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Solving fuzzy Riccati differential equations with Trapezoidal fuzzy functions by Homotopy perturbation method

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Abstract This study innovatively combines the homotopy perturbation method (HPM) with trapezoidal fuzzy functions to solve nonlinear Riccati differential equations under uncertainty. By decomposing fuzzy equations into α-cut systems and solving coupled crisp problems via HPM, the approach preserves trapezoidal fuzzy structures while ensuring computational efficiency and precision, The stability behavior of the fuzzy Riccati differential equation was also studied, and good and highly efficient results were obtained in convergence of α values. This convergence was proven theoretically and numerically. Numerical results validate its rapid convergence and accuracy, as for the numerical results, He's polynomials and the accelerated Adomian polynomials were used. The framework extends HPM's applicability to complex uncertainty modeling, offering a robust tool for control theory and engineering systems with trapezoidal fuzzy parameters.

Keywords Trapezoidal fuzzy number (TFN), Fuzzy Quadratic Riccati differential equation (FQRDE), Homotopy perturbation method (HPM), He's polynomials (HePs), Accelerated Adomian polynomials (AAPs), Stability analysis with trapezoidal fuzzy numbers

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1. Introduction

Ordinary and partial differential equations are frequently employed to tackle initial and boundary value problems [1]. Fuzzy ordinary differential equations are an appropriate mathematical model for dynamical systems with ambiguities or uncertainties. When building the systems, it is crucial to account for the frequent presence of fuzzy numbers in measurements and parameters. This necessitates the development of mathematical models and numerical techniques specifically designed to solve general fuzzy linear systems. Fuzzy systems, rooted in concepts extending classical set theory and logic [3, 2]. In 1965, Zadeh introduced fuzzy sets as an example of knowledge and analysis with nonstatistical uncertainty [4].
The homotopy perturbation method (HPM) constructs a homotopy using an embedding parameter $p \in [0, 1]$, treating it as a small parameter, and expressing the solution as an infinite series converging rapidly to an accurate solution by Prof. Dr. Ji-Huan He in 1999 [5, 6, 7, 8]. Many mathematicians and engineers have used HPM to solve various functional equations [9]. Consequently, many techniques have been devised to solve FIVPs, comprising exact-analytical, semi-analytical, and fuzzy numerical solutions [10]. The variety of possible solution techniques is necessary because an exact solution cannot always be obtained, so a numerical or semi-analytical approach should be used. For every numerical or semi-analytical solution technique for fuzzy systems. For simulating a variety of physical systems, including spring-mass systems, resistor-capacitor-inductor circuits, beam bending, and chemical reactions, nonlinear differential equations are essential tools. These are just a few of the engineering and scientific

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applications of the well-known nonlinear differential equation Riccati [11].
Recently, researchers have become increasingly interested in this type of nonlinear differential equation due to its importance in many applications. File and Aga applied the classical Runge-Kutta fourth-order (RK4) method to solve nonlinear quadratic Riccati differential equations with wide applications [11]. Sabr et al. used the fuzzy homotopy analysis method (HAM) to find the fuzzy series solution; this study offers a precise Analytical approximation of the first-order fuzzy Riccati differential equation [12]. Oqielat et al. proposed the Laplace residual powers series (LRPS), which offers a novel approach to solving fuzzy quadratic Riccati differential equations by combining the Laplace transform with power series techniques [13].
This study aims to establish an effective mathematical model for solving fuzzy Riccati differential equations by applying HPM with α-cut form to obtain exact approximate solutions. Moreover, it endeavors to study the stability of the solutions in the context of Lipschitz condition, which would help illustrate the stability and convergence of the sequences of approximations, the comparative analysis of the results obtained with He's polynomials (HePs), and the accelerated Adomian's polynomials (AAPs) while at the same time analyzing the efficiency and accuracy of the procedure through numerical examples.

2. Basic Concepts

This paper builds on these current methods by combining fuzzy set theory with a multistage fuzzy HPM [14].Let A be a crisp subset of the universe X. It is distinguished by a function $\mu_A : X \to \{0, 1\}$. An interval, A, over X is defined by $\widetilde{A} = [\underline{s}, \overline{s}]$ where $\underline{s}, \overline{s} \in X$ and $\underline{s} \leq \overline{s}$. The interval valued fuzzy set $\widetilde{A}$, over X is defined by $\mu_{\widetilde{A}}(s) = [\underline{u}(s), \overline{u}(s)]$, with a lower membership function $\mu_{\underline{A}}(s) = \underline{u}(s)\ \forall\ s$, and an upper membership function $\mu_{\overline{A}}(s) = \overline{u}(s)\ \forall\ s$.

Definition 1 Fuzzy set [15]. Let X be the universal set, and $\widetilde{A}$ in X be a set of ordered pairs such that $\widetilde{A} = \left\{\left(s, \mu_{\widetilde{A}}(s)\right) : s \in X, \mu_{\widetilde{A}}(s) \in [0, 1]\right\}$, where $\mu_{\widetilde{A}}(s)$ is the membership function.

Definition 2 [3]. Four real numbers make up the trapezoidal fuzzy number A, where $a < b < c < d$. The trapezoid's base is $[a, d]$, and its vertices are located at $s = b$ and $s = c$. The following is the membership function:

$$\mu_A(s) = \begin{cases} \dfrac{s-a}{b-a}, & a \leq s \leq b \\ 1, & b \leq s \leq c \\ \dfrac{d-s}{d-c}, & c \leq s \leq d \\ 0, & o.w. \end{cases}$$

The class of all fuzzy subsets of $\mathbb{R}$ (i.e., $A : \mathbb{R} \to [0, 1]$) that meet the following properties will be denoted by $\mathbb{R}_F$:

(i) $A(s_0) = 1, \forall\ A \in \mathbb{R}_F, \exists\ s_0 \in \mathbb{R}$, A has normalty,
(ii) $A(ts + (1-t)r) \geq min\{A(s), A(r)\},\ \forall\ t \in [0, 1]$, $s, r \in \mathbb{R}$, A has convexty,$\forall\ A \in \mathbb{R}_F$,
(iii) A is upper semi-continuous on $\mathbb{R}, \forall\ A \in \mathbb{R}_F$,
(iv) $\overline{A}$ is the closure of A if $\left\{\overline{s \in \mathbb{R} : A(s) > 0}\right)$ is compact, which is clearly $\mathbb{R} \subset \mathbb{R}_F$. The α-level set, $s \in \mathbb{R}$,is defined;

$$[A]_\alpha = \{s \backslash A\{s\} \geq \alpha\},\ \alpha \in [0, 1],$$

and $[A]_0 = \{s \backslash A\{s\} \geq 0\}$ is compact, and A satisfies the following the requirments [16]:

1. The upper functionis of $\widetilde{A}$ is half–continuous ,
2. $\widetilde{A}(s) = 0$ outside the boundaries of the interva $[a, d]$.

Definition 3 [3, 16]. Suppose that $A : [0,1] \to \mathbb{R}$, the parametric form of the trapezoidal fuzzy number $A = (a, b, c, d)$

$$\mu_{\widetilde{A}} = [\underline{u}, \overline{u}] = [a + (b-a)\,\alpha, d + (c-d)\,\alpha]\,,$$

satisfies the following requirements:

1. The function $\underline{u}\,(\alpha)$ is bounded, left-continuous, and nondecreasing over $(0, 1]$,
2. The function $\overline{u}\,(\alpha)$is bounded, nonincreasing, and right-continuous over $(0, 1]$,
3. $\underline{u}\,(\alpha) \leq \overline{u}\,(\alpha)\ \forall\, \alpha \in (0, 1]$.

Theorem 1 [17]. If $f(t, x)$ is the ODE and $x' = f(t, x)$, it is uniformly Lipschitz in x. If there is a continuous function $V\,(t, x)$ specified for$x \geq 0$ and $|x| < \delta$, then the zero solution of these ODEs is essentially Lipschitz stable.

1. $|x| \leq V\,(t, x) \leq L\,|x|$ *,for some constant* L.
2. $|V\,(t, x) - V\,(t, y)| \leq |x - y|\,, \forall\, t \geq 0,\, x, y \in \mathbb{R}^n$, with $|x| < \delta$, $|y| < \delta$.
3. $V^{'}\,(t, x) \leq 0$.

Theorem 2 Consider the FQRDE

$$\begin{gathered}\widetilde{u}'\,(s,\alpha) = \widetilde{g}\,(s,\alpha) + B\,(s,\alpha)\,\widetilde{u}\,(s,\alpha) + C\,(s,\alpha)\,\widetilde{u}^2\,(s,\alpha)\,,\\ \widetilde{u}\,(0,\alpha) = \widetilde{u}_0, \quad s_0 \leq s \leq T, \quad \alpha \in [0,1]\end{gathered} \tag{1}$$

where $B\,(s,\alpha)$ and $C\,(s,\alpha) \neq 0$, and $\widetilde{g}\,(s,\alpha)$ are continuous $[s_0, T]$ and $\widetilde{u}_0 = [\underline{u}_0, \overline{u}_0]$ are arbitrary constants for $\widetilde{u}\,(s,\alpha) = [\underline{u}\,(s,\alpha)\,, \overline{u}\,(s,\alpha)]$, which is an unknown function, such that:

1. $\forall\, r > 0$ and $|\widetilde{u}(s,\alpha)| \leq r$, then the FQRDE is locally Lipschitz.
2. If $\widetilde{g}\,(s,\alpha) = 0$, then the FQRDE has exponential (asymptotic) stability uniformly.

Proof
To prove 1, we assume that $\widetilde{u}\,(s,\alpha)$ and $\widetilde{z}\,(s,\alpha)$ are two solutions of FQRDE and $|\widetilde{u}\,(s,\alpha)\,| \leq r \textit{ and } |\widetilde{z}(s,\alpha)| \leq r$, $\forall\, s \in [s_0, T]$. By theorem 1

$$\begin{aligned} d(\widetilde{u}(s,\alpha) - \widetilde{z}(s,\alpha)) &= |f(s,\alpha,\widetilde{u}) - f(s,\alpha,\widetilde{z})|\,,\\ &= \left|B(s,\alpha)(\widetilde{u}\,(s,\alpha) - \widetilde{z}(s,\alpha)) + C(s,\alpha)\left(\widetilde{u}^2\,(s,\alpha) - \widetilde{z}^2(s,\alpha)\right)\right|,\\ &= |\,(\widetilde{u}\,(s,\alpha) - \widetilde{z}\,(s,\alpha))\,[B(s,\alpha) + C(s,\alpha)\,(\widetilde{u}\,(s,\alpha) + \widetilde{z}\,(s,\alpha))]\,|, \end{aligned} \tag{2}$$

so that,

$$|f\,(s,\alpha,\widetilde{u}) - f\,(s,\alpha,\widetilde{z})| \leq |\widetilde{u}\,(s,\alpha) - \widetilde{z}\,(s,\alpha)|\,[|B(s,\alpha)| + |C(s,\alpha)|\,|\widetilde{u}\,(s,\alpha) + \widetilde{z}\,(s,\alpha)|]\,. \tag{3}$$

Since $|\widetilde{u}\,(s,\alpha) - \widetilde{z}\,(s,\alpha)|$ satisfies the differential inequality, we have $|\widetilde{u}\,(s,\alpha) + \widetilde{z}\,(s,\alpha)| \leq 2r$. From Eq. (3), we have

$$\begin{aligned} |f(s,\alpha,\widetilde{u}) - f(s,\alpha,\widetilde{z})| &\leq |\widetilde{u}\,(s,\alpha) - \widetilde{z}\,(s,\alpha)|\left[\sup_{s\in[s_0,T]}|B(s,\alpha)| + 2r\sup_{s\in[s_0,T]}|C(s,\alpha)|\right],\\ &\leq L_r\,|\widetilde{u}\,(s,\alpha) - \widetilde{z}\,(s,\alpha)|\,, \end{aligned} \tag{4}$$

where

$$L_r = \sup_{s\in[s_0,T]}|B(s,\alpha)| + 2r\sup_{s\in[s_0,T]}|C(s,\alpha)|\,.$$

From Eqs. (2) and (4), we conclude that

$$\frac{d}{dt}\left(\left|\widetilde{u}\left(s,\alpha\right)-\widetilde{z}\left(s,\alpha\right)\right|\right)\leq L_r\left|\widetilde{u}\left(s,\alpha\right)-\widetilde{z}\left(s,\alpha\right)\right|.$$

By Grönwall's Inequality

$$\left|\widetilde{u}\left(s,\alpha\right)-\widetilde{z}\left(s,\alpha\right)\right|\leq\left|\widetilde{u}\left(s_0,\alpha\right)-\widetilde{z}\left(s_0,\alpha\right)\right|e^{L_r(s-s_0)}.$$

This verifiesthe existenceof a uniform Lipschitz constant on a bounded set.

To prove 2, we assume that $B\left(s,\alpha\right)<-\beta<0,\ \forall\ s\geq s_0$, $C\left(s,\alpha\right)$ is bounded on $[s_0,\infty]$ and $\widetilde{g}\left(s,\alpha\right)=0$. Let $V\left(\widetilde{u}\right)=\widetilde{u}^2$: we have

$$\begin{aligned}\dot{V}\left(\widetilde{u}\right) &= 2\widetilde{u}\widetilde{u}',\\ &= 2\widetilde{u}\left[B\left(s,\alpha\right)\widetilde{u}+C\left(s,\alpha\right)\widetilde{u}^2\right],\\ &= 2B\left(s,\alpha\right)\widetilde{u}^2+2C\left(s,\alpha\right)\widetilde{u}^3.\end{aligned}$$

For all $\varepsilon>0$, $\exists\ \delta>0$ such that $\left|\widetilde{u}\left(s,\alpha\right)\right|<\delta$. Since $C\left(s,\alpha\right)$ is bounded then there exist a constant $M_c>0$, then $\left|C\left(s,\alpha\right)\right|\leq M_c$ and $\widetilde{u}^3$ has $O\left(\widetilde{u}^2\right)$ as $\widetilde{u}\to 0$, we get

$$\left|2C\left(s,\alpha\right)\widetilde{u}^3\right|\leq 2M_c\delta\widetilde{u}^2,\ \forall\ s\geq s_{0,}\ \alpha\in[0,1]$$

$$\begin{aligned}\dot{V}\left(\widetilde{u}\right) &\leq 2B\left(s,\alpha\right)\widetilde{u}^2+2M_c\delta\widetilde{u}^2,\\ &\leq 2\beta\widetilde{u}^2+2\varepsilon\widetilde{u}^2,\ \varepsilon=M_c\delta\\ &\leq -\gamma V\left(\widetilde{u}\right),\ \gamma=2\beta-2\varepsilon.\end{aligned}\tag{5}$$

Integrating both sides of the Eq. (5) from s_0 to s, we get

$$V\left(\widetilde{u}\left(s,\alpha\right)\right)\leq V\left(\widetilde{u}\left(s_0,\alpha\right)\right)e^{-\gamma(s-s_0)},\ s\geq s_0$$

$$\left|\widetilde{u}\left(s,\alpha\right)\right|\leq\widetilde{u}\left(s_0,\alpha\right)e^{-\gamma(s-s_0)},\ s\geq s_0\tag{6}$$

Thus, the equilibrium point $\widetilde{u}^*\left(s,\alpha\right)=0$ is stable; this means that FQRDE has asymptotic exponential stability uniformly. ∎

3. The fundamental concept of HPM

The following non-linear differential equation is assumed to illustrate the basic idea of the HPM. [5, 6, 7, 8].

$$G\left(u\right)-f\left(s\right)=0,\ s\in\Sigma\tag{7}$$

with the condition

$$E\left(u,\frac{\partial u}{\partial s}\right)=0,\ s\in\Lambda\tag{8}$$

where G, a general differential operator, is composed of linear L and nonlinear terms related to Eq. (7), E is a boundary operator; $f(s)$ is a known analytic function and Λ is the boundary of the domain Σ. Eq. (7) can be rewritten as

$$L(u)+N(u)-f(s)=0.\tag{9}$$

Using the homotopy method, we create a homotopy $v(r, p) : \Sigma \times [0, 1] \to \mathbb{R}$ that satisfies

$$H(v, p) = (1 - p)[L(v) - L(v_0)] + p[G(v) - f(s)] = 0, \tag{10}$$

or

$$H(v, p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(s)] = 0, \tag{11}$$

where v_0 is an initial approximation, the initial approximation is typically chosen to satisfy the prescribed initial or boundary conditions and to represent the simplest approximation of the exact solution of Eq. (7) that fulfills the boundary conditions

$$H(v, 0) = L(v) - L(v_0) = 0, \tag{12}$$

and

$$H(v, 1) = G(v) - f(r) = 0. \tag{13}$$

The transformation of p from zero to unity is identical to the transformation of $u(s, p)$ from $u_0(s)$ to $u(s)$. This is referred to as ”deformation” in topology, and ”homptopic” refers to $L(u) - L(v_0)$ and $G(u) - f(s)$.

Here, the authors will first use the embedding parameter $0 \leq p \leq 1$ as a “small parameter” and assume that the solution of Eqs. (10) and (11) can be written as a power series in p :

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + \cdots , \tag{14}$$

and the nonlinear term Nu can be decomposed as

$$Nu = \sum_{n=0}^{\infty} p^n H_n(u), \tag{15}$$

where the $H_n(u)$ are He’s polynomials of $u_0, u_1, \ldots, u_n$ and are calculated by the definitional formula [18]

$$H_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0, 1, 2, \ldots .$$

Setting $p = 1$ in (14), the approximate solution of Eq. (7) is

$$u = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n y_n = u_0 + u_1 + u_2 + \cdots . \tag{16}$$

The series solution (16) is shown to converge in [**?**].

4. Fuzzy HPM applied to RDE

Applying the fuzzy HPM, Eq. (1) can be written as

$$\widetilde{u}'(s, \alpha) - \widetilde{v}_0(s, \alpha) + p\left[\widetilde{v}_0(s, \alpha) + B\widetilde{u}(s, \alpha) + C\widetilde{u}^2(s, \alpha) - \widetilde{g}\right] = 0, \tag{17}$$

Substituting the Eqs. (14) and (15) into (17), we obtain

$$\sum_{n=0}^{\infty} p^n \widetilde{u}'_n(s, \alpha) - \widetilde{v}_0(s, \alpha) + p\left[\widetilde{v}_0(s, \alpha) + B\sum_{n=0}^{\infty} p^n \widetilde{u}_n(s, \alpha) + C\sum_{n=0}^{\infty} p^n \widetilde{H}_n(u) - \widetilde{g}\right] = 0, \tag{18}$$

and equating the terms in the same power of p in Eq. (18), we have a fuzzy system IVPs of first order using HePs

$$\begin{array}{lll} p^0: & \widetilde{u}_0'(s,\alpha)-\widetilde{v}_0(s,\alpha)=0, & \widetilde{u}_0(0,\alpha)=\widetilde{u}_0 \\ p^1: & \widetilde{u}_1'(s,\alpha)+\widetilde{v}_0(s,\alpha)+B\widetilde{u}_0(s,\alpha)+C\widetilde{H}_0(u)-\widetilde{g}=0, & \widetilde{u}_1(0,\alpha)=0 \\ \vdots & & \\ p^n: & \widetilde{u}_n'(s,\alpha)+B\widetilde{u}_{n-1}(s,\alpha)+C\widetilde{H}_{n-1}(u)=0, \ \ n=2,3,\ldots & \widetilde{u}_n(0,\alpha)=0, \end{array} \tag{19}$$

where the non-linear terms $\widetilde{u}^2(s,\alpha)$ defined by the series

$$\widetilde{u}^2(s,\alpha)=\sum_{n=0}^{\infty}p^n\widetilde{H}_n(u),$$

the corresponding He's polynomials $\widetilde{H}_n=\left[\underline{H}_n,\overline{H}_n\right]$ [18] are

$$\underline{H}_n=\sum_{i=0}^{n}\underline{u}_i\underline{u}_{n-i}, \quad \overline{H}_n=\sum_{i=0}^{n}\overline{u}_i\overline{u}_{n-i}, \quad n\geq i, \quad n=0,1,2,\ldots$$

Now, in the fuzzy system (19), replace HePs by the AAPs [19]

$$\begin{array}{lll} p^0: & \widetilde{u}_0'(s,\alpha)-\widetilde{v}_0(s,\alpha)=0, & \widetilde{u}_0(0,\alpha)=\widetilde{u}_0 \\ p^1: & \widetilde{u}_1'(s,\alpha)+\widetilde{v}_0(s,\alpha)+B\widetilde{u}_0(s,\alpha)+C\widetilde{A}_0(u)-\widetilde{g}=0, & \widetilde{u}_1(0,\alpha)=0 \\ \vdots & & \\ p^n: & \widetilde{u}_n'(s,\alpha)+B\widetilde{u}_{n-1}(s,\alpha)+C\widetilde{A}_{n-1}(u)=0, \ \ n=2,3,\ldots & \widetilde{u}_n(0,\alpha)=0, \end{array} \tag{20}$$

where the non-linear terms $\widetilde{u}^2(s,\alpha)$ defined by the series

$$\widetilde{u}^2(s,\alpha)=\sum_{n=0}^{\infty}p^n\widetilde{A}_n(u),$$

and the accelerated Adomian polynomials $\widetilde{A}_n=\left[\underline{A}_n,\overline{A}_n\right]$ [19] are

$$\begin{aligned} &\underline{A}_0=\underline{u}_0^2, && \overline{A}_0=\overline{u}_0^2 \\ &\underline{A}_n=\underline{u}_n^2+2\underline{u}_n\sum_{i=0}^{n-1}\underline{u}_i, && \overline{A}_n=\overline{u}_n^2+2\overline{u}_n\sum_{i=0}^{n-1}\overline{u}_i, \quad n=1,2,\ldots \end{aligned}$$

5. Applications

We show the remarkable accuracy and precision of the outcomes produced by this approach in this section. We compare the maximum absolute error (MAE) by presenting the solutions of three different examples where the exact solution given with the fuzzy approximate solution by HPM is defined as follows:

$$\begin{aligned} MAE_L&=\left\|\underline{u}_{Exact}(s,\alpha)-\underline{u}(s,\alpha)\right\|_\infty, \\ MAE_U&=\left\|\overline{u}_{Exact}(s,\alpha)-\overline{u}(s,\alpha)\right\|_\infty, \end{aligned}$$

where

$$\begin{array}{ll} \left(\underline{u}_{Exact}(s,\alpha),\overline{u}_{Exact}(s,\alpha)\right): & \text{Exact solution Lower and Upper,} \\ \left(\underline{u}_{Appro}(s,\alpha),\overline{u}_{Appro}(s,\alpha)\right): & \text{Approximate solution Lower and Upper.} \end{array}$$

The computations associated with the problems were performed using a Maple 24 package with a precision of 20 digits.

Example 1. Consider the following FQRDE:

$$\widetilde{u}'(s,\alpha) - 8s\widetilde{u}(s,\alpha) - \widetilde{u}^2(s,\alpha) = \widetilde{g}(s,\alpha), \quad 0 \le s \le 1 \tag{21}$$

with the initial conditions (ICs)

$$\widetilde{u}(0,\alpha) = [0.85 + 0.10\alpha, 1.15 - 0.1\alpha], \tag{22}$$

where $\widetilde{g}(s,\alpha) = \left[\underline{g}(s,\alpha), \overline{g}(s,\alpha)\right]$ are chosen such that the exact solutions will be

$$\widetilde{u}_{Exact}(s,\alpha) = [0.85 + 0.10\alpha - (3.90 + 0.05\alpha)s, 1.15 - 0.10\alpha - (4.10 - 0.05\alpha)s]. \tag{23}$$

In Tables 1 and 2 comparison the MAE between the exact solutions given $[\underline{u}_{Exact}(s,\alpha), \overline{u}_{Exact}(s,\alpha)]$ (23) and the fuzzy approximate solutions by HPM $[\underline{u}(s,\alpha), \overline{u}(s,\alpha)]$ of the Eqs. (21) with (22) using the Eqs .(19) and (20) for $0 \le \alpha \le 1$ at $s = 0.1$ and $s = 0.2$. Table 3 represents the CPU time in seconds between the MAE_L and MAE_U within the interval $0 \le \alpha \le 1$ at $s = 0.1$ and $s = 0.2$.

Table 1. Comparison MAE_L for for example 1.

	HePs		AAPs	
n	$s=0.1$	$s=0.2$	$s=0.1$	$s=0.2$
2	4.2857×10^{-2}	1.9276×10^{-1}	4.2857×10^{-2}	1.9276×10^{-1}
3	2.1714×10^{-3}	8.1177×10^{-3}	2.6755×10^{-3}	2.4629×10^{-2}
4	9.1426×10^{-4}	1.4391×10^{-2}	1.2460×10^{-4}	2.2454×10^{-3}
5	5.4219×10^{-5}	2.8842×10^{-3}	4.6747×10^{-6}	1.6534×10^{-4}
6	1.1861×10^{-5}	5.0633×10^{-4}	1.4681×10^{-7}	1.0258×10^{-5}
7	2.0417×10^{-6}	2.9828×10^{-4}	3.9632×10^{-9}	5.4940×10^{-7}
8	6.9274×10^{-8}	3.4734×10^{-5}	9.3789×10^{-11}	2.5864×10^{-8}

Table 2. Comparison MAE_U for example 1.

	HePs		AAPs	
n	$s=0.1$	$s=0.2$	$s=0.1$	$s=0.2$
2	5.2013×10^{-2}	2.2937×10^{-1}	5.2013×10^{-2}	2.2937×10^{-1}
3	1.2749×10^{-3}	6.1948×10^{-3}	3.8798×10^{-3}	3.4802×10^{-2}
4	1.0627×10^{-3}	1.5823×10^{-2}	2.1569×10^{-4}	3.7670×10^{-3}
5	1.0122×10^{-4}	4.4274×10^{-3}	9.6534×10^{-6}	3.2870×10^{-4}
6	1.1197×10^{-5}	3.0237×10^{-4}	3.6159×10^{-7}	2.4158×10^{-5}
7	3.0306×10^{-6}	3.7874×10^{-4}	1.1640×10^{-8}	1.5326×10^{-6}
8	1.3138×10^{-7}	8.2515×10^{-5}	3.2849×10^{-10}	8.5466×10^{-8}

where n represents the number of iterations.

Table 3. CPU time in seconds for example 1.

	HePs		AAPs	
n	$s=0.1$	$s=0.2$	$s=0.1$	$s=0.2$
2	0.484	0.406	0.485	0.578
3	0.641	0.563	0.922	1.000
4	0.938	1.015	2.171	2.297
5	1.109	1.062	6.391	6.094
6	1.719	1.625	28.157	28.218
7	2.219	2.125	216.828	221.281
8	2.672	2.844	3293.515	5194.813

Figures 1 and 2 below display the absolute error between the exact solutions given $[\underline{u}_{Exact}(s,\alpha), \overline{u}_{Exact}(s,\alpha)]$ (23) and the fuzzy approximate solutions by seventh-order HPM $[\underline{u}(s,\alpha), \overline{u}(s,\alpha)]$ of the Eqs. (21) with (22) for $0 \le \alpha \le 1$ at $s = 0.1$. In Figure 3 illustrate the exact solutions $(\widetilde{u}_{Exact}(s,\alpha))$, and the fuzzy approximate solutions by seventh-order HPM $(\widetilde{u}(s,\alpha))$ in the form of TFN for any $0 \le \alpha \le 1$ at $s = 0.1$. Figure 4 the stability analysis with TFN along $0 \le s \le 1$ at $\alpha = 0, 0.5, 1.0$.

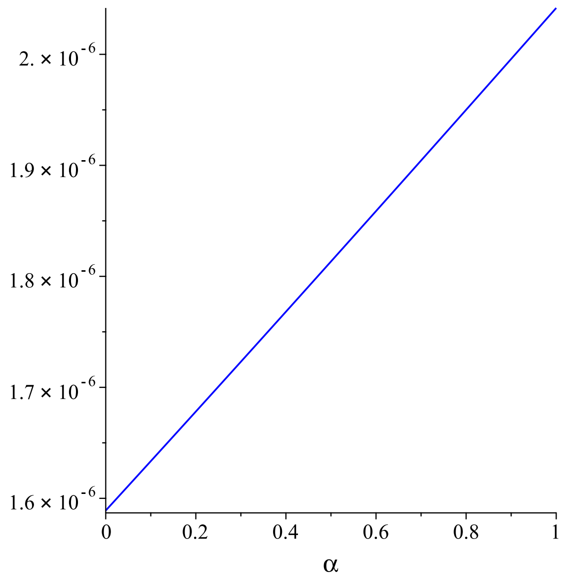

Figure 1. Graph the lower absolute error for example 1.

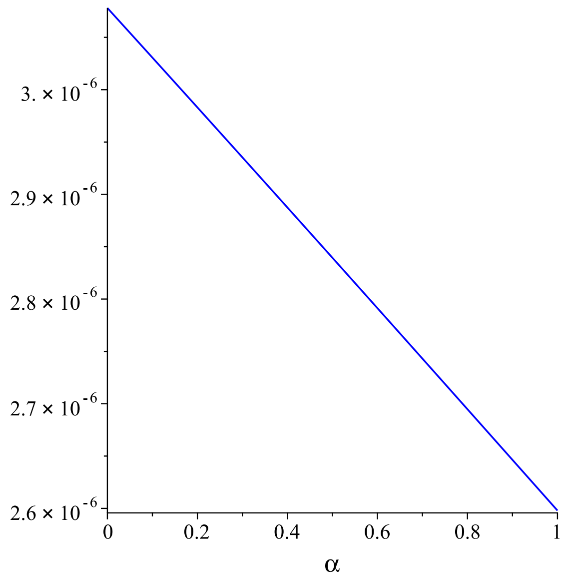

Figure 2. Graph the upper absolute error for example 1.

Example 2. Consider the following FQRDE:

$$\widetilde{u}'(s,\alpha) - 2e^{2s}\widetilde{u}(s,\alpha) + e^{s}\widetilde{u}^2(s,\alpha) = \widetilde{g}(s,\alpha), \quad 0 \le s \le 1 \tag{24}$$

with the ICs

$$\widetilde{u}(0,\alpha) = [0.80 + 0.15\alpha, 1.20 - 0.15\alpha], \tag{25}$$

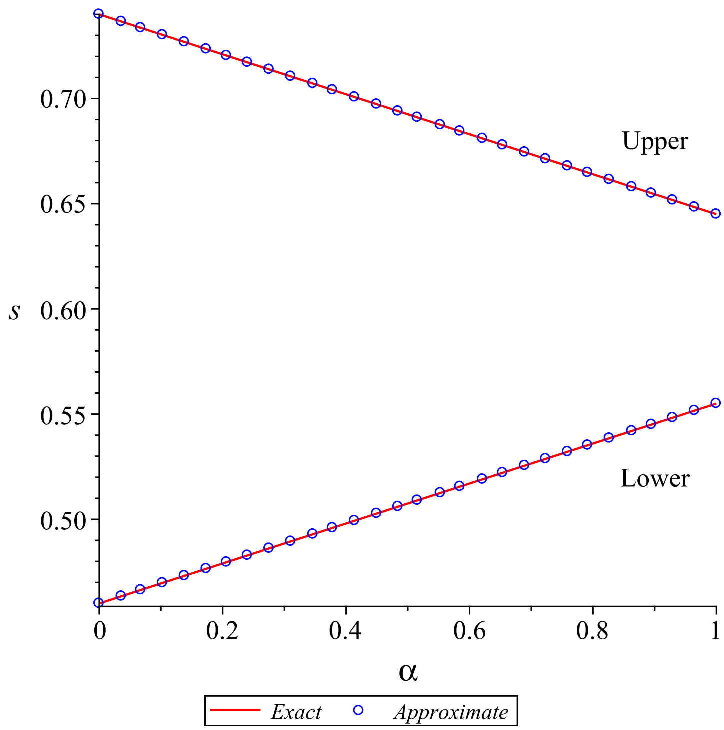

Figure 3. Graph of $\tilde{u}_{Exact}\left(s,\alpha\right)$ and $\tilde{u}\left(s,\alpha\right)$ $(n=7)$ at $s=0.1$ for example 1.

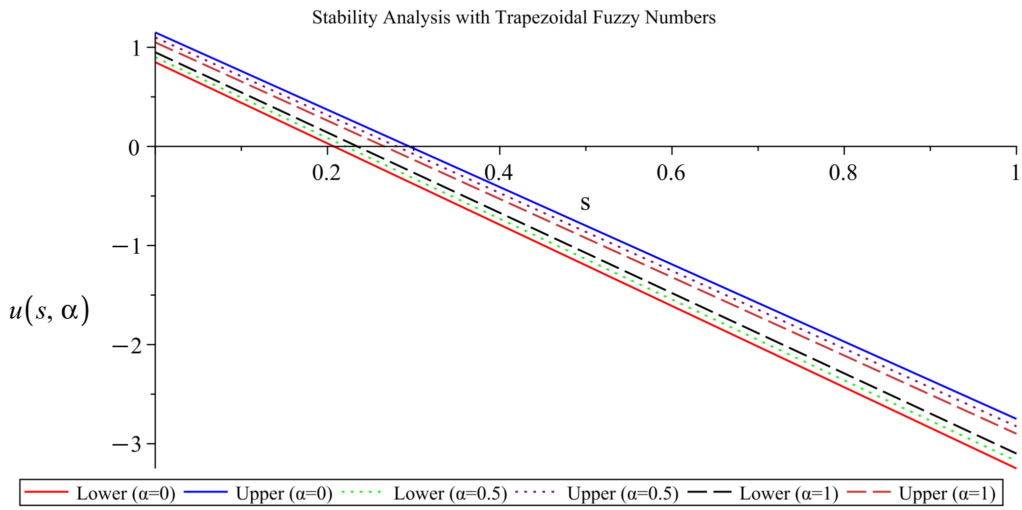

Figure 4. Stability analysis with trapezoidal fuzzy number for example 1.

where $\widetilde{g}\left(s,\alpha\right)=\left[\underline{g}\left(s,\alpha\right),\overline{g}\left(s,\alpha\right)\right]$ are chosen such that the exact solutions will be

$$\widetilde{u}_{Exact}\left(s,\alpha\right)=\left[\left(0.80+0.15\alpha\right)e^{s},\left(1.20-0.15\alpha\right)e^{s}\right]. \tag{26}$$

Tables 4 and 5 comparison the MAE between the exact solutions given $\left[\underline{u}_{Exact}\left(s,\alpha\right),\overline{u}_{Exact}\left(s,\alpha\right)\right]$ (26) with the fuzzy approximate solutions by HPM $\left[\underline{u}\left(s,\alpha\right),\overline{u}\left(s,\alpha\right)\right]$ of the Eqs. (24) with (25) using the Eqs. (19) and (20) for $0 \leq \alpha \leq 1$ at $s = 0.1$ and $s = 0.2$. Table 6 represents the CPU time in seconds between the MAE_L and MAE_U for $0 \leq \alpha \leq 1$ at $s = 0.1$ and $s = 0.2$.

Table 4. Comparison MAE_L for for example 2.

	HePs		AAPs	
n	$s = 0.1$	$s = 0.2$	$s = 0.1$	$s = 0.2$
2	2.0422×10^{-3}	1.0884×10^{-2}	2.0422×10^{-3}	1.0884×10^{-2}
3	3.4247×10^{-4}	3.0999×10^{-3}	2.7370×10^{-5}	3.2426×10^{-4}
4	1.1960×10^{-5}	2.7683×10^{-4}	2.7616×10^{-7}	7.1837×10^{-6}
5	1.3821×10^{-6}	5.2083×10^{-5}	2.2393×10^{-9}	1.2826×10^{-7}
6	5.6724×10^{-8}	5.8119×10^{-6}	1.5173×10^{-11}	1.9178×10^{-9}

Table 5. Comparison MAE_U for for example 2.

	HePs		AAPs	
n	$s = 0.1$	$s = 0.2$	$s = 0.1$	$s = 0.2$
2	2.0503×10^{-3}	7.2523×10^{-3}	2.0503×10^{-3}	7.2523×10^{-3}
3	5.2844×10^{-4}	5.0530×10^{-3}	3.0077×10^{-5}	2.6066×10^{-4}
4	1.6807×10^{-5}	2.3787×10^{-4}	3.2039×10^{-7}	6.4720×10^{-6}
5	2.6310×10^{-6}	1.1662×10^{-4}	2.6944×10^{-9}	1.2475×10^{-7}
6	1.2040×10^{-7}	7.2524×10^{-6}	1.8742×10^{-11}	1.9705×10^{-9}

Table 6. CPU time in seconds for example 2.

	HePs		AAPs	
n	$s = 0.1$	$s = 0.2$	$s = 0.1$	$s = 0.2$
2	0.407	0.422	0.469	0.500
3	0.563	0.547	1.250	1.234
4	0.875	0.875	5.047	4.953
5	1.250	1.266	35.453	35.734
6	1.844	1.890	481.984	399.937

Figures 5 and 6 below display the absolute error between the exact solutions given $[\underline{u}_{Exact}(s,\alpha), \overline{u}_{Exact}(s,\alpha)]$ (26) and the fuzzy approximate solutions by fifth-order HPM $[\underline{u}(s,\alpha), \overline{u}(s,\alpha)]$ of the Eqs. (24) with (25) for $0 \leq \alpha \leq 1$ at $s = 0.1$. In Figure 7 illustrates the exact solutions $(\widetilde{u}_{Exact}(s,\alpha))$, and the fuzzy approximate solutions by fifth-order HPM $(\widetilde{u}(s,\alpha))$ in the form of TFN for any $0 \leq \alpha \leq 1$ at $s = 0.1$. In Figure 8 studying the stability analysis with TFN along $0 \leq s \leq 1$ at $\alpha = 0, 0.5, 1.0$.

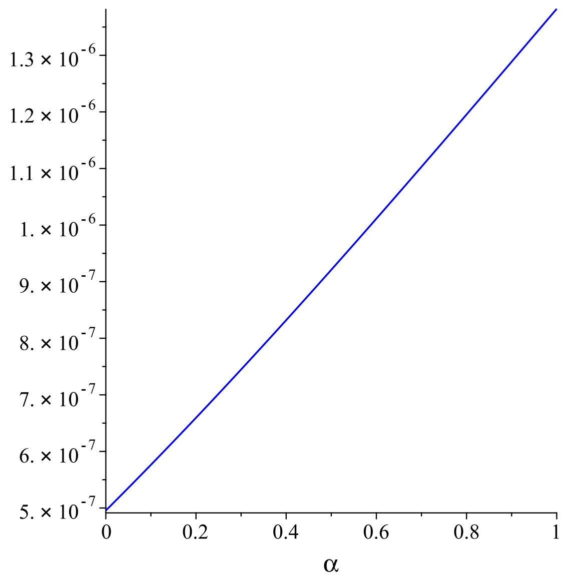

Figure 5. Graph the lower absolute error for example 2.

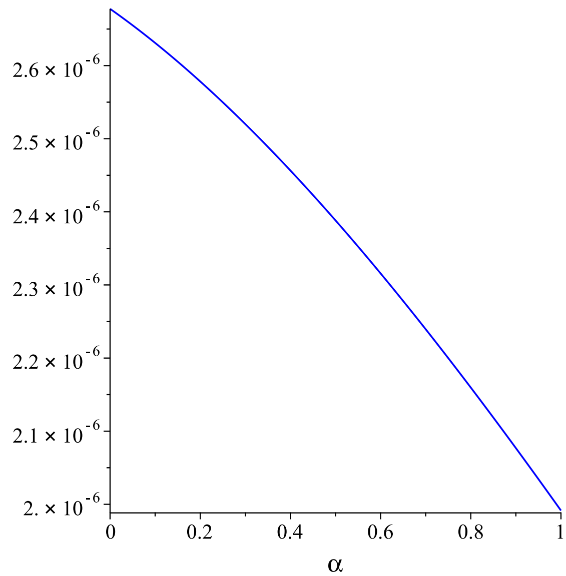

Figure 6. Graph the upper absolute error for example 2.

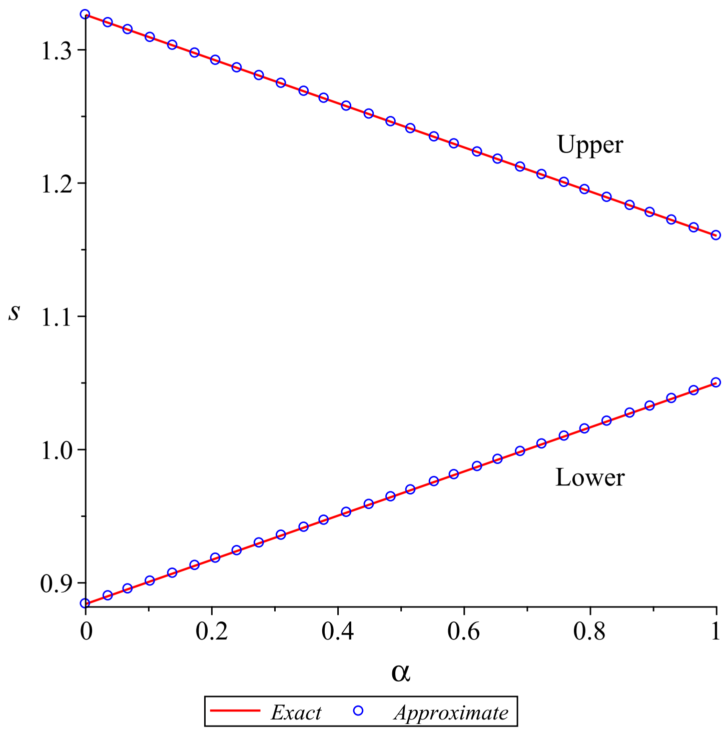

Figure 7. Graph of $\tilde{u}_{Exact}(s,\alpha)$ and $\tilde{u}(s,\alpha)$ $(n=5)$ at $s=0.1$ for example 2.

Example 3. Consider the following FQRDE:

$$\widetilde{u}'(s,\alpha) - \widetilde{u}(s,\alpha) + \widetilde{u}^2(s,\alpha) = \widetilde{g}(s,\alpha), \quad 0 \le s \le 1 \tag{27}$$

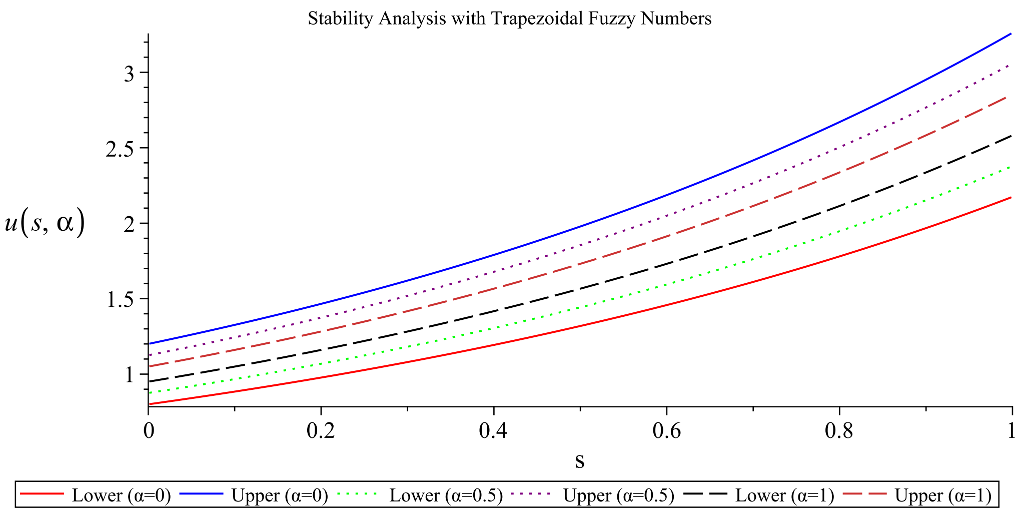

Figure 8. Stability analysis with trapezoidal fuzzy numbers for example 2.

with the ICs

$$\widetilde{u}\,(0,\alpha) = [0.90 + 0.05\alpha, 1.10 - 0.05\alpha]\,, \tag{28}$$

where $\widetilde{g}\,(s,\alpha) = \left[\underline{g}\,(s,\alpha)\,,\overline{g}\,(s,\alpha)\right]$ are chosen such that the exact solutions will be

$$\widetilde{u}_{Exact}\,(s,\alpha) = \left[\frac{0.90 + 0.05\alpha}{1+s}, \frac{1.10 - 0.05\alpha}{1+s}\right]. \tag{29}$$

In Tables 7 and 8 comparison the MAE between the exact solutions given $[\underline{u}_{Exact}\,(s,\alpha)\,,\overline{u}_{Exact}\,(s,\alpha)]$ (29) and the fuzzy approximate solutions by HPM $[\underline{u}\,(s,\alpha)\,,\overline{u}\,(s,\alpha)]$ of the Eqs. (27) with (28) using the Eqs. (19) and (20) for $0 \leq \alpha \leq 1$ at $s = 0.1$ and $s = 0.2$. Table 9 represents the CPU time in seconds between the MAE_L and MAE_U along the interval $0 \leq \alpha \leq 1$ at $s = 0.1$ and $s = 0.2$.

Table 7. Comparison MAE_L for for example 3.

	HePs		AAPs	
n	$s = 0.1$	$s = 0.2$	$s = 0.1$	$s = 0.2$
2	3.7492×10^{-3}	1.3288×10^{-2}	3.7492×10^{-3}	1.3288×10^{-2}
3	3.7674×10^{-4}	2.6738×10^{-3}	9.8945×10^{-5}	6.1099×10^{-4}
4	2.5918×10^{-5}	3.6285×10^{-4}	1.9838×10^{-6}	2.1899×10^{-5}
5	2.1703×10^{-6}	6.0165×10^{-5}	3.1939×10^{-8}	6.3210×10^{-7}

Figures 9 and 10 below display the absolute error between the exact solutions given $[\underline{u}_{Exact}\,(s,\alpha)\,,\overline{u}_{Exact}\,(s,\alpha)]$ (29) with the fuzzy approximate solutions by fifth-order HPM $[\underline{u}\,(s,\alpha)\,,\overline{u}\,(s,\alpha)]$ of the Eqs. (27) with (28) for $0 \leq \alpha \leq 1$ at $s = 0.1$. In Figure 11 illustrates the exact solutions $(\widetilde{u}_{Exact}\,(s,\alpha))$, and the fuzzy approximate solutions by fifth-order HPM $(\widetilde{u}\,(s,\alpha))$ in the form of TFN for any $0 \leq \alpha \leq 1$ at $s = 0.1$. In Figure 12 studying the stability analysis with TFN along $0 \leq s \leq 1$ at $\alpha = 0, 0.5, 1.0$.

Table 8. Comparison MAE_U for for example 3.

	HePs		AAPs	
n	$s = 0.1$	$s = 0.2$	$s = 0.1$	$s = 0.2$
2	5.7648×10^{-3}	2.0609×10^{-2}	5.7648×10^{-3}	2.0609×10^{-2}
3	5.8184×10^{-4}	4.1567×10^{-3}	2.0500×10^{-4}	1.3043×10^{-3}
4	4.8373×10^{-5}	6.8491×10^{-4}	5.5294×10^{-6}	6.4115×10^{-5}
5	4.3850×10^{-6}	1.2327×10^{-4}	1.1960×10^{-7}	2.5300×10^{-6}

Table 9. CPU time in seconds for example 3.

	HePs		AAPs	
n	$s = 0.1$	$s = 0.2$	$s = 0.1$	$s = 0.2$
2	0.360	0.453	0.531	0.484
3	0.735	0.766	1.532	1.484
4	1.437	1.531	4.313	4.125
5	4.969	4.453	506.328	642.562

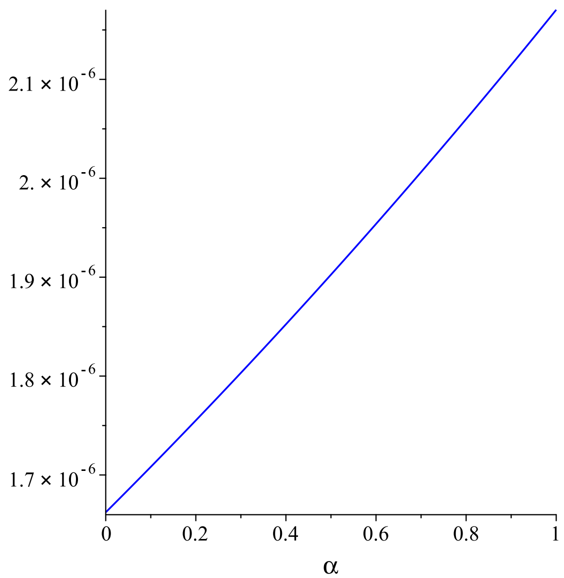

Figure 9. Graph the lower absolute error for example 3.

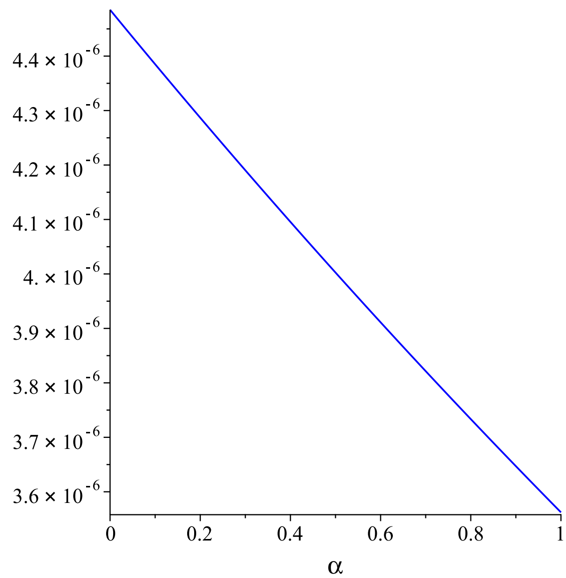

Figure 10. Graph the upper absolute error for example 3.

6. Conclusions

This study presents an effective approach for solving Riccati differential equations involving trapezoidal fuzzy functions. By integrating the homotopy perturbation method with α-cut analysis, the methodology achieves a

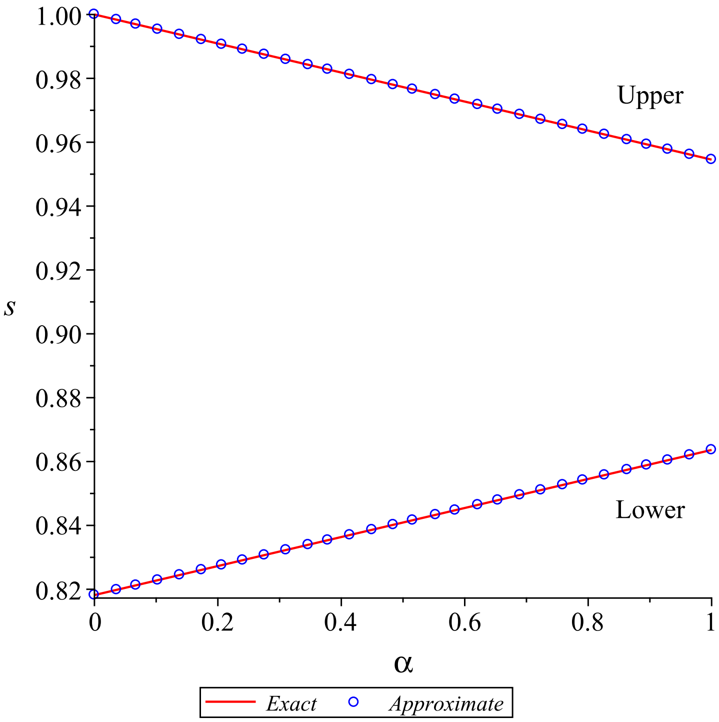

Figure 11. Graph of $\tilde{u}_{Exact}\,(s,\alpha)$ and $\tilde{u}\,(s,\alpha)$ $(n=5)$ at $s=0.1$ for example 5.

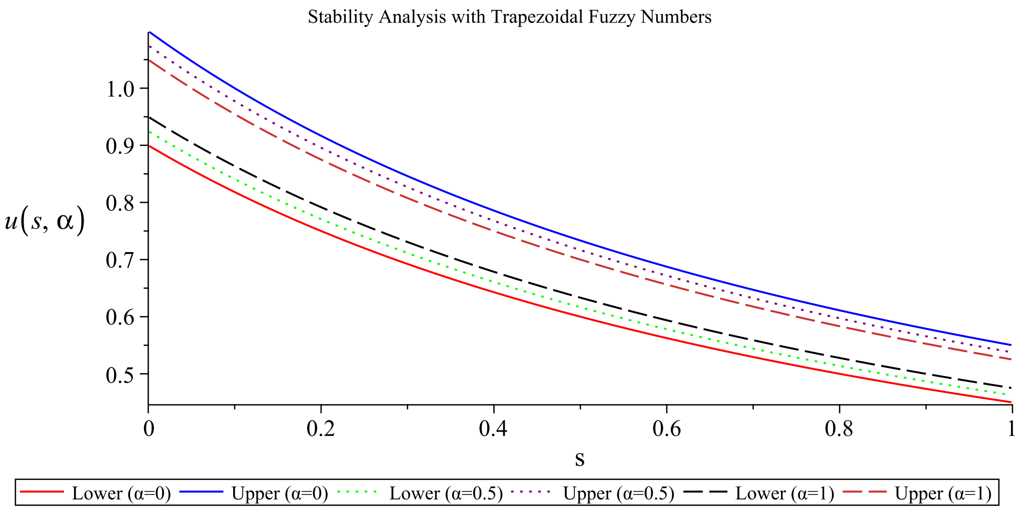

Figure 12. Stability analysis with trapezoidal fuzzy numbers for example 3.

crucial balance between computational accuracy and the preservation of the trapezoidal fuzzy structure. The α-cut technique systematically transforms inherent fuzzy uncertainties into well-defined, solvable interval systems. Simultaneously, the numerical-analytical synergy of fuzzy HPM directly addresses the equations' nonlinearities without approximating or distorting the fundamental trapezoidal fuzzy shapes.

Numerical validation demonstrates that this combined approach ensures rapid convergence and high solution accuracy, outperforming conventional methods, which often suffer from structural distortion or computational inefficiency in fuzzy contexts. Furthermore, comparative error analysis against established techniques like He's polynomial and the accelerated Adomian polynomials confirms a significantly reduced error rate. Finally, the application of the method to the FQRDE yields notable stability results, with solutions remaining consistent and stable across different values of α.

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