

## A certain algebraic structure of bipolar neutrosophic subgroups

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**Abstract** The notion of a bipolar neutrosophic set (BNS) was created as an expansion of a neutrosophic set when every single-value neutrosophic membership function has two poles (U positive pole and a negative pole). The BNS environment is an innovative tool for addressing ambiguous situations in several decision-making problems, especially in situations where the human mind perceives two patterns of thinking: positive and negative. In this study, we apply this idea in an algebraic environment when we initiate the novel concept of bipolar neutrosophic subgroups and prove that every BN subgroup generates two BN subgroups. We explain several ideas based on this environment, such as the level set, support, kernel for BNS, BN characteristic function, and BN-point. Then, we illuminate the BN-subgroup, BN-normal subgroup, BSVN-conjugate, normalizer for BN-subgroup, BN-abelian subgroup, and BN-factor group. Furthermore, we present the linked theorems and examples and prove these theorems. Moreover, we discussed the image and pre-image of BN-subgroups under homomorphism and proved the related theorems. Continuing our application of algebraic concepts to BNS, we will present a new mathematical framework for the concept of BN-matrics and some related mathematical properties. In addition, we will provide a practical application of these tools in dealing with a decision-making (DM) problem. Finally, in the conclusion section, we recommended that these results be a qualitative addition to the fuzzy algebraic environment and that they open up research doors for other future research work that takes advantage of the bipolar environment.

**Keywords** Neutrosophic set; bipolar neutrosophic set; bipolar neutrosophic subgroup; bipolar neutrosophic normal subgroup; homomorphism; Bipolar neutrosophic matrix

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### 1. Introduction

Subgroups are a fundamental concept in set theory, a fundamental branch of mathematics. A subgroup is defined as a set composed of elements of another set, such that all of its elements are present in that original set. Subgroups are used in many areas of mathematics, including algebra and set theory, where they help organize relationships between elements and better understand mathematical structures. Subgroups have several properties, such as the existence of an empty set, the original set itself, and the interaction properties of elements. Identifying and analyzing subgroups is an important step in the study of mathematical systems, contributing to the development of mathematical theories and their applications in everyday life. Subgroups deserve in-depth study to understand how they can be used to solve mathematical problems and enhance general understanding of mathematics. Zadeh [1] in

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1965 developed the initial results of fuzzy sets (FSs) theory. This theory has been successfully employed in handling uncertainty and ambiguous information in our real world. The mathematical structure of this theory is characterized by the inclusion of a membership function that represents the truth that an object belongs to a set of knowledge called the universal set. For example, FSs provide suitable solutions to many problems related to bioinformatics and computational biology, such as medical image processing, protein structure analysis, gene expression analysis, and medical data classification. Atanassov [2] then came up with the idea of the intuitionistic fuzzy set (IFS) to address uncertainty issues that the truth function of the fuzzy set alone cannot handle. Atanassov's idea is an extension of the mathematical structure of the FS by adding a second membership function called the falsity function. The mathematical structure of both FS and IFS lacks a neutral perspective, as these concepts fail to explain the neutral nature of the human mind. Sometimes, the user reaches a point where they cannot accept, reject, support, or disapprove of a particular issue. To address this situation, Smarandache [3] initiated the idea of neutrosophy in order to handle the indeterminate information that exists commonly in real situations. The neutrosophy idea considers an extension of the mathematical structure when Smarandache adds a third function, named the 'indeterminacy-membership function', to study the beginning nature and look of neutralities, in addition to their interactions with individual ideational spectra that are common in real situations. To adjust Smarandache's idea and put it into practice, Wang et al. [4] founded a new notion called neutrosophic sets in single-valued (SVNSs), which is more specified from an engineering point of view. Researchers have been working on employing the concept of SVNS in different mathematical directions. Researchers have been working on employing the concept of SVNS in different mathematical directions. Smarandache and Zhang [5] provided a comprehensive explanation of how this concept will be implemented. Şahin and Küçük [6] introduced axioms for subethood measure for SVNS, and they show the importance of these measures in application regarding multicriteria decision-making problems. Shahzadi et al [7] presented an application of SVNS in medical diagnosis. Arora et al [8] set up a novel similarity measure for SVNS. Alqahtani et al. [9] studied neutrosophic graphs under complex numbers and demonstrated their use in the design of hospital infrastructure. Al-Quran et al. [10, 11] made remarkable efforts to put out the notion of SVNS in DM-problems. Palanikumar et al. [12] discuss a new aggregation operator in this environment. Abed [13, 14] formulates and studies a new property, namely, the indeterminacy of the hollow module. Romdhini et al. [15, 16] present the Q-CNSFs and examine their unique algebraic properties. Al-Qudah et al. [17, 18] studied this concept from many mathematical aspects. Bataihah and Azaymeh [19]-[21] studied topological aspects and presented many important applications.

On the other hand, these research contributions of SVN cannot avail the positive pole and negative opinion of human thinking. To cover this barrier, Deli et al. [22] developed the concept of bipolar single value neutrosophic sets (BSVNSs) by extending the notion of bipolar fuzzy sets (BFSs) [23] where every SVN-membership has two poles, i.e., positive pole and negative pole. The positive ideas of the user (decision-maker) are covered by positive SVN functionality, while the negative ideas are covered by negative SVN functionality. This concept has been studied by many researchers, and many results have been presented about it [24]-[31]. Fahmi et al. [32] proposed some operational laws of BSVNS and examined its applications. Svičević et al. [33] analysed decision-making processes that employ the energy of BSVNS.

In addition, Roenfeld [34] discovered a concept for fuzzy subgroup notation (FSG) where each component belonging to the group has a degree of fuzzy relevance ranging between 0 and 1. Researchers considered this contribution as the beginning of opening the platform to integrating algebraic structures with the renewed fuzzy concepts mentioned above. Therefore, Dib and Hassan [35] investigated concepts of fuzzy space and fuzzy normal subgroup (FNSG) and their properties. The fuzzy coset, conjugate classes, and normality of an FSG of a group were established by Mukherjee and Bhattacharya [36] when they improved that FSG to be determined to be fuzzy normal iff it is (constant) on classes of conjugates of group  $G$ . Also, the idea of intuitionistic FSGs on IFSs was developed by Biswas [37] when the truth and false functions are invested in IFS to express the interaction of the binary operation of the two components belonging to the group  $G$ . Çetkin and Aygün [38] examined the algebraic approach to neutrosophic subgroups and established some of their basic characterisations. Abuqamar and Hassan [39] defined the notion of neutrosophic normal soft groups under  $Q$  effect and discussed some related properties. Al-Masarwah et al. [40] introduced the idea of the direct product of sets that include Fermat neutrosophic (FN). Moreover, Hameed et al. [41] discussed some characterisation SVN subgroups. Zhang and Hu [42] examined the

subgroup and normal subgroup structures of single-valued neutrosophic sets. Zhang et al. [43] generalized of the notion of the standard neutrosophic triplet group. But until now, the algebraic concepts and properties have not been studied with BNSs, and this prompted us to present this study.

However, to date, algebraic concepts and properties have not been studied in the context of BSVNSs, which prompted us to follow the aforementioned studies and present this study, which aims to apply the advantages available in BSVNSs with algebraic structures, where the role of the positive and negative aspects of BNSs becomes clear with the interaction between the components of the group.

**The main contributions of this work are put out as follows:**

1. We introduce and study a new algebraic structure named BSVNSGs and present their key properties. This study demonstrates the effect of bipolarity in the human mind when dealing with negative and positive everyday life problem data with algebraic structures.
2. We got a new structure called a BSVNS image and a BSVNS preimage, and examined the advanced features of the homomorphic image of a BSVN-(normal) subgroup. and preimage of a BSVN-(normal) subgroup.
3. We have employed these new ideas to create and prove many theories and results, which will constitute a new addition to knowledge in the mathematics environment.
4. In this method, we derive the generalised form of the FSG and the intuitionistic fuzzy subgroups defined on a traditional group. In addition, the next figure, Figure 1, shows the relationship between established and existing methods.
5. We will put on some new forms about algebraic structures, namely BN-matrices. We also defined some related mathematical properties. In addition, we will provide a practical application of these tools in dealing with a decision-making problem.

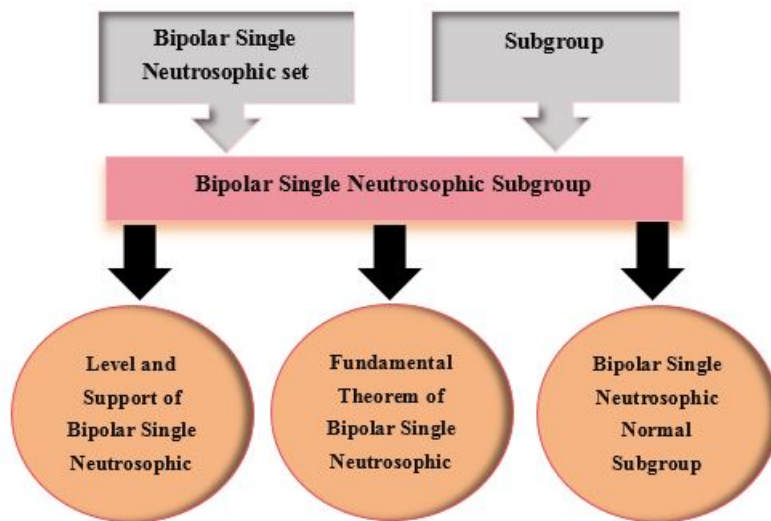


Figure 1. The relationship between established and existing methods.

The residual paper is organised into five sections: Section 2 presents the preliminary terms and definitions that we used to propose the results in this work. The main results, i.e., the elementary notions of BNSG and its elementary relevant theorems and examples, are present in Section 3. Also, the main results, i.e., the elementary notions of BNSNG and its elementary relevant theorems and examples, are present in Section 4. Section 5 hands over a summary, limitations, and future extent of the results that are present in this work.

## 2. Preliminaries

**Definition 2.1.** [4] The following mathematical frame that is known on  $X$ :

$$A = \{\check{m}, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle, m \in X\}$$

is named an NS, where its three memberships  $\mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \in [0, 1]$  denote the truth-NS membership, indeterminacy-NS membership, and non-truth-NS membership, respectively. for  $v \in X$  with the standard condition:  $0 \leq \mathcal{T}_U(v) + \mathcal{I}_U(v) + \mathcal{F}_U(v) \leq 3$ .

**Definition 2.2.** [4] Let

$$A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle, m \in X\}$$

and

$$B = \{m, \langle \mathcal{T}_V(v), \mathcal{I}_V(v), \mathcal{F}_V(v) \rangle, m \in X\}$$

be two neutrosophic sets (NSs). Then the fundamental set theory on both NSs is given as follows:

1. The subset between two NSs is denoted as  $U \subseteq V$  where

$$\mathcal{T}_U(v) \leq \mathcal{T}_V(v), \quad \mathcal{I}_U(v) \geq \mathcal{I}_V(v), \quad \mathcal{F}_U(v) \geq \mathcal{F}_V(v).$$

2. The equality between two NSs is denoted as  $U = V$  where

$$\mathcal{T}_U(v) = \mathcal{T}_V(v), \quad \mathcal{I}_U(v) = \mathcal{I}_V(v), \quad \mathcal{F}_U(v) = \mathcal{F}_V(v).$$

3. The union between two NSs is denoted as  $U \cup V$  where:

$$A \cup B = \{\langle \max\{\mathcal{T}_U(v), \mathcal{T}_V(v)\}, \min\{\mathcal{I}_U(v), \mathcal{I}_V(v)\}, \min\{\mathcal{F}_U(v), \mathcal{F}_V(v)\} \rangle\}.$$

4. The intersection between two NSs is denoted as  $U \cap V$  where:

$$A \cap B = \{\langle \min\{\mathcal{T}_U(v), \mathcal{T}_V(v)\}, \max\{\mathcal{I}_U(v), \mathcal{I}_V(v)\}, \max\{\mathcal{F}_U(v), \mathcal{F}_V(v)\} \rangle\}.$$

5. The complement of NS is denoted as  $U^C$  where:

$$U^C = \{\langle \mathcal{F}_U(v), 1 - \mathcal{I}_U(v), \mathcal{T}_U(v) \rangle\}.$$

**Definition 2.3.** [5] A neutrosophic structure  $U$  given in Definition 2.1 on a non-empty classical group  $G, *$  is named an NS-subgroup of  $G$  if  $\tilde{\rho}, \tilde{z} \in \tilde{G}$  we have the following:

$$\begin{aligned} \mathcal{T}_U(\tilde{\rho} * \tilde{z}) &\geq \min\{\mathcal{T}_U(\tilde{\rho}), \mathcal{T}_U(\tilde{z})\}, \\ \mathcal{I}_U(\tilde{\rho} * \tilde{z}) &\geq \min\{\mathcal{I}_U(\tilde{\rho}), \mathcal{I}_U(\tilde{z})\}, \\ \mathcal{F}_U(\tilde{\rho} * \tilde{z}) &\leq \max\{\mathcal{F}_U(\tilde{\rho}), \mathcal{F}_U(\tilde{z})\}, \\ \mathcal{T}_U(\tilde{\rho}^{-1}) &\geq \mathcal{T}_U(\tilde{\rho}), \\ \mathcal{I}_U(\tilde{\rho}^{-1}) &\geq \mathcal{I}_U(\tilde{\rho}), \\ \mathcal{F}_U(\tilde{\rho}^{-1}) &\leq \mathcal{F}_U(\tilde{\rho}). \end{aligned}$$

In this work, we denote by  $NS(\tilde{G})$  the set of all terms of NS-subgroups of  $G$ .

**Definition 2.4.** [24] The following mathematical frame that is known on  $X$ :

$$\ddot{A} = \{v \in X, \mathcal{T}_{(\ddot{A})}^+ s, \mathcal{I}_{(\ddot{A})}^+ s, \mathcal{F}_{(\ddot{A})}^+ s, \mathcal{T}_{(\ddot{A})}^- s, \mathcal{I}_{(\ddot{A})}^- s, \mathcal{F}_{(\ddot{A})}^- s\}$$

is named a bipolar neutrosophic aet (BNS) on non-empty  $X$  where the following memberships

$\mathcal{T}_{(\ddot{A})}^+ s, \mathcal{I}_{(\ddot{A})}^+ s, \mathcal{F}_{(\ddot{A})}^+ s, \mathcal{T}_{(\ddot{A})}^- s, \mathcal{I}_{(\ddot{A})}^- s, \mathcal{F}_{(\ddot{A})}^- s \in [-1, 1]$  denote the positive and negative of three NS memberships mentioned in definition 2.1 and for every element  $v \in X$ .

### 3. A new structure of bipolar neutrosophic subgroups

Here in this part, we going to diagnose the mathematical structure of Bipolar single value neutrosophic subgroups and their fundamentals properties.

**Definition 3.1.** Assume that A be a BN-subset on a classical non-empty group  $(\tilde{G}, *)$  and  $\beta \in [-1, 1]$  .Then the bipolar neutrosophic  $\beta$ -level set denotes as  $(\tilde{A})_\beta$  and given as following:

$$U_\beta = \{s \in \tilde{G}, \mathcal{T}_{U_\beta}^+(s), \mathcal{I}_{U_\beta}^+(s), \mathcal{F}_{U_\beta}^+(s), \mathcal{T}_{U_\beta}^-(s), \mathcal{I}_{U_\beta}^-(s), \mathcal{F}_{U_\beta}^-(s)\}$$

$$\mathcal{T}_{(\tilde{A})_\beta}^+(s) = \{s \in \tilde{G} \mid \mathcal{T}_{(\tilde{A})}^+ \geq \beta, \beta \in [0, 1]\}$$

$$\mathcal{T}_{(\tilde{A})_\beta}^-(s) = \{s \in \tilde{G} \mid \mathcal{T}_{(\tilde{A})}^- \leq \beta, \beta \in [-1, 0]\}$$

$$\mathcal{I}_{(\tilde{A})_\beta}^+(s) = \{s \in \tilde{G} \mid \mathcal{I}_{(\tilde{A})}^+ \geq \beta, \beta \in [0, 1]\}$$

$$\mathcal{I}_{(\tilde{A})_\beta}^-(s) = \{s \in \tilde{G} \mid \mathcal{I}_{(\tilde{A})}^- \leq \beta, \beta \in [-1, 0]\}$$

$$\mathcal{F}_{(\tilde{A})_\beta}^+(s) = \{s \in \tilde{G} \mid \mathcal{F}_{(\tilde{A})}^+ \leq \beta, \beta \in [0, 1]\}$$

$$\mathcal{F}_{(\tilde{A})_\beta}^-(s) = \{s \in \tilde{G} \mid \mathcal{F}_{(\tilde{A})}^- \geq \beta, \beta \in [-1, 0]\}.$$

*Remark 3.2.* For two bipolar neutrosophic subsets of the classical group  $\tilde{G}$ , i.e., U and V, if  $U \subseteq V$  and  $\beta \in [-1, 1]$ , then for the positive side:

$$\mathcal{T}_U^+ \leq \mathcal{T}_V^+, \quad \mathcal{I}_U^+ \leq \mathcal{I}_V^+, \quad \mathcal{F}_U^+ \geq \mathcal{F}_V^+$$

and for the negative side:

$$\mathcal{T}_U^- \geq \mathcal{T}_V^-, \quad \mathcal{I}_U^- \geq \mathcal{I}_V^-, \quad \mathcal{F}_U^- \leq \mathcal{F}_V^-.$$

**Definition 3.3.** Assume that U be a BN-subset on a classical non-empty group  $(\tilde{G}, *)$  and  $\beta \in [-1, 1]$  .Then the BN support set denotes as  $(T)_S$  and given as following:

$$(T)_S = \{t \in \tilde{G}, \mathcal{T}_{(T)_S}^+(t), \mathcal{I}_{(T)_S}^+(t), \mathcal{F}_{(T)_S}^+(t), \mathcal{T}_{(T)_S}^-(t), \mathcal{I}_{(T)_S}^-(t), \mathcal{F}_{(T)_S}^-(t)\}$$

where:

$$\mathcal{T}_{(T)_S}^+(t) = \{t \in \tilde{G} \mid \mathcal{T}_{(T)}^+ \geq 0\}$$

$$\mathcal{T}_{(T)_S}^-(t) = \{t \in \tilde{G} \mid \mathcal{T}_{(T)}^- \leq 0\}$$

$$\mathcal{I}_{(T)_S}^+(t) = \{t \in \tilde{G} \mid \mathcal{I}_{(T)}^+ \geq 0\}$$

$$\mathcal{I}_{(T)_S}^-(t) = \{t \in \tilde{G} \mid \mathcal{I}_{(T)}^- \leq 0\}$$

$$\mathcal{F}_{(T)_S}^+(t) = \{t \in \tilde{G} \mid \mathcal{F}_{(T)}^+ \leq 0\}$$

$$\mathcal{F}_{(T)_S}^-(t) = \{t \in \tilde{G} \mid \mathcal{F}_{(T)}^- \geq 0\}.$$

**Definition 3.4.** Assume that  $(\tilde{G}, *)$  be a classical fundamental group and U be a BN subset of  $\tilde{G}$  with  $\beta \in [-1, 1]$ . Then the set

$$U_K = \{v \in \tilde{G}, \mathcal{T}_{U_K}^+ s, \mathcal{I}_{U_K}^+ s, \mathcal{F}_{U_K}^+ s, \mathcal{T}_{U_K}^- s, \mathcal{I}_{U_K}^- s, \mathcal{F}_{U_K}^- s\}$$

where:

$$\begin{aligned} \mathcal{T}_{U_K}^+ s &= \{v \in \tilde{G} \mid \mathcal{T}_{(K)}^+ = 1\} \\ \mathcal{T}_{U_K}^- s &= \{v \in \tilde{G} \mid \mathcal{T}_{(K)}^- = -1\} \\ \mathcal{I}_{U_K}^+ s &= \{v \in \tilde{G} \mid \mathcal{I}_{(K)}^+ = 1\} \\ \mathcal{I}_{U_K}^- s &= \{v \in \tilde{G} \mid \mathcal{I}_{(K)}^- = -1\} \\ \mathcal{F}_{U_K}^+ s &= \{v \in \tilde{G} \mid \mathcal{F}_{(K)}^+ = 1\} \\ \mathcal{F}_{U_K}^- s &= \{v \in \tilde{G} \mid \mathcal{F}_{(K)}^- = -1\}. \end{aligned}$$

Here we denote  $U_K$  as the kernel of the bipolar neutrosophic subset of the classical group  $\tilde{G}$ .

**Definition 3.5.** Suppose that  $(\tilde{G}, *)$  is a classical fundamental group and for any subset  $K$  of the classical group  $\tilde{G}$ . Then the bipolar characteristic function neutrosophic (BCN) is denoted as

$$U_K = \{\mathcal{T}_{U_K}, \mathcal{I}_{U_K}, \mathcal{F}_{U_K}\}$$

where

$$\begin{aligned} \mathcal{T}_{U_K} s &= \begin{cases} \mathcal{T}_{U_K}^+ s = 1 & \text{if } s \in G \\ \mathcal{T}_{U_K}^+ s = 0 & \text{if } s \notin G \\ \mathcal{T}_{U_K}^- s = -1 & \text{if } s \in G \\ \mathcal{T}_{U_K}^- s = 0 & \text{if } s \notin G \end{cases} \\ \mathcal{I}_{U_K} s &= \begin{cases} \mathcal{I}_{U_K}^+ s = 1 & \text{if } s \in G \\ \mathcal{I}_{U_K}^+ s = 0 & \text{if } s \notin G \\ \mathcal{I}_{U_K}^- s = -1 & \text{if } s \in G \\ \mathcal{I}_{U_K}^- s = 0 & \text{if } s \notin G \end{cases} \\ \mathcal{F}_{U_K} s &= \begin{cases} \mathcal{F}_{U_K}^+ s = 1 & \text{if } s \in G \\ \mathcal{F}_{U_K}^+ s = 0 & \text{if } s \notin H \\ \mathcal{F}_{U_K}^- s = -1 & \text{if } s \in G \\ \mathcal{F}_{U_K}^- s = 0 & \text{if } s \notin H \end{cases} \end{aligned}$$

*Remark 3.6.* Evidently, a generalization of a BCN is a BN-subset of  $\tilde{G}$ .

**Definition 3.7.** Suppose that  $(\tilde{G}, *)$  is a classical group and for any  $v, n \in \tilde{G}$ ,  $\rho \in [-1, 1]$ . Then the bipolar neutrosophic subset of  $\tilde{G}$  is given as follows:

$$A = \{\mathcal{T}_U, \mathcal{I}_U, \mathcal{F}_U\}$$

where

$$\mathcal{T}_U s = \begin{cases} \mathcal{T}_U^+ s = \rho & \text{if } v = n \\ \mathcal{T}_U^+ s = 0 & \text{if } v \neq n \\ \mathcal{T}_U^- s = \rho & \text{if } v = n \\ \mathcal{T}_U^- s = 0 & \text{if } v \neq n \end{cases}$$

$$\mathcal{I}_U s = \begin{cases} \mathcal{I}_U^+ s = \rho & \text{if } s = n \\ \mathcal{I}_U^+ s = 0 & \text{if } s \neq n \\ \mathcal{I}_U^- s = \rho & \text{if } s = n \\ \mathcal{I}_U^- s = 0 & \text{if } s \neq n \end{cases} \mathcal{F}_U s = \begin{cases} \mathcal{F}_U^+ s = \rho & \text{if } s = n \\ \mathcal{F}_U^+ s = 0 & \text{if } s \neq n \\ \mathcal{F}_U^- s = \rho & \text{if } s = n \\ \mathcal{F}_U^- s = 0 & \text{if } s \neq n \end{cases}$$

is said to be a BN point with value  $\rho \in [-1, 1]$  and support  $v$ .

**Definition 3.8.** Suppose that  $(\tilde{G}, *)$  is a classical group and  $U$  is a BNS on the classical group  $\tilde{G}$  with  $\beta \in [-1, 1]$ . Then the set

$$U_\beta = \{v \in \tilde{G}, \mathcal{T}_{U_\beta}^+ s, \mathcal{I}_{U_\beta}^+ s, \mathcal{F}_{U_\beta}^+ s, \mathcal{T}_{U_\beta}^- s, \mathcal{I}_{U_\beta}^- s, \mathcal{F}_{U_\beta}^- s\}$$

where:

$$\begin{aligned} \mathcal{T}_{U_\beta}^+ s &= \{v \in \tilde{G} \mid \mathcal{T}_U^+ \geq \beta, \beta \in [0, 1]\} \\ \mathcal{T}_{U_\beta}^- s &= \{v \in \tilde{G} \mid \mathcal{T}_U^- \leq \beta, \beta \in [-1, 0]\} \\ \mathcal{I}_{U_\beta}^+ s &= \{v \in \tilde{G} \mid \mathcal{I}_U^+ \geq \beta, \beta \in [0, 1]\} \\ \mathcal{I}_{U_\beta}^- s &= \{v \in \tilde{G} \mid \mathcal{I}_U^- \leq \beta, \beta \in [-1, 0]\} \\ \mathcal{F}_{U_\beta}^+ s &= \{v \in \tilde{G} \mid \mathcal{F}_U^+ \leq \beta, \beta \in [0, 1]\} \\ \mathcal{F}_{U_\beta}^- s &= \{v \in \tilde{G} \mid \mathcal{F}_U^- \geq \beta, \beta \in [-1, 0]\}. \end{aligned}$$

Here we denote  $U_\beta$  as the bipolar neutrosophic  $\beta$ -level set.

**Definition 3.9.** Assume that  $(\tilde{G}, *)$  is a traditional group and  $U$  is a BN-subset on  $\tilde{G}$ . Then  $U$  is named a BN-subgroup of  $\tilde{G}$  if the next conditions are satisfied:

1. For all  $v, n \in \tilde{G}$ :

$$U(v * n) \geq \min\{As, U(n)\}$$

such that:

$$\begin{aligned} \mathcal{T}_U^+(v * n) &\geq \min\{\mathcal{T}_U^+ s, \mathcal{T}_U^+(n)\}, \\ \mathcal{I}_U^+(v * n) &\geq \min\{\mathcal{I}_U^+ s, \mathcal{I}_U^+(n)\}, \\ \mathcal{F}_U^+(v * n) &\leq \max\{\mathcal{F}_U^+ s, \mathcal{F}_U^+(n)\}. \end{aligned}$$

2. For all  $v \in \tilde{G}$ :

$$U(v^{-1}) \geq As$$

such that:

$$\begin{aligned} \mathcal{T}_U^+(v^{-1}) &\geq \mathcal{T}_U^+ s, \\ \mathcal{I}_U^+(v^{-1}) &\geq \mathcal{I}_U^+ s, \\ \mathcal{F}_U^+(v^{-1}) &\leq \mathcal{F}_U^+ s. \end{aligned}$$

3. For all  $v, n \in \tilde{G}$ :

$$U(v * n) \leq \min\{As, U(n)\}$$

such that:

$$\begin{aligned} \mathcal{T}_U^-(v * n) &\leq \max\{\mathcal{T}_U^- s, \mathcal{T}_U^-(n)\}, \\ \mathcal{I}_U^-(v * n) &\leq \max\{\mathcal{I}_U^- s, \mathcal{I}_U^-(n)\}, \\ \mathcal{F}_U^-(v * n) &\geq \min\{\mathcal{F}_U^- s, \mathcal{F}_U^-(n)\}. \end{aligned}$$

4. For all  $v \in \tilde{G}$ :

$$U(v^{-1}) \leq As$$

such that:

$$\begin{aligned} \mathcal{T}_U^-(v^{-1}) &\leq \mathcal{T}_U^- s, \\ \mathcal{I}_U^-(v^{-1}) &\leq \mathcal{I}_U^- s, \\ \mathcal{F}_U^-(v^{-1}) &\geq \mathcal{F}_U^- s. \end{aligned}$$

Here the summation of all bipolar neutrosophic subgroups of  $\tilde{G}$  is indicated as  $BNS(\tilde{G})$ .

**Example 3.10.** Take  $\tilde{G} = K_4 = \{1, v, n, v * n\}$  as a classical group defined as the following Cayley table, where  $*$  is natural multiplication:

Table 1. The Cayley table of  $K_4$ .

$*$	1	$v$	$n$	$v * n$
1	1	$v$	$n$	$v * n$
$v$	$v$	1	$v * n$	$n$
$n$	$n$	$v * n$	1	$v$
$v * n$	$v * n$	$n$	$v$	1

with following properties:

$$v^2 = n^2 = (v * n)^2 = 1 \quad \text{and} \quad v * n = n * v.$$

Now take the BN subset values of  $\tilde{G} = K_4$  as follows:

$$A = \left\{ \frac{\langle 0.3, 0.4, 0.5, -0.4, -0.6, -0.7 \rangle}{1}, \frac{\langle 0.3, 0.4, 0.5, -0.4, -0.6, -0.7 \rangle}{\tilde{m}}, \frac{\langle 0.3, 0.2, 0.5, -0.4, -0.6, -0.7 \rangle}{\tilde{n}}, \frac{\langle 0.3, 0.4, 0.5, -0.4, -0.6, -0.7 \rangle}{\tilde{m}\tilde{n}} \right\}$$

Then we aim to investigate:

1.  $U(v * n) \geq \min\{As, U(n)\}$  for all  $v, n \in K_4$ :

$$\begin{aligned} \mathcal{T}_U^+(v * n) &\geq \min\{\mathcal{T}_U^+ s, \mathcal{T}_U^+(n)\}, \\ \mathcal{I}_U^+(v * n) &\geq \min\{\mathcal{I}_U^+ s, \mathcal{I}_U^+(n)\}, \\ \mathcal{F}_U^+(v * n) &\leq \max\{\mathcal{F}_U^+ s, \mathcal{F}_U^+(n)\}. \end{aligned}$$

2.  $U(v^{-1}) \geq As$  :

$$\begin{aligned} \mathcal{T}_U^+(v^{-1}) &\geq \mathcal{T}_U^+ s, \\ \mathcal{I}_U^+(v^{-1}) &\geq \mathcal{I}_U^+ s, \\ \mathcal{F}_U^+(v^{-1}) &\leq \mathcal{F}_U^+ s. \end{aligned}$$

3.  $U(v * n) \leq \min\{As, U(n)\}$ :

$$\mathcal{T}_U^-(v * n) \leq \max\{\mathcal{T}_U^- s, \mathcal{T}_U^-(n)\},$$

$$\begin{aligned} \mathcal{I}_U^-(v * n) &\leq \max\{\mathcal{I}_U^- s, \mathcal{I}_U^-(n)\}, \\ \mathcal{F}_U^-(v * n) &\geq \min\{\mathcal{F}_U^- s, \mathcal{F}_U^-(n)\}. \end{aligned}$$

4.  $U(v^{-1}) \leq As$ :

$$\begin{aligned} \mathcal{T}_U^-(v^{-1}) &\leq \mathcal{T}_U^- s, \\ \mathcal{I}_U^-(v^{-1}) &\leq \mathcal{I}_U^- s, \\ \mathcal{F}_U^-(v^{-1}) &\geq \mathcal{F}_U^- s. \end{aligned}$$

Therefore, we will set  $v = 1$  and  $n = v$ . To prove:

$$U(1 * v) \geq \min\{U(1), As\}$$

for all  $1, v \in K_4$ .

To prove:

$$U(1 * v) \geq \min\{U(1), As\}$$

for all  $1, v \in G$ , then:

$$\mathcal{T}_U^+(1 * v) = \mathcal{T}_U^+ s = 0.3, \quad \mathcal{T}_U^+(1) = 0.3, \quad \mathcal{T}_U^+ s = 0.3$$

then

$$\mathcal{T}_U^+(1 * v) \geq \min\{\mathcal{T}_U^+(1), \mathcal{T}_U^+ s\} \text{ is true.}$$

$$\mathcal{I}_U^+(1 * v) = \mathcal{I}_U^+ s = 0.4, \quad \mathcal{I}_U^+(1) = 0.4, \quad \mathcal{I}_U^+ s = 0.4$$

then

$$\mathcal{I}_U^+(1 * v) \geq \min\{\mathcal{I}_U^+(1), \mathcal{I}_U^+ s\} \text{ is true.}$$

$$\mathcal{F}_U^+(1 * v) = \mathcal{F}_U^+ s = 0.5, \quad \mathcal{F}_U^+(1) = 0.5, \quad \mathcal{F}_U^+ s = 0.5$$

then

$$\mathcal{F}_U^+(1 * v) \leq \max\{\mathcal{F}_U^+(1), \mathcal{F}_U^+ s\} \text{ is true.}$$

For the negative side:

$$\mathcal{T}_U^-(1 * v) = \mathcal{T}_U^- s = -0.4, \quad \mathcal{T}_U^-(1) = -0.4, \quad \mathcal{T}_U^- s = -0.4$$

then

$$\mathcal{T}_U^-(1 * v) \leq \max\{\mathcal{T}_U^-(1), \mathcal{T}_U^- s\} \text{ is true.}$$

$$\mathcal{I}_U^-(1 * v) = \mathcal{I}_U^- s = -0.6, \quad \mathcal{I}_U^-(1) = -0.6, \quad \mathcal{I}_U^- s = -0.6$$

then

$$\mathcal{I}_U^-(1 * v) \leq \max\{\mathcal{I}_U^-(1), \mathcal{I}_U^- s\} \text{ is true.}$$

$$\mathcal{F}_U^-(1 * v) = \mathcal{F}_U^- s = -0.7, \quad \mathcal{F}_U^-(1) = -0.7, \quad \mathcal{F}_U^- s = -0.7$$

then

$$\mathcal{F}_U^-(1 * v) \geq \min\{\mathcal{F}_U^-(1), \mathcal{F}_U^- s\} \text{ is true.}$$

Thus,  $U(1 * v) \geq \min\{U(1), As\}$  for all  $1, v \in G$ .

To prove:

$$U(v^{-1}) \geq As \text{ such that}$$

$$\mathcal{T}_U^+(1^{-1}) = \mathcal{T}_U^+(1) = 0.3 \text{ and } \mathcal{T}_U^+(1) = 0.3 \text{ then } \mathcal{T}_U^+(1^{-1}) \geq \mathcal{T}_U^+(1).$$

$$\mathcal{I}_U^+(1^{-1}) = \mathcal{I}_U^+(1) = 0.4 \text{ and } \mathcal{I}_U^+(1) = 0.4 \text{ then } \mathcal{I}_U^+(1^{-1}) \geq \mathcal{I}_U^+(1).$$

$$\mathcal{F}_U^+(1^{-1}) = \mathcal{F}_U^+(1) = 0.5 \text{ and } \mathcal{F}_U^+(1) = 0.5 \text{ then } \mathcal{F}_U^+(1^{-1}) \leq \mathcal{F}_U^+(1).$$

For the negative side:

$$\mathcal{T}_U^-(1^{-1}) = \mathcal{T}_U^-(1) = -0.6 \text{ and } \mathcal{T}_U^-(1) = -0.6 \text{ then } \mathcal{T}_U^-(1^{-1}) \leq \mathcal{T}_U^-(1).$$

$$\mathcal{I}_U^-(1^{-1}) = \mathcal{I}_U^-(1) = -0.6 \text{ and } \mathcal{I}_U^-(1) = -0.6 \text{ then } \mathcal{I}_U^-(1^{-1}) \leq \mathcal{I}_U^-(1).$$

$$\mathcal{F}_U^-(1^{-1}) = \mathcal{F}_U^-(1) = -0.7 \text{ and } \mathcal{F}_U^-(1) = -0.7 \text{ then } \mathcal{F}_U^-(1^{-1}) \geq \mathcal{F}_U^-(1).$$

Thus,  $U(v^{-1}) \geq U(v)$ .

And so on for the rest of the elements in  $K_4$ , thus the BN subset  $U$  of  $K_4$  is a BNSG.

*Theorem 3.11*

Assume that  $U$  is a BN-subgroup of  $\tilde{G}$ . Then, the next properties are fulfilled:

- i.  $U(e) \geq U(v)$  where  $e$  given as a identity element in  $\tilde{G}$  and  $m \in \tilde{G}$ .
- ii.  $U(v^{-1}) \geq U(v)$ , where  $v^{-1}$  is given as a inverse element in  $\tilde{G}$  and  $m \in \tilde{G}$ .

*Proof*

(i) For  $e \in \tilde{G}$  and  $v \in \tilde{G}$ . Then by definition 3.1, we get for the positive side:

$$\mathcal{T}_U^+(e) = \mathcal{T}_U^+(v * v^{-1}) \geq \min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(v^{-1})\} \geq \min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(v)\} = \mathcal{T}_U^+(v).$$

$$\mathcal{F}_U^+(e) = \mathcal{F}_U^+(v * v^{-1}) \leq \max\{\mathcal{F}_U^+(v), \mathcal{F}_U^+(v^{-1})\} \leq \max\{\mathcal{F}_U^+(v), \mathcal{F}_U^+(v)\} = \mathcal{F}_U^+(v).$$

From similar steps, it is clearly shown that

$$\mathcal{I}_U^+(e) \geq \mathcal{I}_U^+(v).$$

Now for the negative side, we get:

$$\mathcal{T}_U^-(e) = \mathcal{T}_U^-(v * v^{-1}) \leq \max\{\mathcal{T}_U^-(v), \mathcal{T}_U^-(v^{-1})\} \leq \max\{\mathcal{T}_U^-(v), \mathcal{T}_U^-(v)\} = \mathcal{T}_U^-(v).$$

$$\mathcal{F}_U^-(e) = \mathcal{F}_U^-(v * v^{-1}) \geq \min\{\mathcal{F}_U^-(v), \mathcal{F}_U^-(v^{-1})\} \geq \min\{\mathcal{F}_U^-(v), \mathcal{F}_U^-(v)\} = \mathcal{F}_U^-(v).$$

From similar steps, it is clearly shown that

$$\mathcal{I}_U^-(e) \leq \mathcal{I}_U^-(v).$$

Hence, we get the desired inequality in (i):

$$U(e) \geq U(v).$$

(ii) Suppose that  $v \in \tilde{G}$  and  $v^{-1}$  is the inverse element of  $v \in \tilde{G}$ . Again, by using (ii) in definition 3.1 and utilizing the some properties of  $\tilde{G}$ , the other side of the inequality, it is proved as next:

$$\mathcal{T}_U^+(v) = \mathcal{T}_U^+(v^{-1})^{-1} \geq \mathcal{T}_U^+(v^{-1}),$$

$$\mathcal{T}_U^-(v) = \mathcal{T}_U^-(v^{-1})^{-1} \leq \mathcal{T}_U^-(v^{-1}),$$

$$\mathcal{I}_U^+(v) = \mathcal{I}_U^+(v^{-1})^{-1} \geq \mathcal{I}_U^+(v^{-1}),$$

$$\mathcal{I}_U^-(v) = \mathcal{I}_U^-(v^{-1})^{-1} \leq \mathcal{I}_U^-(v^{-1}),$$

$$\begin{aligned}\mathcal{F}_U^+(v) &= \mathcal{F}_U^+(v^{-1})^{-1} \leq \mathcal{F}_U^+(v^{-1}), \\ \mathcal{F}_U^-(v) &= \mathcal{F}_U^-(v^{-1})^{-1} \geq \mathcal{F}_U^-(v^{-1}).\end{aligned}$$

Therefore,

$$\begin{aligned}U(v^{-1}) &= (\mathcal{T}_U^+(v^{-1}), \mathcal{I}_U^+(v^{-1}), \mathcal{F}_U^+(v^{-1}), \mathcal{T}_U^-(v^{-1}), \mathcal{I}_U^-(v^{-1}), \mathcal{F}_U^-(v^{-1})) \\ &= (\mathcal{T}_U^+(v), \mathcal{I}_U^+(v), \mathcal{F}_U^+(v), \mathcal{T}_U^-(v), \mathcal{I}_U^-(v), \mathcal{F}_U^-(v)) = U(v).\end{aligned}$$

□

*Theorem 3.12*

Assume that  $U$  is a BNS of  $\tilde{G}$ . Then  $U \in BNS(\tilde{G})$  iff

$$U^+(v * n^{-1}) \geq \min\{U^+(v), U^+(n^{-1})\} = \min\{U^+(v), U^+(n)\}$$

and

$$U^-(v * n^{-1}) \leq \max\{U^-(v), U^-(n^{-1})\} = \max\{U^-(v), U^-(n)\}$$

where each  $v, n \in \tilde{G}$ .

*Proof*

Let  $U$  is a BNSG of  $\tilde{G}$  and for  $v, n \in \tilde{G}$ . Then it's clear based on the above discussion we get

$$\mathcal{T}_U^+(v * n^{-1}) \geq \min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(n^{-1})\} = \min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(n)\}$$

and

$$\mathcal{T}_U^-(v * n^{-1}) \leq \max\{\mathcal{T}_U^-(v), \mathcal{T}_U^-(n^{-1})\} = \min\{\mathcal{T}_U^-(v), \mathcal{T}_U^-(n)\}.$$

Thus for the other terms,

$$\mathcal{I}_U^+(v * n^{-1}) \geq \min\{\mathcal{I}_U^+(v), \mathcal{I}_U^+(n^{-1})\} = \min\{\mathcal{I}_U^+(v), \mathcal{I}_U^+(n)\}$$

and

$$\mathcal{I}_U^-(v * n^{-1}) \leq \max\{\mathcal{I}_U^-(v), \mathcal{I}_U^-(n^{-1})\} = \max\{\mathcal{I}_U^-(v), \mathcal{I}_U^-(n)\},$$

$$\mathcal{F}_U^+(v * n^{-1}) \leq \max\{\mathcal{F}_U^+(v), \mathcal{F}_U^+(n^{-1})\} = \max\{\mathcal{F}_U^+(v), \mathcal{F}_U^+(n)\}$$

and

$$\mathcal{F}_U^-(v * n^{-1}) \geq \min\{\mathcal{F}_U^-(v), \mathcal{F}_U^-(n^{-1})\} = \min\{\mathcal{F}_U^-(v), \mathcal{F}_U^-(n)\}.$$

Hence,

$$\begin{aligned}U(v * n^{-1}) &= (\mathcal{T}_U^+(v * n^{-1}), \mathcal{I}_U^+(v * n^{-1}), \mathcal{F}_U^+(v * n^{-1}), \mathcal{T}_U^-(v * n^{-1}), \mathcal{I}_U^-(v * n^{-1}), \mathcal{F}_U^-(v * n^{-1})) \\ &\geq (\min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(n)\}, \min\{\mathcal{I}_U^+(v), \mathcal{I}_U^+(n)\}, \max\{\mathcal{F}_U^+(v), \mathcal{F}_U^+(n)\}, \max\{\mathcal{T}_U^-(v), \mathcal{T}_U^-(n)\}), \\ &\max\{\mathcal{I}_U^-(v), \mathcal{I}_U^-(n)\}, \min\{\mathcal{F}_U^-(v), \mathcal{F}_U^-(n)\} \\ &= \min\{U^+(v), U^+(n)\} \quad \text{and} \quad \max\{U^-(v), U^-(n)\}.\end{aligned}$$

Conversely, for  $e$  in  $\tilde{G}$ . Then

$$\begin{aligned}\mathcal{I}_U^+(n^{-1}) &= \mathcal{I}_U^+(e * n^{-1}) \geq \min\{\mathcal{I}_U^+(e), \mathcal{I}_U^+(n^{-1})\} = \min\{\mathcal{I}_U^+(n * n^{-1}), \mathcal{I}_U^+(n^{-1})\} \\ &\geq \min\{\mathcal{I}_U^+(n), \mathcal{I}_U^+(n), \mathcal{I}_U^+(n)\} = \mathcal{I}_U^+(n).\end{aligned}$$

With similar steps for the rest of the bounds, we get  $U \in BNS(\tilde{G})$ .

□

*Theorem 3.13*

Assume that  $U$  and  $V$  are two BNSGs of  $\tilde{G}$ , then  $U \cap V$  is also a BNSG of  $\tilde{G}$ .

*Proof*

Let  $v, n \in \tilde{G}$  and based theorem 3.4, then we have:

$$U \cap V(v * n^{-1}) \geq \min\{U \cap V(v), U \cap V(n)\},$$

such that:

$$\mathcal{T}_{(U \cap V)}^+(v * n^{-1}) \geq \min\{\mathcal{T}_{(U \cap V)}^+(v), \mathcal{T}_{(U \cap V)}^+(n)\},$$

$$\mathcal{I}_{(U \cap V)}^+(v * n^{-1}) \geq \min\{\mathcal{I}_{(U \cap V)}^+(v), \mathcal{I}_{(U \cap V)}^+(n)\},$$

$$\mathcal{F}_{(U \cap V)}^+(v * n^{-1}) \leq \max\{\mathcal{F}_{(U \cap V)}^+(v), \mathcal{F}_{(U \cap V)}^+(n)\}.$$

Now, reflect on the positive and negative truth membership degree of the intersection as follows. The other inequalities are likewise proved:

$$\begin{aligned} \mathcal{T}_{(U \cap V)}^+(v * n^{-1}) &= \min\{\mathcal{T}_U^+(v * n^{-1}), \mathcal{T}_V^+(v * n^{-1})\} \\ &\geq \min(\min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(n)\}, \min\{\mathcal{T}_V^+(v), \mathcal{T}_V^+(n)\}) \\ &= \min(\min\{\mathcal{T}_U^+(v), \mathcal{T}_V^+(v)\}, \min\{\mathcal{T}_U^+(n), \mathcal{T}_V^+(n)\}) \\ &= \min\{\mathcal{T}_{(U \cap V)}^+(v), \mathcal{T}_{(U \cap V)}^+(n)\}. \end{aligned}$$

$$\begin{aligned} \mathcal{T}_{(U \cap V)}^-(v * n^{-1}) &= \max\{\mathcal{T}_U^-(v * n^{-1}), \mathcal{T}_V^-(v * n^{-1})\} \\ &\geq \min(\max\{\mathcal{T}_U^-(v), \mathcal{T}_U^-(n)\}, \max\{\mathcal{T}_V^-(v), \mathcal{T}_V^-(n)\}) \\ &= \max(\max\{\mathcal{T}_U^-(v), \mathcal{T}_V^-(v)\}, \max\{\mathcal{T}_U^-(n), \mathcal{T}_V^-(n)\}) \\ &= \max\{\mathcal{T}_{(U \cap V)}^-(v), \mathcal{T}_{(U \cap V)}^-(n)\}. \end{aligned}$$

The remaining parts are proven similarly based on the definition of intersection of BNSs. Therefore,  $U \cap V \in BNS(\tilde{G})$ . □

*Proposition 3.14*

Let  $U$  be a BNS of  $\tilde{G}$  if and only if for all  $\beta \in [-1, 1]$ , the notions of  $\beta$ -level sets of  $U$  given by  $\mathcal{T}_{U_\beta}^+, \mathcal{T}_{U_\beta}^-, \mathcal{I}_{U_\beta}^+, \mathcal{I}_{U_\beta}^-, \mathcal{F}_{U_\beta}^+, \mathcal{F}_{U_\beta}^-$  are classical subgroups of  $\tilde{G}$ .

*Proof*

In this proof, we will only deal with the positive and negative true membership sides, and the proofs for the other sides are similar based on the above definitions.

Let  $U$  be a BNSG of  $\tilde{G}$ ,  $\beta \in [-1, 1]$ , and for  $v, n \in \mathcal{T}_{U_\beta}^+$ . Then, by the assumption, we have:

$$\mathcal{T}_U^+(v * n^{-1}) \geq \min\{\mathcal{T}_U^+(v), \mathcal{T}_U^+(v * n^{-1})\} \geq \min\{\beta, \beta\} = \beta.$$

And

$$\mathcal{T}_U^-(v * n^{-1}) \leq \max\{\mathcal{T}_U^-(v * n^{-1}), \mathcal{T}_U^-(v * n^{-1})\} \leq \max\{\beta, \beta\} = \beta.$$

Hence, we conclude that  $\mathcal{T}_{U_\beta}^+, \mathcal{T}_{U_\beta}^-$  are classical subgroups of  $\tilde{G}$  for each  $\beta \in [-1, 1]$ .

Conversely, assume that  $\mathcal{T}_{U_\beta}^+, \mathcal{T}_{U_\beta}^-$  are classical subgroups of  $\tilde{G}$  for each  $\beta \in [-1, 1]$ . Let  $v, n \in \tilde{G}$ ,  $\beta = \min\{\mathcal{T}_{U_\beta}^+(v), \mathcal{T}_{U_\beta}^+(n)\}$  and  $\delta = \mathcal{T}_{U_\delta}^+(v)$ .

Since  $\mathcal{T}_{U_\beta}^+$  and  $\mathcal{T}_{U_\delta}^+$  are classical subgroups of  $\tilde{G}$ , then  $v * n \in \mathcal{T}_{U_\beta}^+$  and  $v^{-1} \in \mathcal{T}_{U_\delta}^+$ .

Thus,

$$\mathcal{T}_U^+(v * n) \geq \beta = \min\{\mathcal{T}_{U_\beta}^+(v), \mathcal{T}_{U_\beta}^+(n)\}$$

and

$$\mathcal{T}_U^+(v^{-1}) \geq \beta = \mathcal{T}_U^+(v).$$

Also, since  $\mathcal{T}_{U_\beta}^-$  and  $\mathcal{T}_{U_\delta}^-$  are classical subgroups of  $\tilde{G}$ , then  $v * n \in \mathcal{T}_{U_\beta}^-$  and  $v^{-1} \in \mathcal{T}_{U_\delta}^-$ .

Thus,

$$\mathcal{T}_U^-(v * n) \leq \beta = \max\{\mathcal{T}_{U_\beta}^-(v), \mathcal{T}_{U_\beta}^-(n)\}$$

and

$$\mathcal{T}_U^-(v^{-1}) \leq \beta = \mathcal{T}_U^-(v).$$

□

The remaining BNS parts are proved similarly. Hence, the conditions given in Definition 3.1 are satisfied.

**Definition 3.15.** Assume that a mapping  $F : G_1 \rightarrow G_2$  is a function from  $G_1$  into  $G_2$ . Let:

$$U_1 = \{s \in G_1, \mathcal{T}_{(U_1)}^+(s), \mathcal{I}_{(U_1)}^+(s), \mathcal{F}_{(U_1)}^+(s), \mathcal{T}_{(U_1)}^-(s), \mathcal{I}_{(U_1)}^-(s), \mathcal{F}_{(U_1)}^-(s)\}$$

$$U_2 = \{s \in G_2, \mathcal{T}_{(U_2)}^+(s), \mathcal{I}_{(U_2)}^+(s), \mathcal{F}_{(U_2)}^+(s), \mathcal{T}_{(U_2)}^-(s), \mathcal{I}_{(U_2)}^-(s), \mathcal{F}_{(U_2)}^-(s)\}$$

are two BN-subsets of  $G_1$  and  $G_2$  respectively. Then the set

$$F(U_1) = \{s \in G_1, F(\mathcal{T}_{(U_1)}^+(s)), F(\mathcal{I}_{(U_1)}^+(s)), F(\mathcal{F}_{(U_1)}^+(s)), F(\mathcal{T}_{(U_1)}^-(s)), F(\mathcal{I}_{(U_1)}^-(s)), F(\mathcal{F}_{(U_1)}^-(s))\}$$

is a BN-subset of  $G_2$  and is given as next:

$$F(\mathcal{T}_{(U_1)}^+)(t) = \begin{cases} \max\{\mathcal{T}_{(U_1)}^+(s) : s \in F^{-1}(t)\} & \text{if } F^{-1}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mathcal{T}_{(U_1)}^-)(t) = \begin{cases} \max\{\mathcal{T}_{(U_1)}^-(s) : s \in F^{-1}(t)\} & \text{if } F^{-1}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mathcal{F}_{(U_1)}^+)(t) = \begin{cases} \min\{\mathcal{F}_{(U_1)}^+(s) : s \in F^{-1}(t)\} & \text{if } F^{-1}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mathcal{F}_{(U_1)}^-)(t) = \begin{cases} \min\{\mathcal{F}_{(U_1)}^-(s) : s \in F^{-1}(t)\} & \text{if } F^{-1}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mathcal{I}_{(U_1)}^+)(t) = \begin{cases} \min\{\mathcal{I}_{(U_1)}^+(s) : s \in F^{-1}(t)\} & \text{if } F^{-1}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mathcal{I}_{(U_1)}^-)(t) = \begin{cases} \min\{\mathcal{I}_{(U_1)}^-(s) : s \in F^{-1}(t)\} & \text{if } F^{-1}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

*Theorem 3.16*

Suppose that  $F$  is a homomorphism mapping from  $\tilde{G}_1$  to  $\tilde{G}_2$ , and let  $U$  be a BN-subgroup of  $\tilde{G}_1$ . Then the image of  $U$ , i.e.,  $F(U)$ , is a BN-subgroup of  $\tilde{G}_2$ .

*Proof*

In this proof, we will only deal with the positive membership side, and the rest of the proof is similar based on the above definitions. Therefore, assume that  $v_1, m_2 \in \tilde{G}_1$  such that  $F(v_1) = n_1$  and  $F(v_2) = n_2$ . Since  $F$  is a group homomorphism, then:

$$\begin{aligned} F(\mathcal{T}_U^+)(n * n^{-1}) &= \min_{(n * n^{-1} = F(v))} \mathcal{T}_U^+(v) \geq \mathcal{T}_U^+(v * v^{-1}), \\ F(\mathcal{I}_U^+)(n * n^{-1}) &= \min_{(n * n^{-1} = F(v))} \mathcal{I}_U^+(v) \geq \mathcal{I}_U^+(v * v^{-1}), \\ F(\mathcal{F}_U^+)(n * n^{-1}) &= \max_{(n * n^{-1} = F(v))} \mathcal{F}_U^+(v) \leq \mathcal{F}_U^+(v * v^{-1}). \end{aligned}$$

Now we will work to prove that

$$FU(n_1 * n_2^{-1}) \geq \min\{FU(n_1), FU(n_2)\}.$$

Then,

$$FU(n_1 * n_2^{-1}) = (F(\mathcal{T}_U^+)(n_1 * n_2^{-1}), F(\mathcal{I}_U^+)(n_1 * n_2^{-1}), F(\mathcal{F}_U^+)(n_1 * n_2^{-1})).$$

This rewrites to:

$$= \left( \min_{(n * n^{-1} = F(v))} \mathcal{T}_U^+(v), \min_{(n * n^{-1} = F(v))} \mathcal{I}_U^+(v), \max_{(n * n^{-1} = F(v))} \mathcal{F}_U^+(v) \right)$$

From the above, we have:

$$\geq (\mathcal{T}_U^+(v * v^{-1}), \mathcal{I}_U^+(v * v^{-1}), \mathcal{F}_U^+(v * v^{-1})).$$

Thus,

$$\geq (\min\{\mathcal{T}_U^+(v_1), \mathcal{T}_U^+(v_2)\}, \min\{\mathcal{I}_U^+(v_1), \mathcal{I}_U^+(v_2)\}, \max\{\mathcal{F}_U^+(v_1), \mathcal{F}_U^+(v_2)\}).$$

This holds for each  $v_1, m_2 \in \tilde{G}_1$  with  $F(v_1) = n_1$  and  $F(v_2) = n_2$ . Then we get the following:

$$FU(n_1 * n_2^{-1}) \geq \min\{\mathcal{T}_U^+(v_1), \mathcal{I}_U^+(v_1), \mathcal{F}_U^+(v_1)\}, \{\mathcal{T}_U^+(v_2), \mathcal{I}_U^+(v_2), \mathcal{F}_U^+(v_2)\}.$$

The negative part is proven in a similar way based on the definition 3.15. Hence, the image of a BN-subgroup is also a BN-subgroup. □

*Theorem 3.17*

Let  $F$  be a homomorphism mapping from classical groups  $\tilde{G}_1$  to classical groups  $\tilde{G}_2$ , and let  $V$  be a BN-subgroup of  $\tilde{G}_2$ . Then the preimage  $F^{-1}V$  is a BN-subgroup of  $\tilde{G}_1$ .

*Proof*

In this proof, we will only deal with the positive membership side, and the rest of the proof is similar based on the above definitions. Therefore, assume that  $v_1, m_2 \in \tilde{G}_1$  such that  $F(v_1) = n_1$  and  $F(v_2) = n_2$ . Since  $F$  is a group homomorphism, we have:

$$\begin{aligned} F^{-1}V(v_1 * m_2^{-1}) &= (\mathcal{T}_V^+(F(v_1 * m_2^{-1})), \mathcal{I}_V^+(F(v_1 * m_2^{-1})), \mathcal{F}_V^+(F(v_1 * m_2^{-1}))) \\ &= (\mathcal{T}_V^+(F(v_1) * F(v_2^{-1})), \mathcal{I}_V^+(F(v_1) * F(v_2^{-1})), \mathcal{F}_V^+(F(v_1) * F(v_2^{-1}))) \end{aligned}$$

This implies:

$$\geq (\min\{\mathcal{T}_V^+(F(v_1)), \mathcal{T}_V^+(F(v_2))\}, \min\{\mathcal{I}_V^+(F(v_1)), \mathcal{I}_V^+(F(v_2))\},$$

$$\max\{\mathcal{F}_V^+(F(v_1)), \mathcal{F}_V^+(F(v_2))\})$$

Thus:

$$\begin{aligned} &= \min(\mathcal{T}_V^+(F(v_1)), \mathcal{I}_V^+(F(v_1)), \mathcal{F}_V^+(F(v_1)), \mathcal{T}_V^+(F(v_2)), \mathcal{I}_V^+(F(v_2)), \mathcal{F}_V^+(F(v_2))) \\ &= \min\{F^{-1}V(v_1), F^{-1}V(v_2)\}. \end{aligned}$$

The negative part is proven in a similar way based on the definition 3.15.

Therefore,  $F^{-1}V \in BN(\tilde{G}_1)$ . □

*Corollary 3.18*

Suppose that  $F$  be a homomorphism mapping from classical groups  $\tilde{G}_1$  to classical groups  $\tilde{G}_2$ , and let  $U$  be a BN-subgroup of  $\tilde{G}_1$ . Then we have next  $F^{-1}(FU) = A$ .

**4. Bipolar neutrosophic normal subgroup (BNNSG)**

**Definition 4.1.** Assume that  $(\tilde{G}, *)$  is a traditional group and  $U$  is a BNS on classical group  $\tilde{G}$ . Then  $U$  is named a bipolar neutrosophic normal subgroup (BNNSG) of  $\tilde{G}$  if the following conditions are satisfied:

$$U(v * n * v^{-1}) \geq U(n) \quad \text{for all } m, n \in \tilde{G}.$$

This means:

$$\begin{aligned} \mathcal{T}_U^+(v * n * v^{-1}) &\geq \mathcal{T}_U^+(n), & \mathcal{I}_U^+(v * n * v^{-1}) &\geq \mathcal{I}_U^+(n), & \mathcal{F}_U^+(v * n * v^{-1}) &\leq \mathcal{F}_U^+(n), \\ \mathcal{T}_U^-(v * n * v^{-1}) &\leq \mathcal{T}_U^-(n), & \mathcal{I}_U^-(v * n * v^{-1}) &\leq \mathcal{I}_U^-(n), & \mathcal{F}_U^-(v * n * v^{-1}) &\geq \mathcal{F}_U^-(n). \end{aligned}$$

Here, the collection of all the BNNSGs of classical group  $\tilde{G}$  is indicated by  $BNNS(\tilde{G})$ .

**Example 4.2.** Assume that the traditional-group  $\tilde{G} = \{1, -1, v, -v\}$  with multiplication operation. Then the bipolar neutrosophic set  $U$  on  $\tilde{G}$  is given as next value:

Table 2. The Cayley table of  $K_4$ .

$ *$	1	-1	$v$	$-v$
1	1	-1	$v$	$-v$
-1	-1	1	$-v$	$v$
$v$	$v$	$-v$	-1	1
$-v$	$-v$	$v$	1	-1

$$A = \left\{ \frac{\langle 0.3, 0.4, 0.5, -0.4, -0.6, -0.7 \rangle}{1}, \frac{\langle 0.3, 0.4, 0.5, -0.4, -0.6, -0.7 \rangle}{\tilde{m}}, \frac{\langle 0.3, 0.2, 0.5, -0.4, -0.6, -0.7 \rangle}{\tilde{n}}, \frac{\langle 0.3, 0.4, 0.5, -0.4, -0.6, -0.7 \rangle}{\tilde{m}\tilde{n}} \right\}$$

Based on Definition 4.1, it is clear that the bipolar neutrosophic set  $U$  is a bipolar neutrosophic normal subgroup of  $\tilde{G}$ .

*Theorem 4.3*

Let  $\tilde{G}$  be a traditional group and  $U, B \in BNNS(\tilde{G})$ . Then  $U \cap B \in BNNS(\tilde{G})$  also.

*Proof*

In this proof, we will only deal with the positive membership side, and the rest of the proof is similar based on the above definitions. Therefore, since  $U, B \in BNNS(\tilde{G})$ , we have:

$$\mathcal{T}_U^+(\rho * n * \rho^{-1}) \geq \mathcal{T}_U^+(n) \quad \text{and} \quad \mathcal{T}_V^+(\rho * n * \rho^{-1}) \geq \mathcal{T}_V^+(n).$$

Now, based on the definition of intersection, we get:

$$\begin{aligned} \mathcal{T}_{(U \cap V)}^+(v * n * v^{-1}) &= \min\{\mathcal{T}_U^+(v * n * v^{-1}), \mathcal{T}_V^+(v * n * v^{-1})\} \\ &\geq \min\{\mathcal{T}_U^+(n), \mathcal{T}_V^+(n)\} = \mathcal{T}_{(U \cap V)}^+(n). \end{aligned}$$

By a similar argument, we obtain:

$$\mathcal{I}_{(U \cap V)}^+(v * n * v^{-1}) \geq \mathcal{I}_{(U \cap V)}^+(n) \quad \text{and} \quad \mathcal{F}_{(U \cap V)}^+(v * n * v^{-1}) \geq \mathcal{F}_{(U \cap V)}^+(n).$$

Therefore, the intersection of two BNNSGs is also a BNNSG. □

**Proposition 4.4**

Assume that  $(\tilde{G}, *)$  is a traditional group and  $U$  is a bipolar neutrosophic set on classical group  $\tilde{G}$ . Then the following points are fulfilled:

- i.  $U \in BNNS(\tilde{G})$ .
- ii.  $U(v * n * v^{-1}) \geq U(n)$  for all  $v, n \in \tilde{G}$ .
- iii.  $U(v * n) = U(n * m)$  for all  $v, n \in \tilde{G}$ .

*Proof*

In this proof, we will only deal with the positive membership side, and the rest of the proof is similar based on the above definitions.

(i)  $\Rightarrow$  (ii): Let  $U$  be a bipolar neutrosophic normal subgroup of classical group  $\tilde{G}$  and for all  $v, n \in \tilde{G}$ . Then by definition:

$$\mathcal{T}_U^+(v * n * v^{-1}) \geq \mathcal{T}_U^+(n), \quad \mathcal{I}_U^+(v * n * v^{-1}) \geq \mathcal{I}_U^+(n), \quad \mathcal{F}_U^+(v * n * v^{-1}) \leq \mathcal{F}_U^+(n).$$

Thus, utilizing the arbitrary property of  $v$ , we obtain:

$$\mathcal{T}_U^+((v^{-1}) * n * m) = \mathcal{T}_U^+(v^{-1} * n * (v^{-1})^{-1}) \geq \mathcal{T}_U^+(n).$$

Therefore,

$$\mathcal{T}_U^+(n) = \mathcal{T}_U^+(v^{-1} * (v * n * v^{-1}) * m) \leq \mathcal{T}_U^+(v * n * v^{-1}) \leq \mathcal{T}_U^+(n),$$

i.e.,

$$\mathcal{T}_U^+(v * n * v^{-1}) = \mathcal{T}_U^+(n).$$

With similar steps for the other terms, we get:

$$\mathcal{I}_U^+(v * n * v^{-1}) = \mathcal{I}_U^+(n) \quad \text{and} \quad \mathcal{F}_U^+(v * n * v^{-1}) = \mathcal{F}_U^+(n).$$

Now, (ii)  $\Rightarrow$  (iii): This can be directly accessed by substituting  $n$  for  $n * m$  in the proof steps of (ii).

Now, (iii)  $\Rightarrow$  (i): Regarding  $U(v * n) = U(n * m)$  for all  $v, n \in \tilde{G}$ . Then:

$$U(v * n * v^{-1}) = U(v * v^{-1} * n) = U(n) \geq U(n).$$

Hence, it is satisfied. □

**Proposition 4.5**

Assume that  $\tilde{G}$  is a traditional group and  $U$  is a BNNSG of  $\tilde{G}$  if and only if for all  $\beta \in [-1, 1]$ , the  $\beta$ -level set of  $U$  given by  $\mathcal{T}_{U_\beta}^+, \mathcal{T}_{U_\beta}^-, \mathcal{I}_{U_\beta}^+, \mathcal{I}_{U_\beta}^-, \mathcal{F}_{U_\beta}^+, \mathcal{F}_{U_\beta}^-$  are classical subgroups of traditional group  $\tilde{G}$ .

*Proof*

The proof is clear and similar to the proof steps mentioned in the above proposition. □

**Definition 4.6.** Assume that  $(\tilde{G}, *)$  is a traditional group and  $U$  is a BNS on traditional group  $\tilde{G}$ . The normalizer of a BN-subgroup  $U$  in classical group  $\tilde{G}$  is given as follows:

$$NU = \{n \in \tilde{G} \mid U(n^{-1} * m * n) \geq \min\{U(v), U(n^{-1} * m * n)\} \leq U(v), \quad \forall m \in \tilde{G}\}.$$

*Theorem 4.7*

Let  $\tilde{G}$  be a traditional group and  $U \in BNSG(\tilde{G})$ . Then:

1.  $NU$  is a subgroup of  $\tilde{G}$ .
2.  $U$  is a BNNSG of  $\tilde{G}$  if and only if  $NU = \tilde{G}$ .

*Proof*

(i) In this proof, we will only deal with the positive and negative true memberships side, and the rest of the proof is similar based on the above definitions. Therefore, suppose that  $n_1, n_2 \in NU$  and we need to prove that  $n_1 n_2^{-1} \in NU$ .

For any  $v \in \tilde{G}$ , we have:

$$\mathcal{I}_U^+((n_1 n_2^{-1})^{-1} m (n_1 n_2^{-1})) = \mathcal{I}_U^+((n_2 n_1^{-1}) m (n_1 n_2^{-1})).$$

$$\mathcal{I}_U^+(n_2 (n_1^{-1} m n_1) n_2^{-1}) \geq \mathcal{I}_U^+(n_1^{-1} m n_1) \geq \mathcal{I}_U^+(v).$$

And,

$$\mathcal{I}_U^-((n_1 n_2^{-1})^{-1} m (n_1 n_2^{-1})) = \mathcal{I}_U^-((n_2 n_1^{-1}) m (n_1 n_2^{-1})).$$

$$\mathcal{I}_U^-(n_2 (n_1^{-1} m n_1) n_2^{-1}) \leq \mathcal{I}_U^-(n_1^{-1} m n_1) \leq \mathcal{I}_U^-(v).$$

Thus, we conclude that  $n_1 n_2^{-1} \in NU$ , which implies that  $NU$  is a subgroup of  $\tilde{G}$ .

(ii) This part follows directly from the definition of BNNSG of  $\tilde{G}$  mentioned above. □

**Definition 4.8.** Assume that  $(\tilde{G}, *)$  is a traditional group and for two BN subgroups  $U_1$  and  $U_2$  of traditional group  $\tilde{G}$ . We say that

$$U_1 = \{m, \langle \mathcal{T}_{(U_1)}(v), \mathcal{I}_{(U_1)}(v), \mathcal{F}_{(U_1)}(v) \rangle\}$$

is a BN conjugate of

$$U_2 = \{m, \langle \mathcal{T}_{(U_2)}(v), \mathcal{I}_{(U_2)}(v), \mathcal{F}_{(U_2)}(v) \rangle\}$$

if for any  $v \in \tilde{G}$  we have the following conditions:

$$\mathcal{T}_{(U_1)}(n) = \mathcal{T}_{(U_2)}(v^{-1} n m), \quad \mathcal{I}_{(U_1)}(n) = \mathcal{I}_{(U_2)}(v^{-1} n m), \quad \mathcal{F}_{(U_1)}(n) = \mathcal{F}_{(U_2)}(v^{-1} n m)$$

for both sides (positive and negative).

**Definition 4.9.** Assume that  $(\tilde{G}, *)$  is a traditional group and a BNSG

$$A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle\}$$

of traditional group  $\tilde{G}$ . The BN left coset in traditional group  $\tilde{G}$  is given as a BN subset:

$$U^l = \{m, \langle \mathcal{T}_{(U^l)}(v), \mathcal{I}_{(U^l)}(v), \mathcal{F}_{(U^l)}(v) \rangle\}$$

of traditional group  $\tilde{G}$ , such that:

$$\mathcal{T}_{(U^l)}^+(n) = \mathcal{T}_{(U^l)}^+(v^{-1} n), \quad \mathcal{I}_{(U^l)}^+(n) = \mathcal{I}_{(U^l)}^+(v^{-1} n), \quad \mathcal{F}_{(U^l)}^+(n) = \mathcal{F}_{(U^l)}^+(v^{-1} n)$$

For the negative side:

$$\mathcal{T}_{(U^l)}^-(n) = \mathcal{T}_{(U^l)}^-(v^{-1} n), \quad \mathcal{I}_{(U^l)}^-(n) = \mathcal{I}_{(U^l)}^-(v^{-1} n), \quad \mathcal{F}_{(U^l)}^-(n) = \mathcal{F}_{(U^l)}^-(v^{-1} n).$$

**Definition 4.10.** Assume that  $(\tilde{G}, *)$  is a traditional group and a BNSG

$$A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle\}$$

of traditional group  $\tilde{G}$ . The BN right coset in traditional group  $\tilde{G}$  is given as a BN subset:

$$U^r = \{m, \langle \mathcal{T}_{(U^r)}(v), \mathcal{I}_{(U^r)}(v), \mathcal{F}_{(U^r)}(v) \rangle\}$$

of traditional group  $\tilde{G}$ , such that:

$$\mathcal{T}_{(U^r)}^+(n) = \mathcal{T}_{(U^r)}^+(nv^{-1}), \quad \mathcal{I}_{(U^r)}^+(n) = \mathcal{I}_{(U^r)}^+(nv^{-1}), \quad \mathcal{F}_{(U^r)}^+(n) = \mathcal{F}_{(U^r)}^+(nv^{-1})$$

For the negative side:

$$\mathcal{T}_{(U^r)}^-(n) = \mathcal{T}_{(U^r)}^-(nv^{-1}), \quad \mathcal{I}_{(U^r)}^-(n) = \mathcal{I}_{(U^r)}^-(nv^{-1}), \quad \mathcal{F}_{(U^r)}^-(n) = \mathcal{F}_{(U^r)}^-(nv^{-1}).$$

*Theorem 4.11*

Assume that  $\tilde{G}$  is a traditional group and a BNSG

$$A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle\}$$

of traditional group  $\tilde{G}$  called a BN normal subgroup of  $\tilde{G}$  if and only if  $\forall m \in \tilde{G}$ :

$$\mathcal{T}_{(U^l)}^+(v) = \mathcal{T}_{(U^r)}^+(v), \quad \mathcal{I}_{(U^l)}^+(v) = \mathcal{I}_{(U^r)}^+(v), \quad \mathcal{F}_{(U^l)}^+(v) = \mathcal{F}_{(U^r)}^+(v),$$

$$\mathcal{T}_{(U^l)}^-(v) = \mathcal{T}_{(U^r)}^-(v), \quad \mathcal{I}_{(U^l)}^-(v) = \mathcal{I}_{(U^r)}^-(v), \quad \mathcal{F}_{(U^l)}^-(v) = \mathcal{F}_{(U^r)}^-(v).$$

*Proof*

Assume that

$$A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle\}$$

is a BNNSG of traditional group  $\tilde{G}$  and we need to prove (1) and (2).

For all  $v \in \tilde{G}$ , we have:

$$\begin{aligned} \mathcal{T}_{(U^l)}^+(n) &= \mathcal{T}_{(U^l)}^+(v^{-1}n) = \mathcal{T}_{(U^l)}^+(n^{-1}nv^{-1}n) \\ &= \mathcal{T}_U^+(n^{-1}(nv^{-1})n) \geq \mathcal{T}_U^+(nv^{-1}) = \mathcal{T}_{(U^r)}^+(n). \end{aligned}$$

Thus,

$$\mathcal{T}_{(U^l)}^+(n) \geq \mathcal{T}_{(U^r)}^+(n).$$

Similarly, we can apply the same procedures for the remaining positive terms. Now, again  $\forall m \in \tilde{G}$ :

$$\begin{aligned} \mathcal{T}_{(U^r)}^+(n) &= \mathcal{T}_{(U^r)}^+(nv^{-1}) = \mathcal{T}_{(U^l)}^+(n^{-1}nv^{-1}n) \\ &= \mathcal{T}_U^+(n^{-1}(nv^{-1})n) \geq \mathcal{T}_U^+(v^{-1}n) = \mathcal{T}_{(U^l)}^+(n). \end{aligned}$$

Thus,

$$\mathcal{T}_{(U^r)}^+(n) \geq \mathcal{T}_{(U^l)}^+(n).$$

For the negative terms, we have:

$$\begin{aligned} \mathcal{T}_{(U^l)}^-(n) &= \mathcal{T}_{(U^l)}^-(v^{-1}n) = \mathcal{T}_{(U^l)}^-(n^{-1}nv^{-1}n) \\ &= \mathcal{T}_U^-(n^{-1}(nv^{-1})n) \leq \mathcal{T}_U^-(nv^{-1}) = \mathcal{T}_{(U^r)}^-(n). \end{aligned}$$

Thus,

$$\mathcal{T}_{(\bar{U}^l)}^-(n) \leq \mathcal{T}_{(\bar{U}^r)}^-(n).$$

Similarly, we can apply the same procedures for the remaining negative terms. Now, again  $\forall m \in \tilde{G}$ :

$$\begin{aligned} \mathcal{T}_{(\bar{U}^r)}^-(n) &= \mathcal{T}_{(\bar{U}^r)}^-(nv^{-1}) = \mathcal{T}_{(\bar{U}^l)}^-(n^{-1}nv^{-1}n) \\ &= \mathcal{T}_{\bar{U}}^-(n^{-1}(nv^{-1})n) \leq \mathcal{T}_{\bar{U}}^-(v^{-1}n) = \mathcal{T}_{(\bar{U}^l)}^-(n). \end{aligned}$$

Thus,

$$\mathcal{T}_{(\bar{U}^r)}^-(n) \leq \mathcal{T}_{(\bar{U}^l)}^-(n).$$

□

**Definition 4.12.** Assume that  $(\tilde{G}, *)$  is a traditional group and a BNSG

$$A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle\}$$

of  $\tilde{G}$ . Then  $U$  is called an abelian subgroup of  $\tilde{G}$  if and only if  $\forall m, n \in \tilde{G}$ :

$$\mathcal{T}_U(vn) = \mathcal{T}_U(nm), \quad \mathcal{I}_U(vn) = \mathcal{I}_U(nm), \quad \mathcal{F}_U(vn) = \mathcal{F}_U(nm).$$

This implies:

$$\begin{aligned} \mathcal{T}_U^+(vn) &= \mathcal{T}_U^+(nm), \quad \mathcal{I}_U^+(vn) = \mathcal{I}_U^+(nm), \quad \mathcal{F}_U^+(vn) = \mathcal{F}_U^+(nm), \\ \mathcal{T}_U^-(vn) &= \mathcal{T}_U^-(nm), \quad \mathcal{I}_U^-(vn) = \mathcal{I}_U^-(nm), \quad \mathcal{F}_U^-(vn) = \mathcal{F}_U^-(nm). \end{aligned}$$

**Example 4.13.** Consider the values given in the previous example. It is clear that the BNSG on  $\tilde{G} = K_4$  is an abelian subgroup.

*Theorem 4.14*

Assume that  $A = \{m, \langle \mathcal{T}_U(v), \mathcal{I}_U(v), \mathcal{F}_U(v) \rangle\}$  is a BN subgroup of  $\tilde{G}$ . Then  $\forall m, n \in \tilde{G}$ , the following axioms are equivalent:

1.  $U$  is called an abelian subgroup of  $\tilde{G}$ .
2.  $U$  is called a normal subgroup of  $\tilde{G}$ .

*Proof*

This following prof is directly from the above definition.

**Proof (1).** For  $t, s \in G$

$$\begin{aligned} \mathcal{T}_A(s * t) = \mathcal{T}_A(t * s) &\Rightarrow \mathcal{T}_U^+(s * t) = \mathcal{T}_U^+(t * s) \quad \text{and} \quad \mathcal{T}_U^-(s * t) = \mathcal{T}_U^-(t * s) \\ \mathcal{I}_U^-(t * s) = \mathcal{I}_A(t * s) &\Rightarrow \mathcal{I}_U^+(s * t) = \mathcal{I}_U^+(t * s) \quad \text{and} \quad \mathcal{I}_U^-(s * t) = \mathcal{I}_U^-(t * s) \end{aligned}$$

$$\mathcal{F}_U^+(s * t) = \mathcal{F}_U^+(t * s) \Rightarrow \mathcal{F}_U^+(s * t) = \mathcal{F}_U^+(t * s) \quad \text{and} \quad \mathcal{F}_U^-(s * t) = \mathcal{F}_U^-(t * s)$$

Then we take the positive true term. We will only cover this part, and the rest of the parts will be done in the same way:

$$\begin{aligned} \mathcal{T}_U^+(s * t) &= \mathcal{T}_U^+(s * t * s * s^{-1}) = \mathcal{T}_U^+(s * (t * s) * s^{-1}) \geq \mathcal{T}_U^+(s * t) \\ &\Rightarrow \mathcal{T}_U^+(s * t) \geq \mathcal{T}_U^+(t * s). \end{aligned}$$

And

$$\begin{aligned} \mathcal{T}_U^+(t * s) &= \mathcal{T}_U^+(t * s * t * t^{-1}) = \mathcal{T}_U^+(t * (s * t) * t^{-1}) \geq \mathcal{T}_U^+(t * s) \\ &\Rightarrow \mathcal{T}_U^+(t * s) \geq \mathcal{T}_U^+(s * t). \end{aligned}$$

Thus

$$\mathcal{T}_U^+(t * s) = \mathcal{T}_U^+(s * t).$$

Next, we take the positive true term

$$\begin{aligned} \mathcal{T}_U^-(s * t) &= \mathcal{T}_U^-(s * t * s * s^{-1}) = \mathcal{T}_U^-(s * (t * s) * s^{-1}) \leq \mathcal{T}_U^-(s * t) \\ &\Rightarrow \mathcal{T}_U^-(s * t) \leq \mathcal{T}_U^-(t * s). \end{aligned}$$

And

$$\begin{aligned} \mathcal{T}_U^-(t * s) &= \mathcal{T}_U^-(t * s * t * t^{-1}) = \mathcal{T}_U^-(t * (s * t) * t^{-1}) \leq \mathcal{T}_U^-(t * s) \\ &\Rightarrow \mathcal{T}_U^-(t * s) \leq \mathcal{T}_U^-(s * t). \end{aligned}$$

Thus,

$$\mathcal{T}_U^-(t * s) = \mathcal{T}_U^-(s * t).$$

$$\therefore \mathcal{T}_A(s * t) = \mathcal{T}_A(t * s).$$

**Proof (2).** It is proven in a similar way based on the definition 4.1. □

### 5. More results about BN-Matrices

In this part, we will put on some new forms about algebraic structures, namely BN-matrices. We study some related mathematical properties. In addition, we will provide a practical application of these tools in dealing with a decision-making problem.

**Definition 5.1.** The mathematical structure of BNS mentioned in definition 2.4 is given in matrix form as follows:

$$\Omega = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

Where

$$\langle \Omega_{(i,j)} \rangle = \langle \mathcal{T}_{((i,j))}^+(v), \mathcal{I}_{((i,j))}^+(v), \mathcal{F}_{((i,j))}^+(v), \mathcal{T}_{((i,j))}^-(v), \mathcal{I}_{((i,j))}^-(v), \mathcal{F}_{((i,j))}^-(v) \rangle$$

and each boundary retains its fuzzy properties such that every term belongs to the closed interval  $[-1, 1]$ .

**Example 5.2.** The following matrix form represent the BN-matrix:

$$\Omega = \begin{bmatrix} (0.3 & 0.7 & 0.7 & -0.6 & -0.2 & -0.1) & (0.8 & 0.4 & 0.7 & -0.8 & -0.9 & -0.5) \\ (0.1 & 0.2 & 0.2 & -0.1 & -0.1 & -0.1) & (0.9 & 0.2 & 0.8 & -0.5 & -0.6 & -0.7) \\ (0.6 & 0.5 & 0.3 & -0.2 & -0.9 & -0.2) & (0.3 & 0.9 & 0.4 & -0.6 & -0.3 & -0.3) \end{bmatrix}_{3 \times 2}$$

**Definition 5.3.** A BNM  $\Omega$  denotes as BN-empty matrix  $\langle \Phi_{(i,j)} \rangle$  and given as following:

$$\Phi_{(i,j)} = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

where

$$\langle \Omega_{(i,j)} \rangle = \langle 0, 1, 1, 0, -1, -1 \rangle$$

**Definition 5.4.** A BNM  $\Omega$  denotes as BN-obsolete matrix  $\langle \Omega_{(i,j)} \rangle$  and given as following:

$$\Omega = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

where

$$\langle \Omega_{(i,j)} \rangle = \langle 1, 0, 0, -1, 0, 0 \rangle$$

**Definition 5.5.** For two BNMs given as following:

$$\Omega_1 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

and

$$\Omega_2 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

Then  $\Omega_1 + \Omega_2 = \Omega_3$

where

$$\Omega_3 = \left[ \begin{array}{ccc} \frac{\mathcal{T}_{((i,j))}^{+,1} + \mathcal{T}_{((i,j))}^{+,2}}{2}, & \frac{\mathcal{F}_{((i,j))}^{+,1} + \mathcal{F}_{((i,j))}^{+,2}}{2}, & \frac{\mathcal{I}_{((i,j))}^{+,1} + \mathcal{I}_{((i,j))}^{+,2}}{2}, \\ \frac{\mathcal{T}_{((i,j))}^{-,1} + \mathcal{T}_{((i,j))}^{-,2}}{2}, & \frac{\mathcal{F}_{((i,j))}^{-,1} + \mathcal{F}_{((i,j))}^{-,2}}{2}, & \frac{\mathcal{I}_{((i,j))}^{-,1} + \mathcal{I}_{((i,j))}^{-,2}}{2} \end{array} \right], \tag{5.1}$$

**Definition 5.6.** For two BNMs given as following:

$$\Omega_1 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

and

$$\Omega_2 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

Then  $\Omega_1 \times \Omega_2 = \Omega_3$

where

$$\Omega_3 = \left[ \begin{array}{ccc} \mathcal{T}_{((i,j))}^{+,1} \times \mathcal{T}_{((i,j))}^{+,2}, & \mathcal{F}_{((i,j))}^{+,1} \times \mathcal{F}_{((i,j))}^{+,2}, & \mathcal{I}_{((i,j))}^{+,1} \times \mathcal{I}_{((i,j))}^{+,2}, \\ -(\mathcal{T}_{((i,j))}^{-,1} \times \mathcal{T}_{((i,j))}^{-,2}), & -(\mathcal{F}_{((i,j))}^{-,1} \times \mathcal{F}_{((i,j))}^{-,2}), & -(\mathcal{I}_{((i,j))}^{-,1} \times \mathcal{I}_{((i,j))}^{-,2}) \end{array} \right] \tag{5.2}$$

**Definition 5.7.** For two BNMs given as following:

$$\Omega_1 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

and

$$\Omega_2 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

Then  $\Omega_1 - \Omega_2 = \Omega_3$

where

$$\Omega_3 = \left[ \begin{array}{ccc} |\mathcal{T}_{((i,j))}^{+,1} - \mathcal{T}_{((i,j))}^{+,2}|, & |\mathcal{F}_{((i,j))}^{+,1} - \mathcal{F}_{((i,j))}^{+,2}|, & |\mathcal{I}_{((i,j))}^{+,1} - \mathcal{I}_{((i,j))}^{+,2}| \\ \frac{(\mathcal{T}_{((i,j))}^{-,1} + \mathcal{T}_{((i,j))}^{-,2})}{2}, & \frac{\mathcal{F}_{((i,j))}^{-,1} + \mathcal{F}_{((i,j))}^{-,2}}{2}, & \frac{\mathcal{I}_{((i,j))}^{-,1} + \mathcal{I}_{((i,j))}^{-,2}}{2} \end{array} \right] \quad (5.3)$$

**Definition 5.8.** For a BNM given as following:

$$\Omega_1 = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

Then scalar multiplication  $K\Omega_1$  given as following  
where

$$K\Omega_1 = \begin{bmatrix} \langle K\Omega_{1,1} \rangle & \langle K\Omega_{1,2} \rangle & \cdots & \langle K\Omega_{1,j} \rangle \\ \langle K\Omega_{2,1} \rangle & \langle K\Omega_{2,2} \rangle & \cdots & \langle K\Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle K\Omega_{i,1} \rangle & \langle K\Omega_{i,2} \rangle & \cdots & \langle K\Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

where  $K \in [-1, 1]$

**Example 5.9.** Assume that  $\Omega_1$  and  $\Omega_2$  are two BNMs and given as next

$$\Omega_1 = \begin{bmatrix} (0.6 & 0.1 & 0.1 & -0.6 & -0.2 & -0.1) \\ (0.3 & 0.2 & 0.2 & -0.1 & -0.1 & -0.1) \\ (0.8 & 0.5 & 0.3 & -0.2 & -0.9 & -0.2) \end{bmatrix}_{3 \times 1}$$

$$\Omega_2 = \begin{bmatrix} (0.1 & 0.4 & 0.3 & -0.5 & -0.2 & -0.6) \\ (0.2 & 0.2 & 0.2 & -0.1 & -0.1 & -0.1) \\ (0.4 & 0.5 & 0.4 & -0.2 & -0.4 & -0.5) \end{bmatrix}_{3 \times 1}$$

Then

$$\Omega_1 + \Omega_2 = \begin{bmatrix} (0.35 & 0.25 & 0.2 & -0.55 & -0.2 & -0.35) \\ (0.25 & 0.2 & 0.2 & -0.1 & -0.1 & -0.1) \\ (0.6 & 0.5 & 0.35 & -0.2 & -0.65 & -0.35) \end{bmatrix}_{3 \times 1}$$

$$\Omega_1 - \Omega_2 = \begin{bmatrix} (0.5 & 0.3 & 0.2 & -0.55 & -0.2 & -0.35) \\ (0.1 & 0 & 0.2 & -0.1 & -0.1 & -0.1) \\ (0.4 & 0 & 0.1 & -0.2 & -0.65 & -0.35) \end{bmatrix}_{3 \times 1}$$

For  $K = -1$

$$K\Omega_1 = \begin{bmatrix} (-0.1 & -0.4 & -0.3 & 0.5 & 0.2 & 0.6) \\ (-0.2 & -0.2 & -0.2 & 0.1 & 0.1 & 0.1) \\ (-0.4 & -0.5 & -0.4 & 0.2 & 0.4 & 0.5) \end{bmatrix}_{3 \times 1}$$

$$\Omega_1 \times \Omega_2 = \begin{bmatrix} (0.06 & 0.04 & 0.03 & -0.3 & -0.04 & -0.06) \\ (0.25 & 0.2 & 0.2 & -0.01 & -0.01 & -0.01) \\ (0.6 & 0.5 & 0.35 & -0.04 & -0.36 & -0.1) \end{bmatrix}_{3 \times 1}$$

**Proposition 5.10**

Assume that  $\Omega_1, \Omega_2$  and  $\Omega_3$  are three BN-matrices. Then the next points are fulfilled:

1.  $\Omega_1 + \Omega_2 = \Omega_2 + \Omega_1$ .
2.  $\Omega_1 - \Omega_2 = \Omega_2 - \Omega_1$ .
3.  $K(\Omega_1 + \Omega_2) = (K\Omega_1 + K\Omega_2)$
4.  $K(\Omega_1 - \Omega_2) = (K\Omega_1 - K\Omega_2)$

*Proof*

The proof of these issues is directly based on the definitions provided above. □

Now we know the mechanism for finding the BN-matrix energy. This definition will help us deal with some everyday applications that are characterized by an environment of uncertainty.

**Definition 5.11.** The mathematical structure of BNS mentioned in definition 2.4 is given in matrix form as follows:

$$\Omega = \begin{bmatrix} \langle \Omega_{1,1} \rangle & \langle \Omega_{1,2} \rangle & \cdots & \langle \Omega_{1,j} \rangle \\ \langle \Omega_{2,1} \rangle & \langle \Omega_{2,2} \rangle & \cdots & \langle \Omega_{2,j} \rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle \Omega_{i,1} \rangle & \langle \Omega_{i,2} \rangle & \cdots & \langle \Omega_{i,j} \rangle \end{bmatrix}_{m \times n}$$

Where

$$\langle \Omega_{(i,j)} \rangle = \langle \mathcal{T}_{((i,j))}^+(v), \mathcal{I}_{((i,j))}^+(v), \mathcal{F}_{((i,j))}^+(v), \mathcal{T}_{((i,j))}^-(v), \mathcal{I}_{((i,j))}^-(v), \mathcal{F}_{((i,j))}^-(v) \rangle$$

and each boundary retains its fuzzy properties such that every term belongs to the closed interval  $[-1, 1]$ .

Then the energy of BNM is given as follows formally:

$$E(\Omega) = \sum_{i=1, j=1}^{n, m} \left( \left( \mathcal{T}_{((i,j))}^+ + \mathcal{F}_{((i,j))}^+ - \mathcal{I}_{((i,j))}^+ \right) - \left( \mathcal{T}_{((i,j))}^- + \mathcal{F}_{((i,j))}^- - \mathcal{I}_{((i,j))}^- \right) \right) \tag{5.4}$$

**5.1 Application in a decision-making environment**

Here, we will apply the BN-matrix energy-extraction mechanism to help the user (decision-maker) make the best choice.

**Example 5.12.** To help Mr. Xua choose the right car from among three cars that can be represented by  $X = \{q_1, q_2, q_3\}$  according to certain criteria: price and model. We will evaluate each of these three cars according to these criteria, with the evaluation taking the form of a numerical matrix. Next, we will extract the matrix energy for each matrix and compare them to find the best one.

Below are three matrices  $\Omega_1, \Omega_2, \Omega_3$  representing the value of each car.

$$\Omega_1 = \begin{bmatrix} (0.6 & 0.1 & 0.1 & -0.6 & -0.2 & -0.1) & (0.4 & 0.7 & 0.7 & -0.8 & -0.3 & -0.5) \\ (0.3 & 0.2 & 0.2 & -0.1 & -0.1 & -0.1) & (0.8 & 0.1 & 0.8 & -0.4 & -0.5 & -0.9) \\ (0.8 & 0.5 & 0.3 & -0.2 & -0.9 & -0.2) & (0.9 & 0.9 & 0.4 & -0.6 & -0.3 & -0.3) \end{bmatrix}_{3 \times 2}$$

$$\Omega_2 = \begin{bmatrix} (0.1 & 0.4 & 0.3 & -0.5 & -0.2 & -0.6) & (0.4 & 0.1 & 0.5 & -0.7 & -0.7 & -0.3) \\ (0.2 & 0.2 & 0.2 & -0.1 & -0.1 & -0.1) & (0.2 & 0.1 & 0.8 & -0.4 & -0.5 & -0.6) \\ (0.4 & 0.5 & 0.4 & -0.2 & -0.4 & -0.5) & (0.3 & 0.4 & 0.6 & -0.7 & -0.4 & -0.8) \end{bmatrix}_{3 \times 2}$$

$$\Omega_3 = \begin{bmatrix} (0.1 & 0.1 & 0.1 & -0.6 & -0.2 & -0.1) & (0.2 & 0.5 & 0.7 & -0.8 & -0.3 & -0.3) \\ (0.8 & 0.3 & 0.2 & -0.1 & -0.1 & -0.1) & (0.4 & 0.3 & 0.8 & -0.4 & -0.5 & -0.2) \\ (0.9 & 0.5 & 0.3 & -0.2 & -0.9 & -0.2) & (0.5 & 0.9 & 0.4 & -0.6 & -0.3 & -0.1) \end{bmatrix}_{3 \times 2}$$

Then, using equation 3, we obtain the matrix energy values given as following:

$$E(\Omega_1) = 2.873, E(\Omega_2) = 2.504, E(\Omega_3) = 1.942$$

We conclude from the above that the first car  $q_1$  is the best choice according to the accepted criteria.

## 6. Conclusion

In continuation of the recent research work presented in the fuzzy algebraic environment, and to explain the apparatus of operation of the bipolar system in the fuzzy algebraic environment. In this study, we applied this idea in an algebraic environment when we initiated the novel concept of bipolar neutrosophic subgroups and proved that every bipolar neutrosophic subgroup generates two bipolar neutrosophic subgroups. We explained the level set, support, kernel for bipolar single value neutrosophic set, bipolar neutrosophic characteristic function, and bipolar neutrosophic point. Then, we illuminated the bipolar neutrosophic subgroup, bipolar neutrosophic normal subgroup, bipolar neutrosophic conjugate, normalizer for bipolar neutrosophic subgroup, bipolar neutrosophic abelian subgroup, and bipolar neutrosophic factor group. Furthermore, we presented the linked theorems and examples and prove these theorems. We presented a new mathematical framework for the concept of BN-matrices and some related mathematical properties. In addition, we provided a practical application of these tools in dealing with a decision-making (DM) problem. Finally, we discussed the image and pre-image of bipolar neutrosophic subgroups under homomorphism and proved the related theorems. However, the result presented in this work has some limitations as well, and the investigated work cannot be employed for the examination of molecule structures. BN subgroups can be generalised to bipolar complex neutrosophic subgroups, bipolar complex picture fuzzy subgroups, bipolar complex neutrosophic soft groups, etc. This work is limited to the structure of real numbers, and therefore, from a future perspective, we aim to expand this research to complex number environments when these results expand to bipolar complex neutrosophic soft sets [44, 45], and complex bipolar complex neutrosophic N-soft sets [46]. We hope that these concepts will form the basis for innovative research on subgroups.

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