchases, as severity, this model is formulated in discrete-time form to be able to model the dynamics of business surplus in periodically and structurally.

2.2. Binomial Distribution

Claim frequency N is one of the main components in the compound claim process, representing the number of incidents in a time period. In this study, the random variable N is defined as the number of daily fishermen who sell their catch to the productive waqf-based crab cracker industry.

In actuarial literature terms, the frequency of claims is generally modeled using the Poisson distribution [9], which assumes the population is infinite and events occur independently at continuous time intervals. However, in the operational context of the crab cracker industry, the number of fishermen who supply raw materials is limited n and identified in a local fishing group. As a result, the assumption variance value and its expectations value are

not always met, especially when empirical data shows underdispersion characteristics. Therefore, the Binomial distribution is a more representative choice both structurally and empirically.

Based on exploratory analysis and statistical testing of distribution fit, the daily claim frequency is modeled using the Binomial distribution [10]. Mathematically,

$$N \sim \text{Binomial}(n, p),$$

with the probability mass function expressed by:

$$P(N = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n, \tag{2}$$

where n states the maximum number of fishermen who sell their catch per day and p is the probability of a fisherman making a transaction on that day, and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Expected value and variance from claim frequency N each is formulated as:

$$E[N] = np, \tag{3}$$

$$\text{Var}(N) = np(1-p). \tag{4}$$

2.3. Log-Normal Distribution

Claim severity X_i represents the value of crab purchases from fishermen i within a daily transaction period. This value is continuous and positive, and exhibits significant variation due to differences in catch volume. In the actuarial literature, various right-tailed distributions are commonly used to model claim severity, such as Gamma, Weibull, and Log-Normal.

Results of exploratory analysis and statistical testing of distribution fit show that the severity data of crab purchase values follows a Log-Normal distribution, which is denoted as $X_i \sim \text{Log-Normal}(\mu, \sigma^2)$ [11]. The probability density function is expressed X_i by:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), \quad x > 0. \tag{5}$$

Expected value and variance of severity X_i each is formulated as:

$$E[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right), \tag{6}$$

$$\text{Var}(X) = \left(\exp(\sigma^2) - 1\right) \exp(2\mu + \sigma^2). \tag{7}$$

2.4. Compound Claim Process

Frequency model N_t and severity X_i has set out in Subsections 2.2 and 2.3. Based on second component said, total expenditure purchase crab (aggregate claim) in the period daily to-t ($t = 1, 2, 3, \ldots$) formulated as compound claim [12, 13], namely:

$$S_t = \sum_{i=1}^{N_t} X_{i,t}. \tag{8}$$

Distribution from S_t is compound distribution, namely combination stochastic between distribution of frequency and distribution of severity. In this study, the basic assumptions are as follows:

1. $N_t \sim \text{Binomial}(n, p)$.
2. $X_{i,t} \sim \text{Log-Normal}(\mu, \sigma^2)$.
3. $\{X_{i,t}\}$ naturally independently and identically distributed (i.i.d).
4. N_t independent to $X_{i,t}$.

Based on the law of total expectation and the law of total variance, expectation and variance from aggregate claim daily S_t each formulated as:

$$E[S_t] = E[N_t]E[X], \tag{9}$$

$$\text{Var}(S_t) = E[N_t]\text{Var}(X) + \text{Var}(N_t)(E[X])^2. \tag{10}$$

Expectation $E[S_t]$ and variance $\text{Var}(S_t)$ the used as component subtraction cash outflow in formation of a surplus time process model discrete, namely:

$$U_t = U_{t-1} + C_t - S_t. \tag{11}$$

Where U_t state surplus position at the end period to-t, U_{t-1} is the surplus balance in the previous period, and C_t state cash inflow during the period to-t [14].

In actuarial science, the compound claim process implemented to analyse stability of finance of conventional and sharia companies insurances. On the other hand, finance modeling in business waqf productive is still dominated by the deterministic approach [15, 16]. Therefore, this research bridges literature gaps by applying distribution parameters of frequency and severity based on empirical data in discrete time surplus modelling, to provide a quantitative foundation for sustainability of real business unit based on productive waqf.

3. Data and Research Methodology

This research uses a quantitative approach that combines operational cost analysis and surplus calculations, deterministic, stochastic modeling, and numerical simulation. The research stage begins with the collection of empirical data and deterministic calculations of cash flow and surplus of the crab cracker industry, which includes the allocation of raw materials, operational costs, labor, and the determination of revenue from product sales. This calculation serves as a deterministic baseline under ideal conditions before incorporating stochastic uncertainty elements [17]. Next, the expenditure components are modeled through statistical testing of frequency and severity distributions to form a compound claim process. Based on the distribution parameters obtained, Monte Carlo simulations are conducted to generate frequency and severity data randomly, followed by testing the validity of the model using Mean Absolute Percentage Error (MAPE) [18]. In the final stage, a discrete-time surplus process model with multiple threshold levels is formulated, which is equipped with a sensitivity analysis to variations in threshold values and the percentage of unsold products to test the robustness of the model and verify its theorem numerically.

3.1. Research Data and Deterministic Surplus Calculation

The empirical data for this research were obtained from daily data collection of crab catches by fishermen for 30 days through coordination with the head of the fishermen's group, involving 140 crab fishermen in Mundu Pesisir Village, Mundu District, Cirebon Regency, West Java. The information collected included catch volume (kilograms) and purchase value (rupiah), which are presented in Tables 1 and 2.

Based on the daily catch volume data, deterministic calculations are performed to determine all production variable requirements. For each day t, $(t = 1, 2, \ldots, 30)$, total catch volume crab become reference in count proportion raw material requirements, operational cost, as well as labor wages. Every component is noted based on amount of usage and price per unit so that obtained daily total cost production. As for daily income is determined by the total number of crab crackers produced on the day to-t multiplied with price selling per kilogram. With assumptions all over results production on the day to-t sold out, daily deterministic surplus counted as difference between income reduced expenditure of productions [17]. The results of these daily calculations produce a deterministic baseline surplus under ideal conditions before including stochastic uncertainty elements.

Table 1. Catch volume (kilograms)

Day	**Fisherman**						
	1	**2**	**3**	**4**	**5**	$\cdots$	**140**
1	20	35	30	40	30	$\cdots$	25
2	25	25	20	33	50	$\cdots$	15
3	15	30	35	25	35	$\cdots$	30
4	30	23	40	50	40	$\cdots$	50
5	25	15	60	25	43	$\cdots$	23
6	0	0	70	15	70	$\cdots$	32
7	60	20	15	0	60	$\cdots$	27
8	40	21	19	13	30	$\cdots$	31
9	25	30	16	20	25	$\cdots$	22
10	17	33	25	30	21	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
30	0	32	20	26	32	$\cdots$	15

Table 2. Purchase value (rupiah)

Day	**Fisherman**						
	1	**2**	**3**	**4**	**5**	$\cdots$	**140**
1	600,000	1,050,000	900,000	1,200,000	900,000	$\cdots$	750,000
2	750,000	750,000	600,000	990,000	1,500,000	$\cdots$	450,000
3	450,000	900,000	1,050,000	750,000	1,050,000	$\cdots$	900,000
4	900,000	690,000	1,200,000	1,500,000	1,200,000	$\cdots$	1,500,000
5	750,000	450,000	1,800,000	750,000	1,290,000	$\cdots$	690,000
6	0	0	2,100,000	450,000	2,100,000	$\cdots$	960,000
7	1,800,000	600,000	450,000	0	1,800,000	$\cdots$	710,000
8	1,200,000	630,000	570,000	390,000	900,000	$\cdots$	930,000
9	750,000	900,000	480,000	600,000	750,000	$\cdots$	660,000
10	510,000	990,000	750,000	900,000	630,000	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
30	0	660,000	600,000	780,000	930,000	$\cdots$	450,000

As an illustration, the calculation on the first day ($t = 1$) is done by adding up all cost components, consisting of raw material costs, labor wages, and operational costs. The details of the cost components on the first day are presented in Table 3.

Table 3. Total production costs of crab crackers on the first day

No	Cost Components	Quantity	Cost (Rupiah)
1	Crab	4,067 kg	122,010,000
2	Tapioca flour	7,321 kg	95,173,000
3	Sugar	732 kg	10,248,000
4	Garlic	244 kg	6,100,000
5	Egg	81 tray	4,860,000
6	Flavoring	81 packs	10,125,000
7	Salt	325 packs	1,625,000
8	Gas	569 tubes	17,070,000
9	Plastic	651 packs	22,785,000
10	Label	163 packs	6,520,000
11	Wages for crab picking	62 people	4,650,000
12	Wages for Cutting, Drying and Wrapping	312 people	5,460,000
Total cost			**306,626,000**

Based on the total crab obtained on the first day of 4,067 kg, the production process produced 8,541 kg of crab crackers with a selling price of IDR 60,000 per kilogram [17]. Assuming that all products were sold on the same day, the first day's income is presented in Table 4.

Table 4. Total income from crab cracker production on the first day

Day	Production Volume (Kg)	Crab Cracker Price (Rupiah/Kg)	Total Gross Income (Rupiah)
1	8,541	60,000	512,460,000

Based on total production costs of IDR 306,626,000 and total production revenue of IDR 512,460,000, a surplus of IDR 205,834,000 was obtained on the first day. The same calculation procedure was then applied consistently for the second day to the 30th day, so that the daily surplus was obtained from the difference between total revenue and total expenditure on each day. A recapitulation of the values of daily revenue, expenditure, and surplus for 30 days is presented in Table 5.

The first day surplus of IDR 205,834,000 indicates that crab cracker production generates a relatively high profit margin under normal operating conditions. This indicates that revenue from product sales not only covers all production costs but also provides a significant surplus. Within the framework of productive waqf, the availability of these funds opens up opportunities for gradual allocation, both for business development and for fishermen's welfare programs, without disrupting the integrity of the waqf's principal assets.

Table 5. Recapitulation of daily gross income, cost, and surplus of the crab cracker industry

Day	Gross Income (Rupiah)	Total Cost (Rupiah)	Surplus (Rupiah)
1	512,460,000	306,626,000	205,834,000
2	530,100,000	316,837,000	213,263,000
3	497,460,000	298,312,000	199,148,000
4	570,660,000	340,446,000	230,214,000
5	486,840,000	291,839,000	195,001,000
6	479,820,000	287,082,000	192,738,000
7	512,460,000	305,926,000	206,534,000
8	463,080,000	278,218,000	184,862,000
9	478,920,000	287,262,000	191,658,000
10	463,920,000	278,441,000	185,479,000
⋮	⋮	⋮	⋮
30	374,880,000	224,864,000	150,016,000
Total	**13,752,240,000**	**8,262,543,000**	**5,489,697,000**

3.2. Modeling Frequency Distribution, Severity, and Compound Claim Process

As a development from calculation of surplus in deterministic modeling, the cost components of purchasing raw crab materials are modeled stochastically using two main components: claim frequency and claim severity. However, before proceeding to stochastic modeling, statistical testing will be conducted to ensure that the research sample data is randomly drawn from a population.

3.2.1. Randomness and Autocorrelation Test

The independence assumption test was conducted to ensure that fishermen's daily transactions were random and free from inter-period dependence (autocorrelation). The test was conducted using two statistical methods: the Runs Test to test the randomness of the data sequence and the Ljung-Box test to test for the presence of autocorrelation at various time lags.

A summary of the test statistics and p-values for the claim frequency data are presented in Table 6.

Table 6. Results of randomness test and autocorrelation test of claim frequency data

Test	Statistics	p-value
Runs Test (manual)	-1.1767	0.2393
Runs Test (statsmodels)	-0.9762	0.3290
Ljung-Box lag=1	1.9376	0.1639
Ljung-Box lag=2	1.9733	0.3728
Ljung-Box lag=3	2.4265	0.4887
Ljung-Box lag=5	4.6742	0.4569
Ljung-Box lag=10	7.2061	0.7059

Based on the results in Table 6, the Runs Test on the claim frequency data yielded p-values of 0.2393 and 0.3290 respectively. Since both p-values exceeded the significance level $\alpha = 0.05$, the Null Hypothesis was not rejected. This proves that the occurrence of fishermen transacting from day to day is randomly distributed and independent

of each other. Furthermore, the Ljung-Box autocorrelation test of lag=1 until lag=10 consistently produces p-values above 0.05. These results show that there is no significant autocorrelation in the claim frequency time series.

Next, Table 7 presents a summary of the test statistics and p-values for the claim severity data.

Table 7. Results of randomness test and autocorrelation test of claim severity data

Test	**Statistics**	**p-value**
Runs Test (manual)	-0.0371	0.9704
Runs Test (statsmodels)	-0.0368	0.9706
Ljung-Box lag=1	0.0842	0.7717
Ljung-Box lag=2	0.3125	0.8553
Ljung-Box lag=3	0.7841	0.8553
Ljung-Box lag=5	1.8421	0.8706
Ljung-Box lag=10	4.2105	0.9374

Based on the results in Table 7, the Runs Test for claim severity data yielded p-values of 0.9704 and 0.9706 respectively, which are above the significance level $\alpha = 0.05$. Similarly, in the Ljung-Box autocorrelation test, the p-values for all lags ranged from 0.7717 to 0.9374. These results indicate that the daily transaction value per fisherman is also proven to be randomly distributed, independent of each other, and does not have time-series dependence.

Final stage of independence testing is done between claim frequency and claim severity. Testing done using the Spearman Correlation Test to ensure fulfilment assumptions that both of them does not affect each other. Test results show value of coefficient correlation of 0.2393 with p-value of 0.2029. Therefore p-values higher from level significance $\alpha = 0.05$, then the Null Hypothesis (H_0) is not rejected. This informs that statistically there is no linear relationship or significant correlation between the daily number of fishermen and the crab purchase value. Therefore, fulfilment assumptions randomness, no autocorrelation, and independence between these data give enough theoretical justification for modelling claim frequency and claim severity as a random variable independently and identically distributed (i.i.d).

3.2.2. Claim Frequency Distribution Model

The frequency of claims in this study is represented as the number of daily fishermen selling crab catch. Based on the daily data in Table 1, histograms and descriptive statistics for claim frequency data are presented in Table 8 and Figure 1.

Table 8. Descriptive statistics of claim frequency data

Mean	136.5	Skewness	-1.134608208
Standard Error	0.934338539	Range	14
Median	140	Minimum	126
Mode	140	Maximum	140
Standard Deviation	5.117582942	Sum	4095
Sample Variance	26.18965517	Count	30
Kurtosis	-0.089093702	Confidence Level (95.0%)	1.910936876

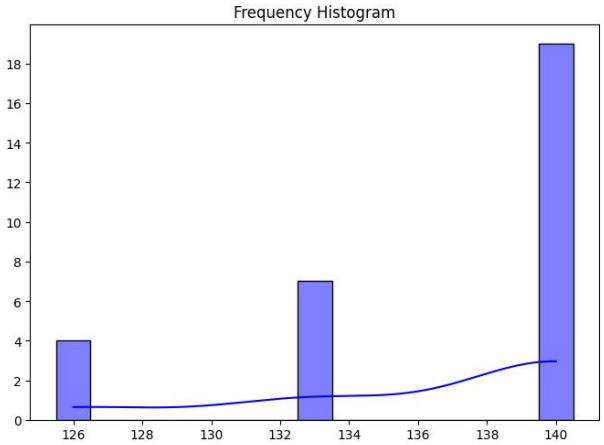


Figure 1. Histogram of claim frequency data

Based on results statistics descriptively in Table 8, the average number of daily fisherman selling crab catch is 136.5 with variance of 26.19. The variance value is far smaller compared to with the average value shows existence symptom of underdispersion in frequency data claim. Next, to determine the best distribution, a goodness of fit test was conducted between Binomial, Poisson, and Negative Binomial distributions.

Table 9. Results test the suitability of the distribution of claim frequency data

Distribution	**AIC**	**BIC**	p**-values**
Binomial	182.5574	183.9586	0.1751
Poisson	210.2831	211.6843	0.0642
Negative Binomial	212.2845	215.0869	0.0641

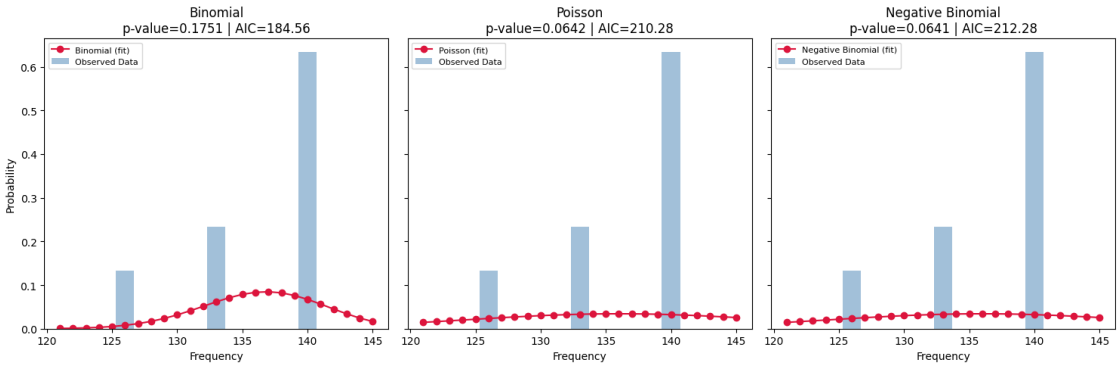


Figure 2. Test of goodness of fit of frequency claim data distribution

Based on Table 9 and Figure 2, the Binomial distribution is the most appropriate distribution for modeling the frequency of daily claims.

3.2.3. Claim Severity Distribution Model

Claim severity represents the value of crab purchases from the i-th fishermen on the t-th day. Based on the data in Table 2, the histogram and descriptive statistics for the claim severity data are presented in Table 10 and Figure 3.

Table 10. Descriptive statistics of claim severity data

Mean	799,273.59	Skewness	0.454492024
Standard Error	16,798.62	Range	345,315.79
Median	801,842.11	Minimum	653,684.21
Mode	#N/A	Maximum	999,000
Standard Deviation	92,009.83	Sum	23,978,207.60
Sample Variance	8,465,807,915	Count	30
Kurtosis	-0.492836438	Confidence Level (95.0%)	34,357.03

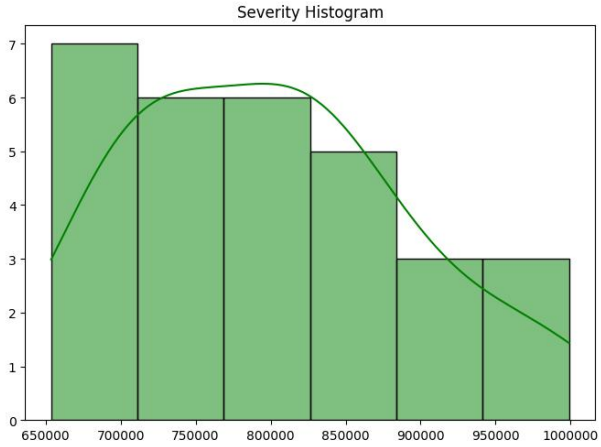


Figure 3. Histogram of claim severity data

Based on the descriptive statistics in Table 10, the average daily purchase value of crab from fishermen was IDR 799,273.59, with a variance of 8.47×10^9. A skewness of 0.45 indicates a slight skew to the right, while a kurtosis of -0.49 indicates a distribution that is flatter than normal. To determine the best distribution, a goodness of fit test was conducted between Log-Normal, Gamma, and Weibull distributions.

Table 11. Results test the suitability of the distribution of claim severity data

Distribution	**AIC**	**BIC**	p**-values**
Log-Normal	772.7307	775.5331	0.9336
Gamma	773.0295	775.8319	0.9177
Weibull	777.7542	780.5566	0.7791

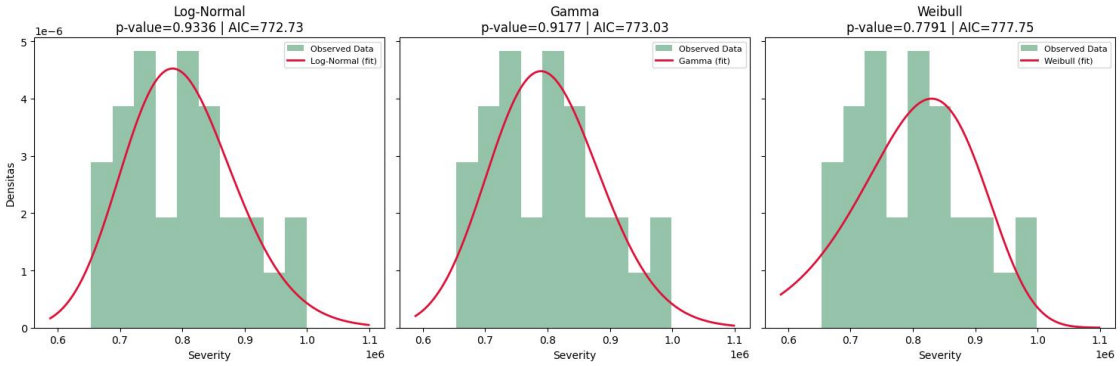


Figure 4. Test of goodness of fit of claim severity data distribution

Based on Table 11 and Figure 4, the Log-Normal distribution is the most appropriate distribution for modeling daily claim severity data. Furthermore, the QQ-Plot results in Figure 5 show that most data points follow the diagonal line, indicating that the data generally approximates the theoretical distribution, but has a right-skewed tendency, a common characteristic in positive-value economic data. Therefore, the claim severity data is appropriately modeled using the Log-Normal distribution.

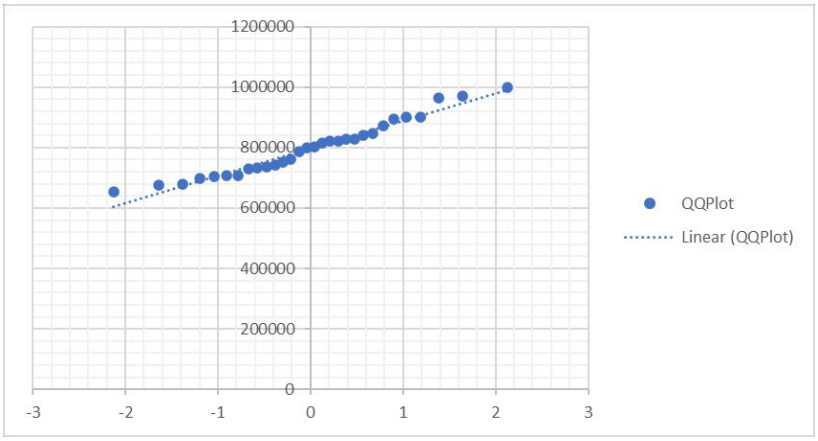


Figure 5. QQ-plot of claim severity data

3.2.4. Parameter Estimation for Claim Frequency and Claim Severity Distributions

After the distribution models of frequency and severity determined, the next step is to estimate the parameters of each distribution use moment method. Expected value and variance from Binomial distribution are formulated in equations (3) and (4), while values of expectations and variance from Log-Normal distribution are formulated in equations (6) and (7).

i. *Estimation of Binomial distribution parameters*
Based on statistics descriptive in Table 8, the average value of the sample is 136.5 with amount test that is amount fishermen involved as many as 140 people. The moment method equates the first moment of the population (expected value) with the first moment of the sample (sample mean).

$$E[N] = \overline{N},$$

$$np = 136.5,$$

$$\hat{p} = \frac{136.5}{140} = 0.975.$$

Value of $\hat{p} = 0.975$ represents the probability of a fisherman selling crab daily is 0.975.

ii. *Estimation of Log-Normal distribution parameters*
Based on statistics descriptive in Table 10, the average value of the sample is 799,274 and the sample variance is 8.47×10^9. The moment method equalizes first moment (average) and second moment (variance) of population with mean and sample variance.

$$E[X] = \overline{X},$$

$$\exp\left(\mu + \frac{\sigma^2}{2}\right) = \overline{X},$$

and

$$\text{Var}(X) = S^2,$$

$$\left(\exp(\sigma^2) - 1\right)\exp(2\mu + \sigma^2) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Obtained:

$$\hat{\sigma}^2 = \ln\left(1 + \frac{S^2}{(\overline{X})^2}\right) = \ln\left(1 + \frac{8.47 \times 10^9}{(799,273.59)^2}\right) = 0.013,$$

$$\hat{\mu} = \ln(\overline{X}) - \frac{1}{2}\hat{\sigma}^2 = \ln(799273.5867) - \frac{1}{2}(0.013) = 13.585.$$

Parameter estimates show that severity data claim distributed leaning to right with median is IDR 794,128. This means that the majority of purchase value of crab is under IDR 800,000 with a few extreme claims of very large values, so this Log-Normal model is in accordance to analysis financial risk and claims predictions in future.

3.3. Formulation and Estimation Characteristics of Compound Claim Process

Based on results of distributions parameter estimations of Binomial and Log-Normal, total daily industry expenditure to purchase raw materials of crabs modelled as compound claim process S_t in equation (8) with expectations and variance formulated in equations (9) and (10). The calculation results expectations and variance for frequency, severity, and compound claim process presented in Table 12.

Table 12. Compound claim process expectations and variances

	Claim frequency	**Claim severity**	**Compound claim**
Expectation	136.5	799,274	109,100,901
Variance	3.4125	8.47×10^9	3.34×10^{12}

Based on Table 12, the average daily claim frequency is 136 fishermen with level of relative small variance. The average expenditure on crab purchases by each fisherman was IDR 799,274 with a fairly large variance. The combination of these two components resulted in an expected compound claim value of IDR 109,100,901 per day. This value represents the average daily expenditure on crab purchases by all fishermen. Based on the data in Table 5 and Table 12, assumptions that product crab crackers sold out every day, obtained results that average daily income exceeds the average expenditure daily, namely $E[C] > E[S]$.

4. Formulation of Discrete-Time Surplus Process Model with Multiple Threshold Levels

A discrete-time surplus process model with multiple threshold levels is formulated to represent the dynamics of the crab cracker industry surplus over time within a single period. In this study, one time period is defined as one operational day, so that the surplus movement can be observed discretely based on daily transaction activity.

Within the framework of actuarial risk theory, the surplus process is generally modeled as the difference between the income earned and the expenses that must be paid in a certain time period. Therefore, the discrete-time surplus process model in this study can be expressed as in equation (11), namely:

$$U_t = U_{t-1} + C_t - S_t, \quad t = 1, 2, \ldots, 30,$$

with U_t states the surplus in the period t, while U_{t-1} is the surplus balance in the previous period with initial conditions $U_0 = u > 0$. Component C_t represent income net profit obtained from activity operational crab cracker industry, while S_t is compound claim which describes the total value purchase crab from fishermen in the period t.

Since the crab cracker industry is based on productive waqf, business surplus or known as benefit waqf must allocated for compensation fisherman welfare with still maintain integrity of the principal waqf assets as well as sustainability of industry development. Therefore, the allocation of surplus in this study is arranged using two levels threshold, namely δ_1 and δ_2, with provision $0 < \delta_1 < \delta_2$. Based on surplus value in the period previously, the surplus process in the period t formulated as following:

$$U_t = \begin{cases} U_{t-1} + C_t - S_t, & \text{if } U_{t-1} < \delta_1, \\ U_{t-1} + C_t - S_t - \gamma_1, & \text{if } \delta_1 \leq U_{t-1} < \delta_2, \\ U_{t-1} + C_t - S_t - \gamma_2, & \text{if } U_{t-1} \geq \delta_2, \end{cases} \tag{12}$$

with $\gamma_2 > \gamma_1 > 0$. Values γ_1 and γ_2 state the amount of surplus allocated for certain objective in accordance with the level of surplus available. In Indonesia, the amount of surplus allocated regulated in Article 12 of Law Number 41 of 2004 concerning waqf and Government Regulation Number 42 of 2006 concerning the Implementation of Law Number 41 of 2004 concerning waqf [19].

Rule allocation of this surplus in line with threshold dividend strategy concept or multiple threshold strategy in actuarial risk theory. Provisions of threshold are used to arrange allocation priority of surplus as following:

1. If $U_{t-1} < \delta_1$, no surplus allocation is made for either development of crab cracker industry and for compensation of fisherman welfare. This condition aims to maintain stability of industry financial so as not to experience deficit.
2. If $\delta_1 \leq U_{t-1} < \delta_2$, some of the surplus is allocated for development crab cracker industry, which is represented by the reduction as amount as γ_1.
3. If $U_{t-1} \geq \delta_2$, some of the surplus is allocated for compensation fisherman welfare, which is represented by the reduction as amount as γ_2, with value γ_2 bigger than γ_1.

Provision of threshold functions as financial risk control system which is the allocation of surplus is only done when condition of industry financial is at a safe level. Thus, this approach does not only aim for preventing occurrence of deficit risk, but also ensure integrity of the principal waqf assets as well as support sustainability of industry development in long term. To represent this mechanism mathematically, the surplus process model in equation (12) contains components of income and expenses. Net income component crab cracker industry stated as:

$$C_t = R_t - B_t, \tag{13}$$

with R_t represent gross income daily in the period to-t and B_t is the total of operational cost daily in the same period. Meanwhile, components expenditure raw materials purchases modelled through compound claim process as in equation (8).

To clarify the formulated model, the assumptions used in this study are as follows:

1. Random variables C_t and S_t is assumed to be independent and identically distributed (i.i.d), so that the industry's net income and expenditure on purchasing raw materials in a period are not affected by the value in the previous period.
2. Daily claim frequency N_t and claim severity variables $X_{t,i}$ each is i.i.d. Based on the results of the independence test, both are proven to be independent of each other.
3. Surplus allocation at the level γ_1 and γ_2 is limited in such a way as to prevent the surplus value from becoming negative. In other words, the surplus allocation policy maintains a sufficient level of reserves to ensure the operational sustainability of the crab cracker industry.

5. Validation of the Surplus Process Model Using Monte Carlo Simulation

To test the reliability and accuracy of the discrete-time surplus process model using multiple threshold levels that have been formulated, the next step is to validate the model. This testing is carried out through a parametric approach using Monte Carlo Simulation because the distribution of claim frequency (Binomial) and claim severity (Log-Normal) has been obtained through distribution fit testing. This approach was chosen because it is able to regenerate a random process continuously based on the theoretical population parameter estimates that have been obtained, thus producing more representative random variability than non-parametric resampling methods.

The validation procedure begins by randomly generating daily claim frequency and claim severity data based on estimated Binomial and Log-Normal parameter values. The simulated data is then used to estimate compound claim values and calculate surplus values for each time period t, with the average daily net income used as the income component. Next, the simulated surplus value is compared with the surplus value from the deterministic calculation. The difference between the two is defined as the error values. To measure the overall accuracy and precision of the model, performance evaluation is calculated using the Mean Absolute Percentage Error (MAPE).

As a stage of implementation of the surplus process model, the threshold level in this research determined to use percentile approach from the empirical surplus values data in Table 5. The threshold value δ_1 set at the 60th percentile as the initial limit financial stability, whereas threshold δ_2 set at the 80th percentile as the high surplus limit. This determination aims to ensure that surplus allocation only done when condition of industry financial is at a safe, appropriate level in accordance with concept of multiple threshold strategy.

The amount of surplus allocated to each threshold stated as following:

$$\gamma_{1,t} = U_t - \delta_1,$$

$$\gamma_{2,t} = U_t - \delta_2.$$

This approach ensures that the surplus continues to grow positively while allowing the allocation of the surplus to be carried out sustainably [20]. In its implementation, the value δ_1 set amounting to IDR 1,468,647,200, while δ_2 amounting to IDR 1,826,719,200. Based on this limit, the new surplus allocation started when surplus value exceeds δ_1 and increasing gradually when it passes δ_2. Determination of value δ_1 amounting to IDR 1,468,647,200 consistently with provision sufficient reserves of industry financial to finance operational up to 3 months ahead [21].

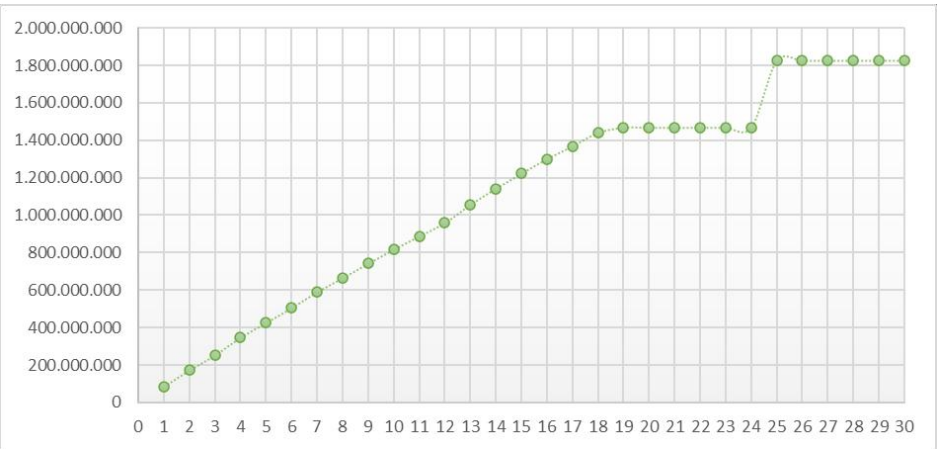


Figure 6. Surplus value (deterministic) after applying the value δ_1 and δ_2

The calculation results show that from day 1 to day 18, there is not surplus allocated because the surplus value was still below δ_1. Allocation began on day 19 and continued to increase until day 24 as the surplus value exceeded δ_1. In the following period, from day 25 to day 30, the allocation pattern adjusted to the surplus condition that had been above δ_2, so that it was allocated in a larger amount but controlled proportion. This pattern indicates that the application of multiple threshold levels is able to control surplus growth to keep it stable.

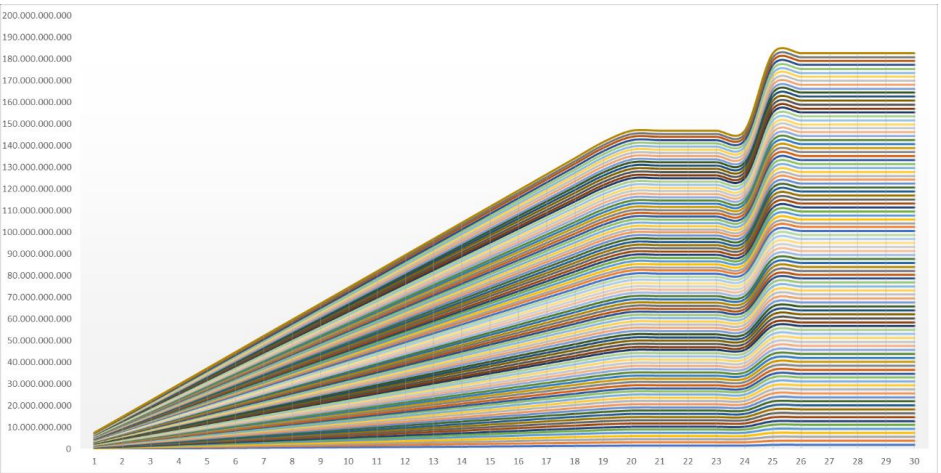


Figure 7. Surplus value (Monte Carlo simulation) after applying the values δ_1 and δ_2

The graph in Figure 7 shows a similar pattern to Figure 6. The similarity in the dynamics of the surplus value indicates that the formulated model is consistent. Furthermore, a comparison between the deterministic surplus value and the Monte Carlo simulation surplus value was performed to measure the model's error rate. The calculation results show a Mean Absolute Percentage Error (MAPE) value of 0.059, indicating that the model has a high level of accuracy in representing empirical data.

Table 13. Surplus value deterministic, simulation, and error

Day	Deterministic	Simulation	Error	Day	Deterministic	Simulation	Error
1	74,317,330	83,824,000	9,506,670	16	1,192,222,329	1,300,677,000	108,454,671
2	148,675,321	170,877,000	22,201,679	17	1,266,955,886	1,368,533,000	101,577,114
3	222,898,230	251,305,000	28,406,770	18	1,341,420,953	1,440,670,000	99,249,047
4	297,757,246	345,649,000	47,891,754	19	1,416,341,674	1,468,647,200	52,305,526
5	371,769,824	424,730,000	52,960,176	20	1,468,622,381	1,468,647,200	24,819
6	446,242,954	503,858,000	57,615,046	21	1,468,647,200	1,468,647,200	0
7	520,720,985	589,082,000	68,361,015	22	1,468,647,200	1,468,647,200	0
8	595,174,982	663,694,000	68,519,018	23	1,468,647,200	1,468,647,200	0
9	670,190,058	741,322,000	71,131,942	24	1,468,647,200	1,468,647,200	0
10	744,976,461	816,551,000	71,574,539	25	1,826,719,200	1,826,719,200	0
11	819,760,797	887,554,000	67,793,203	26	1,826,719,200	1,826,719,200	0
12	894,263,081	958,430,000	64,166,919	27	1,826,719,200	1,826,719,200	0
13	968,930,429	1,055,948,000	87,017,571	28	1,826,719,200	1,826,719,200	0
14	1,043,251,594	1,142,179,000	98,927,406	29	1,826,719,200	1,826,719,200	0
15	1,117,851,780	1,222,629,000	104,777,220	30	1,826,719,200	1,826,719,200	0

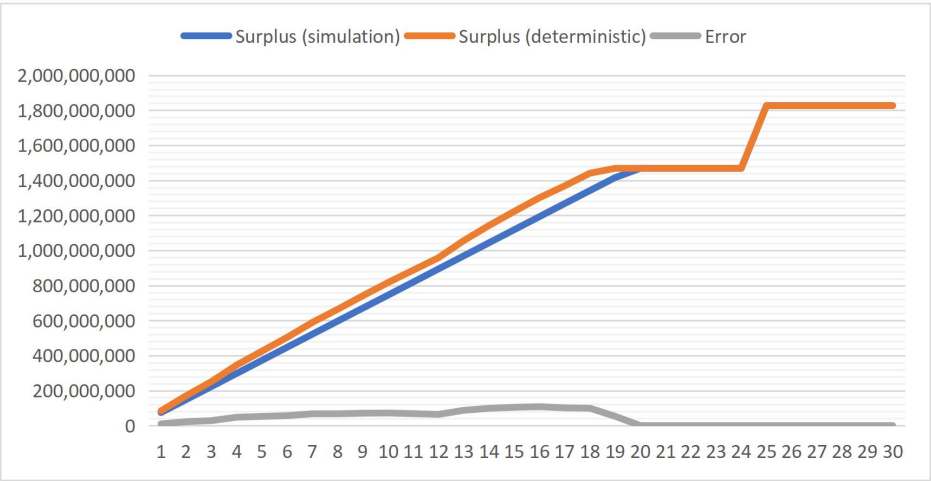


Figure 8. Surplus value deterministic, Monte Carlo simulation and error for 30 days

Figure 8 shows that the two curves are close together with a relatively small gap and exhibit no significant deviations. This nearly identical movement pattern confirms that the simulation results are able to follow the dynamics of the empirical data well, while also reinforcing that the formulated model is not only consistent but also empirically accurate.

6. Proof of the Surplus Stability Theorem and Model Sensitivity Analysis

The robustness of the surplus process model was tested through a sensitivity analysis to determine how the surplus dynamics respond to changes in several parameters. This test focused on two scenarios. First, variations in threshold

value pairs δ_1 and δ_2, namely the percentile pair scenarios (P50, P70), (P55, P75), (P60, P80), (P65, P85), and (P70, P90). Second, variations in the percentage of unsold crab cracker products have a direct impact on the decrease in daily net income (a decrease of 1%, 5%, and 10%).

Before conducting numerical simulations for both scenarios, a theorem is formulated to theoretically prove the stability characteristics of the discrete-time surplus process model with multiple threshold levels that have been formulated.

6.1. Proof of the Surplus Stability Theorem

Theorem 1 (Stability and Control of Surplus Growth)

Let $\{U_t\}_{t\geq 0}$ that is a discrete-time surplus process defined by a surplus model with multiple threshold levels. If $E[C_t] = \mu_C$ and $E[S_t] = \mu_S$ and satisfy the conditions

$$\mu_C - \mu_S > 0,$$

then the surplus process has a tendency to increase in low surplus conditions and the multiple threshold level mechanism ensures that surplus growth remains under control as the surplus value increases.

Proof:

Using equation (12) it is obtained

$$U_t = U_{t-1} + C_t - S_t - D(U_{t-1})$$

where $D(U_{t-1})$ is the surplus allocation function,

$$D(U_{t-1}) = \begin{cases} 0, & \text{if } U_{t-1} < \delta_1, \\ \gamma_1, & \text{if } \delta_1 \leq U_{t-1} < \delta_2, \\ \gamma_2, & \text{if } U_{t-1} \geq \delta_2. \end{cases}$$

It is assumed that the income and expenses in period t are not directly influenced by the surplus balance in the previous period. Therefore,

$$E[U_t \mid U_{t-1}] = E\left[U_{t-1} + C_t - S_t - D(U_{t-1}) \mid U_{t-1}\right]$$

where

$$\begin{aligned} E[U_{t-1} \mid U_{t-1}] &= U_{t-1} \\ E[C_t \mid U_{t-1}] &= E[C_t] \\ E[S_t \mid U_{t-1}] &= E[S_t] \\ E[D(U_{t-1}) \mid U_{t-1}] &= D(U_{t-1}) \end{aligned}$$

so that

$$\begin{aligned} E[U_t \mid U_{t-1}] &= U_{t-1} + E[C_t] - E[S_t] - D(U_{t-1}) \\ E[U_t \mid U_{t-1}] &= U_{t-1} + \mu_C - \mu_S - D(U_{t-1}) \end{aligned}$$

1. Conditions when $U_{t-1} < \delta_1$ and $D(U_{t-1}) = 0$:
   So
   $$E[U_t \mid U_{t-1}] = U_{t-1} + \mu_C - \mu_S$$
   Because $\mu_C - \mu_S > 0$ so $E[U_t \mid U_{t-1}] > U_{t-1}$. It means surplus value increases.
2. Conditions when $\delta_1 \leq U_{t-1} < \delta_2$ and $D(U_{t-1}) = \gamma_1$:
   So
   $$E[U_t \mid U_{t-1}] = U_{t-1} + \mu_C - \mu_S - \gamma_1 = U_{t-1} + (\mu_C - \mu_S - \gamma_1)$$

Because $\mu_C - \mu_S > \gamma_1 > 0$ so $E[U_t \mid U_{t-1}] > U_{t-1}$. It means surplus value increases but its growth more slow down because $(\mu_C - \mu_S - \gamma_1) < (\mu_C - \mu_S)$

$$E[U_t - U_{t-1} \mid U_{t-1}] = \mu_C - \mu_S \quad \text{(condition 1)}$$

$$E[U_t - U_{t-1} \mid U_{t-1}] = \mu_C - \mu_S - \gamma_1 \quad \text{(condition 2)}$$

3. Conditions $U_{t-1} \geq \delta_2$, that is $D(U_{t-1}) = \gamma_2$:
So
$$E[U_t \mid U_{t-1}] = U_{t-1} + \mu_C - \mu_S - \gamma_2 = U_{t-1} + (\mu_C - \mu_S - \gamma_2)$$
Because $\mu_C - \mu_S > \gamma_2 > 0$ so $E[U_t \mid U_{t-1}] > U_{t-1}$. It means surplus value increases but its growth more slowdown from second condition.

Combining the three conditions above, we obtain:

$$E[U_t \mid U_{t-1}] = \begin{cases} U_{t-1} + \mu_C - \mu_S, & \text{if } U_{t-1} < \delta_1, \\ U_{t-1} + \mu_C - \mu_S - \gamma_1, & \text{if } \delta_1 \leq U_{t-1} < \delta_2, \\ U_{t-1} + \mu_C - \mu_S - \gamma_2, & \text{if } U_{t-1} \geq \delta_2 \end{cases}$$

and

$$E[U_t - U_{t-1} \mid U_{t-1}] = \begin{cases} \mu_C - \mu_S, & \text{if } U_{t-1} < \delta_1, \\ \mu_C - \mu_S - \gamma_1, & \text{if } \delta_1 \leq U_{t-1} < \delta_2, \\ \mu_C - \mu_S - \gamma_2, & \text{if } U_{t-1} \geq \delta_2. \end{cases}$$

Hence, Theorem 1 is proven. The mechanism multiple threshold levels gives two main effects of stabilization, namely retaining allocation in low surplus conditions to maintain reserves is safe, and surplus distribution proportional in high surplus conditions so that growth still controlled. As a result, the surplus process is proven stable and secure from deficit risk systematically. □

6.2. Sensitivity and Robustness Analysis of the Model

After mathematical stability is proven, numerical testing is carried out to observe the level of robustness of the model to changes in operational parameters.

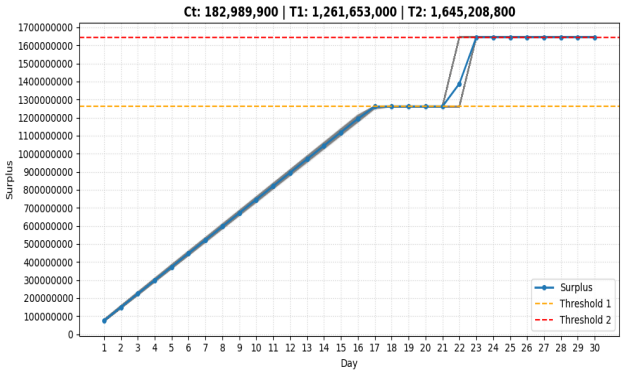
i. Scenario 1: Threshold Pair Variation

In the first scenario, the threshold value pair δ_1 and δ_2 changed gradually assuming daily net income remains constant. A summary of the tested threshold pairs is presented in Table 14.

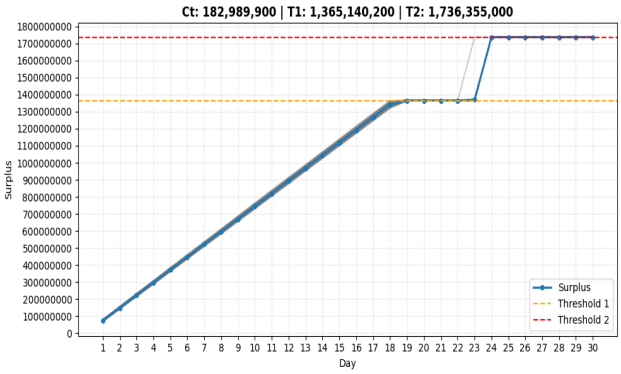
Table 14. Combination of threshold value pairs δ_1 and δ_2 (in rupiah)

Threshold Pair	Threshold δ_1	Threshold δ_2	Information
1	1,261,653,000	1,645,208,800	(P50, P70)
2	1,365,140,200	1,736,355,000	(P55, P75)
3	1,468,647,200	1,826,719,200	(P60, P80)
4	1,569,235,800	1,928,920,100	(P65, P85)
5	1,645,208,800	2,024,573,300	(P70, P90)

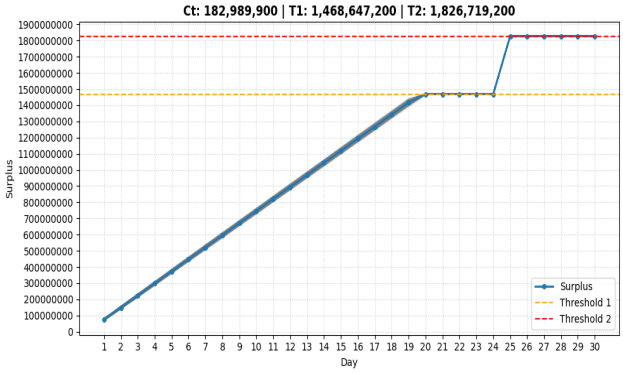
Based on testing scenario 1 above, the following simulation of surplus value dynamics was obtained:



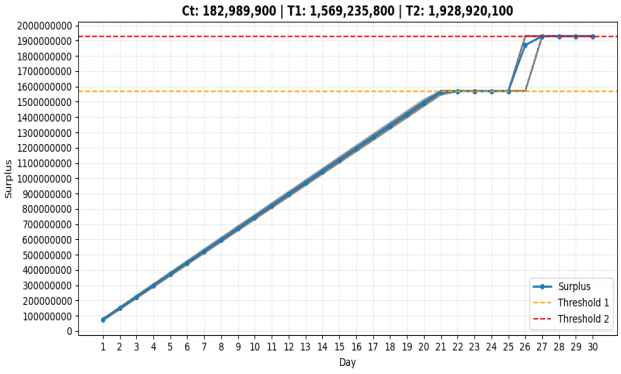
(a) $\delta_1 = 1,261,653,000$, $\delta_2 = 1,645,208,800$



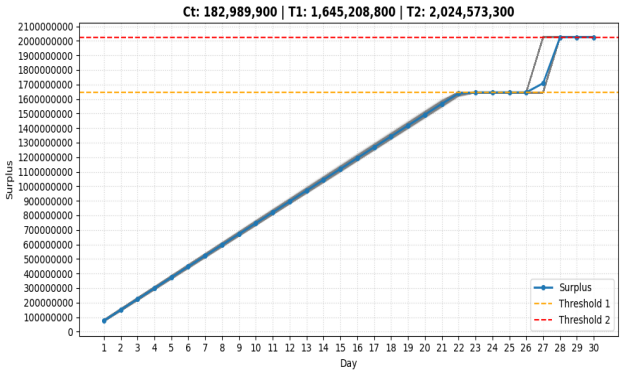
(b) $\delta_1 = 1,365,140,200$, $\delta_2 = 1,736,355,000$



(c) $\delta_1 = 1,468,647,200$, $\delta_2 = 1,826,719,200$



(d) $\delta_1 = 1,569,235,800$, $\delta_2 = 1,928,920,100$

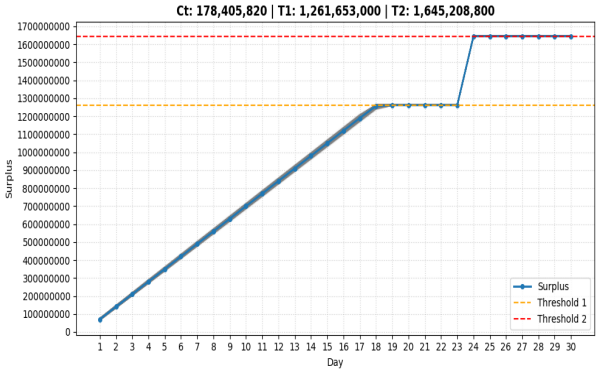


(e) $\delta_1 = 1,645,208,800$, $\delta_2 = 2,024,573,300$

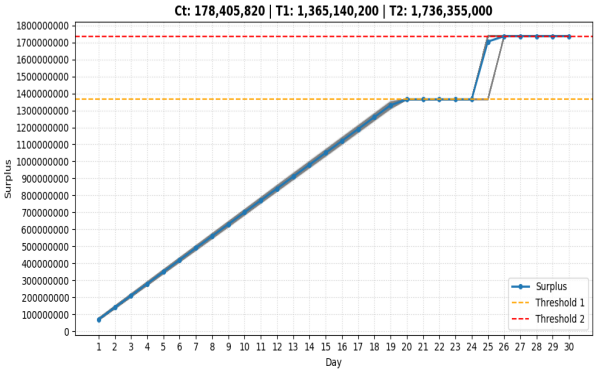
Figure 9. Dynamics of surplus value from testing results using scenario 1

ii. Scenario 2: Variation in Percentage of Unsold Products (Revenue Decrease)

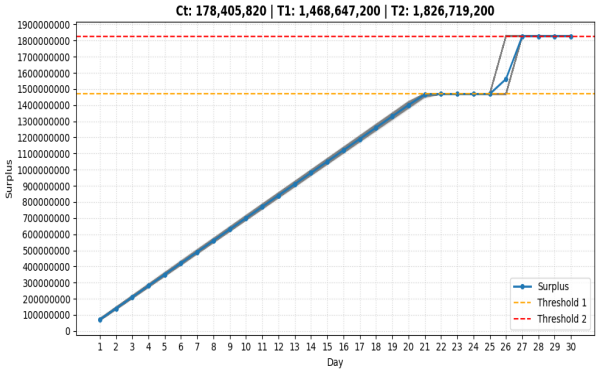
The second scenario tested the impact of a decrease in daily revenue due to unsold crab crackers. The test was conducted with variations in revenue decreases of 1%, 5%, and 10% combined with changes in the threshold pair.



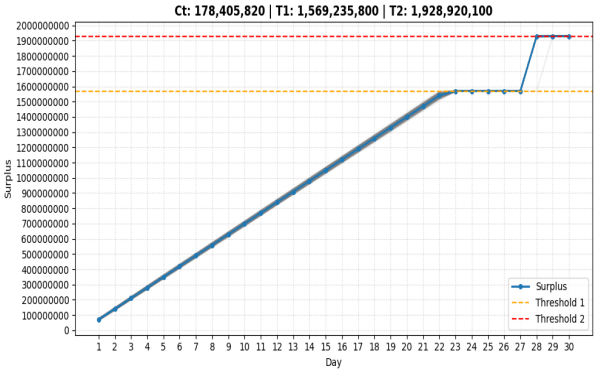
(a) $\delta_1 = 1,261,653,000$, $\delta_2 = 1,645,208,800$



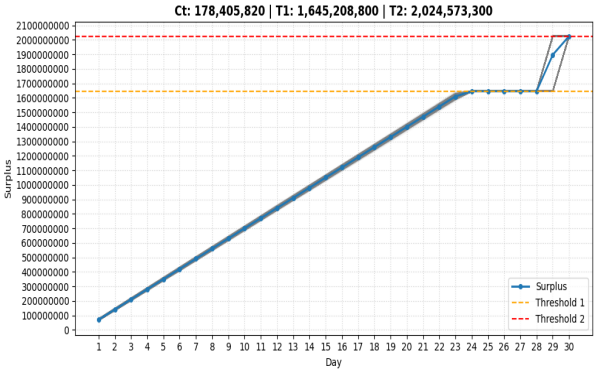
(b) $\delta_1 = 1,365,140,200$, $\delta_2 = 1,736,355,000$



(c) $\delta_1 = 1,468,647,200$, $\delta_2 = 1,826,719,200$

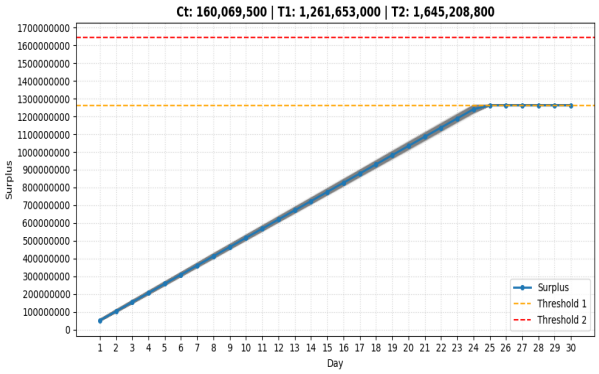


(d) $\delta_1 = 1,569,235,800$, $\delta_2 = 1,928,920,100$

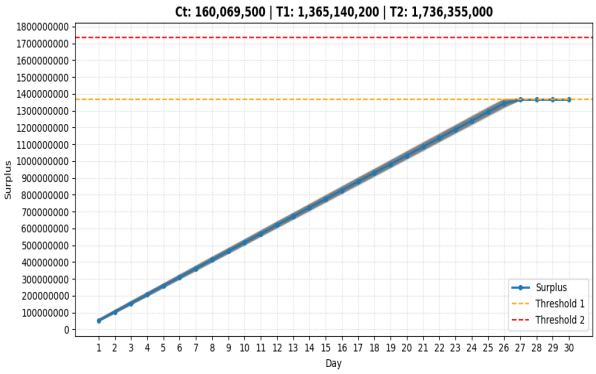


(e) $\delta_1 = 1,645,208,800$, $\delta_2 = 2,024,573,300$

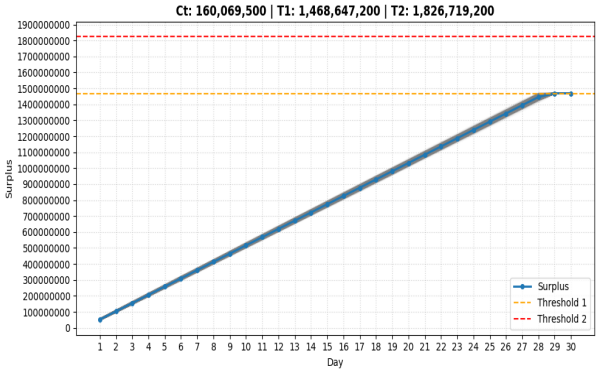
Figure 10. Dynamics of the surplus value of the test results using scenario 2 with a 1% decrease in income



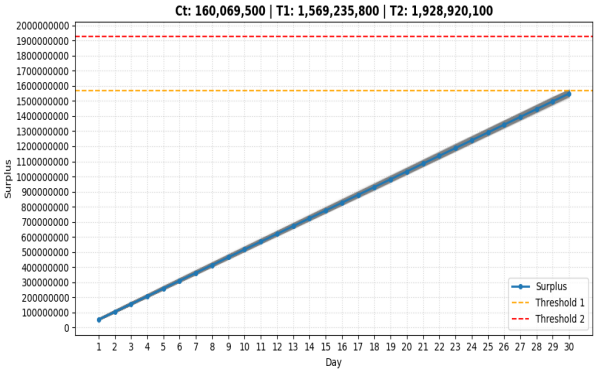
(a) $\delta_1 = 1,261,653,000$, $\delta_2 = 1,645,208,800$



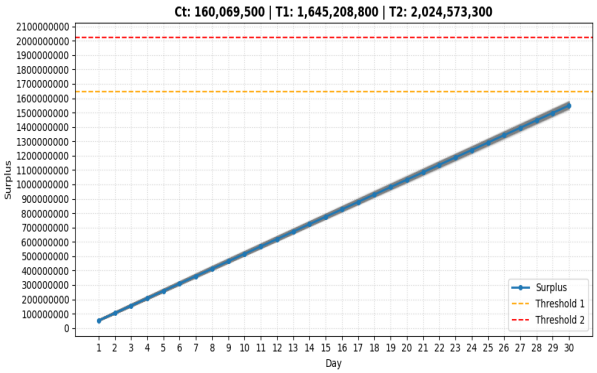
(b) $\delta_1 = 1,365,140,200$, $\delta_2 = 1,736,355,000$



(c) $\delta_1 = 1,468,647,200$, $\delta_2 = 1,826,719,200$

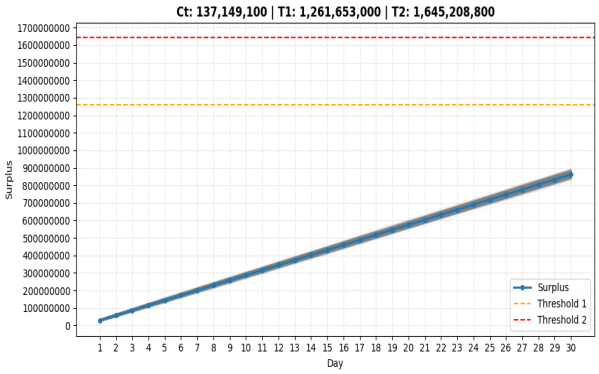


(d) $\delta_1 = 1,569,235,800$, $\delta_2 = 1,928,920,100$

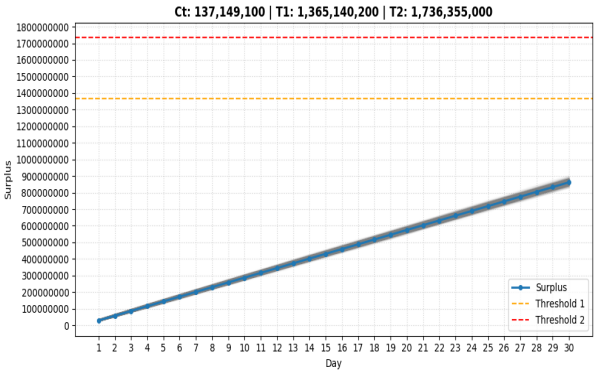


(e) $\delta_1 = 1,645,208,800$, $\delta_2 = 2,024,573,300$

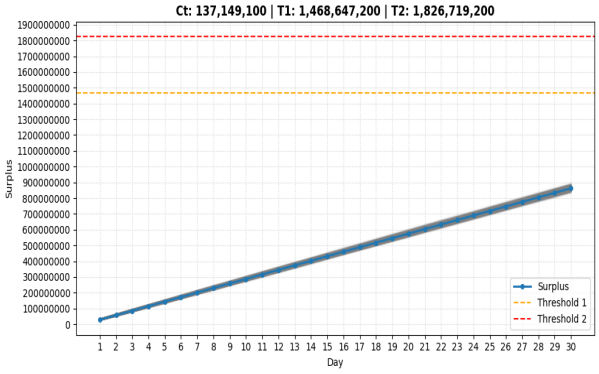
Figure 11. Dynamics of the surplus value of the test results using scenario 2 with a 5% decrease in income



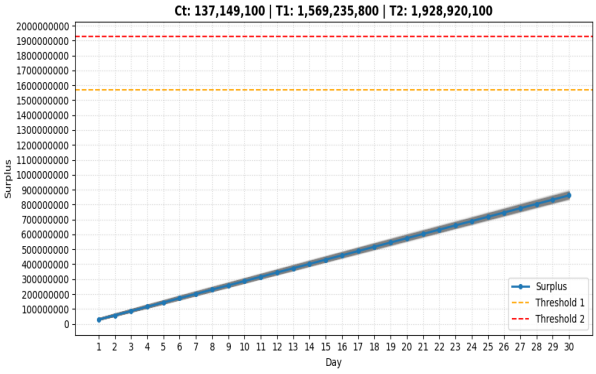
(a) $\delta_1 = 1,261,653,000$, $\delta_2 = 1,645,208,800$



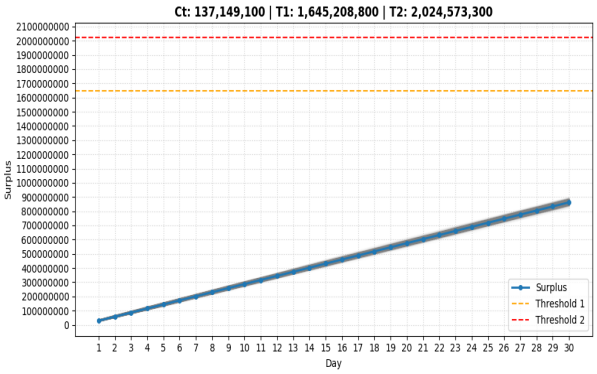
(b) $\delta_1 = 1,365,140,200$, $\delta_2 = 1,736,355,000$



(c) $\delta_1 = 1,468,647,200$, $\delta_2 = 1,826,719,200$



(d) $\delta_1 = 1,569,235,800$, $\delta_2 = 1,928,920,100$



(e) $\delta_1 = 1,645,208,800$, $\delta_2 = 2,024,573,300$

Figure 12. Dynamics of the surplus value of the test results using scenario 2 with a 10% decrease in income

The results of the model robustness test show that the formulated surplus process model is very robust against changes in threshold levels. Shifts in the values of δ_1 and δ_2 only affect the amount of surplus allocated, without disrupting the stability of long-term reserve fund accumulation. Conversely, the model proved highly sensitive to changes in the percentage of unsold crab crackers. The tolerance level for unsold products was set at a maximum of 1% so that all threshold scenario options could still facilitate optimal surplus allocation. When the percentage of unsold products increased to 5% or more, the surplus position failed to exceed δ_2, so that the surplus allocation for fishermen's welfare benefits was completely halted. This finding confirms that sales efficiency is an absolute

requirement for maintaining the continuity of the allocation of surplus (benefits) from productive waqf on a sustainable basis.

7. Conclusion

This research succeeds in formulating and validating a discrete-time surplus process model with multiple threshold levels in crab cracker industry based on productive waqf. Research results make two contributions main:

1. **Theoretical and Numerical Findings**
   Through Theorem 1, the model is proven mathematically own double stability characteristic, namely: staying away from deficit risk in low surplus conditions and control growth in high surplus conditions. Validation based Monte Carlo simulation on combinations compound claim (Binomial and Log-Normal) produces level high accuracy with Mean Absolute Percentage Error (MAPE) is 0.059. Testing of robustness shows that the model is resistant to variation threshold, however sensitive to sale efficiency.
2. **Managerial Implications**
   i. *Surplus allocation decision*: threshold δ_1 of IDR 1,468,647,200 serves as the operational safety limit. Managers must not allocate surplus if the surplus is below δ_1 and permitted to allocate surplus when beyond δ_2 amounting to IDR 1,826,719,200.
   ii. *Business development decisions*: consistent surplus stays above δ_2 indicates that financial are very healthy, so become baseline for manager to allocate funds for development business.
   iii. *Risk tolerance limits*: Manager must maintain level of unsold product maximum 1%, while at the level of 5% or more require termination temporary of surplus allocation to maintain integrity of the principal waqf assets.

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