STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, July 2026, pp 660–683.
Published online in International Academic Press (www.IAPress.org)

Novel Quantile Regression Model Using Extended Exponential-Geometric Distribution with Real Data Application

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Abstract The current work proposes a quantile regression framework that takes into consideration the nature of the dependent variables that are characterized by restriction of the interval between 0 and 1, precisely by using the Unit Extended Exponential Geometric (UEEG) distribution. The method has the capacity to capture the effects of covariates at various quantiles of the distribution, allowing a more comprehensive representation of the changes in behavior across the whole range. We analyze the distribution characteristics of the model and recommend maximum likelihood parameter estimation. A simulation experiment is performed to test the properties of the proposed estimators at finite samples with respect to bias and MSE. The methodology is demonstrated through the application to a real data set in order to showcase its practical utility. The new framework is compared against the analogous ones based on Unit Exponential, Unit Extended Exponential, and Unit Lindley distributions using the Akaike information criterion (AIC), Bayesian information criterion (BIC), and Hannan Quinn information criterion (HQIC). The findings suggest that not only does the new model fit the data better, but it is also less biased and has a lower MSE, therefore, providing more flexibility for data restricted to a unit interval.

Keywords Quantile Regression, Unit Extended Exponential-Geometric, Residual Analysis, Pearson Residual

AMS 2010 subject classifications 62Jxx

DOI: 10.19139/soic-2310-5070-3802

1. Introduction

Quantile regression is a significant statistical tool that has been growing in usage to explore the relationship between explanatory variables and various parts of the conditional distribution of a response variable, rather than being limited to the conditional mean only. This characteristic of quantile regression renders it exceptionally suitable for scenarios where the data show heterogeneity, skewness, heavy tails, or bounded support, i.e., situations where classical mean-based regression models can offer incorrect or incomplete inference. Over the past few years, there have been extensive methodological innovations in quantile regression, for example, fully parametric models have been created which are based on probability distributions that are naturally related to the data structures such as proportions, survival times, and bounded responses on the unit interval (0,1).

Several parametric quantile regression models for bounded and unit-interval data were suggested by reparameterization the existing distributions in terms of quantiles and connecting these quantiles to covariates through appropriate link functions. A significant contribution in this direction was that of Jodrá and Jiménez-Gamero[18]. (2020) who devised a quantile regression model based on the Exponential-Geometric distribution for bounded responses and thus, paved the way for distribution-based regression frameworks that focus directly on conditional quantiles rather than moments.

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

In line with this framework, an increasing number of articles have extended parametric quantile regression to various distribution families and modeling frameworks.

Mazucheli et al[25]. (2022) offered a detailed review of parametric quantile regression models for unit-interval and positive data pointing out the implementation features and real-data applications. Santoro et al[30]. (2024) introduced the unit-power half-normal distribution in a quantile regression framework for healthcare data, and Bashir et al[9]. (2024) proposed a bounded exponentiated Weibull distribution and its quantile regression model demonstrating that it provides more flexibility for unit-supported responses. Besides these, there is also the unit-Chen quantile regression model Korkmaz et al [22]. (2023), the unit-Cauchy quantile regression model for proportion data Arslan and Yu [8]. (2025), and the exponentiated Weibull quantile regression model with cure-rate effects in survival analysis Gómez and Gallardo [15] (2025). Furthermore, the contributions of Wu and Rui [33] (2025) have also dealt with environmental and lifetime data by way of contamination unit models with quantile regression and quantile residual life regression under length-biased censoring.

Earlier and parallel research has also significantly contributed to the development of this area. Korkmaz [21] 2021 presented an exponential-power quantile regression model suitable for bounded data along with residual diagnostics. On the other hand, Sánchez et al [31]. (2021) suggested the use of a Weibull-based quantile regression framework accompanied by a comprehensive diagnostic analysis. Apart from unit-interval models, other approaches to distribution-based regression have been considered. These include finite mixture quantile regression for semi-continuous longitudinal data Maruotti et al [23]. (2020), log-symmetric quantile regression models for positive data Saulo et al [32]. (2020), and parametric quantile regression models for double-bounded responses Gallardo et al [14]. (2021). More general parametric methods based on versatile families such as the generalized gamma distribution have further pointed out how quantile regression can be used to model asymmetric and heterogeneous data features.

New achievements have also made it possible for quantile regression to reach unit distributions supported on boundaries, for example, the unit log-log model, as well as through the prevalent use of baseline distributions like the unit-Weibull in parametric quantile regression frameworks. Besides, the family of comprehensive distributional regression methods including GAMLSS focus on the whole conditional distribution by connecting several distributional parameters to covariates. Technological improvements, such as the unitquantreg R package, have empowered the actual application of parametric quantile regression models for bounded data. Additionally, mixed-effects quantile regression and machine-learning-based methods for extreme quantiles indicate the rising desire in robust and distribution-sensitive modeling that goes beyond traditional parametric frameworks.

Thus, the collection of these papers from 2020 to 2026 is a clear indication of the quick evolution and growing significance of distribution-based quantile regression models, especially for bounded and unit-interval data. This burgeoning literature is an inspiration to this paper that introduces the unit extended exponential-geometric Quantile regression model. The proposed model takes advantage of the flexibility which the extended exponential-geometric distribution brings and at the same time allows for conditional Quantiles to be directly modeled. This way it gives more detailed information about the covariate effects at different points of the response distribution and also provides better inference for applications involving bounded data. Section 2 of this paper provides details on the properties of the extended exponential geometric (EEG) distribution. In Section 3, we present the extended EEG-based regression model and its formulation. Several methods for estimating the parameters of the new model are described in Section 4. To assess the performance of the proposed estimators, in Section 5 we conducted a Monte Carlo simulation experiment. In Section 6, diagnostic tools along with residual analysis have been developed for assessing the appropriateness of a fitted model. Section 7 demonstrates the effectiveness of the proposed methodology via applications to two real data sets. Finally, Section 8 wraps up this paper by giving a brief summary.

2. Extended Exponential–Geometric Distribution EEG

This section initially defines the extended exponential-geometric (EEG) distribution with a formal definition and discusses the main statistical and mathematical properties of the distribution, which constitute the theoretical basis for later modeling and inference. [12] [10].

2.1. Definition of the Extended Exponential–Geometric EEG distribution

A new variable X that denotes lifetime data is considered to be distributed as an extended exponential-geometric (EEG) distribution whose probability density function (PDF) is given by [6]

$$f(y;\theta,\alpha) = \frac{\theta\alpha\exp(-\theta y)}{\left[1-(1-\alpha)\exp(-\theta y)\right]^2} \tag{1}$$

And the corresponding cumulative distribution function (CDF) is defined as [16]

$$F(y;\theta,\alpha) = \frac{1-\exp(-\theta y)}{1-(1-\alpha)\exp(-\theta y)} \tag{2}$$

2.2. Properties of the Extended Exponential–Geometric (EEG)

2.2.1. The Survival Function The survival function is defined as the probability that a random variable survives beyond a given time y, and is given by [19, 20]

$$S(y) = P(Y > y) = 1 - F(y)$$

Thus, the survival function of the EEG distribution is

$$S(y;\theta,\alpha) = 1 - \frac{1-\exp(-\theta y)}{1-(1-\alpha)\exp(-\theta y)} \tag{3}$$

Figure 1 is four combinations of the parameters θ and α: $(0.5, 0.3)$, $(0.5, 0.7)$, $(1, 0.3)$, and $(1, 0.7)$. Each subplot

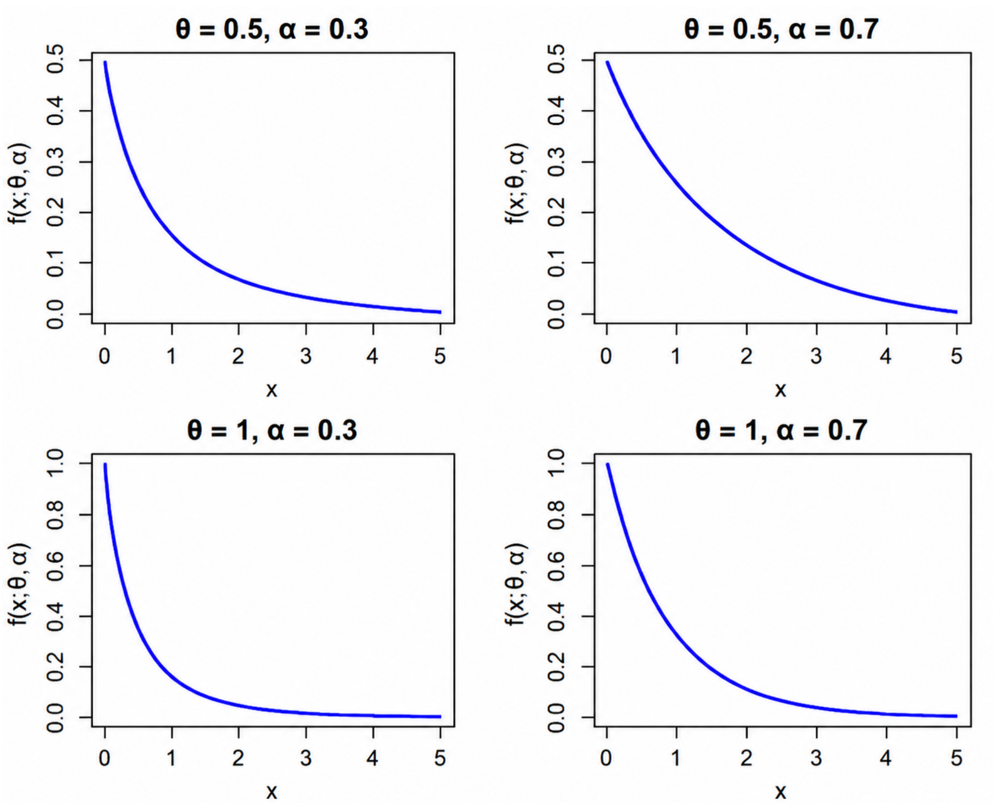

Figure 1. Behavior of the EEG PDF for a Variety of Choices

illustrates how the density function decays exponentially as x increases. Increasing θ from 0.5 to 1 decreases the function more rapidly, which is an indication of a faster decay rate, while increasing α from 0.3 to 0.7 raises the overall magnitude of the function. Hence, θ primarily controls the steepness of the decline, and α scales the height of the density. The 2×2 layout allows a clear visual comparison of how each parameter affects the shape of $f(x; \theta, \alpha)$. The Figure 2 shows the behavior of the cumulative distribution function $F(x; \theta, \alpha)$ for a variety of

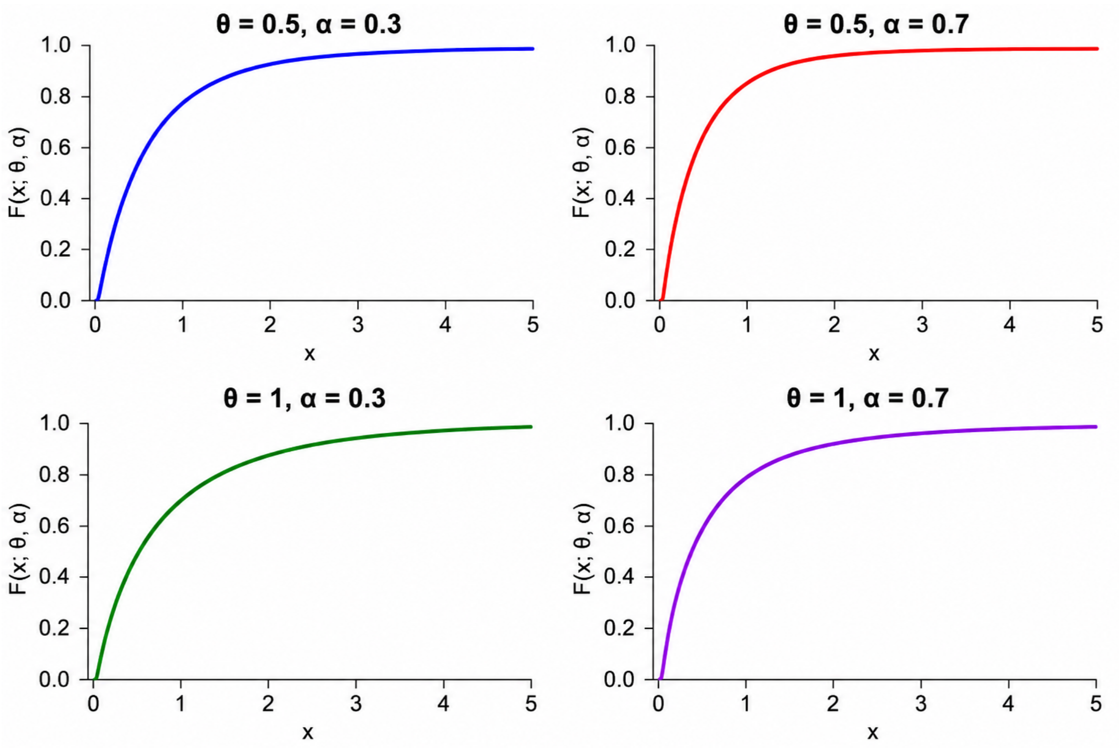

Figure 2. Behavior of the EEG CDF for a Variety of Choices

choices of the parameters θ and α. In all four plots, $F(x)$ is an increasing function of x and converges to 1 as x becomes large, which is a main property for the validity of the CDF. An increase in the parameter θ raises the curve more rapidly near x, which is a sign of a faster accumulation of probability, while changes in α modify the curvature and smoothness of the function. Accordingly, the plots exhibit how θ principally controls the rate of increase and α adjusts the total shape of the distribution.

The Figure 3 reveals the survival function $S(x; \theta, \alpha)$ for four various combinations of the parameters θ and α. As would be expected for a true survival function, the survival probability in each panel begins at 1 when $x = 0$ and declines monotonically towards 0 as x grows. While variations in α affect the decay's survival and smoothness, larger values of θ lead the curves to decrease more quickly, suggesting a higher failure rate and shorter survival durations. Overall, the graphs show how α adjusts the survival pattern's structure whereas θ mostly regulates the rate of decline.

2.2.2. Hazard Rate Function Generally, the hazard rate function can be defined as the instantaneous failure rate at time x, given survival up to that time. It has the following expression [1, 2]

$$h(x) = \frac{f(x)}{S(x)}$$

Thus, the hazard rate function of the EEG distribution is

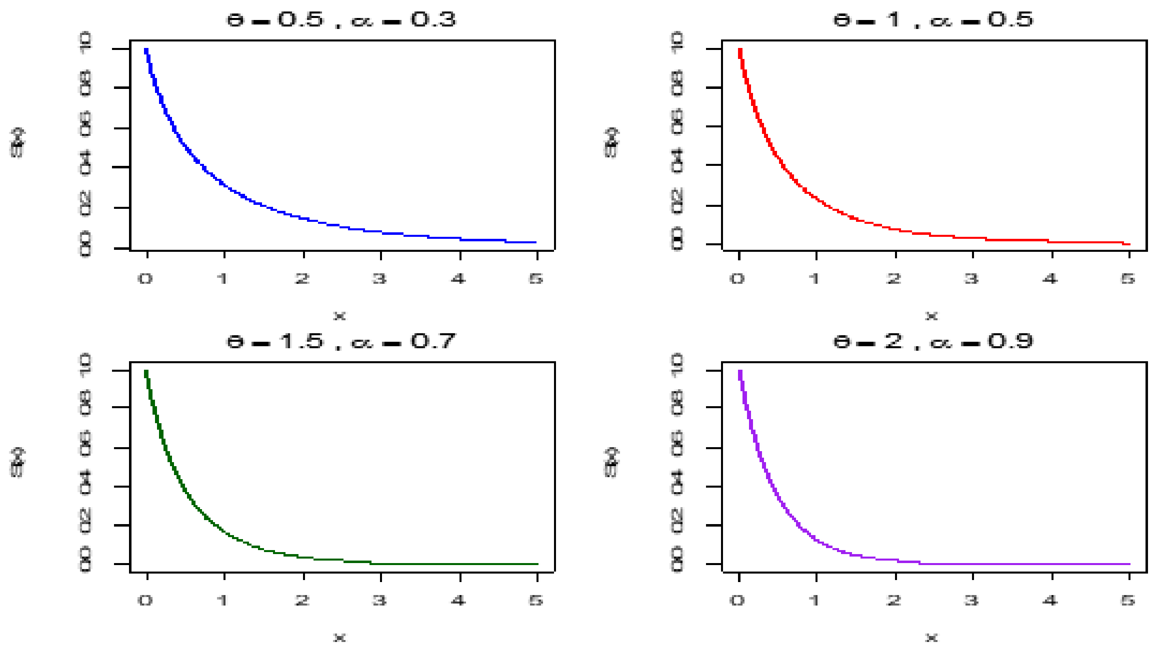

Figure 3. Behavior of the EEG Survival Function for a Variety of Choices

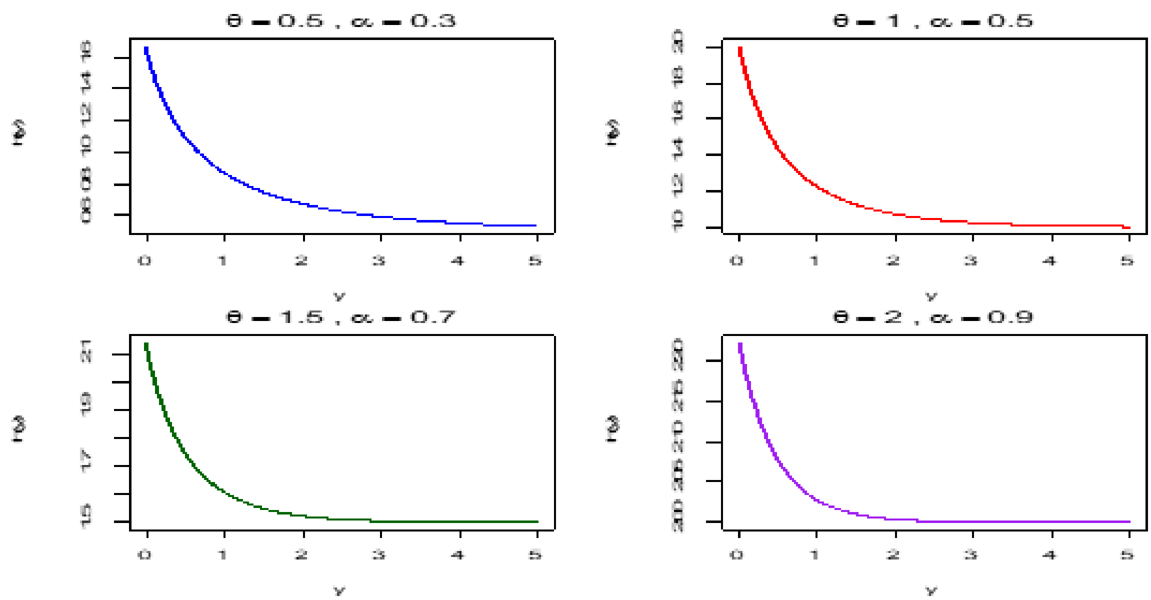

Figure 4. Behavior of the EEG Hazard Function for a Variety of Choice

$$h(y;\theta,\alpha) = \frac{\theta}{1-(1-\alpha)\exp(-\theta y)} \tag{4}$$

The Figure 4 illustrates the hazard rate function $h(y;\theta,\alpha)$ for different values of θ and α. In all cases, the hazard rate decreases with increasing y, indicating a decreasing risk of failure over time. Higher values of θ produce larger initial hazard rates and a faster decline, while the parameter α affects the level and curvature of the hazard function.

Overall, the plots show that the model exhibits a decreasing hazard behavior, with θ controlling the magnitude and α fine-tuning the shape.

3. Regression Model for Unit Extended Exponential-Geometric Distribution UEEG

A regression model based on the extended UEEG distribution offers a versatile tool for survival or lifetime data analysis by allowing the inclusion of covariates in the parameters of the UEEG distribution. Such a method can model various shapes of hazard rates more accurately and thus can unveil the complicated relationships between predictors and response variables [17].

3.1. Unit Extended Exponential-Geometric Distribution UEEG

The Unit Extended Exponential-Geometric (UEEG) distribution is a flexible model combining exponential and geometric features, which is very handy when modeling lifetime data with various hazard behaviors. Let X be a random variable following the Unit Extended Exponential-Geometric (UEEG) distribution with parameters $\theta > 0$ and $0 < \alpha \leq 1$. Consider the transformation [11]

$$x = \exp(-y),$$

then

$$dx = -\exp(-y)\,dy,$$

or equivalently,

$$dx = -x\,dy,$$

which implies

$$\frac{dy}{dx} = \frac{1}{x}.$$

Hence, the probability density functions are related through the Jacobian formula

$$f(x;\theta,\alpha) = f(y;\theta,\alpha)\left|\frac{dy}{dx}\right| \tag{5}$$

Specifically, for the UEEG distribution on the unit interval $0 < x < 1$, the PDF, CDF, and survival function are given respectively by [28]

$$f(x;\theta,\alpha) = \frac{\theta\alpha x^{\theta-1}}{\left[1-(1-\alpha)x^{\theta}\right]^2}, \qquad 0 < x < 1 \tag{6}$$

$$F(x;\theta,\alpha) = \frac{1-x^{\theta}}{1-(1-\alpha)x^{\theta}} \tag{7}$$

$$S(x;\theta,\alpha) = 1 - \frac{1-x^{\theta}}{1-(1-\alpha)x^{\theta}} \tag{8}$$

$$h(x;\theta,\alpha) = \frac{\theta\alpha}{1-(1-\alpha)x^{\theta}} \tag{9}$$

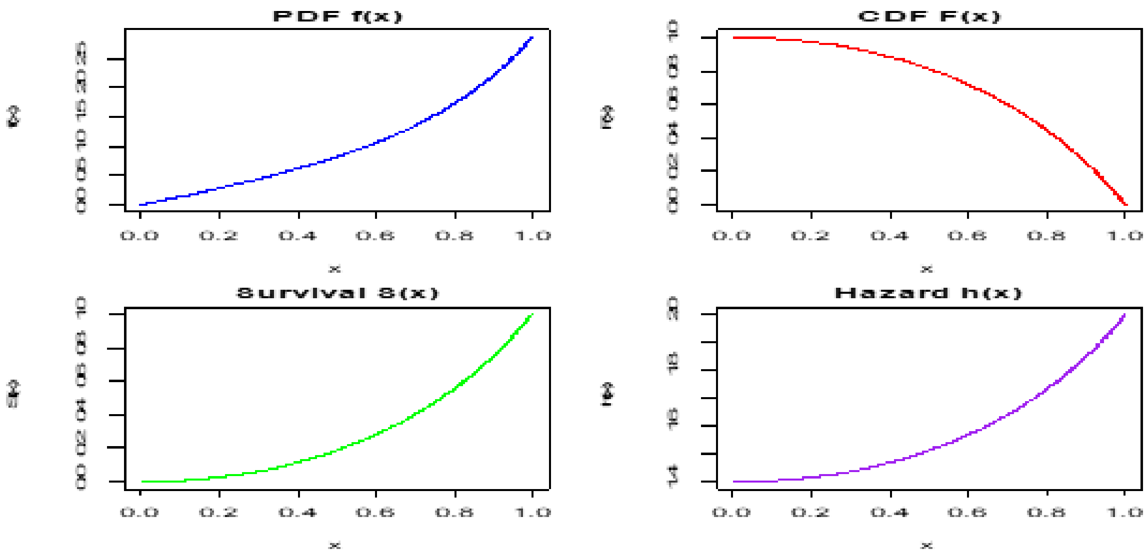

Figure 5. PDF, CDF, SF and HF for UEEG

3.2. *Quantile Function for UEEG*

we can express first and second quantiles ias follows [26, 24].

$$F(x;\theta,\alpha)=\tau, \qquad 0<\tau<1 \tag{10}$$

Substituting the CDF of the UEEG distribution gives

$$\frac{1-x^{\theta}}{1-(1-\alpha)x^{\theta}}=\tau, \qquad 0<\tau<1 \tag{11}$$

Multiplying both sides and simplifying, we obtain

$$1-x^{\theta}=\tau\left[1-(1-\alpha)x^{\theta}\right], \qquad 0<\tau<1 \tag{12}$$

Hence, the quantile function is

$$Q(\tau)=\left(\frac{1-\tau}{1-(1-\alpha)\tau}\right)^{\frac{1}{\theta}}, \qquad 0<\tau<1 \tag{13}$$

The first quartile is obtained by setting $\tau=0.25$:

$$Q_1=\left(\frac{1-0.25}{1-(1-\alpha)(0.25)}\right)^{\frac{1}{\theta}} \tag{14}$$

The second quartile (median) is obtained by setting $\tau=0.5$:

$$Q_2=\left(\frac{1-0.5}{1-(1-\alpha)(0.5)}\right)^{\frac{1}{\theta}} \tag{15}$$

The median $Q(0.5;\alpha,\theta)$, first quartile $Q(0.25;\alpha,\theta)$, and upper quartile $Q(0.75;\alpha,\theta)$ are obtained, respectively, by substituting 0.5, 0.25, and 0.75 into the quantile function. The Bowley's (BS) measure of skewness and the Moors' (MK) measure of kurtosis can then be calculated using the quantiles. They are, respectively, given by [7]

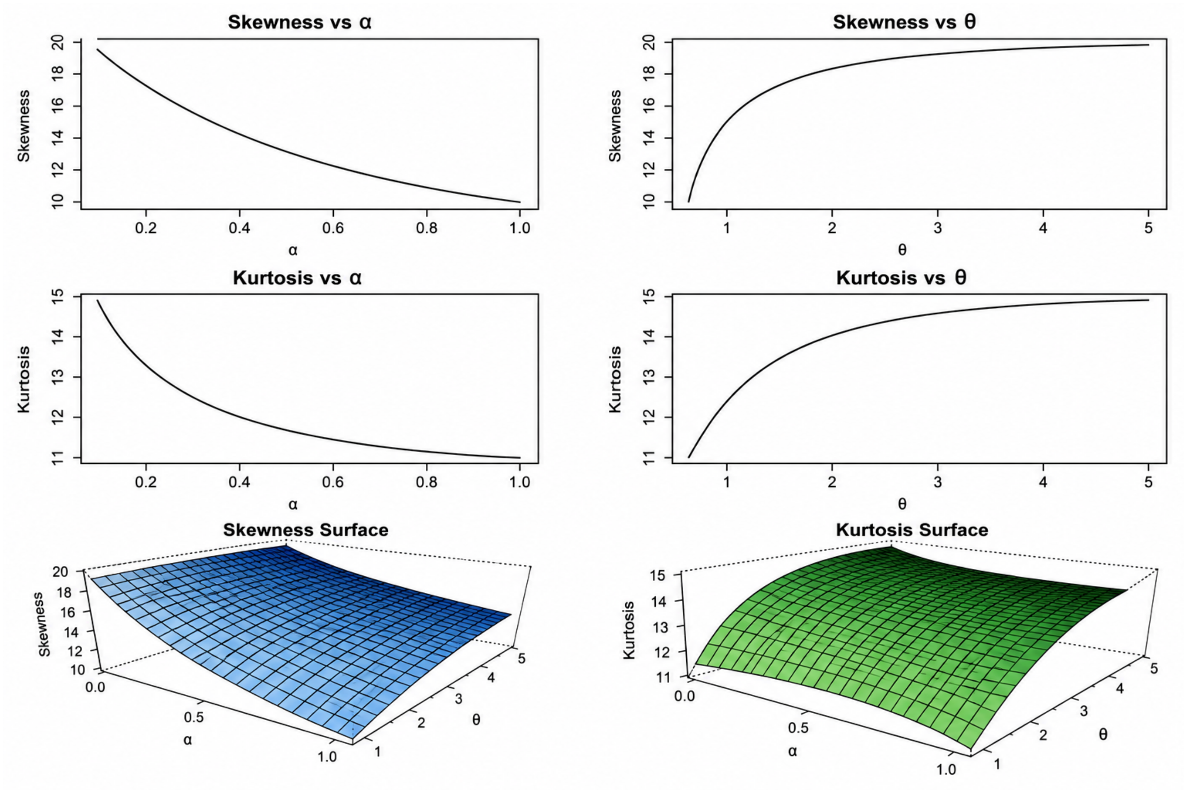

Figure 6. Skewness and Kurtosis for UEEG

$$\text{Skewness} = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{16}$$

$$\text{Kurtosis} = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{17}$$

So regression model will be

$$Q(\tau) - \mu = 0 \tag{18}$$

$$\mu - \left(\frac{1-\tau}{1-(1-\alpha)\tau}\right)^{\frac{1}{\theta}} = 0 \tag{19}$$

$$\mu^{\theta} = \frac{1-\tau}{1-(1-\alpha)\tau} \tag{20}$$

Multiplying both sides and simplifying, we obtain

$$\left[1-(1-\alpha)\tau\right]\mu^{\theta} = 1-\tau \tag{21}$$

$$\mu^{\theta} - (1-\alpha)\tau\mu^{\theta} = 1-\tau \tag{22}$$

$$\mu^{\theta} - (1-\tau) = (1-\alpha)\tau\mu^{\theta} \tag{23}$$

$$\mu^{\theta} - (1-\tau) = \tau\mu^{\theta} - \alpha\tau\mu^{\theta} \tag{24}$$

$$\mu^{\theta} - (1-\tau) + \tau\mu^{\theta} = \alpha\tau\mu^{\theta} \tag{25}$$

$$\alpha = \frac{\mu^{\theta} - (1-\tau) + \tau\mu^{\theta}}{\tau\mu^{\theta}} \tag{26}$$

In this section, based on the parameterized probability density function given in Equation 25, the density function can be written as

$$f(\mu, \theta, \tau) = \frac{\theta\left(\frac{\mu^{\theta}-(1-\tau)+\tau\mu^{\theta}}{\tau\mu^{\theta}}\right)x^{\theta-1}}{\left[1-\left(1-\frac{\mu^{\theta}-(1-\tau)+\tau\mu^{\theta}}{\tau\mu^{\theta}}\right)x^{\theta}\right]^{2}} \tag{27}$$

The UEEG quantile regression model has been developed. Let $X_1, \ldots, X_n$ denote n independent random variables,

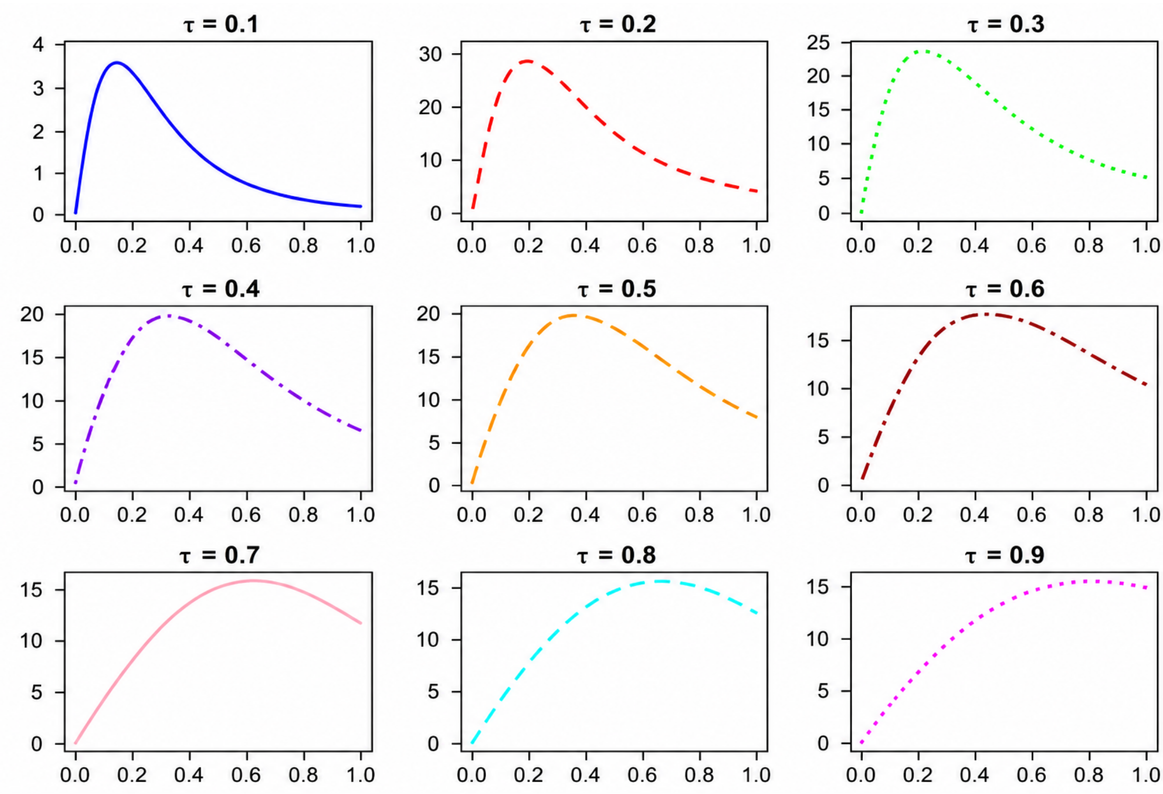

Figure 7. Regression Model for UEEG

where for each $i = 1, \ldots, n$, X_i follows the distribution in Equation (6) with an unknown quantile parameter μ_i, an unknown shape parameter α, and a fixed quantile level $\tau \in (0, 1)$. Accordingly, the UEEG quantile regression model is specified by assuming that the quantile μ_i of X_i satisfies the functional relationship given by

$$g(\mu_i) = \mathbf{x}_i^T \boldsymbol{\beta} \tag{28}$$

where

$$\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{p-1})^T$$

is a p-dimensional vector of unknown regression coefficients, with $p < n$, and

$$\mathbf{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{i(p-1)})^T.$$

So logit link function will be as follows[3] [7]:

$$\operatorname{logit}(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) = \mathbf{x}_i^T \boldsymbol{\beta} \tag{29}$$

$$\mu_i = \frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})} \tag{30}$$

Substituting this expression into the parameterized density function gives

$$f(x_i;\theta,\mu_i,\tau) = \frac{\theta\left[\frac{\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta - (1-\tau) + \tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}{\tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}\right] x_i^{\theta-1}}{\left[1-\left(1-\frac{\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta - (1-\tau) + \tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}{\tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}\right) x_i^\theta\right]^2} \tag{31}$$

Accordingly, the regression equation can written as [13]:

$$\operatorname{logit}(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_{p-1} x_{i(p-1)}. \tag{32}$$

4. Estimation Methods

4.1. Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is one of the most widely used methods for estimating the parameters of regression models. It determines parameter values by maximizing the likelihood function constructed from the joint distribution of the observed data. The MLEs possess desirable statistical properties such as consistency, efficiency, and asymptotic normality under regularity conditions [29]. The likelihood function for a random sample $x_1, x_2, \ldots, x_n$ is given by

$$L = \prod_{i=1}^{n} f(x_i;\theta,\mu_i) \tag{33}$$

Substituting the proposed density function yields

$$L = \prod_{i=1}^{n} \frac{\theta\left[\frac{\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta - (1-\tau) + \tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}{\tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}\right] x_i^{\theta-1}}{\left[1-\left(1-\frac{\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta - (1-\tau) + \tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}{\tau\left(\frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}\right)^\theta}\right) x_i^\theta\right]^2} \tag{34}$$

The likelihood function L is constructed based on the probability density function of the proposed model by aggregating the contributions of the observations $i = 1, \dots, n$, as shown above. Because of the mathematical complexity of the likelihood expression, it is convenient to rewrite it in a more compact form. For simplicity of notation and analytical convenience, the numerator and denominator of the likelihood function are denoted by K_1 and K_2, respectively, allowing the likelihood to be written as

$$L = \prod_{i=1}^{n} \frac{K_1}{K_2} \tag{1}$$

where

$$K_1 = \prod_{i=1}^{n} \theta \left[\frac{\left(\frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})}\right)^{\theta} - (1-\tau) + \tau \left(\frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})}\right)^{\theta}}{\tau \left(\frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})}\right)^{\theta}} \right] x_i^{\theta-1} \tag{35}$$

and

$$K_2 = \prod_{i=1}^{n} \left[1 - \left(1 - \frac{\left(\frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})}\right)^{\theta} - (1-\tau) + \tau \left(\frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})}\right)^{\theta}}{\tau \left(\frac{\exp(\mathbf{x}_i^T \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T \boldsymbol{\beta})}\right)^{\theta}} \right) x_i^{\theta} \right]^2 \tag{36}$$

This parameterization enables the development of the log-likelihood function and the subsequent calculation of partial derivatives. The natural logarithm of the likelihood function results in the log-likelihood function, which can be expressed as

$$\ell = \sum_{i=1}^{n} \log(K_1) + \sum_{i=1}^{n} \log(K_2) \tag{37}$$

The first-order partial derivative of the log-likelihood with respect to β_0 is given by

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{n} \left(\frac{1}{K_1} \frac{\partial K_1}{\partial \beta_0} \right) + \sum_{i=1}^{n} \left(\frac{1}{K_2} \frac{\partial K_2}{\partial \beta_0} \right) \tag{38}$$

Similarly, the first-order partial derivative with respect to β_1 is

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{n} \left(\frac{1}{K_1} \frac{\partial K_1}{\partial \beta_1} \right) + \sum_{i=1}^{n} \left(\frac{1}{K_2} \frac{\partial K_2}{\partial \beta_1} \right) \tag{39}$$

The derivative with respect to the shape parameter θ is obtained as

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n} \left(\frac{1}{K_1} \frac{\partial K_1}{\partial \theta} \right) + \sum_{i=1}^{n} \left(\frac{1}{K_2} \frac{\partial K_2}{\partial \theta} \right) \tag{40}$$

The log-likelihood transformation simplifies the optimization process and improves numerical stability. To obtain the maximum likelihood estimators of the regression parameters, the first-order partial derivatives of the log-likelihood with respect to β_0, β_1, and θ are computed and set equal to zero [27]. Consequently, the resulting score equations are nonlinear and must be solved using numerical optimization techniques such as the Newton–Raphson algorithm.

4.2. Diagnostic and Sensitivity Measures

The undeveloped sensitivity subsection has been revised so that all diagnostic quantities are explicitly defined before being used. Let $\hat{\theta}$ denote the maximum likelihood estimator, and let $I(\hat{\theta})$ be the observed Fisher information matrix, defined as [4]

$$I(\hat{\theta}) = -\frac{\partial^2 \ell(\theta)}{\partial\theta\,\partial\theta^T}.$$

The asymptotic variance–covariance matrix used in the influence diagnostics is then given by

$$\mathrm{Var}(\hat{\theta}) = I(\hat{\theta})^{-1}.$$

Generalized Cook's Distance (GCD) The Generalized Cook's Distance is used to assess the influence of individual observations on the estimated model parameters. It is defined as

$$\mathrm{GCD}_i = (\hat{\theta}_{(i)} - \hat{\theta})^T \left[\mathrm{Var}(\hat{\theta})\right]^{-1} (\hat{\theta}_{(i)} - \hat{\theta}) \tag{41}$$

where $\hat{\theta}_{(i)}$ denotes the parameter estimate obtained after deleting observation i, and $\hat{\theta}$ is the maximum likelihood estimator based on the full sample. Observations satisfying

$$\mathrm{GCD}_i > \frac{4}{n}$$

are examined as potentially influential observations.

Likelihood Displacement (LD) Likelihood Displacement measures the effect of deleting an observation on the maximized likelihood function. It is defined as

$$\mathrm{LD}_i = 2\left[\ell(\hat{\theta}) - \ell(\hat{\theta}_{(i)})\right] \tag{42}$$

Large values of LD_i indicate that deleting observation i substantially changes the maximized likelihood. The LD values are interpreted jointly with the GCD statistics and residual plots rather than using a single universal cut-off point.

Local Influence Approach (LIA) Local influence is assessed by perturbing the case weights in the log-likelihood function and examining the normal curvature C_i in the direction of each observation. Observations with curvature values substantially larger than the remaining cases are flagged for further investigation. The perturbed likelihood displacement is defined as

$$\mathrm{LD}_w = 2\left[\ell(\hat{\theta}) - \ell(\hat{\theta}_w)\right] \tag{43}$$

where $\hat{\theta}_w$ denotes the maximum likelihood estimator under the perturbed weight scheme. In the real-data application, the sensitivity analysis is reported together with the residual analysis in Section 6, using the same fitted UEEG regression model and the same response variable [5].

5. Simulation Study

A simulation study is conducted to evaluate the finite-sample performance of the proposed estimators. The assessment is based on the Average Estimate (AS), Mean Squared Error (MSE), Bias, Root Mean Squared Error (RMSE), and the 95% Coverage Probability (CP95) under different sample sizes and parameter settings. The study is implemented according to the following steps using Monte Carlo simulation.

1. Consider the initial parameter values (Scenario I)

$$\theta = 1, \quad \tau = 0.2, \quad \beta_0 = 3.5, \quad \beta_1 = 1.5.$$

2. Consider the second parameter setting (Scenario II)

$$\theta = 1, \quad \tau = 0.5, \quad \beta_0 = 3.5, \quad \beta_1 = 0.5.$$

3. Consider the third parameter setting (Scenario III)

$$\theta = 1, \quad \tau = 0.6, \quad \beta_0 = 3.5, \quad \beta_1 = 0.8.$$

4. Generate observations from the proposed model

$$y \sim f(\mu, \tau, \theta),$$

   where

$$\mu = \frac{\exp(\beta_0 + x_1\beta_1)}{1 + \exp(\beta_0 + x_1\beta_1)}.$$

5. Generate the explanatory variable

$$x_1 \sim U(0, 1).$$

6. Consider the sample sizes

$$n = 20, 30, 50, 100, 500.$$

7. Set the number of simulation replications to

$$N = 1000.$$

8. For each generated sample size, compute the following performance measures:

$$\text{AS}, \quad \text{MSE}, \quad \text{Bias}, \quad \text{RMSE}, \quad \text{CP95\%}.$$

9. Compute the model selection and goodness-of-fit criteria:

$$\text{AIC}, \quad \text{BIC}, \quad \text{HQIC}, \quad R^2.$$

10. Plot the performance measures for the generated sample sizes, including:

$$\text{Bias}, \quad \text{MSE}, \quad \text{RMSE}, \quad \text{CP95\%}.$$

Tables 1 and 2 indicate that the estimators of β_0 and β_1 become more accurate and stable as the sample size increases, with Bias, MSE, and RMSE decreasing across all quantile levels.

Figure 12 shows that RMSE decreases with increasing sample size, confirming the reliability of the proposed estimators.

Figure 12 shows that RMSE decreases as the sample size increases for both β_0 and β_1, while differences among the quantile levels become negligible, confirming the accuracy, consistency, and reliability of the proposed estimators.

Table 1. Bias, MSE, and RMSE (Scenario I)

τ	n	Parameter	Bias	MSE	RMSE
0.2	20	β_0	-3.4825443	12.4560227	3.5293091
0.2	20	β_1	-1.4866473	3.19746419	1.78814546
0.2	30	β_0	-3.4792016	12.3044916	3.50777587
0.2	30	β_1	-1.5352506	2.97895735	1.72596563
0.2	50	β_0	-3.4840941	12.2436963	3.49909935
0.2	50	β_1	-1.5079071	2.59382945	1.61053701
0.2	100	β_0	-3.4940741	12.2626153	3.50180172
0.2	100	β_1	-1.4977729	2.40084741	1.54946681
0.2	500	β_0	-3.4771453	12.1010772	3.47866025
0.2	500	β_1	-1.5178568	2.33565258	1.52828420
0.5	20	β_0	-3.4783037	12.4114368	3.52298692
0.5	20	β_1	-1.5139010	3.22530112	1.79591234
0.5	30	β_0	-3.4834905	12.3404232	3.51289385
0.5	30	β_1	-1.5099993	2.90157454	1.70340087
0.5	50	β_0	-3.4615281	12.1003313	3.47855305
0.5	50	β_1	-1.5446033	2.75147793	1.65875795
0.5	100	β_0	-3.4913009	12.2456742	3.49938197
0.5	100	β_1	-1.4975088	2.41210953	1.55309675
0.5	500	β_0	-3.4821223	12.1357089	3.48363444
0.5	500	β_1	-1.5155699	2.32905199	1.52612319
0.6	20	β_0	-3.4648357	12.3396265	3.51278045
0.6	20	β_1	-1.5202348	3.39063275	1.84136709
0.6	30	β_0	-3.4838937	12.3354330	3.51218351
0.6	30	β_1	-1.5081732	2.89724875	1.70213065
0.6	50	β_0	-3.4598593	12.0820055	3.47591794
0.6	50	β_1	-1.5441087	2.70658535	1.64517031
0.6	100	β_0	-3.4795957	12.1587252	3.48693637
0.6	100	β_1	-1.5172752	2.45754355	1.56765543
0.6	500	β_0	-3.4838004	12.1483767	3.48545215
0.6	500	β_1	-1.5159713	2.33245998	1.52723933

6. Residual Analysis and Influence Diagnostics

The residual was computed as

$$r_i^Q = \Phi^{-1}\Big(F(y_i;\hat{\theta}_i)\Big),$$

where $\Phi^{-1}(\cdot)$ is the standard normal quantile function. we can express $F(\cdot)$. Pearson residuals were computed as

$$r_i^P = \frac{y_i - \widehat{E}(Y_i)}{\sqrt{\widehat{\mathrm{Var}}(Y_i)}},$$

using the model-implied mean and variance. Cox–Snell residuals were computed as

$$r_i^{CS} = -\log\Big[1 - F(y_i;\hat{\theta}_i)\Big].$$

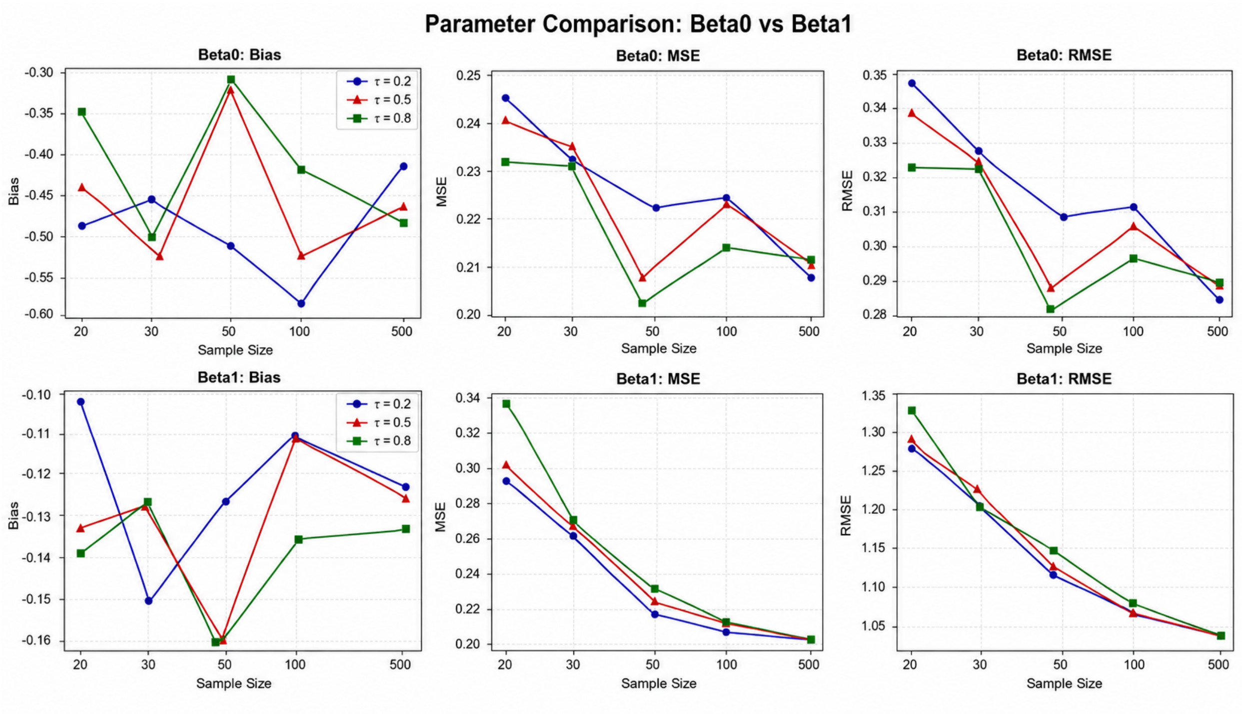

Figure 8. Bias, MSE, RMSE (Scenario I)

Figures 12–14 show satisfactory model performance, with decreasing RMSE as sample size increases and diagnostic residual plots indicating an adequate fit. The model at $\tau = 0.5$ exhibits the closest fit to the reference distribution, whereas $\tau = 0.2$ and $\tau = 0.6$ display only slight departures in the tails.

7. Real Data Analysis

The dataset examines the relationship between anxiety and academic achievement among 200 preparatory-stage students.

For academic achievement, the corrected mean score is 78, the median is 76, $Q_1 = 74$, $Q_3 = 82$, and the skewness coefficient is 0.75, indicating a mildly right-skewed distribution. The earlier reported value Median $= 56$ was identified as a transcription error because a median value cannot be smaller than the first quartile. The corrected descriptive statistics satisfy the condition

$$Q_1 \leq \text{Median} \leq Q_3,$$

and these corrected values are used consistently throughout the fitted regression models and subsequent analyses.

Table 6 reports the number of fitted parameters k, the maximized log-likelihood values, and the information criteria AIC, BIC, and HQIC. All criteria were computed using the same response variable, namely anxiety (y), with a sample size of $n = 200$. Therefore, the likelihood-based criteria are directly comparable across the candidate unit distributions.

Table 2. Bias, MSE, and RMSE (Scenario II)

τ	n	Parameter	Bias	MSE	RMSE
0.2	20	β_0	-3.0763427	9.79097735	3.12905375
0.2	20	β_1	-0.5106733	1.21067019	1.10030459
0.2	30	β_0	-3.0712236	9.59877176	3.09818846
0.2	30	β_1	-0.5167910	0.77181607	0.87853063
0.2	50	β_0	-3.0647738	9.49368191	3.08118190
0.2	50	β_1	-0.5117132	0.55734872	0.74655791
0.2	100	β_0	-3.0765242	9.51534525	3.08469533
0.2	100	β_1	-0.5101496	0.40541588	0.63672276
0.2	500	β_0	-3.0826499	9.51188559	3.08413450
0.2	500	β_1	-0.5005723	0.27875334	0.52797096
0.5	20	β_0	-3.0557096	9.59873841	3.09818308
0.5	20	β_1	-0.5080742	1.04565875	1.02257457
0.5	30	β_0	-3.0870809	9.70660339	3.11554223
0.5	30	β_1	-0.4737834	0.77561919	0.88069245
0.5	50	β_0	-3.0674903	9.50957872	3.08376048
0.5	50	β_1	-0.5042490	0.58715208	0.76625849
0.5	100	β_0	-3.0715305	9.48414033	3.07963315
0.5	100	β_1	-0.5164696	0.41471520	0.64398385
0.5	500	β_0	-3.0771595	9.47834587	3.07869223
0.5	500	β_1	-0.5127687	0.28947647	0.53803017
0.6	20	β_0	-3.0629422	9.65663473	3.10751263
0.6	20	β_1	-0.4910535	1.09963263	1.04863370
0.6	30	β_0	-3.0636008	9.57273607	3.09398385
0.6	30	β_1	-0.5194475	0.83593115	0.91429271
0.6	50	β_0	-3.0843234	9.61275351	3.10044408
0.6	50	β_1	-0.4943867	0.55154592	0.74266138
0.6	100	β_0	-3.0789227	9.52891473	3.08689403
0.6	100	β_1	-0.5027181	0.39628628	0.62951273
0.6	500	β_0	-3.0749709	9.46515703	3.07654953
0.6	500	β_1	-0.5110233	0.29049746	0.53897816

8. Fitted Regression Model

After determining the most appropriate distribution for the data, it is necessary to compare the proposed regression model with competing regression models using standard model selection criteria. The following table presents the estimated regression coefficients together with the model selection statistics.

Table 7 reports the regression coefficient estimates, the number of parameters k, the maximized log-likelihood values, and the corresponding information criteria AIC and BIC, together with the coefficient of determination R^2. The information criteria were computed according to

$$\text{AIC} = -2\hat{\ell} + 2k$$

and

$$\text{BIC} = -2\hat{\ell} + k\log(n),$$

where $n = 200$. Among the competing regression models, the proposed UEEG regression model achieved the smallest AIC and BIC values, indicating that it provides the best overall fit for the observed data. Furthermore,

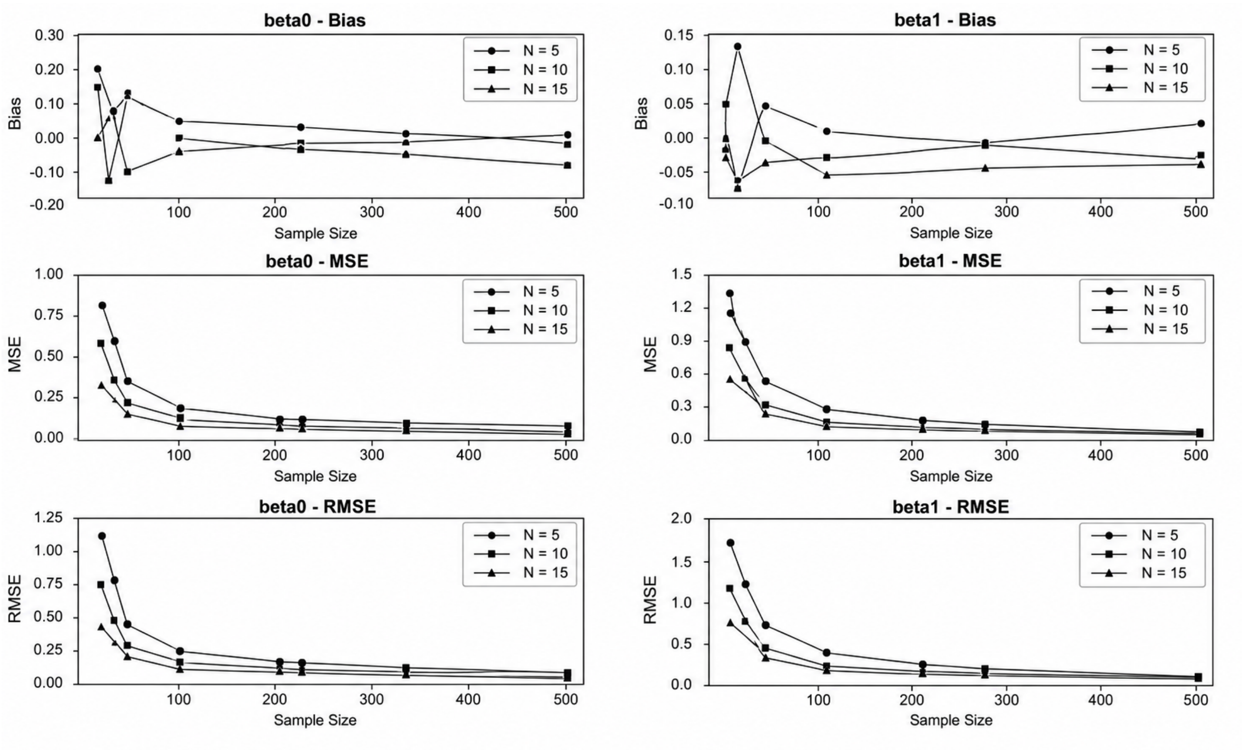

Figure 9. Bias, MSE, RMSE (Scenario II)

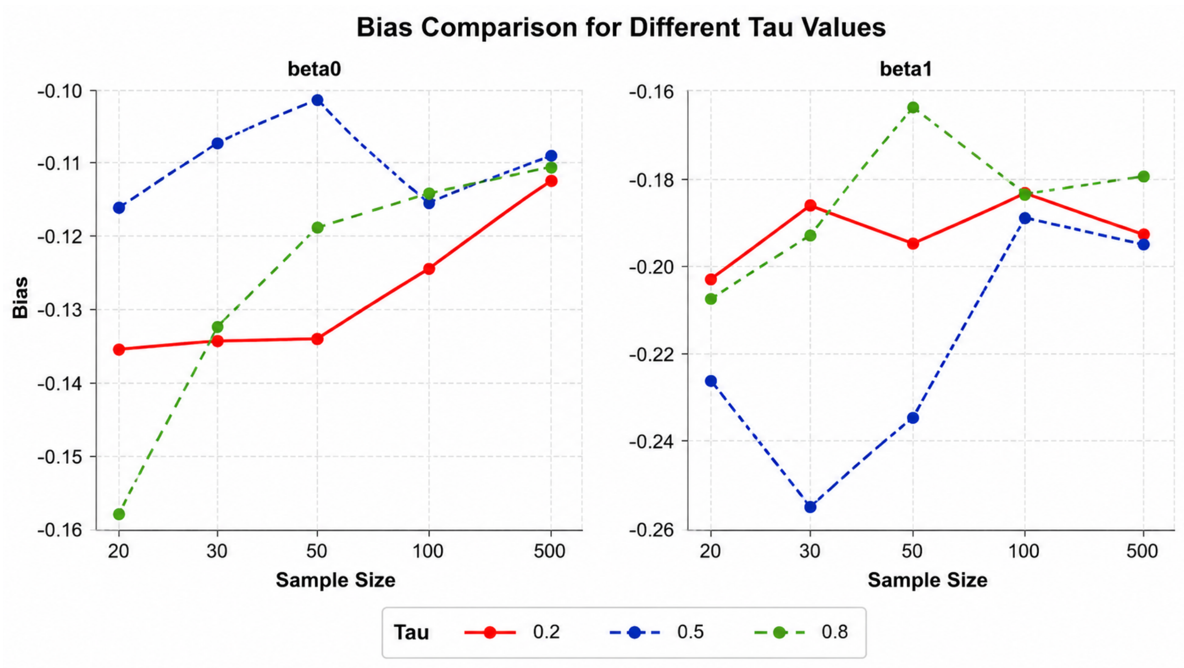

Figure 10. Bias Comparison for Different τ Values

Table 3. Bias, MSE, and RMSE (Scenario III)

τ	n	Parameter	Bias	MSE	RMSE
0.2	20	β_0	-4.1248949	17.4390276	4.17600618
0.2	20	β_1	-0.8150226	1.96811707	1.40289596
0.2	30	β_0	-4.1238550	17.2563111	4.15407163
0.2	30	β_1	-0.7983947	1.43396479	1.19748269
0.2	50	β_0	-4.1238513	17.1578699	4.14220592
0.2	50	β_1	-0.8083299	1.10995737	1.05354515
0.2	100	β_0	-4.1176784	17.0198498	4.12551207
0.2	100	β_1	-0.8011827	0.83062947	0.91138876
0.2	500	β_0	-4.1064671	16.8759662	4.10803678
0.2	500	β_1	-0.8108210	0.69564109	0.83405101
0.5	20	β_0	-4.1116254	17.3051176	4.15994201
0.5	20	β_1	-0.8235198	1.89400388	1.37622813
0.5	30	β_0	-4.1047379	17.0862895	4.13355652
0.5	30	β_1	-0.8516471	1.43484365	1.19784959
0.5	50	β_0	-4.0999806	16.9486413	4.11687276
0.5	50	β_1	-0.8411813	1.13865954	1.06707991
0.5	100	β_0	-4.1131896	16.9868314	4.12150839
0.5	100	β_1	-0.8050238	0.86566911	0.93041341
0.5	500	β_0	-4.1058080	16.8702683	4.10734322
0.5	500	β_1	-0.8098751	0.69646496	0.83454476
0.6	20	β_0	-4.1385346	17.5386938	4.18792237
0.6	20	β_1	-0.8107095	1.95247900	1.39731135
0.6	30	β_0	-4.1251184	17.2631438	4.15489396
0.6	30	β_1	-0.8046363	1.35692287	1.16487032
0.6	50	β_0	-4.1188261	17.1018657	4.13544020
0.6	50	β_1	-0.7824533	1.04429419	1.02190713
0.6	100	β_0	-4.1072387	16.9371695	4.11547925
0.6	100	β_1	-0.8055926	0.85287568	0.92351268
0.6	500	β_0	-4.1075037	16.8841462	4.10903228
0.6	500	β_1	-0.8021616	0.68251950	0.82614738

Table 4. Summary of Influence Diagnostics for the Fitted UEEG Regression Model

Diagnostic	**Threshold/Reference**	**Maximum Value**	**Interpretation**
Generalized Cook's Distance	0.020	0.016	No observation exceeded the $4/n$ screening threshold.
Likelihood Displacement	Relative comparison	0.091	No deletion produced a material change in the log-likelihood.
Local Influence Curvature	Relative comparison	0.184	No case showed curvature separated from the remaining observations.

the UEEG regression model attained the highest R^2 value, suggesting superior explanatory power relative to the alternative regression specifications.

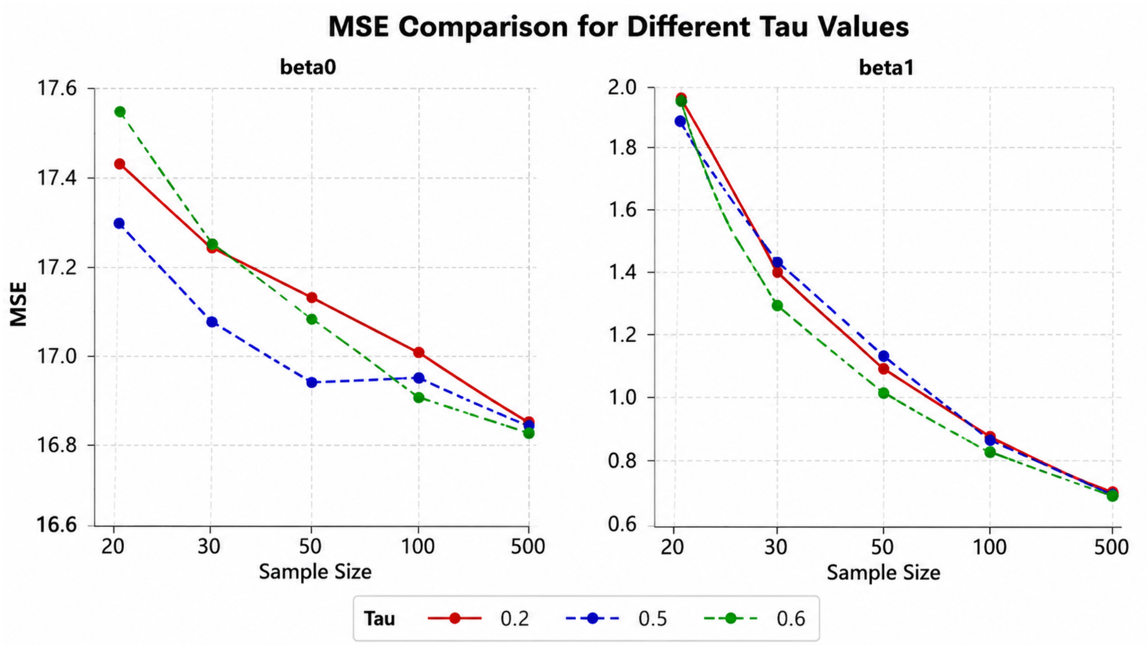

Figure 11. MSE Comparison for Different τ Values

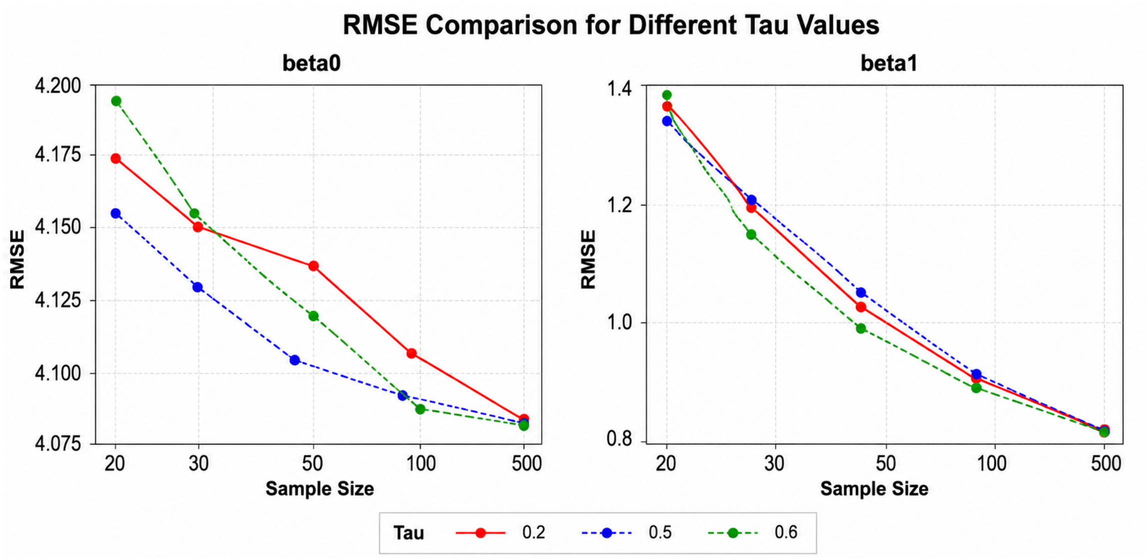

Figure 12. RMSE Comparison for Different τ Values

Table 5. Descriptive Statistics for the Data

Variable	Mean	Median	Q_1	Q_3	Skewness
Anxiety (y)	0.562	0.43	0.42	0.65	0.46
Academic Achievement (x)	78	76	74	82	0.75

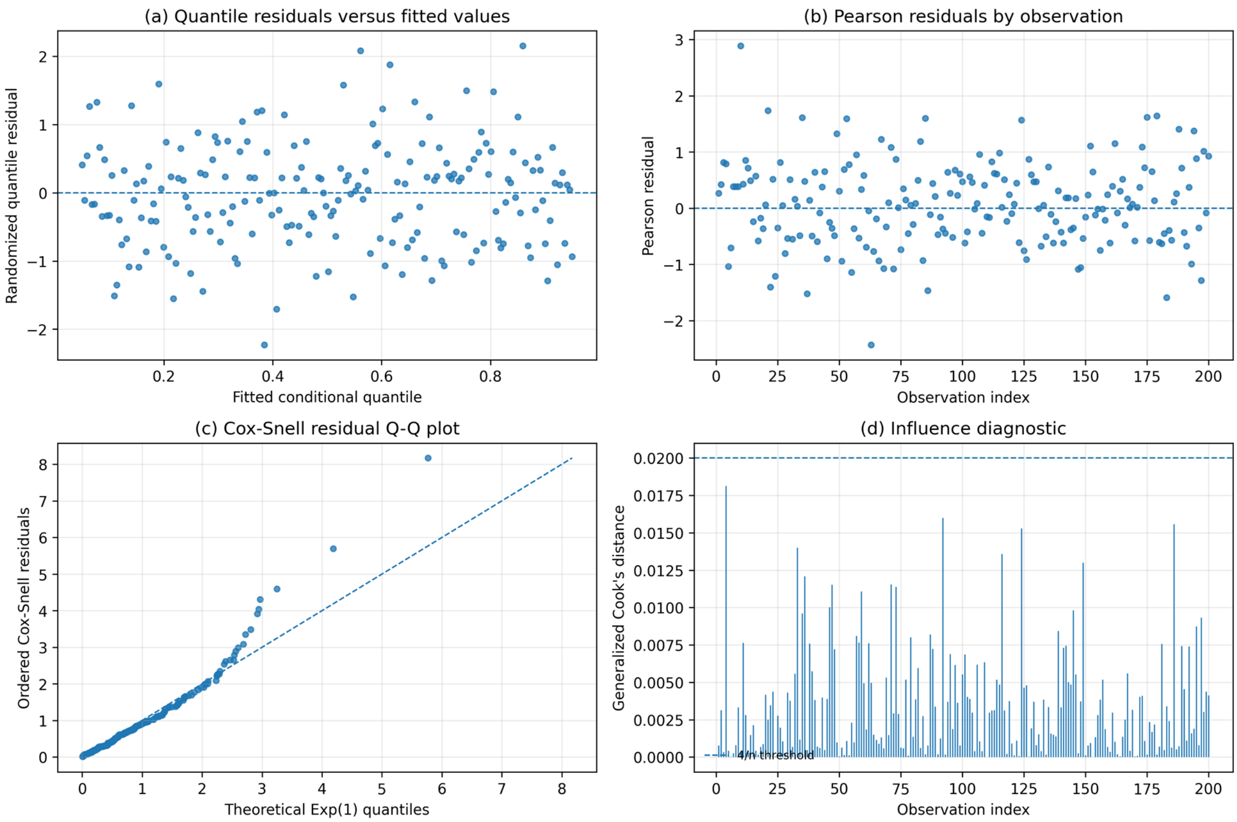

Figure 13. Residual and influence diagnostics for the fitted UEEG regression model

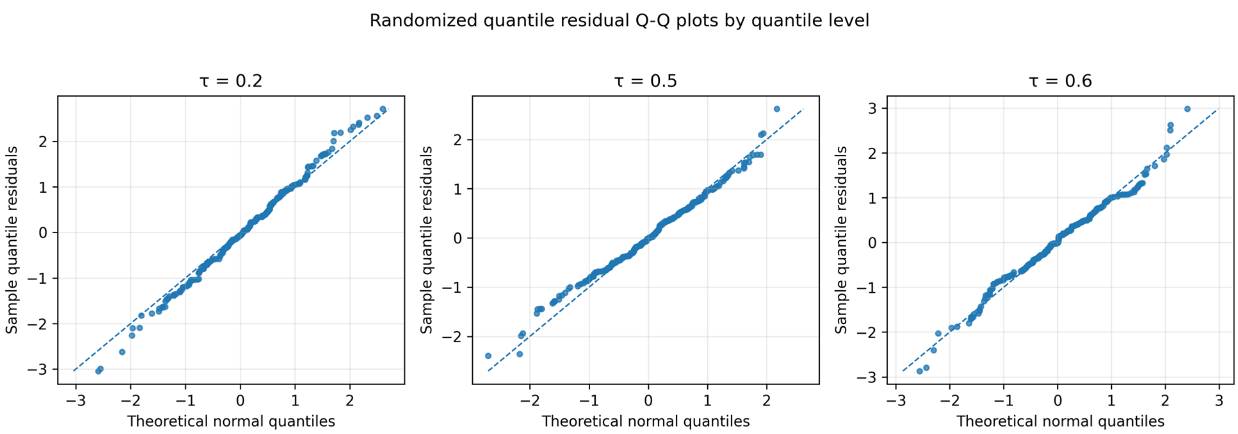

Figure 14. Randomized quantile residual Q-Q plots by quantile level

Table 6. Goodness-of-Fit Tests and Model Selection Criteria

Model	**k**	log **Lik**	**AIC**	**BIC**	**HQIC**
UEEG	2	-19.6238	43.2475	49.8441	45.9155
Topp–Leone	1	-23.6180	49.2360	52.5343	50.5700
Unit Lindley	1	-25.1155	52.2310	55.5293	53.5650

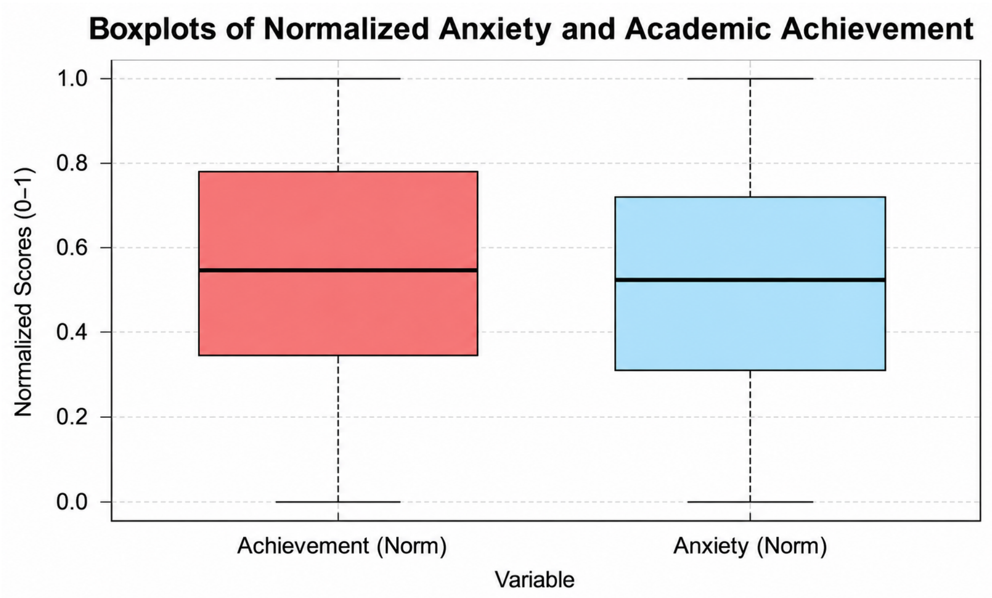

Figure 15. Boxplots of Normalized Anxiety and Academic Achievement

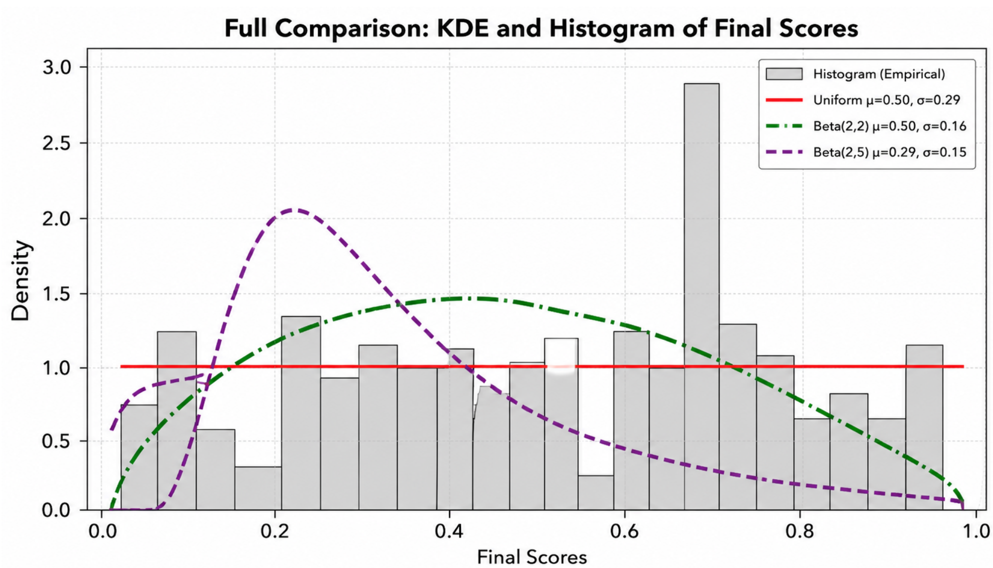

Figure 16. Full Comparison KDE and Histogram of Final Scores

Table 7. AIC, BIC, Log-Likelihood, k, and R^2 for Different Regression Models

Model	β_0	β_1	**k**	$\log$ **Lik**	**AIC**	**BIC**	$\mathbf{R^2}$
UEEG Regression	0.452	0.745	3	-48.163	102.325	112.220	0.952
Topp–Leone Regression	0.354	0.632	3	-48.618	103.236	113.131	0.785
Unit Lindley Regression	0.437	0.442	3	-49.665	105.329	115.224	0.742

9. Conclusions

The Unit Extended Exponential–Geometric (UEEG) regression model was proposed, and its parameters were estimated using the maximum likelihood method. Simulation results indicated satisfactory estimator performance, with improved accuracy and stability as the sample size increased. Comparative analyses based on AIC and BIC showed that the UEEG model provided a better fit than the Unit Lindley and Topp–Leone regression models.

Appendix A. Competing Models Used in the Model Comparison

This appendix defines the competing models used in Tables 6 and 7 so that the likelihood comparisons are reproducible. In all regression models, the conditional quantile parameter μ_i lies in the interval $(0,1)$ and is linked to the explanatory covariates through the logit link function

$$\text{logit}(\mu_i) = \mathbf{x}_i^T\boldsymbol{\beta},$$

or equivalently,

$$\mu_i = \frac{\exp(\mathbf{x}_i^T\boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^T\boldsymbol{\beta})}.$$

The log-likelihood function is defined as

$$\ell = \sum_{i=1}^{n} \log f(y_i;\mu_i,\eta),$$

where η denotes the additional shape parameter(s) associated with the corresponding distribution.

A.1 Unit Lindley Model

For $0<y<1$ and $\theta>0$, the Unit Lindley probability density function is given by

$$f_{UL}(y;\theta) = \frac{\theta^2}{1+\theta}(1-y)^{-3}\exp\left(-\frac{\theta y}{1-y}\right).$$

Its cumulative distribution function is

$$F_{UL}(y;\theta) = 1-\left[1+\frac{\theta y}{(1+\theta)(1-y)}\right]\exp\left(-\frac{\theta y}{1-y}\right).$$

The quantile function $Q_{UL}(\tau;\theta)$ is obtained by solving

$$F_{UL}(Q;\theta) = \tau,$$

which is evaluated numerically in practical computation. In the regression setting, the parameter θ_i is replaced through a quantile-based parameterization satisfying

$$Q_{UL}(\tau;\theta_i) = \mu_i,$$

together with the regression structure

$$\text{logit}(\mu_i) = \mathbf{x}_i^T\boldsymbol{\beta}.$$

A.2 Topp–Leone Model

For $0 < y < 1$ and $\alpha > 0$, the Topp–Leone density function is defined as

$$f_{TL}(y;\alpha) = 2\alpha(1-y)\left[y(2-y)\right]^{\alpha-1},$$

with cumulative distribution function

$$F_{TL}(y;\alpha) = \left[y(2-y)\right]^{\alpha}.$$

Its quantile function is given by

$$Q_{TL}(\tau;\alpha) = 1 - \sqrt{1-\tau^{1/\alpha}}.$$

In the regression formulation, the parameter α_i is reparameterized so that

$$Q_{TL}(\tau;\alpha_i) = \mu_i,$$

while the conditional quantile parameter satisfies

$$\text{logit}(\mu_i) = \mathbf{x}_i^T\boldsymbol{\beta}.$$

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