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Efficient Parameter Estimation for the Inverse Power Moment Exponential Model under Ranked Set and Simple Random Sampling with Entropy Measures and Applications

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Abstract It is challenging to model real-world data with complex features, such as skewness, heavy tails, and high variability, because each one demands a distinct probability density function. For these kinds of data, a variety of statistical models can be applied; the right data type should be chosen. When working with units in a population is costly, the ranked set sampling (RSS) technique is crucial for obtaining data. Nevertheless, they can be easily categorized by the relevant variable. Moreover, entropy information quantifies how unpredictable or uncertain a random variable or mechanism is. It is essential to many different fields, notably reliability engineering, environmental studies, medical sciences, economics, actuarial science, finance, and insurance. In this study, we applied the inverse power moment exponential (IPME) model to estimate model parameters from RSS and simple random sampling (SRS) designs using various estimation methods. We additionally introduced several entropy measures, including Rényi, Shanon, Havrda and Charvat, Tsalis, Arimoto, and Mathai-Haubold entropies for the IPME model. A thorough simulation investigation is performed to verify the effectiveness of the proposed estimators under RSS and SRS schemes by calculating the mean, mean squared error, and relative efficiency. The use of real-world datasets further emphasizes the proposed RSS design compared with the SRS scheme. Overall, the findings highlight the potential of the RSS technique as a robust and flexible tool for estimating the model parameters and for obtaining data sets.

Keywords Entropy information, Heavy tails, Inverse power, Ranked set sampling, Simulation, Skewness, Shanon

AMS 2010 subject classifications 62F40, 62F15, 62F10

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1. Introduction

The most common method for developing statistical analysis methods is simple random sampling (SRS). Nevertheless, non-reflective samples could result from small sample sizes. To reduce the number of metrics while maintaining reliability and cost-effective sampling, ranked set sampling (RSS) is a more appropriate design for

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this type of study. McIntyre [1] developed the RSS , and it is a contemporary method for SRS in areas where the variable of advantage is expensive or difficult to implement.

An RSS sample with a fixed set size m and number of cycles, the following algorithm can obtain r:

1. Select a simple random sample of size v^2 from the desired population. random divide it into v groups of arbitrary size v. v is called the set size.
2. The members within each sample are manually organized via visual inspection or a cost-effective alternative method.
3. The minimum value from the first sample is documented, followed by the second minimum value from the second sample, and so on. This sequential procedure continues until the maximum observation in the v-th sample is recorded. (for $s = 1, \dots, v$).
4. Repeat the previous steps, d times (cycles), to get a sample of size $n = dv$.

To increase the versatility of the RSS design and the performance of the RSS framework against the SRS, statisticians have introduced several studies. A few of these are Hassan et al. [2], who introduced an efficient entropy estimation under RSS design for the inverted exponentiated Pareto distribution and provided several entropy-related results for the proposed model. Al Kadiri and Al-Omari [3] analyzed the performance of the double RSS scheme for the power-function distribution. Deka et al. [4] established several estimation methods for the exponentiated inverted Weibull distribution based on RSS and SRS strategies. Also, Hiba [5] defined Bayesian and traditional estimation procedures using the RSS design for the bivariate generalized Burr distribution, and the model parameters for the extended XLindley distribution, also under RSS and SRS schemes, were provided by Alqasem et al. [6]. For mor details, see the jobs of Aljohani [7], Hassan et al. [8], Al-Saleh et al. [9], Bantan et al. [10], Abd El-Raouf and AbaOud [11], Alsadat et al. [12], Chen et al. [13], Hassan et al. [14], Dutta and Kayal [15], and Sharma and Dutta [16].

Tashkandy et al. [17] introduced the inverse power moment exponential (IPME) model as a modified XLindley distribution. The IPME model with two parameters is an adjustable instrument for analyzing real-world events across a variety of domains, as it displays a right-skewed, unimodal, and decreasing density function.

The construction of the IPME model can be described with the following steps:

Let X be a random variable with the moment exponential distribution with probability density function (PDF) given as

$$f_1(x;\eta) = \frac{xe^{-\frac{x}{\eta}}}{\eta^2};\ x,\ \eta > 0. \tag{1}$$

The inverse power transformation $Y = X^{\frac{-1}{\delta}}$ is applied to the PDF given in Eq. (1) to have IPME model with PDF

$$g(y;\eta,\delta) = \left|y^{-\delta}\right|'\ f_1(y^{-\delta};\eta) = \frac{\delta y^{-2\delta-1}\, e^{-\frac{y^{-\delta}}{\eta}}}{\eta^2};\ y,\ \eta,\ \delta > 0. \tag{2}$$

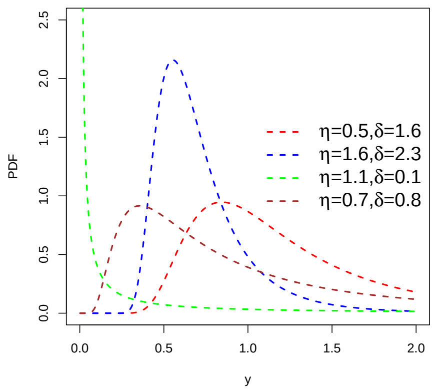
The corresponding cumulative distribution function (CDF) of the IPME model is defined as follows

$$G(y;\eta,\delta) = e^{-\frac{y^{-\delta}}{\eta}}\left(\frac{y^{-\delta}}{\eta} + 1\right). \tag{3}$$

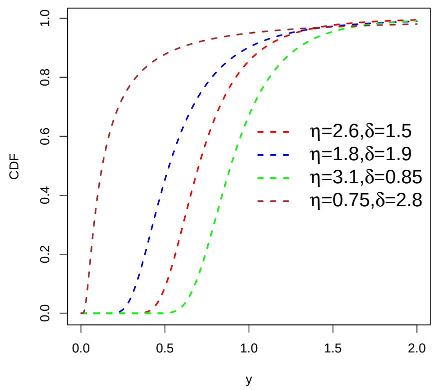
For $\delta = 1$, the inverted moment exponential distribution (IMED) is obtained as a special case of our proposed model, which Almutiry presents [18].

The IPME model can be used in various cases as a quick approach and can capture the features of the exponential distribution. It is used in many real-life data sets quite effectively in analyzing more sets. It adequately fits too many datasets, such as lifetime, medical science, and the coronavirus. Additionally, the extra parameter provides substantially greater flexibility in the baseline distribution, allowing it to capture varying degrees of skewness, tail behavior, and hazard rate shapes.

Figure (1) displays the PDF and CDF of the IPME model. Clearly, the PDF of our proposed model reveals a unimodal, decreasing, and right-skewed shape. These forms imply that our suggested distribution can capture data with a single peak and a rightward-extending tail, as is frequently observed in biological processes, income distributions, and reliability data.



(a) PDF



(b) CDF

Figure 1. Plot of the IPME PDF and CDF for different parametric values.

The Survival function (SF) and hazard rate (HRF) function of our proposed IPME distribution are investigated, respectively, as

$$S\left(y;\eta,\delta\right)=1-G(y;\eta,\delta)=1-\left(\frac{y^{-\delta}}{\eta}+1\right)e^{-\frac{y^{-\delta}}{\eta}}, \tag{4}$$

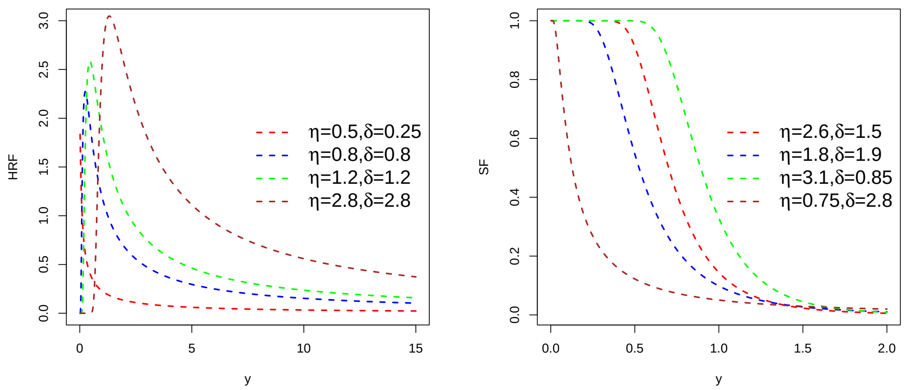
$$h\left(y;\eta,\delta\right)=\frac{g(y;\eta,\delta)}{S(y;\eta,\delta)}=\frac{\delta y^{-2\delta-1}\, e^{-\frac{y^{-\delta}}{\eta}}}{\eta^{2}\left[1-\left(\frac{y^{-\delta}}{\eta}+1\right)e^{-\frac{y^{-\delta}}{\eta}}\right]}. \tag{5}$$

Next, the cumulative hazard rate rate function (CHRF) of the random variable Y wich followd the IPME model is

$$H\left(y,\eta,\delta\right)=-log(S(y;\eta,\delta))=-\log\left(1-\left(\frac{y^{-\delta}}{\eta}+1\right)e^{-\frac{y^{-\delta}}{\eta}}\right), \tag{6}$$

Figure (2) displayed the HRF and SF plots of our IPME model. The HRF curves reveal that the model exhibits a decreasing, unimodal failure-rate pattern. In numerous real-world instances in which developing or fatigue effects are observed, this pattern of behavior suggests that the failure decreases with time. These illustration patterns ensure the model's adaptability in simulating multiple real datasets.

This paper aims to obtain the estimation of the two unknown parameters for the IPME distribution by applying various estimation tools, notably the maximum likelihood estimation (MLE), maximum product of spacing (MPS), least square estimation (LSE), weighted least square (WLS) procedures, Cramer-von-Mises (CV), and Anderson-Darling (AD) estimators based on the two recommended SRS and RSS schemes. Researchers, scientists, practitioners, and engineers are especially interested in these designs because they help select the best methodology for the IPME model parameters. Another objective of this paper is to present various entropy measures for the IPME distribution, which are essential in many fields, such as communication, physics, and chemistry. It is a crucial idea



(a) HRF (b) SF

Figure 2. Plot of the IPME HRF and SF for different parametric values.

in determining the degree of uncertainty surrounding a random variable—a quick simulation investigation using various sample sizes and parameter values to assess the proposed entropy estimators.

The structure of this research study is as follows: Section 2 examined several mathematical properties of the IPME distribution. Section 3 presents methods for estimating IPME model parameters using two different designs. In Section 4, we conduct a simulation study to assess the accuracy and reliability of the resulting sampling schemes. Section 5 presents the mathematical structure of the suggested entropy metrics for the IPME model, along with a brief simulation study to assess the performance of the entropy estimators. Section 6 applies the IPME model to three real-world data sets, demonstrating the practical utility of the IPME distribution and the superiority of the RSS over SRS designs. Finally, Section 7 concludes the study with a summary and closing remarks.

2. Statistical properties

2.1. *Quantile function*

By obtaining the CDF given in Eq. (3) of the IPMED, the quantile function (QF) of the IPMED is obtained by calculating the inverse function of Eq. (3) as follows

$$Q(p) = \left(-\frac{1}{\eta\left(W\left(-\frac{p}{e}\right)+1\right)}\right)^{1/\delta}, \qquad 0 < p < 1, \tag{7}$$

where $W(\cdot)$ is Lambert function.

2.2. *Moments and Generating Functions*

The mathematical computation of the r^{th} moment from the IPME model is described as

$$\mu_r^{'} = \frac{\Gamma\left(2-\frac{r}{\delta}\right)}{\eta^{\frac{r}{\delta}}} \tag{8}$$

Consequently, the expected value $\mu_1^{'}$ and second moment $\mu_2^{'}$ of the IPME distribution are obtained by setting $r = 1$ and $r = 2$, respectively:

$$\mu_1^{'} = \eta^{-\frac{1}{\delta}}\, \Gamma\left(2 - \tfrac{1}{\delta}\right),$$

and

$$\mu_2^{'} = \eta^{-\frac{2}{\delta}}\, \Gamma\left(2 - \tfrac{2}{\delta}\right)$$

Next, the variance (var_Y) and index of dispersion (ID) of the IPME model can be written as follows, respectively.

$$var_Y = \mu_2^{'} - (\mu_1^{'})^2 = \eta^{-\frac{2}{\delta}} \left\{\Gamma\left(2 - \tfrac{2}{\delta}\right) - \left[\Gamma\left(2 - \tfrac{1}{\delta}\right)\right]^2\right\},$$

and

$$ID = \frac{var_Y}{\mu_1^{'}}.$$

Now, the mathematical expression of the moment generating function (MGF) of the IPME distribution, say $M_Y(t)$, can be expressed as

$$M_Y(t) = \int_0^{\infty} e^{yt}\, g(y;\eta,\delta)\, dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r\, \Gamma\left(2 - \frac{r}{\delta}\right)}{\eta^{\frac{r}{\delta}}\, r!}.$$

2.3. Order Statistics of the IPME Model

Let $y_1, y_2, ..., y_n$ be a random sample taken from our proposed IPME model. The density function of k^{th} order statistics $Y_{(k)}$ is

$$\begin{aligned} m_{(k)}(y;\eta,\delta) &= \frac{n!\, g(y;\eta,\delta)}{(k-1)!(n-k)!}\, [G(y;\eta,\delta)]^{k-1}\, [1 - G(y;\eta,\delta)]^{n-k} \\ &= \frac{n!\delta y^{-2\delta-1}\, e^{-\frac{y^{-\delta}}{\eta}}}{\eta^2(k-1)!(n-k)!} \left[e^{-\frac{y^{-\delta}}{\eta}}\left(\frac{y^{-\delta}}{\eta} + 1\right)\right]^{k-1} \\ &\times \left[1 - e^{-\frac{y^{-\delta}}{\eta}}\left(\frac{y^{-\delta}}{\eta} + 1\right)\right]^{n-k} \end{aligned}$$

Next, the CDF of the k^{th} order statistic $Y_{(k)}$ can be written as

$$\begin{aligned} M_{(k)}(y;\eta,\delta) &= \sum_{s=k}^{n} [G(y;\eta,\delta)]^s\, [1 - Gy;\eta,\delta)]^{n-s} \\ &= \sum_{l=s}^{n} \left[e^{-\frac{y^{-\delta}}{\eta}}\left(\frac{y^{-\delta}}{\eta} + 1\right)\right]^s \left[1 - e^{-\frac{y^{-\delta}}{\eta}}\left(\frac{y^{-\delta}}{\eta} + 1\right)\right]^{n-s}. \end{aligned}$$

3. Parameter Estimation Techniques

In this section, we present various techniques for estimating the two parameters η and δ of the proposed IPME distribution under a ranked set sampling design.

3.1. Maximum Likelihood Technique

Consider that the RSS design in our proposed IPME model has several cycles (d) and a specified size (v). Consequently, the MLE with an RSS scheme is provided through

$$L_{RSS}(\eta,\delta)=\prod_{l=1}^{d}\prod_{k=1}^{v}g(y_{kl}), \tag{9}$$

with

$$\begin{aligned} g(y_{kl}) &= \frac{l!}{(k-1)!(l-k)!}[G(y_{kl})]^{k-1}[S(y_{kl})]^{l-k}g(y_{kl}). \qquad (10)\\ &= \frac{l!}{(k-1)!(l-k)!}\left[e^{-\frac{y_{kl}^{-\delta}}{\eta}}\left(\frac{y_{kl}^{-\delta}}{\eta}+1\right)\right]^{k-1}\\ &\times \left[1-e^{-\frac{y_{kl}^{-\delta}}{\eta}}\left(\frac{y_{kl}^{-\delta}}{\eta}+1\right)\right]^{l-k}\frac{\delta y_{kl}^{-2\delta-1}\,e^{-\frac{y_{kl}^{-\delta}}{\eta}}}{\eta^2}. \end{aligned}$$

The log-likelihood function $L_{RSS}(\eta,\delta)$ is written as follows

$$\begin{aligned} \mathcal{LL}_{RSS}(\eta,\delta) &= \sum_{l=1}^{d}\sum_{k=1}^{v}\log\{g(y_{kl})\}\\ &= \sum_{l=1}^{d}\sum_{k=1}^{v}\log\left(\frac{l!}{(k-1)!(l-k)!}\right)+\sum_{l=1}^{d}\sum_{k=1}^{v}(k-1)\log(G(y_{kl}))\\ &+ \sum_{l=1}^{d}\sum_{k=1}^{v}(l-k)\log(S(y_{kl}))+\sum_{l=1}^{d}\sum_{k=1}^{v}\log(g(y_{kl})). \qquad (11) \end{aligned}$$

Therefore, we may get the estimates $\hat{\eta}_{RSS}$ and $\hat{\delta}_{RSS}$ with regards to η and δ through calculating the derivatives of Eq. (11). Numerous strategies, such as Newton-Raphson and bisection methods, are available to solve these equations numerically.

3.2. Maximum Product of Spacings Technique

The goal of the MPS approach is to maximize the variation in the geometric mean. Let $y_{(1:n)}, y_{(2:n)}, \ldots, y_{(n:n)}$ represent the ordered statistics derived from an RSS sample of size $n = dv$, via the IPME distribution. The uniform distance is currently expressed as

$$P_s(\eta,\delta)=G(y_{(s:n)}|\eta,\delta)-G(y_{(s-1:n)}|\eta,\delta), s=1,\ldots,n+1,$$

where, $G(y_{(0:n)}|\eta,\delta)=0$ and $G(y_{(n+1:n)}|\eta,\delta)=1$. Evidently $\sum_{s=1}^{n+1}P_s(\eta,\delta)=1$.

By optimizing the geometric mean of the distances, the MPS calculations $\hat{\eta}_{MPS}$ and $\hat{\delta}_{MPS}$ of η and δ are computed.

$$Q(\eta,\delta|\mathrm{y})=\left[\prod_{s=1}^{n+1}P_s(\eta,\delta)\right]^{\frac{1}{n+1}}. \tag{12}$$

Ultimately, the two nonlinear systems are solved to yield the MPS estimators $\hat{\eta}_{MPS}$ and $\hat{\delta}_{MPS}$

$$\frac{\partial}{\partial\eta}Q(\eta,\delta|\mathrm{y})=\frac{1}{n+1}\sum_{s=1}^{M+1}\frac{1}{P_s(\eta,\delta)}\left[\Delta_1(y_{(s:n)}|\eta,\delta)-\Delta_1(y_{(s-1:n)}|\eta,\delta)\right]=0,$$

$$\frac{\partial}{\partial \delta} Q(\eta, \delta|\mathrm{y}) = \frac{1}{n+1} \sum_{s=1}^{M+1} \frac{1}{P_s(\eta, \delta)} \left[\Delta_2(y_{(s:n)}|\eta, \delta) - \Delta_2(y_{(s-1:n)}|\eta, \delta) \right] = 0,$$

where

$$\Delta_1(y_{(s-1:n)}|\eta, \delta) = \frac{\partial}{\partial \eta} G(y_{(s:n)}|\eta, \delta), \tag{13}$$

and

$$\Delta_2(y_{(s-1:n)}|\eta, \delta) = \frac{\partial}{\partial \delta} G(y_{(s:n)}|\eta, \delta). \tag{14}$$

3.3. Least Squares and Weighted Least Squares Techniques

Let $y_{(1:n)}, \ldots, y_{(n:n)}$ be considered an ordered sample from the RSS design of size $n = dv$. The LSE estimates $\hat{\eta}_{LSE}$ and $\hat{\delta}_{LSE}$ of η and δ will be achieved, accordingly, by

$$\begin{aligned} \mathcal{LS}(\eta, \delta|\mathrm{y}) &= \sum_{s=1}^{n} \left[G(y_{(s:n)}|\eta, \delta) - \frac{s}{n+1} \right]^2 \\ &= \sum_{s=1}^{n} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1 \right) - \frac{s}{n+1} \right]^2, \end{aligned}$$

The two distinct nonlinear systems are solved to determine the $\hat{\eta}_{LSE}$ and $\hat{\delta}_{LSE}$ of the η and δ, accordingly

$$\sum_{s=1}^{n} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1 \right) - \frac{s}{n+1} \right] \Delta_1(y_{(s:n)}|\eta, \delta) = 0,$$

$$\sum_{s=1}^{n} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1 \right) - \frac{s}{n+1} \right] \Delta_2(y_{(s:n)}|\eta, \delta) = 0.$$

In addition, the WLS estimates of the two parameters η and δ, represented by $\hat{\eta}_{WLS}$ and $\hat{\delta}_{WLS}$ are computed with minimizing, respectively, the function

$$\begin{aligned} \mathcal{WLS}(\eta, \delta|\mathrm{y}) &= \sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[G(y_{(s:n)}|\eta, \delta) - \frac{s}{n+1} \right]^2 \\ &= \sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1 \right) - \frac{s}{n+1} \right]^2. \end{aligned}$$

The two distinct nonlinear systems are solved to determine the $\hat{\eta}_{WLS}$ and $\hat{\delta}_{WLS}$ of the η and δ, accordingly

$$\sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1 \right) - \frac{s}{n+1} \right] \Delta_1(y_{(s:n)}|\eta, \delta) = 0,$$

$$\sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1 \right) - \frac{s}{n+1} \right] \Delta_2(y_{(s:n)}|\eta, \delta) = 0.$$

3.4. Cramer-von-Mises Technique

Let $y_{(1:n)}, \ldots, y_{(n:n)}$ be considered an ordered sample from the RSS design of size $n = dv$. The CV estimates $\hat{\eta}_{CV}$ and $\hat{\delta}_{CV}$ of η and δ may be determined, accordingly, by minimizing the following formula.

$$\begin{aligned}\mathcal{C}(\eta, \delta|y) &= \frac{1}{12n} + \sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[G(y_{(s:n)}|\eta, \delta) - \frac{2s-1}{2n}\right]^2 \\ &= \frac{1}{12n} + \sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1\right) - \frac{2s-1}{2n}\right]^2.\end{aligned}$$

The two distinct nonlinear systems are determined to obtain the $\hat{\eta}_{CV}$ and $\hat{\delta}_{CV}$ of the η and δ, respectively.

$$\sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1\right) - \frac{2s-1}{2n}\right] \Delta_1(y_{(s:n)}|\eta, \delta) = 0,$$

$$\sum_{s=1}^{n} \frac{(n+1)^2(n+2)}{s(n-s+1)} \left[e^{-\frac{y_{(s:n)}^{-\delta}}{\eta}} \left(\frac{y_{(s:n)}^{-\delta}}{\eta} + 1\right) - \frac{2s-1}{2n}\right] \Delta_2(y_{(s:n)}|\eta, \delta) = 0.$$

3.5. Anderson-Darling Technique

Let $y_{(1:n)}, \ldots, y_{(n:n)}$ be considered an ordered sample from the RSS design of size $n = dv$. The AD estimates $\hat{\eta}_{AD}$ and $\hat{\delta}_{AD}$ of η and δ may be solved, respectively, by minimizing the expression below.

$$\mathcal{AD}(\eta,\ \delta|y) = -n - \frac{1}{n} \sum_{s=1}^{n} (2s-1) \left\{\log\ G(y_{(s:n)}|\eta, \delta) + \log(1 - G(y_{(s:n)}|\eta, \delta))\right\},$$

As a result, $\hat{\eta}_{AD}$ and $\hat{\delta}_{AD}$ are computed by solving the two nonlinear systems given as follows

$$\sum_{s=1}^{n} (2s-1) \left\{\frac{\Delta_1(y_{(s:n)}|\eta, \delta)}{G(y_{(s:n)}|\eta, \delta)} - \frac{\Delta_1(y_{(n-s+1:n)}|\eta, \delta)}{1 - G(y_{(n-s+1:n)}|\eta, \delta)}\right\} = 0,$$

and

$$\sum_{s=1}^{n} (2s-1) \left\{\frac{\Delta_2(y_{(s:n)}|\eta, \delta)}{G(y_{(s:n)}|\eta, \delta)} - \frac{\Delta_2(y_{(n-s+1:n)}|\eta, \delta)}{1 - G(y_{(n-s+1:n)}|\eta, \delta)}\right\} = 0.$$

4. Numerical Simulation

This section uses substantial simulated data to evaluate how well various estimation techniques, including MLE, MPS, LSE, WLS, CV, and AD, forecast the two parameters η and δ of the IPME model under SRS and RSS designs. The simulation architecture consists of a systematic arrangement of processes in which $N = 1000$ samples are generated for each design's RSS and SRS using the IPME model, with varying sample sizes. This is conducted for three distinct sets of parameter values, selected with careful consideration: Scenario I: $(\eta, \delta) = (0.4, 0.25)$, Scenario II:$(\eta, \delta) = (1.2, 1.4)$, and Scenario III: $(\eta, \delta) = (1.75, 0.5)$. We selected the set size v to be 4, 8, and 12, and the number of cycles d to be 6 and 12 for the RSS scheme. For the SRS concept to have the same size in every situation, the sample size is $n = dv$. The average estimates (Mean), mean squared errors (MSE), and the efficiency (Ef) of the forecasts for the RSS and SRS concepts, using the three estimation techniques, are calculated with precision.

Table 1. Numerical values of simulation measures using Scenario I.

d	v		Design	MLE			MPS			LSE		
				Mean	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**
6	4	$\hat{\eta}$	RSS	0.3974	0.0012	3.1628	0.4115	0.0012	2.6495	0.4072	0.0020	2.1368
			SRS	0.3894	0.0039		0.4044	0.0032		0.3947	0.0042	
		$\hat{\delta}$	RSS	0.2585	0.0012	1.9406	0.2281	0.0016	1.0836	0.2434	0.0021	1.7908
			SRS	0.2702	0.0024		0.2397	0.0017		0.2591	0.0038	
	8	$\hat{\eta}$	RSS	0.3964	0.0005	3.3656	0.4049	0.0005	2.7964	0.4006	0.0005	3.1198
			SRS	0.4002	0.0016		0.4085	0.0015		0.4040	0.0016	
		$\hat{\delta}$	RSS	0.2548	0.0003	3.3500	0.2378	0.0005	1.7718	0.2475	0.0005	2.8026
			SRS	0.2586	0.0011		0.2409	0.0010		0.2503	0.0014	
	12	$\hat{\eta}$	RSS	0.3988	0.0002	3.3646	0.4058	0.0003	2.6704	0.4023	0.0002	4.9366
			SRS	0.4000	0.0008		0.4063	0.0007		0.4015	0.0012	
		$\hat{\delta}$	RSS	0.2512	0.0001	3.9722	0.2379	0.0003	1.4603	0.2451	0.0002	2.8417
			SRS	0.2591	0.0006		0.2459	0.0005		0.2545	0.0007	
12	4	$\hat{\eta}$	RSS	0.4026	0.0008	1.8956	0.4127	0.0009	1.5284	0.4048	0.0010	1.4323
			SRS	0.3964	0.0015		0.4051	0.0014		0.4019	0.0015	
		$\hat{\delta}$	RSS	0.2502	0.0005	2.0215	0.2328	0.0007	1.2727	0.2425	0.0009	1.2801
			SRS	0.2557	0.0009		0.2378	0.0009		0.2486	0.0012	
	8	$\hat{\eta}$	RSS	0.4018	0.0003	4.1931	0.4079	0.0003	3.4922	0.4033	0.0004	3.4311
			SRS	0.4017	0.0011		0.4066	0.0011		0.4032	0.0013	
		$\hat{\delta}$	RSS	0.2542	0.0002	1.8392	0.2435	0.0003	1.3419	0.2500	0.0004	1.8303
			SRS	0.2547	0.0004		0.2441	0.0004		0.2538	0.0007	
	12	$\hat{\eta}$	RSS	0.4014	0.0001	5.1324	0.4050	0.0001	3.4080	0.4035	0.0001	5.1975
			SRS	0.3987	0.0005		0.4023	0.0005		0.3997	0.0006	
		$\hat{\delta}$	RSS	0.2514	0.0001	3.0273	0.2438	0.0002	1.9077	0.2479	0.0002	2.7263
			SRS	0.2518	0.0003		0.2444	0.0003		0.2492	0.0004	

Table 2. Numerical values of simulation measures using Scenario I.

d	v		Design	**WLS**			**CV**			**AD**		
				Mean	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**
6	4	$\hat{\eta}$	RSS	0.4068	0.0017	2.2507	0.3953	0.0020	2.2520	0.4033	0.0017	2.2511
			SRS	0.4052	0.0039		0.3941	0.0044		0.4035	0.0038	
		$\hat{\delta}$	RSS	0.2424	0.0017	1.4750	0.2626	0.0022	1.7844	0.2479	0.0016	1.4421
			SRS	0.2550	0.0026		0.2770	0.0039		0.2576	0.0023	
	8	$\hat{\eta}$	RSS	0.4020	0.0005	3.3876	0.3964	0.0005	3.4543	0.4007	0.0005	3.4325
			SRS	0.3987	0.0017		0.3929	0.0018		0.3984	0.0016	
		$\hat{\delta}$	RSS	0.2466	0.0005	1.9056	0.2568	0.0006	2.2950	0.2487	0.0005	1.7711
			SRS	0.2580	0.0010		0.2689	0.0014		0.2578	0.0008	
	12	$\hat{\eta}$	RSS	0.4034	0.0003	4.9829	0.3996	0.0003	5.1808	0.4028	0.0003	5.0438
			SRS	0.4048	0.0013		0.4011	0.0013		0.4040	0.0013	
		$\hat{\delta}$	RSS	0.2458	0.0003	2.3134	0.2527	0.0003	2.3241	0.2468	0.0002	2.5958
			SRS	0.2461	0.0006		0.2530	0.0006		0.2473	0.0006	
12	4	$\hat{\eta}$	RSS	0.4017	0.0008	2.0579	0.3959	0.0009	1.9673	0.4010	0.0008	1.9897
			SRS	0.4062	0.0017		0.4005	0.0018		0.4063	0.0016	
		$\hat{\delta}$	RSS	0.2487	0.0008	1.4598	0.2590	0.0010	1.5472	0.2500	0.0007	1.5426
			SRS	0.2539	0.0012		0.2645	0.0015		0.2544	0.0011	
	8	$\hat{\eta}$	RSS	0.4008	0.0002	4.7591	0.3979	0.0002	4.6325	0.4006	0.0002	4.7380
			SRS	0.4028	0.0009		0.3999	0.0003		0.4025	0.0009	
		$\hat{\delta}$	RSS	0.2485	0.0003	1.7654	0.2537	0.0003	1.7296	0.2487	0.0002	1.8139
			SRS	0.2481	0.0005		0.2533	0.0005		0.2482	0.0005	
	12	$\hat{\eta}$	RSS	0.4020	0.0001	4.1133	0.4000	0.0001	4.2406	0.4016	0.000‘1	4.0787
			SRS	0.4013	0.0004		0.3994	0.0004		0.4015	0.0005	
		$\hat{\delta}$	RSS	0.2488	0.0001	2.1178	0.2524	0.0001	2.0843	0.2494	0.0001	2.0663
			SRS	0.2488	0.0002		0.2524	0.0003		0.2486	0.0002	

Table 3. Numerical values of simulation measures using Scenario II.

d	v		Design	MLE			MPS			LSE		
				Mean	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**
6	4	$\hat{\eta}$	RSS	1.2846	0.0607	1.620	1.1732	0.0397	1.5407	1.2254	0.0595	2.3361
			SRS	1.2663	0.0984		1.1469	0.0612		1.2394	0.1391	
		$\hat{\delta}$	RSS	1.4726	0.0485	1.111	1.3105	0.0473	1.0729	1.3902	0.0703	1.4888
			SRS	1.4585	0.0539		1.2925	0.0507		1.3866	0.1047	
	8	$\hat{\eta}$	RSS	1.2081	0.0088	3.6641	1.1420	0.0119	2.2419	1.1812	0.0138	2.669
			SRS	1.2217	0.0322		1.1552	0.0267		1.2025	0.0369	
		$\hat{\delta}$	RSS	1.4184	0.0106	1.7163	1.3219	0.0190	1.0954	1.3701	0.0216	1.5675
			SRS	1.4249	0.0182		1.3270	0.0208		1.3970	0.03390	
	12	$\hat{\eta}$	RSS	1.1971	0.0052	5.0405	1.1502	0.0074	2.8033	1.1762	0.0062	5.2231
			SRS	1.2371	0.0262		1.1845	0.0208		1.2248	0.0323	
		$\hat{\delta}$	RSS	1.3951	0.0053	3.3294	1.3243	0.0126	1.3987	1.3624	0.0086	3.1202
			SRS	1.4270	0.0177		1.3531	0.0177		1.3976	0.0267	
12	4	$\hat{\eta}$	RSS	1.2472	0.0185	2.9943	1.1810	0.0151	2.6303	1.2228	0.0241	2.1517
			SRS	1.2592	0.0554		1.1869	0.0397		1.2265	0.0518	
		$\hat{\delta}$	RSS	1.4296	0.0151	2.0169	1.3345	0.0207	1.4187	1.3877	0.0298	1.3315
			SRS	1.4357	0.0306		1.3365	0.0293		1.3856	0.0396	
	8	$\hat{\eta}$	RSS	1.2012	0.0042	3.5155	1.1639	0.0050	2.6898	1.1906	0.0052	3.2648
			SRS	1.2099	0.0147		1.1705	0.0135		1.1947	0.0171	
		$\hat{\delta}$	RSS	1.4073	0.0055	2.2196	1.3469	0.0092	1.4987	1.3943	0.0085	2.0886
			SRS	1.4083	0.0123		1.3505	0.0137		1.3853	0.0177	
	12	$\hat{\eta}$	RSS	1.2023	0.0035	3.5222	1.1735	0.0046	2.3675	1.1914	0.0037	3.8908
			SRS	1.2233	0.0125		1.1936	0.0108		1.2137	0.0144	
		$\hat{\delta}$	RSS	1.4010	0.0037	2.3887	1.3576	0.0062	1.3103	1.3841	0.0049	2.3989
			SRS	1.4242	0.0088		1.3816	0.0082		1.4068	0.0117	

Table 4. Numerical values of simulation measures using Scenario II.

d	v		Design	WLS			CV			AD		
				Mean	MSE	Ef	Mean	MSE	Ef	Mean	MSE	Ef
6	4	$\hat{\eta}$	RSS	1.2077	0.0385	2.6303	1.2868	0.0613	2.6977	1.2248	0.0442	2.3885
			SRS	1.2751	0.1014		1.3708	0.1653		1.2900	0.1055	
		$\hat{\delta}$	RSS	1.3736	0.0427	1.7922	1.4856	0.0589	2.0770	1.3984	0.0414	1.6594
			SRS	1.4270	0.0765		1.5512	0.1224		1.4469	0.0688	
	8	$\hat{\eta}$	RSS	1.1852	0.0129	3.4338	1.2227	0.0154	3.5686	1.1888	0.0121	3.7250
			SRS	1.2182	0.0442		1.2586	0.0549		1.2206	0.0451	
		$\hat{\delta}$	RSS	1.3680	0.0175	1.9007	1.4248	0.0185	2.1934	1.3747	0.0152	2.1499
			SRS	1.4059	0.0332		1.4641	0.0406		1.4103	0.0327	
	12	$\hat{\eta}$	RSS	1.1929	0.0054	7.5761	1.2179	0.0062	7.6529	1.1978	0.0054	7.5257
			SRS	1.2278	0.0406		1.2548	0.0471		1.2269	0.0405	
		$\hat{\delta}$	RSS	1.3862	0.0068	3.8869	1.4248	0.0076	4.0373	1.3939	0.0063	4.0759
			SRS	1.4125	0.0264		1.4515	0.0307		1.4116	0.0257	
12	4	$\hat{\eta}$	RSS	1.2009	0.0151	2.8453	1.2392	0.0189	2.8697	1.2058	0.0151	2.9079
			SRS	1.2342	0.0429		1.2750	0.0543		1.2374	0.0438	
		$\hat{\delta}$	RSS	1.3921	0.0204	1.1968	1.4499	0.0248	1.2675	1.4001	0.0187	1.2569
			SRS	1.4106	0.0244		1.4686	0.0314		1.4164	0.0236	
	8	$\hat{\eta}$	RSS	1.2001	0.0064	2.7136	1.2195	0.0073	2.7396	1.2036	0.0065	2.6553
			SRS	1.2193	0.0174		1.2393	0.0199		1.2212	0.0173	
		$\hat{\delta}$	RSS	1.3955	0.0080	1.3995	1.4252	0.0090	1.4895	1.4011	0.0078	1.3802
			SRS	1.4138	0.0111		1.4435	0.0133		1.4162	0.0107	
	12	$\hat{\eta}$	RSS	1.2002	0.0031	3.6242	1.2133	0.0034	3.3489	1.2014	0.0032	3.4644
			SRS	1.1834	0.0111		1.1958	0.0114		1.1842	0.0109	
		$\hat{\delta}$	RSS	1.3993	0.0035	2.0938	1.4196	0.0040	1.8625	1.4012	0.0036	2.0061
			SRS	1.3867	0.0074		1.4065	0.0075		1.3890	0.0073	

Table 5. Numerical values of simulation measures using Scenario III.

d	v		Design	MLE			MPS			LSE		
				Mean	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**
6	4	$\hat{\eta}$	RSS	1.8674	0.1459	2.1434	1.6165	0.1067	1.7356	1.7136	0.1763	1.5405
			SRS	1.8895	0.3127		1.6393	0.1852		1.7397	0.2716	
		$\hat{\delta}$	RSS	0.5181	0.0044	1.4994	0.4571	0.0059	1.1343	0.4762	0.0074	1.2980
			SRS	0.5167	0.0066		0.4587	0.0067		0.4836	0.0096	
	8	$\hat{\eta}$	RSS	1.8124	0.0532	3.8576	1.6786	0.0497	2.7131	1.7394	0.0602	3.0993
			SRS	1.8999	0.2052		1.7423	0.1349		1.8340	0.1864	
		$\hat{\delta}$	RSS	0.5109	0.0018	2.5550	0.4773	0.0024	1.5152	0.4922	0.0025	2.0629
			SRS	0.5231	0.0045		0.4872	0.0037		0.5076	0.0052	
	12	$\hat{\eta}$	RSS	1.7819	0.0190	4.0962	1.6796	0.0206	3.3449	1.7302	0.0234	3.5776
			SRS	1.7667	0.0778		1.6614	0.0690		1.7226	0.0838	
		$\hat{\delta}$	RSS	0.5052	0.0007	2.2689	0.4804	0.0011	1.8388	0.4915	0.0011	2.1248
			SRS	0.5007	0.0016		0.4749	0.0021		0.4879	0.0024	
12	4	$\hat{\eta}$	RSS	1.7989	0.0939	2.3100	1.6602	0.0851	1.6642	1.7466	0.1176	1.6739
			SRS	1.9046	0.2169		1.7455	0.1416		1.8395	0.1969	
		$\hat{\delta}$	RSS	0.5061	0.0028	1.8074	0.4721	0.0037	0.9896	0.4903	0.0046	1.1585
			SRS	0.5296	0.0050		0.4931	0.0037		0.5137	0.0053	
	8	$\hat{\eta}$	RSS	1.7390	0.0183	2.8421	1.6649	0.0269	1.8036	1.7060	0.0215	3.5405
			SRS	1.7564	0.0519		1.6718	0.0485		1.7382	0.0759	
		$\hat{\delta}$	RSS	0.4989	0.0008	1.7656	0.4796	0.0013	1.2506	0.4898	0.0012	1.9790
			SRS	0.5012	0.0014		0.4804	0.0017		0.4949	0.0023	
	12	$\hat{\eta}$	RSS	1.7625	0.0097	3.7673	1.7010	0.0144	2.2763	1.7391	0.0105	4.092
			SRS	1.7740	0.0366		1.7117	0.0327		1.7402	0.0428	
		$\hat{\delta}$	RSS	0.5029	0.0004	2.4304	0.4876	0.0007	1.3593	0.4968	0.0006	2.719
			SRS	0.5049	0.0009		0.4898	0.0010		0.4964	0.0015	

Table 6. Numerical values of simulation measures fusing Scenario III.

d	v		Design	WLS			CV			AD		
				Mean	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**	**Mean**	**MSE**	**Ef**
6	4	$\hat{\eta}$	RSS	1.7373	0.1307	2.1054	1.9080	0.2169	1.9813	1.7778	0.1359	2.1140
			SRS	1.7912	0.2751		1.9637	0.4298		1.8235	0.2872	
		$\hat{\delta}$	RSS	0.4845	0.0046	1.5822	0.5244	0.0059	1.6701	0.4952	0.0042	1.5644
			SRS	0.4986	0.0072		0.5384	0.0099		0.5053	0.0065	
	8	$\hat{\eta}$	RSS	1.7319	0.0476	2.0818	1.8155	0.0616	2.0571	1.7458	0.0475	2.1583
			SRS	1.7585	0.0991		1.8432	0.1267		1.7698	0.1024	
		$\hat{\delta}$	RSS	0.4928	0.0020	1.4182	0.5136	0.0024	1.4733	0.4962	0.0018	1.6021
			SRS	0.4991	0.0029		0.5195	0.0035		0.5023	0.0029	
	12	$\hat{\eta}$	RSS	1.7285	0.0142	9.2815	1.7830	0.0167	9.8402	1.7410	0.0130	9.9391
			SRS	1.8299	0.1322		1.8916	0.1639		1.8308	0.1296	
		$\hat{\delta}$	RSS	0.4932	0.0007	3.6785	0.5070	0.0008	4.0715	0.4966	0.0006	4.0713
			SRS	0.5037	0.0027		0.5177	0.0032		0.5042	0.0025	
12	4	$\hat{\eta}$	RSS	1.7838	0.0781	1.4551	1.8730	0.1086	1.3985	1.7813	0.0692	1.7045
			SRS	1.7969	0.1136		1.8854	0.1519		1.8053	0.1180	
		$\hat{\delta}$	RSS	0.5027	0.0031	1.2754	0.5239	0.0040	1.2814	0.5031	0.0028	1.3398
			SRS	0.5076	0.0040		0.5284	0.0051		0.5095	0.0037	
	8	$\hat{\eta}$	RSS	1.7548	0.0207	2.9211	1.7974	0.0252	2.8728	1.7593	0.0211	2.8255
			SRS	1.7976	0.0606		1.8421	0.0723		1.7958	0.0597	
		$\hat{\delta}$	RSS	0.5007	0.0011	1.4500	0.5113	0.0013	1.4944	0.5019	0.0011	1.4066
			SRS	0.5068	0.0016		0.5174	0.0019		0.5064	0.0016	
	12	$\hat{\eta}$	RSS	1.7309	0.0104	4.7573	1.7589	0.0107	5.1603	1.7349	0.0094	5.0875
			SRS	1.7791	0.0494		1.8087	0.0551		1.7754	0.0479	
		$\hat{\delta}$	RSS	0.4943	0.0005	3.1427	0.5014	0.0005	3.6186	0.4954	0.0005	3.3485
			SRS	0.5053	0.0016		0.5125	0.0018		0.5044	0.0015	

The following formulas mathematically describe the proposed metrics:

- Mean

$$\text{Est}\,(\boldsymbol{v}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\boldsymbol{v}}_i$$

- Mean square error

$$\text{MSE}\,(\boldsymbol{v}) = \frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{v}_i - \boldsymbol{v}\right)^2.$$

- Efficiency

$$\text{Ef}_{\boldsymbol{v}} = \frac{MSE_{\boldsymbol{v}}^{SRS}}{MSE_{\boldsymbol{v}}^{RSS}}$$

Tables (1)–(6) summarize the findings. The results show that the produced RSS estimator outperforms the SRS estimators in each of the six process estimations as all the efficiency scores are greater than **1**. Additionally, the MSE for the RSS scheme decreases as the number of cycles increases, and the same holds when the set size v increases. We can infer that under the SRS design, the MSE decreases as $n = dv$ increases.

5. Information Measures

In this section, we investigate several entropy measures for our IPME model, which are important in information theory.

5.1. *Renyi entropy*

In information theory, the Renyi entropy [19] R_1 is an indicator for uncertainty and it is described as

$$\begin{aligned} R_1 &= \frac{1}{1-\varphi}\log\left(\int_0^\infty g^{\varphi}(y;\eta,\delta)\,dy\right) \quad \varphi \neq 1, \varphi > 0. \\ &= \frac{1}{1-\varphi}\log\left(\int_0^\infty \left[\frac{\delta y^{-2\delta-1}\, e^{-\frac{y^{-\delta}}{\eta}}}{\eta^2}\right]^{\varphi} dy\right) \\ &= \frac{1}{1-\varphi}\,\log\left(\frac{\delta^{\varphi-1}\,\eta^{\frac{1}{\delta}(\varphi-1)}\,\Gamma\left(2\varphi+\frac{1}{\delta}(\varphi-1)\right)}{\varphi^{2\varphi+\frac{1}{\delta}(\varphi-1)}}\right). \end{aligned} \tag{15}$$

5.2. *Shannon's entropy*

In short, Shannon's entropy [20] R_2 is the quantity of knowledge in a variable, which can be described as

$$\begin{aligned} R_2 &= \text{E}\left(-\log g(y;\eta,\delta)\right) \\ &= \text{E}\left(-\log\left[\frac{\delta y^{-2\delta-1}\, e^{-\frac{y^{-\delta}}{\eta}}}{\eta^2}\right]\right) \\ &= 2\log(\eta) - \log(\delta) + (2\delta+1)\text{E}[\log(y)] - \frac{1}{\eta}\text{E}[y^{-\delta}]. \end{aligned} \tag{16}$$

5.3. Havrda and Charvat entropy

Additionally, the Havrda and Charvat entropy [21] R_3 is a different uncertainty knowledge indicator that can be stated as

$$\begin{aligned} R_3 &= \frac{1}{2^{1-\varphi}-1}\left[\int_0^\infty g^{\varphi}(y;\eta,\delta)\,dy - 1\right] \quad \varphi \neq 1, \varphi > 0. \\ &= \frac{1}{2^{1-\varphi}-1}\left[\frac{\delta^{\varphi-1}\,\eta^{\frac{1}{\delta}(\varphi-1)}\,\Gamma\left(2\varphi+\frac{1}{\delta}(\varphi-1)\right)}{\varphi^{2\varphi+\frac{1}{\delta}(\varphi-1)}} - 1\right]. \end{aligned} \tag{17}$$

5.4. Tsallis entropy

The Tsallis entropy [22] R_4 can be characterized using theour IPME model as

$$\begin{aligned} R_4 &= \frac{1}{1-\varphi}\left(1-\int_0^\infty g^{\varphi}(y;\eta,\delta)\,dy\right) \quad \varphi \neq 1, \varphi > 0. \\ &= \frac{1}{1-\varphi}\left(1-\frac{\delta^{\varphi-1}\,\eta^{\frac{1}{\delta}(\varphi-1)}\,\Gamma\left(2\varphi+\frac{1}{\delta}(\varphi-1)\right)}{\varphi^{2\varphi+\frac{1}{\delta}(\varphi-1)}}\right). \end{aligned} \tag{18}$$

5.5. Arimoto entropy

We examine the Arimoto entropy [23] R_5 for our suggested IPME model, and it is

$$\begin{aligned} R_5 &= \frac{1}{2^{1-\varphi}-1}\left[\int_0^\infty (g^{1/\varphi}(y;\eta,\delta)\,dy)^{\varphi} - 1\right] \quad \varphi \neq 1, \varphi > 0. \\ &= \frac{1}{2^{1-\varphi}-1}\left\{\frac{\delta^{1-\varphi}}{\eta^2}\left[\frac{\Gamma\left(\frac{1}{\delta\varphi}(2\delta+1)-\frac{1}{\delta}\right)}{(\eta\varphi)^{-\frac{1}{\delta\varphi}(2\delta+1)-\frac{1}{\delta}}}\right]^{\varphi} - 1\right\}. \end{aligned} \tag{19}$$

5.6. Mathai–Haubold entropy

This subsection presents an entirely novel entropy measure, the Mathai–Haubold entropy R_6 [24]. It is expressed as

$$\begin{aligned} R_6 &= \frac{1}{\varphi-1}\left(\int_0^\infty g^{2-\varphi}(y;\eta,\delta)\,dy - 1\right) \quad \varphi \neq 1, \varphi > 0. \\ &= \frac{1}{\varphi-1}\left(\int_0^\infty \left[\frac{\delta y^{-2\delta-1}\,e^{-\frac{y^{-\delta}}{\eta}}}{\eta^2}\right]^{2-\varphi} dy - 1\right) \\ &= \frac{1}{\varphi-1}\left(\frac{\delta^{1-\varphi}}{\eta^{2(2-\varphi)}}\,\frac{\Gamma\left(\frac{1}{\delta}(2-\varphi)(2\delta+1)\right)}{\left(\frac{2-\varphi}{\eta}\right)^{\frac{1}{\delta}(2-\varphi)(2\delta+1)}} - 1\right). \end{aligned} \tag{20}$$

Using multiple parameter values for η and δ, Table (8) presents specific numerical results for the suggested entropy metrics for the IPME model. Further, Figures (3)-(6) present possible shapes (3D and contour plots) of the entropy metrics obtained from the IPME model for various parameter values. Clearly, the results taken from the Table (8) ensured that all the recommended information entropy of the IPME distribution can be either negative or positive. This confirms that the PDF of the IPME model is sharply peaked and widely distributed.

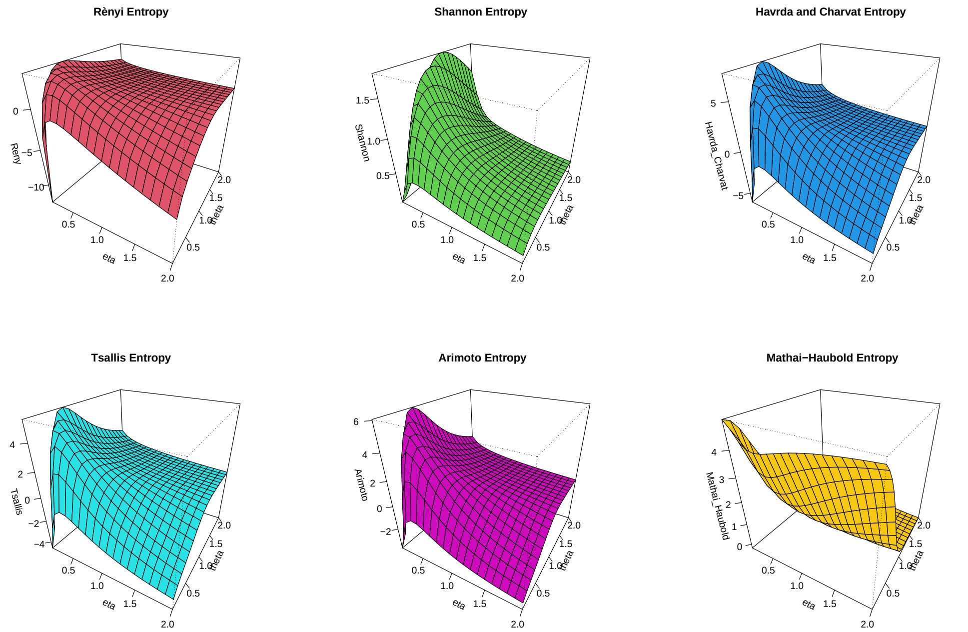


Figure 3. 3-dimensional curves of the recommended entropy information for the IPME model at $\varphi = 0.8$.

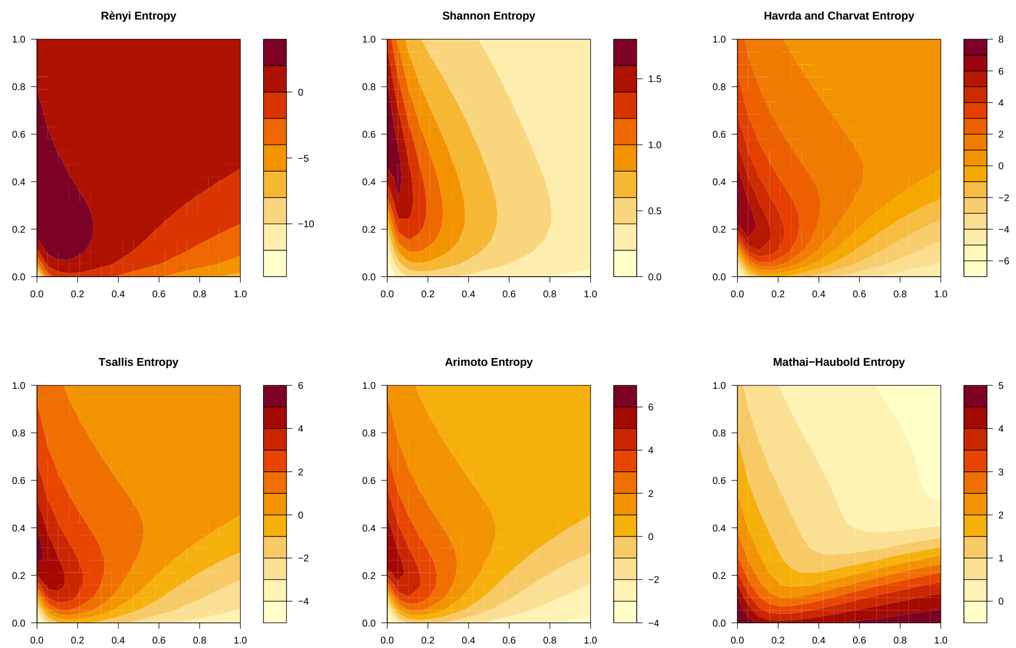


Figure 4. Contour plots of the recommended entropy information for the IPME model at $\varphi = 0.8$.

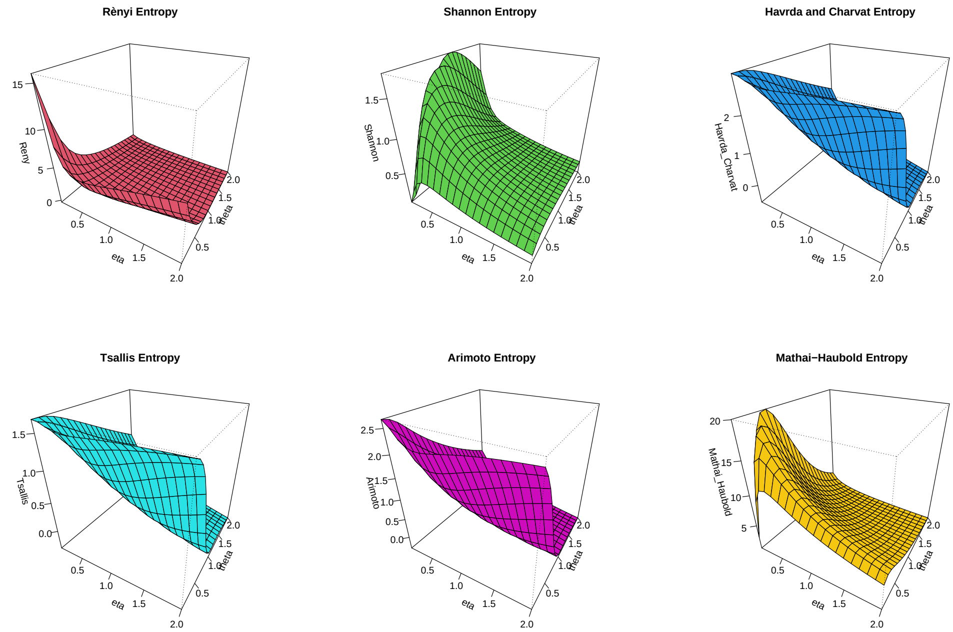


Figure 5. 3-dimensional curves of the recommended entropy information for the IPME model at $\varphi = 1.6$.

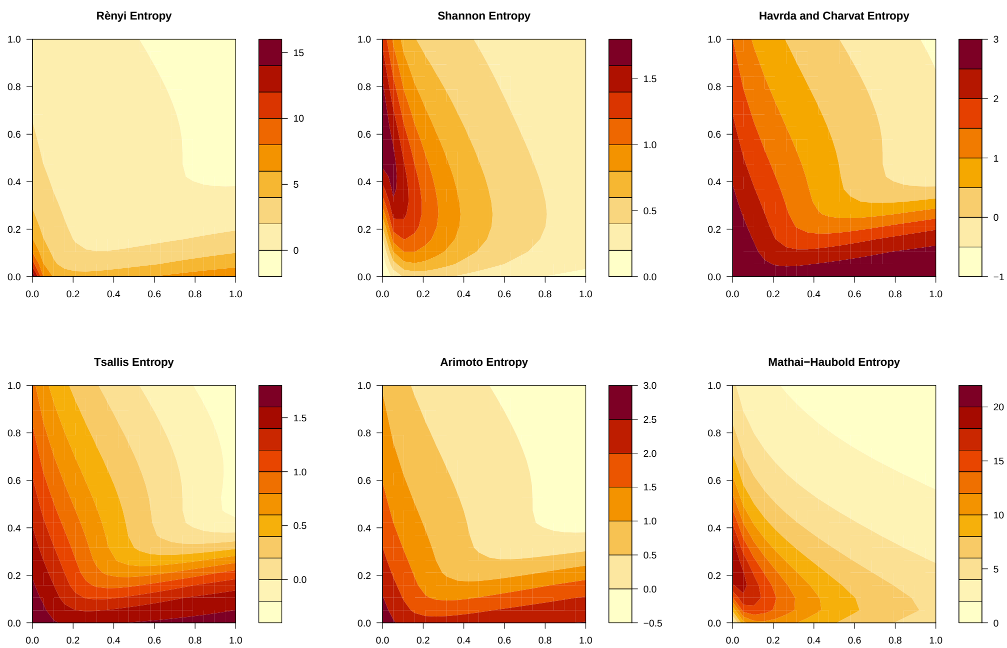


Figure 6. Contour plots of the recommended entropy information for the IPME model at $\varphi = 1.6$.

Table 7. Different numerical records of the proposed R_1, R_2, R_3, R_5, R_5, and R_6 at several parameter values.

φ	η	δ	R_1	R_2	R_3	R_5	R_5	R_6
0.5	0.3	0.4	6.3837	3.9659	56.333	46.668	591.14	-238.07
		0.8	3.7944	2.354	13.681	11.334	43.451	-32.430
		1.2	2.4737	1.6231	5.9022	4.8895	10.8664	-9.6870
		1.6	1.7881	1.1727	3.4886	2.8901	4.9782	-5.5320
0.75	0.6	0.4	3.3384	2.2070	6.8914	5.2156	6.1286	-4.6880
		0.8	1.9008	1.4858	3.2151	2.4333	2.6531	-3.1120
		1.2	1.3028	1.0455	2.0348	1.5400	1.6315	-2.0030
		1.6	0.9375	0.7395	1.3960	1.0565	1.1005	-1.3613
1.5	0.9	0.4	0.3972	1.4505	0.6150	0.3603	0.3721	-0.1891
		0.8	0.6116	0.9683	0.8996	0.5270	0.5533	-0.2836
		1.2	0.4995	0.7705	0.7546	0.4420	0.4601	-0.2348
		1.6	21.328	0.4860	3.4141	2.0000	2.9975	-0.1361
2.5	1.2	0.4	0.6274	0.9992	0.9433	0.4065	0.5228	0.4006
		0.8	-0.0215	0.6046	-0.0508	-0.0219	-0.0217	-0.0107
		1.2	0.0039	0.4676	0.0090	0.0039	0.0039	0.0019
		1.6	-0.0778	0.3062	-0.1914	-0.0825	-0.0796	-0.0378

Table 8. Different numerical records of the proposed R_1, R_2, R_3, R_5, R_5, and R_6 at several parameter values.

φ	η	δ	R_1	R_2	R_3	R_5	R_5	R_6
0.5	0.3	0.4	6.3837	3.9659	56.333	46.668	591.14	-238.07
		0.8	3.7944	2.354	13.681	11.334	43.451	-32.430
		1.2	2.4737	1.6231	5.9022	4.8895	10.8664	-9.6870
		1.6	1.7881	1.1727	3.4886	2.8901	4.9782	-5.5320
0.75	0.6	0.4	3.3384	2.2070	6.8914	5.2156	6.1286	-4.6880
		0.8	1.9008	1.4858	3.2151	2.4333	2.6531	-3.1120
		1.2	1.3028	1.0455	2.0348	1.5400	1.6315	-2.0030
		1.6	0.9375	0.7395	1.3960	1.0565	1.1005	-1.3613
1.5	0.9	0.4	0.3972	1.4505	0.6150	0.3603	0.3721	-0.1891
		0.8	0.6116	0.9683	0.8996	0.5270	0.5533	-0.2836
		1.2	0.4995	0.7705	0.7546	0.4420	0.4601	-0.2348
		1.6	21.328	0.4860	3.4141	2.0000	2.9975	-0.1361
2.5	1.2	0.4	0.6274	0.9992	0.9433	0.4065	0.5228	0.4006
		0.8	-0.0215	0.6046	-0.0508	-0.0219	-0.0217	-0.0107
		1.2	0.0039	0.4676	0.0090	0.0039	0.0039	0.0019
		1.6	-0.0778	0.3062	-0.1914	-0.0825	-0.0796	-0.0378

5.7. Simulation Experiments of the Entropy Measures

Here, we can describe the MLEs of different entropy measures resulted in (15)-(20). Consequently, the MLE of Rény entropy, denoted by $\hat{R}_1$, is determined by replacing η and δ in (15) as

$$\hat{R}_1 = \frac{1}{1-\varphi}\ \log\left(\frac{\hat{\delta}^{\varphi-1}\ \hat{\eta}^{\frac{1}{\hat{\delta}}(\varphi-1)}\ \Gamma\left(2\varphi + \frac{1}{\hat{\delta}}(\varphi-1)\right)}{\varphi^{2\varphi+\frac{1}{\hat{\delta}}(\varphi-1)}}\right).$$

With the same tool and by applying the invariance property of ML estimators, we get other entropy estimators, denoted by $\hat{R}_2$, $\hat{R}_3$, $\hat{R}_4$, $\hat{R}_5$, and $\hat{R}_6$.
Now, we present a Monte Carlo simulation analysis to evaluate the practical behavior of the entropy estimators for the IPME distribution. We produce random numbers of different sizes, namely $n = 20, 40, 80, 100$, and several intial values of η and δ. In this study, each simulation procedure is repeated $M = 1000$ times, and we adopt three established metrics as evaluation criteria, including average mean (Mean), MSE, and mean relative error (MRE), to assess the performance of the recommended entropy measures. Tables (9)-(11) reported numerical results from the analysis of simulation studies. From the obtained simulation analysis, we have observed the following output:

- As the sample size n increases, the estimates of all proposed entropy estimators approach the true value for all parameters. This confirms that the entropy estimators obtained by the MLE technique are consistent and asymptotically unbiased.
- As the sample size n increases, for all entropy measures, the value of MSEs approaches zero. This ensures that the entropy estimators obtained via the MLE technique converge.
- With an increasing value of η and δ, the MSE values increase.

Table 9. Numerical values of simulation R_1, R_2, R_3, R_5, R_5, and R_6 for the IPME model at $\varphi = 0.75$.

n		δ	η	$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$	$\hat{R}_4$	$\hat{R}_5$	$\hat{R}_6$
20	True value	0.25	0.4	5.4537	3.4894	15.3774	11.6381	15.4765	-5.4779
	Mean	0.265	0.3796	5.4657	3.5648	15.5978	11.8049	15.801	-5.5398
	MSE	0.0211	0.0048	0.2497	0.1464	6.3954	3.6632	9.0723	0.5136
	MRE	0.0602	0.051	0.0022	0.0216	0.0143	0.0143	0.021	0.0113
40	Mean	0.255	0.398	5.42	3.4749	15.289	11.5712	15.4042	-5.4356
	MSE	0.0237	0.0018	0.1346	0.0572	3.419	1.9584	4.8087	0.2481
	MRE	0.0201	0.0051	0.0062	0.0042	0.0057	0.0057	0.0047	0.0077
80	Mean	0.2523	0.4021	5.4209	3.4683	15.251	11.5424	15.3426	-5.4501
	MSE	0.0241	0.001	0.0673	0.0303	1.7421	0.9979	2.4587	0.0903
	MRE	0.0093	0.0053	0.006	0.006	0.0082	0.0082	0.0086	0.0051
100	Mean	0.2514	0.4005	5.44	3.4827	15.3358	11.6066	15.4383	-5.4617
	MSE	0.0236	0.0009	0.0454	0.0288	1.1856	0.6791	1.677	0.0691
	MRE	0.0057	0.0014	0.0025	0.0019	0.0027	0.0027	0.0025	0.003

Table 10. Numerical values of simulation R_1, R_2, R_3, R_5, R_5, and R_6 for the IPME model at $\varphi = 1.3$.

n		δ	η	$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$	$\hat{R}_4$	$\hat{R}_5$	$\hat{R}_6$
20	True value	1.4	1.2	0.232	0.3857	0.3581	0.2241	0.2259	-0.1585
	Mean	1.4771	1.2893	0.2132	0.366	0.3143	0.1967	0.2004	-0.1411
	MSE	0.1735	0.1692	0.07	0.0804	0.1611	0.0631	0.0645	0.0318
	MRE	0.0551	0.0744	0.0809	0.0512	0.1223	0.1223	0.1129	0.1101
40	Mean	1.4529	1.2694	0.2023	0.3543	0.3057	0.1913	0.1938	-0.1362
	MSE	0.0838	0.0716	0.036	0.0412	0.0821	0.0321	0.033	0.0163
	MRE	0.0378	0.0578	0.1281	0.0813	0.1464	0.1464	0.1423	0.141
80	Mean	1.44	1.2595	0.1951	0.3465	0.2998	0.1876	0.1893	-0.1329
	MSE	0.0438	0.0276	0.0144	0.0164	0.033	0.0129	0.0132	0.0065
	MRE	0.0285	0.0496	0.1593	0.1016	0.1629	0.1629	0.1621	0.1619
100	Mean	1.4336	1.2314	0.2104	0.3623	0.3235	0.2025	0.2043	-0.1434
	MSE	0.0406	0.0186	0.0107	0.0123	0.0241	0.0095	0.0097	0.0048
	MRE	0.024	0.0262	0.0931	0.0608	0.0966	0.0966	0.0958	0.0955

Table 11. Numerical values of simulation R_1, R_2, R_3, R_5, R_5, and R_6 for the IPME model at $\varphi = 2.1$.

n		δ	η	$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$	$\hat{R}_4$	$\hat{R}_5$	$\hat{R}_6$
20	True value	0.5	1.2	0.4683	0.8721	0.7547	0.366	0.4153	0.0481
	Mean	0.5444	1.3531	0.4007	0.7839	0.5979	0.29	0.3417	0.0415
	MSE	0.0135	0.1936	0.0956	0.1088	0.2296	0.054	0.07	0.001
	MRE	0.0889	0.1276	0.1445	0.1011	0.2077	0.2077	0.1772	0.1368
40	Mean	0.5111	1.2704	0.4676	0.8485	0.7279	0.353	0.407	0.0482
	MSE	0.0039	0.0697	0.0378	0.0627	0.0615	0.0145	0.0235	0.0004
	MRE	0.0221	0.0587	0.0015	0.0271	0.0355	0.0355	0.02	0.003
80	Mean	0.5102	1.2238	0.4521	0.8567	0.7202	0.3493	0.3982	0.0465
	MSE	0.002	0.0276	0.021	0.0259	0.0337	0.0079	0.0131	0.0002
	MRE	0.0203	0.0198	0.0348	0.0177	0.0457	0.0457	0.0411	0.0332
100	Mean	0.5124	1.2514	0.4413	0.8323	0.7091	0.3439	0.3905	0.0453
	MSE	0.0018	0.0259	0.0174	0.0242	0.0298	0.007	0.0112	0.0001
	MRE	0.0249	0.0428	0.0578	0.0456	0.0604	0.0604	0.0597	0.0572

6. Real-Life Data Applications

In this section, we present three real-world applications from the engineering, radiation, and environmental fields to demonstrate the importance and performance of the proposed estimators under RSS and SRS samplings, as well as the computation of the recommended entropy estimators using the two applied data sets.

6.1. Data I

The length of the intervals between times when cars passed a spot on a road made up the suggested data. Jorgensen [25] and Hassan et al. [26] provided the suggested data, and Table (12) stated the data's outcomes.

Table 12. Values of the first data set.

2.50	2.60	2.60	2.70	2.80	2.80	2.90	3.00	3.00	3.10	3.20	3.40	3.70
3.90	3.90	3.90	4.60	4.70	5.00	5.60	5.70	6.00	6.00	6.10	6.60	6.90
6.90	7.30	7.60	7.90	8.00	8.30	8.80	8.80	9.30	9.40	9.50	10.1	11.
11.3	11.9	11.9	12.3	12.9	12.9	13.0	13.8	14.5	14.9	15.3	15.4	15.9
16.2	17.6	20.1	20.3	20.6	21.4	22.8	23.7	23.7	24.7	29.7	30.6	31.0
34.1	34.7	36.8	40.1	40.2	41.3	42.0	44.8	49.8	51.7	55.7	56.5	58.1
70.5	72.6	87.1	88.6	91.7	119.8							

6.2. Data II

The following dataset includes evaluations of indomethacin's blood level (in µg/mL), a substance frequently examined in pharmacokinetic research. The proposed data set was used by Karakas et al. [27], and Table (13) presents the data outcomes.

Table 13. Values of the second data set.

1.50	0.94	0.78	0.48	0.37	0.19	0.12	0.11	0.08	0.07	0.05
2.03	1.63	0.71	0.70	0.64	0.36	0.32	0.20	0.25	0.12	0.08
2.72	1.49	1.16	0.80	0.80	0.39	0.22	0.12	0.11	0.08	0.08
1.85	1.39	1.02	0.89	0.59	0.40	0.16	0.11	0.10	0.07	0.07
2.05	1.04	0.81	0.39	0.30	0.23	0.13	0.11	0.08	0.10	0.06
2.31	1.44	1.03	0.84	0.64	0.42	0.24	0.17	0.13	0.10	0.09

6.3. Data III

The following dataset represents the physicochemical characteristics, specifically the salinity, of groundwater samples collected from the Al-Ahsa Oasis, Saudi Arabia. The proposed data set was studied earlier by Al-Hamad et al. [28], and Table (14) summarizes the data's values.
Table (15) presents descriptive statistics for the length of the intervals within the times data set, and Figures (7)-(9) display a graphical illustration, which includes kernel density, total test time (TTT), probability-probability (PP) plot, quantile-quantile (QQ) plot, histogram, and box graphs.

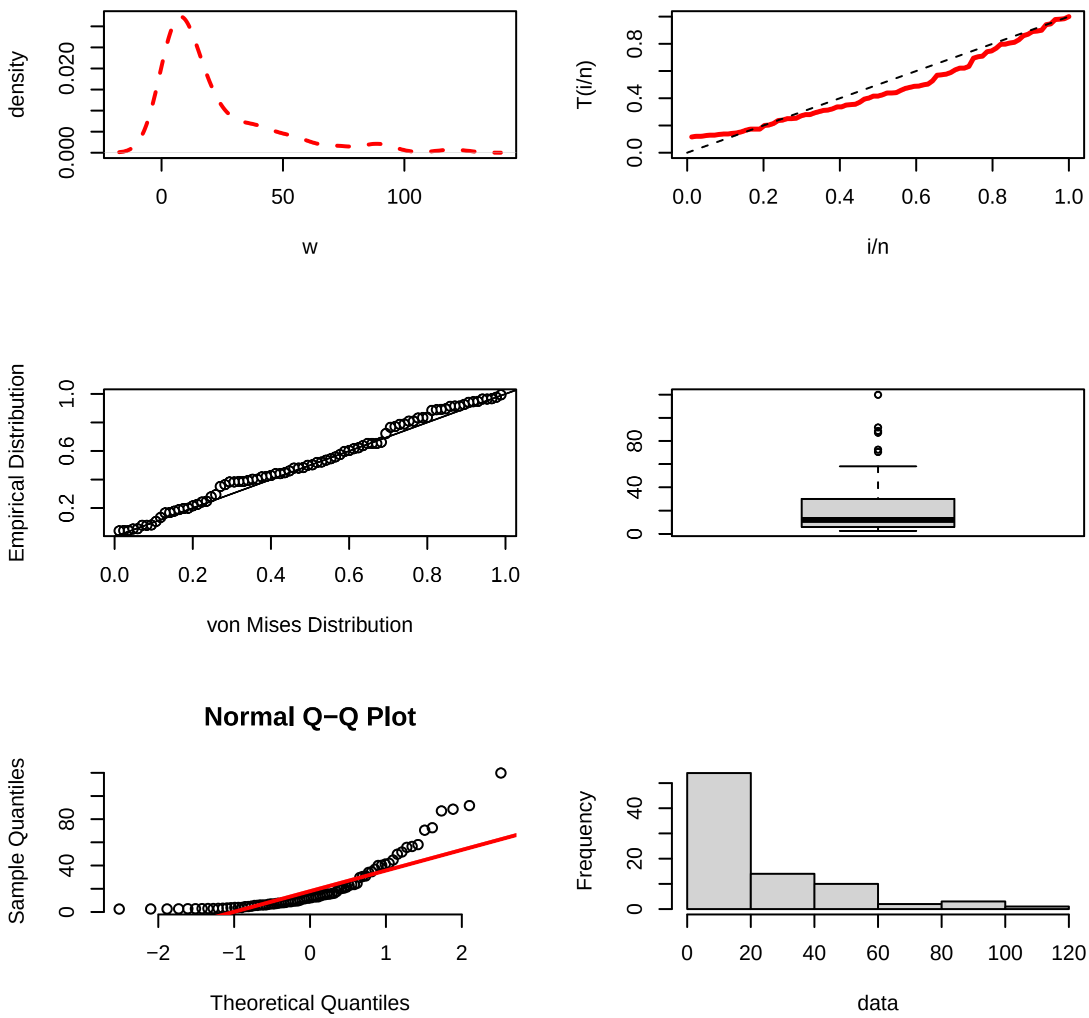


Figure 7. Visual illustrations of the first data set.

Table 14. Values of the third data set.

0.32	0.44	0.49	0.19	0.2	0.32	0.3	0.2	0.2	0.17	0.14
0.45	0.14	0.14	0.14	0.23	0.29	0.47	0.25	0.43	0.74	0.48
0.15	0.14	0.2	0.17	0.17	0.17	0.17	0.17	0.24	0.18	0.17
0.16	0.17									

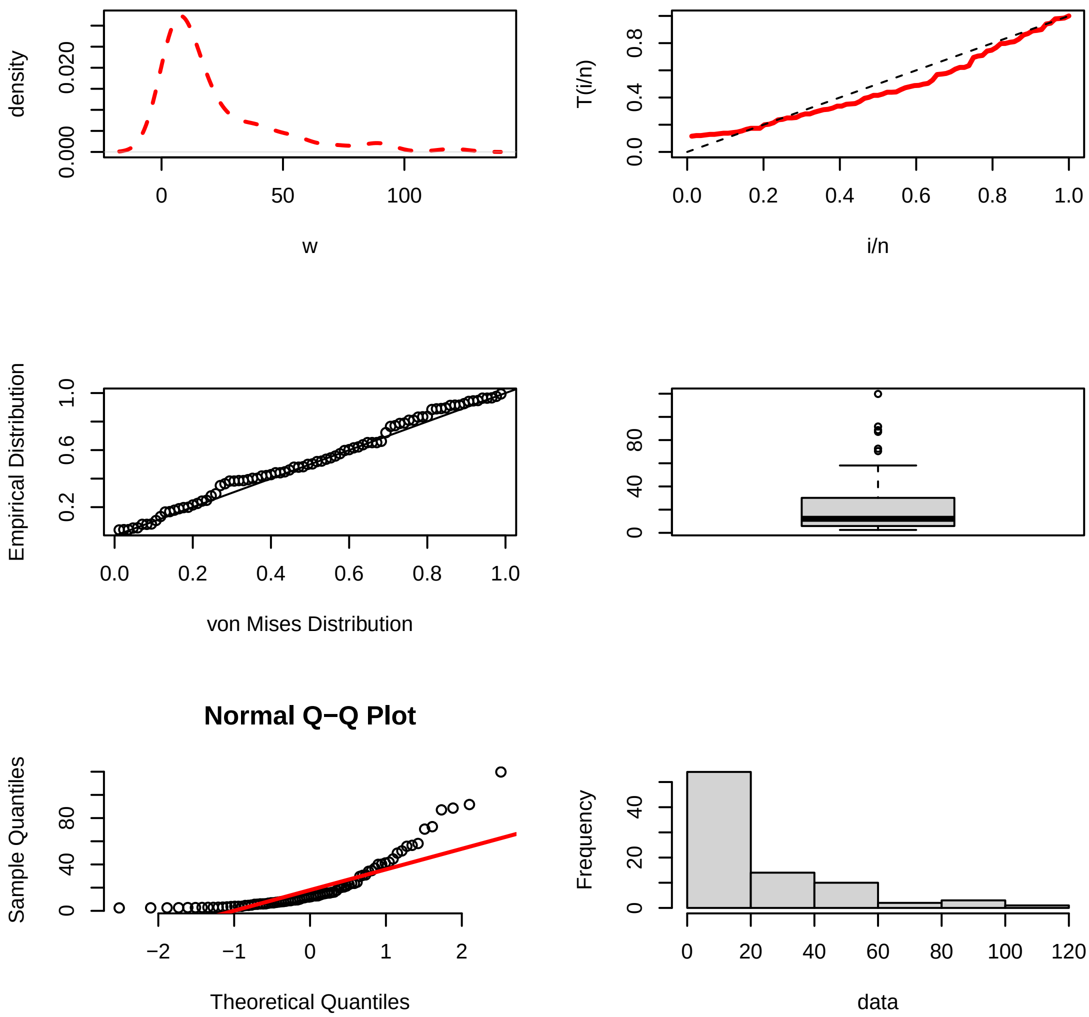


Figure 8. Visual illustrations of the second data set.

Table 15. Summary statistics of the three applied data sets.

Data	n	Mean	Median	Q_1	Q_3	Skewness	Kurtosis
I	84	21.602	12.100	5.925	29.925	1.8813	3.4251
II	66	0.5918	0.3400	0.1100	0.8325	1.4148	1.3364
III	35	0.2569	0.2000	0.1700	0.3100	1.5391	1.9614

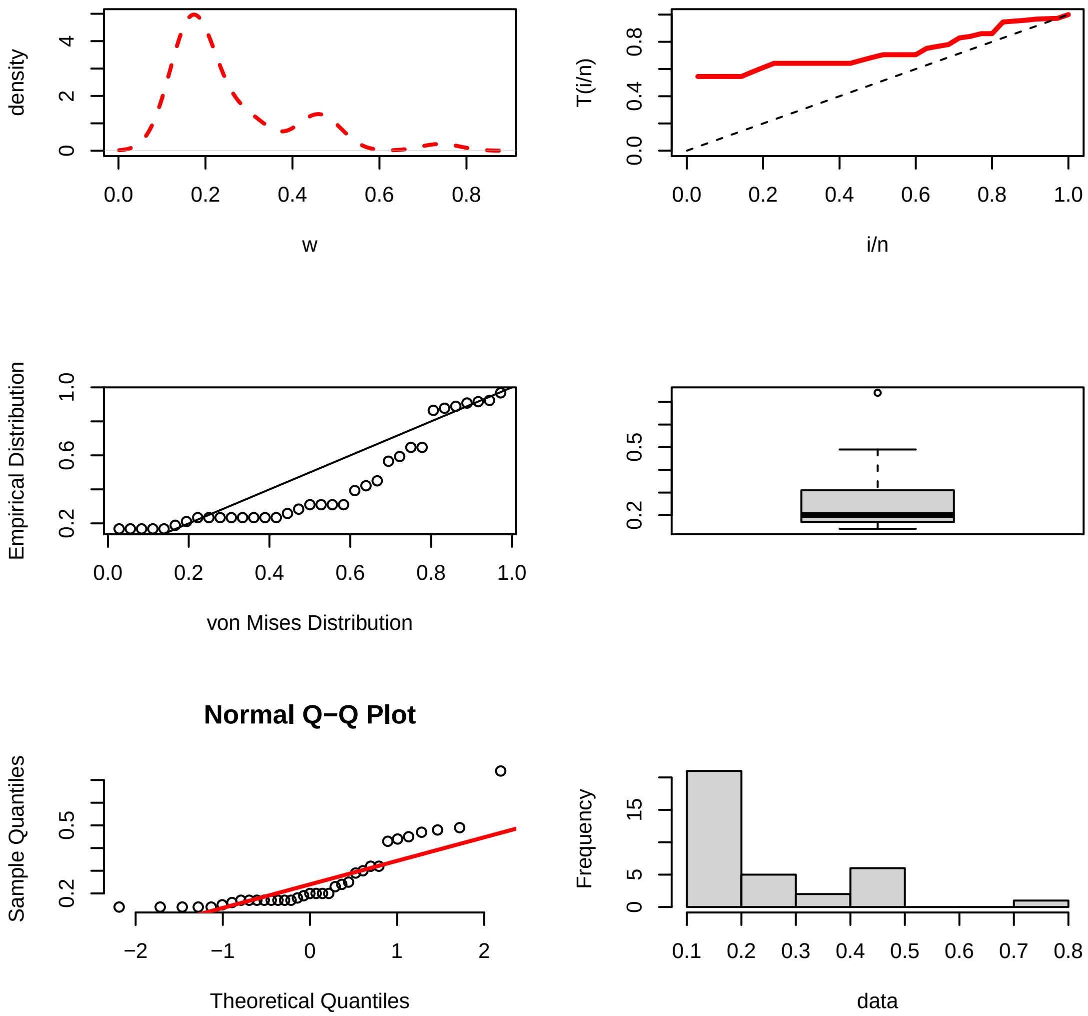


Figure 9. Visual illustrations of the third data set.

Next, to select the best distribution for analyzing the three data sets, we compared the IPME model against several competitor models, notably Then, we evaluate the IPME distribution with some competitor models to choose the most effective distribution to analyze the three data sets, namely, Alpha power transformed cosine moment exponential (APCos-ME), Nadarajah Haghighi (NHD), power Lomax (PLD), moment exponential (ME), Xlindley (XLD) and inverse Weibull (IW) distributions using model selection criteria such as Akaike Information Criterion ($\mathcal{AIC}$), Bayesian Information Criterion ($\mathcal{BIC}$), and Kolmogorov-Smirnov ($\mathcal{KS}$) test. Table (16) presents the results obtained. According to these findings and since the values of ($\mathcal{AIC}$), ($\mathcal{BIC}$), and ($\mathcal{KS}$) are lower than the rest of the values of comparable competitor models, we can infer that our IPME model is an appropriate selection to fit effectively across the three data sets.

Table 16. Estimated parameter values and different statistical measures for proposed fitted models employing the three proposed data sets.

Data	Distribution	$\mathcal{LL}$	$\hat{\delta}$	$\hat{\eta}$	$\mathcal{AIC}$	$\mathcal{BIC}$	$\mathcal{KS}$
I	IPME	-335.416	0.7415	0.0987	674.832	679.694	0.0734
	APCos-ME	-352.290	0.0605	24.022	708.580	713.442	0.1923
	NHD	-341.561	0.7928	0.0685	687.123	689.078	0.1251
	PLD	-341.855	0.7542	0.1981	687.710	692.571	0.1043
	ME	-353.829	10.800		709.659	712.089	0.2204
	XLD	-350.065	0.0856		702.131	704.562	0.1897
	IW	-336.142	9.6596	1.1099	676.284	681.145	0.0819
II	IPME	-28.116	0.6747	1.3886	60.232	64.612	0.1262
	APCos-ME	-45.782	0.0922	0.6243	95.565	99.944	0.2732
	NHD	-30.849	0.7495	2.7772	65.699	70.079	0.1396
	PLD	-31.611	0.9080	2.2028	67.222	71.601	0.1327
	ME	-44.971	0.2959		91.943	94.133	0.2761
	XLD	-31.461	1.8937		64.922	67.111	0.1532
	IW	-28.458	0.1815	1.0195	60.916	65.296	0.1299
III	IPME	33.164	1.9585	11.956	-62.328	-59.217	0.1820
	APCos-ME	24.989	19.796	0.1224	-45.978	-42.867	0.2344
	NHD	18.042	103.58	0.0243	-32.084	-28.974	0.3441
	PLD	23.944	2.0030	12.653	-43.888	-40.778	0.2266
	ME	22.146	0.1284		-42.293	-40.738	0.2974
	XLD	12.592	4.0425		-23.185	-21.629	0.4195
	IW	30.819	0.0343	2.0748	-57.614	-54.503	0.2209

Now, for the first and second data sets, we subsequently utilized an SRS strategy with $dv = 8$. To get estimates using the RSS design, we choose $d = 4$ and $v = 2$ for the following data set, as indicated in Tables (17) and (18). While for the third data set, we subsequently utilized an SRS strategy with $dv = 16$. To get estimates using the RSS design, we choose $d = 4$ and $v = 4$. Table (19) reported the chosen values.
Additionally, Table (20) presents the results of the KS test and the estimated values for the SRS and RSS schemes using the three suggested data sets. According to these findings and using the KS distance as a potential indicator, it is evident that estimates from the RSS scheme have lower KS distances than those from the SRS design in all methods. This suggests that the RSS scheme outperforms the SRS design across the various estimation techniques. For further explanation, Figures (10)-(18) guarantee that the RSS scheme is more effective than the SRS design.

Additionally, we acquired the results of various entropy techniques as previously stated through using various levels of the variable φ, and the MLEs of the IPME parameters for the three data sets. These results are described in Table (21). Consequently, the IPME model can serve as a flexible framework for different data types.

Table 17. RSS design ($d = 4$ and $v = 2$) drawn from the first data set.

Cycle 1	Cycle 2	Cycle 3	Cycle 4
6.9	7.9	24.7	22.8
2.9	8.3	21.4	72.6

Table 18. RSS design ($d = 4$ and $v = 2$) drawn from the second data set.

Cycle 1	Cycle 2	Cycle 3	Cycle 4
0.40	0.09	0.13	1.04
0.11	0.42	0.17	1.39

Table 19. RSS design (for $d = 4$ and $v = 4$) drawn from the third data set.

Cycle 1	Cycle 2	Cycle 3	Cycle 4
0.14	0.15	0.47	0.48
0.14	0.16	0.24	0.45
0.17	0.16	0.17	0.24
0.14	0.44	0.25	0.45

Table 20. The estimate and KS values in SRS and RSS designs employing the three data sets.

Data	Technique	$\hat{\eta}$		$\hat{\delta}$		KS	
		SRS	RSS	SRS	RSS	SRS	RSS
I	MLE	1.1145	0.8719	0.0360	0.0666	0.2039	0.1332
	MPS	0.8377	0.6067	0.0703	0.1289	0.1688	0.0782
	LSE	0.7931	0.6054	0.0822	0.1274	0.1905	0.0809
	WLS	0.8133	0.6131	0.0760	0.1258	0.1719	0.0776
	CV	0.9853	0.7463	0.0507	0.0899	0.1722	0.1008
	AD	0.9503	0.7163	0.0551	0.0973	0.1678	0.0894
II	MLE	0.5555	0.6717	0.9230	1.4829	0.2120	0.1475
	MPS	0.4408	0.6418	0.8370	1.3919	0.2160	0.1304
	LSE	0.4187	0.5660	0.7797	1.2734	0.2069	0.1105
	WLS	0.4337	0.6014	0.8133	1.3198	0.2041	0.1178
	CV	0.5325	0.7087	0.8504	1.5472	0.1793	0.1561
	AD	0.4784	0.6874	0.8356	1.5048	0.1678	0.1501
III	MLE	1.7176	1.6011	8.3415	6.6840	0.2782	0.1853
	MPS	1.7469	1.6451	8.3672	6.7319	0.2197	0.1907
	LSE	1.3105	1.2392	3.9171	1.2734	0.2293	0.2116
	WLS	1.4261	1.4399	5.2816	5.2064	0.2474	0.1801
	CV	1.6505	1.5958	7.7720	6.7145	0.2533	0.1782
	AD	1.3809	1.3496	5.0814	4.5668	0.2254	0.1832

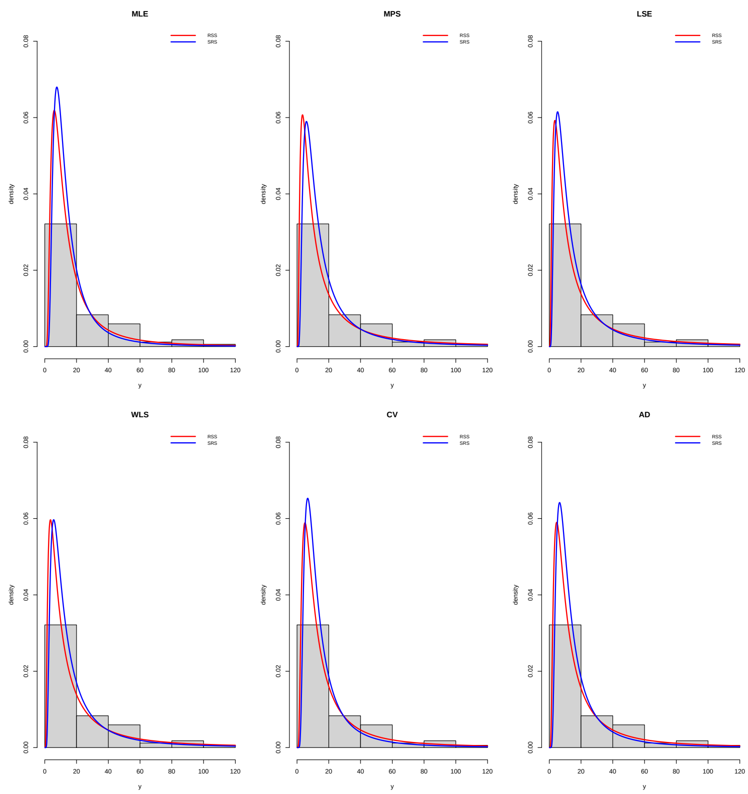


Figure 10. PDF graphs for the first data set according to RSS and SRS schemes.

Table 21. Estimated values of the R_1, R_2, R_3, R_5, R_5, and R_6 based on the three applied data sets at several φ records.

Data Set	Par	φ	R_1	R_2	R_3	R_4	R_5	R_6
		0.5	6.7929	1.7485	69.6687	57.7155	890.4838	-13.0338
I	$\hat{\delta} = 0.7415, \hat{\eta} = 0.0987$	0.75	3.9756	1.7485	8.9941	6.807	8.2888	-0.6834
		1.5	5.4781	1.7485	3.1936	1.8707	2.5169	-1.4946
		2.2	3.3300	1.7485	1.7382	0.818	1.5352	1.0198
		2.5	2.9369	1.7485	1.528	0.6585	1.3805	5.3659
		3.0	2.5051	1.7485	1.3244	0.4967	1.2176	74.4637
		0.5	2.3963	0.5377	5.5863	4.6279	9.9822	-5.8052
II	$\hat{\delta} = 0.6747, \hat{\eta} = 1.3886$	0.75	0.9586	0.5377	1.4313	1.0832	1.1295	-1.3057
		1.5	0.4614	0.5377	0.7034	0.412	0.4277	-0.2179
		2.2	9.6842	0.5377	1.7708	0.8333	1.824	-0.0044
		2.5	9.1108	0.5377	1.5469	0.6667	1.6596	-0.0605
		3.0	8.5373	0.5377	1.3333	0.5	1.4949	-0.1973
		0.5	-51.5206	0.3	-2.4142	-2	-1	1.1608
III	$\hat{\delta} = 1.9585, \hat{\eta} = 11.956$	0.75	161.3124	0.3	-5.2852	-4	-3	1.3527
		1.5	168.0631	0.3	3.4142	2	3	-1.9998
		2.2	104.0178	0.3	1.7708	0.8333	1.8333	190.925
		2.5	94.8685	0.3	1.5469	0.6667	1.6667	3491
		3.0	85.7192	0.3	1.3333	0.5	1.5	70971

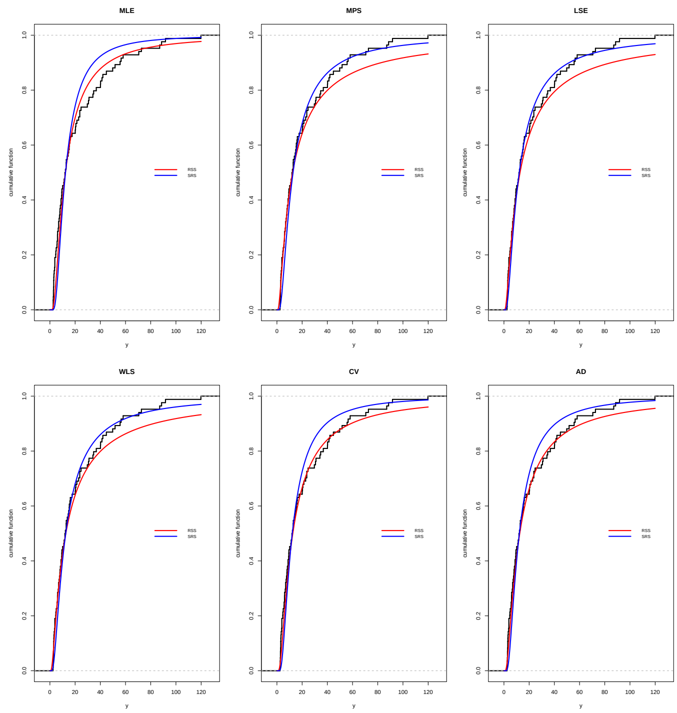


Figure 11. CDF graphs for the first data set according to RSS and SRS schemes.

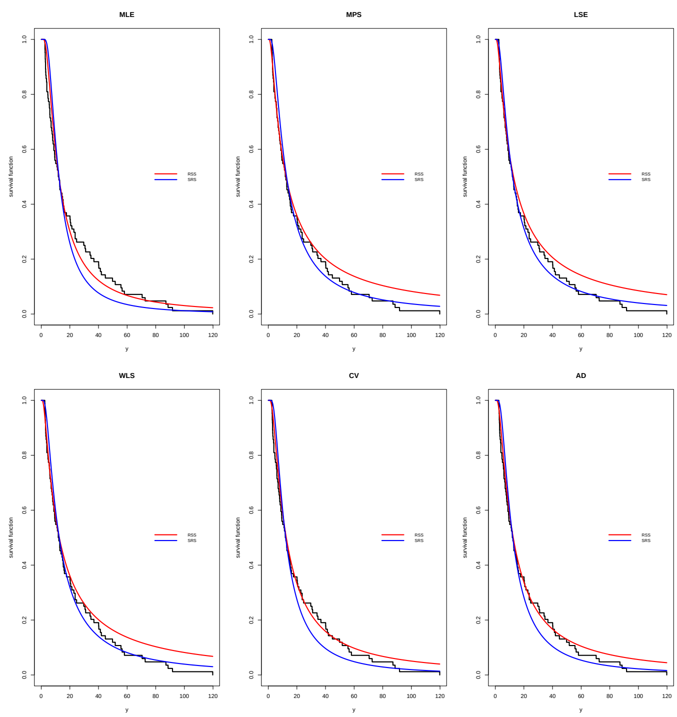


Figure 12. SF graphs for the first data set according to RSS and SRS schemes.

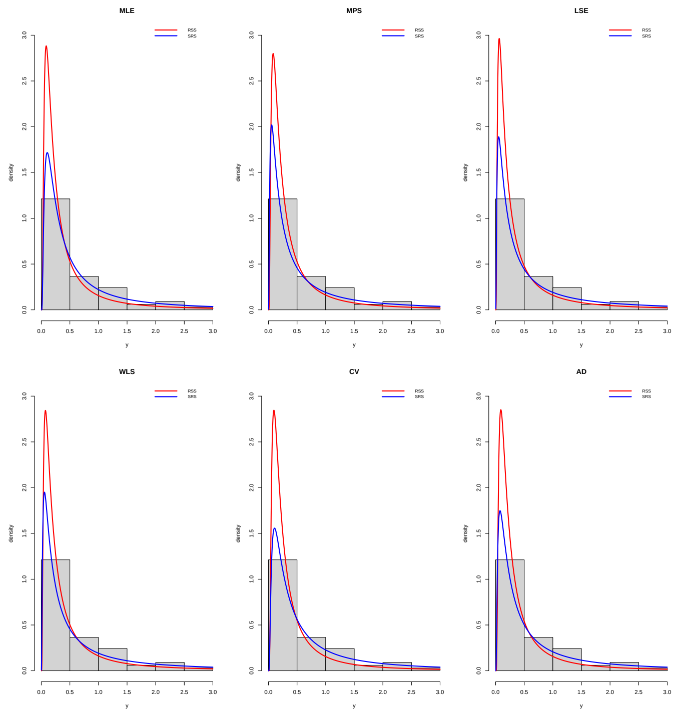


Figure 13. PDF graphs for the second data set according to RSS and SRS schemes.

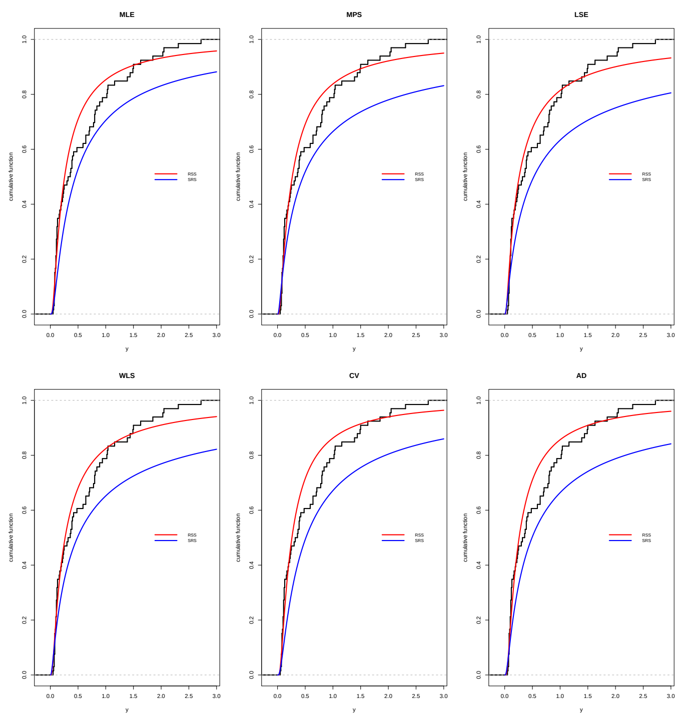


Figure 14. CDF graphs for the second data set according to RSS and SRS schemes.

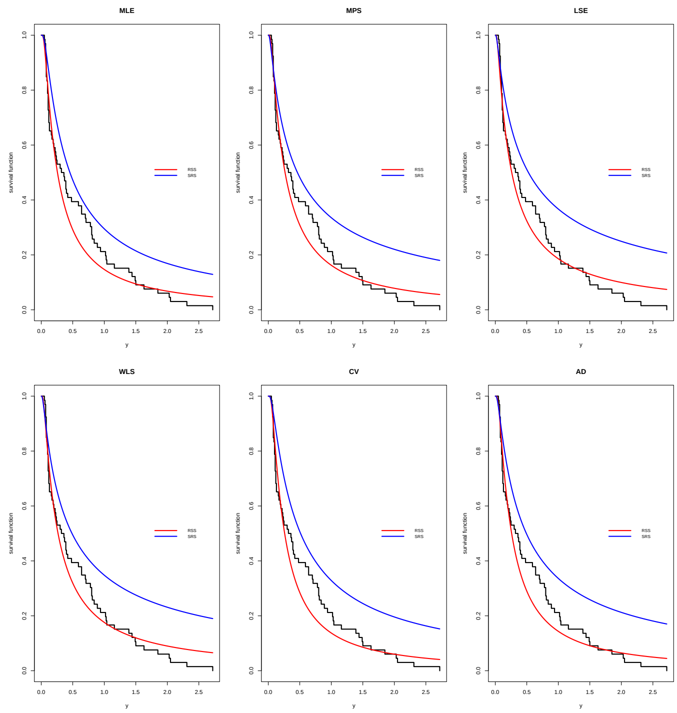


Figure 15. SF graphs for the second data set according to RSS and SRS schemes.

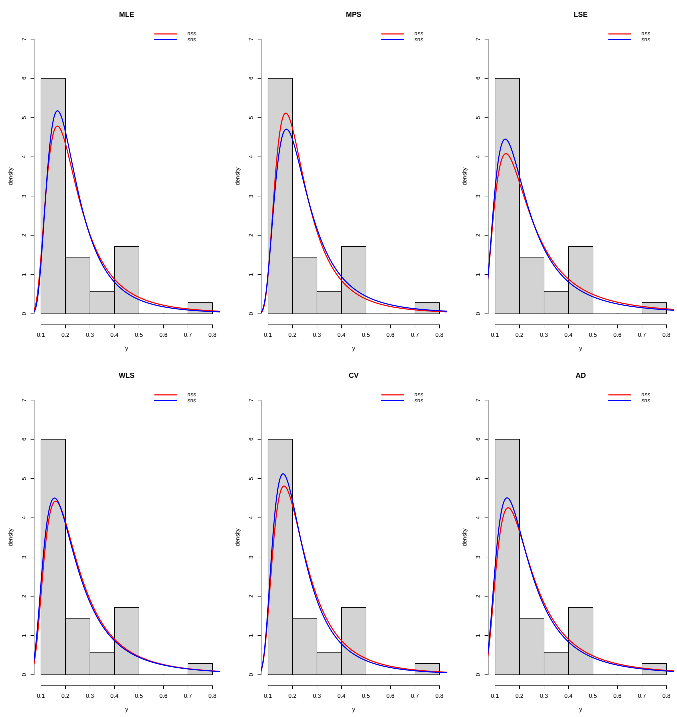


Figure 16. PDF graphs for the third data set according to RSS and SRS schemes.

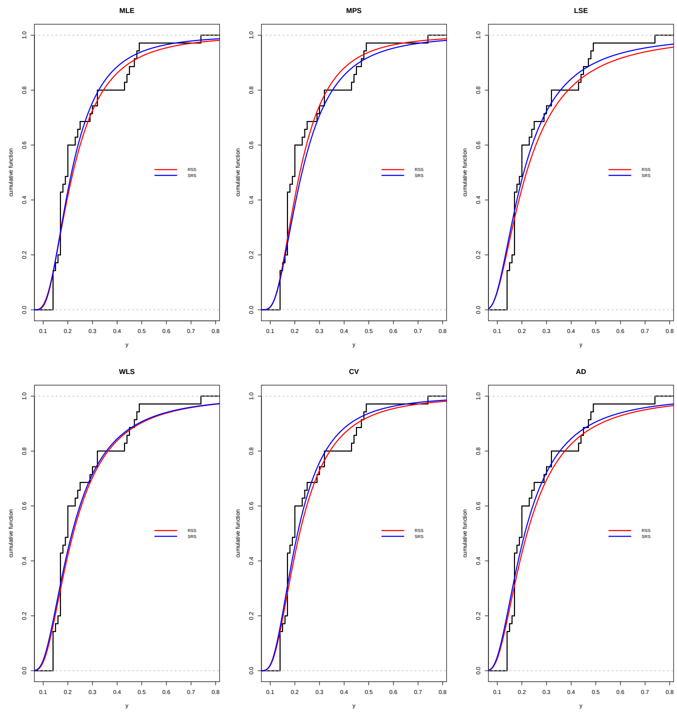


Figure 17. CDF graphs for the third data set according to RSS and SRS schemes.

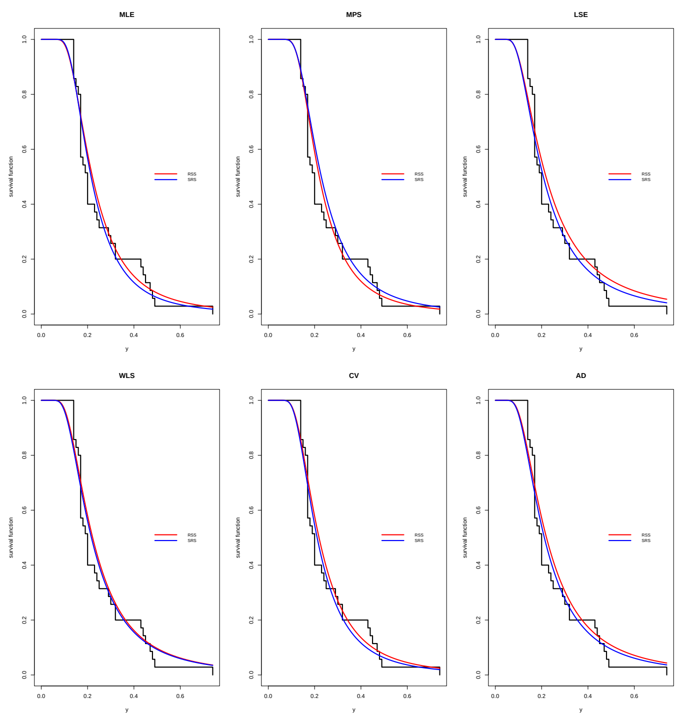


Figure 18. SF graphs for the third data set according to RSS and SRS schemes.

7. Concluding Remarks

The goal of this study is to develop the estimation of the unknown parameters of the IPME model by applying several traditional estimation methods across SRS and RSS designs. The effectiveness of these methodologies for estimation was evaluated through a Monte Carlo simulation experiment. The outcomes consistently showed that the RSS design achieved the lowest mean squared error and outperformed the SRS scheme in all methods. Further, we examine numerous entropy measures and their estimation using the maximum likelihood estimator. We conducted a Monte Carlo simulation analysis using the Mean, MSE, and associated MRE to demonstrate the efficiency of the suggested estimator's entropy information. Three real-world datasets were used to demonstrate the superiority of the RSS over the SRS design based on the KS test. Future research could extend this framework to Bayesian estimation, regression settings, or multivariate generalizations, further broadening its applicability.

Data Availability Statement

The data that supports the findings of this study are available within the article. The data supporting the findings of this study are presented within the article.

Conflicts of Interest

The authors declare no conflict of interest.

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