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HyperInterval-valued and SuperHyperInterval-valued Fuzzy/Neutrosophic Set

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Abstract We study uncertainty models built from interval families over a finite universe. An interval set collects all subsets bounded between a designated lower and upper set. A HyperInterval set assigns to each base interval a nonempty family of admissible refinements, while a SuperHyperInterval set of order n maps elements of the n-fold iterated powerset to $(n-1)$-nested families, enabling hierarchical evidence organization. On the numeric side, an interval-valued fuzzy set attaches to each element an interval of admissible memberships, and an interval-valued neutrosophic set assigns independent intervals for truth, indeterminacy, and falsity. Building on these primitives, we introduce HyperInterval- and SuperHyperInterval-valued fuzzy/neutrosophic sets, define conjunctive "core" (intersection) and disjunctive "hull" semantics, and prove embedding theorems showing that classical interval, fuzzy, and neutrosophic models appear as singleton or degenerate cases. The framework unifies multi-source and hierarchical evidence, offering transparent bounds for conservative and exploratory decision policies.

Keywords Interval set, HyperInterval set, SuperHyperInterval set, Interval-valued Fuzzy Set, Interval-valued Neutrosophic Set

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1. Introduction

1.1. Fuzzy Set and Neutrosophic Set

Over the past several decades, many mathematical frameworks have been proposed for representing and processing uncertainty, such as Fuzzy Sets [1], soft sets [2, 3], rough sets [4, 5], Intuitionistic Fuzzy Sets [6, 7], Uncertain Sets

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[3, 8], and Neutrosophic Sets [9]. In this paper, our focus is on the Neutrosophic Set paradigm. In a Neutrosophic Set, each element is described by three membership functions: a degree of truth T, a degree of indeterminacy I, and a degree of falsity F. These values are typically taken in $[0, 1]$ and satisfy

$$0 \leq T + I + F \leq 3,$$

as introduced and further developed in [10].

A number of important refinements and extensions of the basic Neutrosophic Set have been investigated in the literature, including Bipolar Neutrosophic Sets [11], Hesitant Neutrosophic Sets [12], and q-rung Orthopair Neutrosophic Sets [13]. These generalized models have recently attracted considerable research interest.

It is also well known that the Neutrosophic framework contains several earlier theories of uncertainty as particular cases, including Fuzzy Sets [1], Intuitionistic Fuzzy Sets [7], Vague Sets [14], and Hesitant Fuzzy Sets [15].

For convenience, Table 1 gives a concise comparison of Classical, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets in terms of their membership components and basic characteristics. In view of these developments and applications, the systematic study of Neutrosophic Sets remains an important and active research topic.

Table 1. Compact comparison of Classical, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Sets

Model	Membership components	Short description
Classical Set	$\chi_A(x) \in \{0, 1\}$	Crisp membership given by the characteristic function $\chi_A(x)$; each element either fully belongs to A or does not belong at all.
Fuzzy Set	$\mu_A(x) \in [0, 1]$	Single membership degree $\mu_A(x)$ in $[0, 1]$, allowing graded inclusion between 0 (not in A) and 1 (fully in A).
Intuitionistic Fuzzy Set	$\mu_A(x), \nu_A(x) \in [0, 1]$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$	Membership $\mu_A(x)$ and nonmembership $\nu_A(x)$; the hesitation degree is $1 - \mu_A(x) - \nu_A(x)$.
Neutrosophic Set	$T_A(x), I_A(x), F_A(x) \in [0, 1]$	Separate degrees of truth $T_A(x)$, indeterminacy $I_A(x)$, and falsity $F_A(x)$, handled independently.

1.2. Interval-valued Fuzzy/Neutrosophic Set

An interval set collects all subsets A bounded below and above by A_ℓ and A_u, respectively, satisfying $A_\ell \subseteq A \subseteq A_u$ [16–20]. An interval-valued fuzzy set assigns to each element $u \in U$ an interval $[\alpha, \beta] \subseteq [0, 1]$ of admissible membership degrees, representing uncertainty about the precise membership value [21–23]. As related concepts, interval-valued hesitant fuzzy sets [24–26], interval-valued three-way fuzzy sets [27], and interval-valued plithogenic sets [28, 29] are known in the literature. An interval-valued neutrosophic set assigns to every element intervals for truth, indeterminacy, and falsity, providing independently bounded ranges for all three components [30–32]. These structures can capture complex real-world concepts that cannot be adequately represented by ordinary fuzzy sets or neutrosophic sets. Furthermore, interval-valued fuzzy sets are also referred to as *HyperFuzzy Sets* [33,34], while interval-valued neutrosophic sets are also known as *HyperNeutrosophic Sets* [35,36]. A compact overview of Interval Sets, Interval-valued Fuzzy Sets, and Interval-valued Neutrosophic Sets is provided in Table 2.

Table 2. Compact overview of Interval Set, Interval-valued Fuzzy Set, and Interval-valued Neutrosophic Set

Model	Basic form	Short description
Interval Set	$[A_\ell, A_u] = \{\, A \subseteq U \mid A_\ell \subseteq A \subseteq A_u \,\}$	An interval in the lattice $(\mathcal{P}(U), \subseteq)$ collecting all subsets between fixed bounds A_ℓ and A_u, modeling uncertainty about a crisp subset of U.
Interval-valued Fuzzy Set (IVFS)	$A : U \to L([0,1]),\ A(u) = [\underline{A}(u), \overline{A}(u)]$	Assigns to each $u \in U$ a membership interval $[\underline{A}(u), \overline{A}(u)] \subseteq [0,1]$, representing a range of admissible fuzzy degrees instead of a single grade.
Interval-valued Neutrosophic Set (IVNS)	$A = (T_A, I_A, F_A)$, $T_A, I_A, F_A : U \to L([0,1])$	Assigns to each $u \in U$ three independent intervals $T_A(u)$, $I_A(u)$, $F_A(u)$ for truth, indeterminacy, and falsity, with $T_A^+(u) + I_A^+(u) + F_A^+(u) \leq 3$, giving simultaneous bounds on all three components.

1.3. Our Contributions

From the above discussion, research on Interval-valued Fuzzy Sets and Interval-valued Neutrosophic Sets is highly important. However, hierarchical generalizations in which the membership values themselves are interval-valued at multiple levels have not been extensively investigated. To fill this gap, we introduce HyperInterval- and SuperHyperInterval-valued fuzzy/neutrosophic sets, define conjunctive "core" (intersection) and disjunctive "hull" semantics, and prove embedding theorems showing that classical interval, fuzzy, and neutrosophic models appear as singleton or degenerate cases. Table 3 summarizes the main concepts addressed in this paper.

Table 3. HyperInterval- and SuperHyperInterval-valued fuzzy / neutrosophic sets

Model	Membership components	Short description
HyperInterval-valued fuzzy set	For each x, a finite family of intervals $\{[\alpha_i(x), \beta_i(x)]\}_i \subseteq [0,1]$	Assigns several (typically nested) admissible intervals of fuzzy membership degrees, giving multi-level bounds on $\mu_A(x)$.
SuperHyperInterval-valued fuzzy set	For each x, a hierarchical family of intervals $\mathbb{I}_A(x)$ obtained by iterated powersets of $[0,1]$	Uses higher-order, tree-like collections of intervals to encode layered, hierarchical uncertainty on fuzzy membership.
HyperInterval-valued neutrosophic set	For each x, interval families $\{[T_{\ell,i}(x), T_{u,i}(x)]\}_i$, $\{[I_{\ell,i}(x), I_{u,i}(x)]\}_i$, $\{[F_{\ell,i}(x), F_{u,i}(x)]\}_i$	Provides multi-level interval bounds for truth, indeterminacy, and falsity degrees of x.
SuperHyperInterval-valued neutrosophic set	For each x, hierarchical families $\mathbb{T}_A(x)$, $\mathbb{I}_A(x)$, $\mathbb{F}_A(x)$ of intervals in $[0,1]$	Uses higher-order collections of intervals for each of truth, indeterminacy, and falsity, capturing deeply layered neutrosophic uncertainty.

For reference, Table 4 presents a compact comparison of interval-valued, HyperInterval-valued, and SuperHyperInterval-valued fuzzy sets. The same style of comparison can be carried out in the context of Neutrosophic Sets as well.

Table 4. Comparison of interval-valued, HyperInterval-valued, and SuperHyperInterval-valued fuzzy sets

Model	Membership value space	Membership of an element x	Structural level	Main feature
Interval-valued fuzzy set	Intervals in $[0, 1]$	Single interval $[\alpha(x), \beta(x)]$	Single layer	Basic imprecise degree for x
HyperInterval-valued fuzzy set	Finite families of intervals in $[0, 1]$	$\{[\alpha_i(x), \beta_i(x)]\}_i$	Multi-interval (flat family)	Multi-level bounds on $\mu_A(x)$
SuperHyperInterval-valued fuzzy set	Hierarchical families of intervals in $[0, 1]$	$\mathbb{I}_A(x)$ (higher-order interval system)	Hierarchical / higher-order	Layered, nested uncertainty for $\mu_A(x)$

1.4. Conceptual Positioning of MIVFS, HIVFS, and SHIVFS

We briefly position the proposed MultiInterval-valued, HyperInterval-valued, and SuperHyperInterval-valued fuzzy sets in relation to interval-valued hesitant fuzzy sets and multi-source interval evidence.

Interval-valued hesitant fuzzy sets assign to each element a finite set of admissible membership intervals, representing hesitation among several interval-valued assessments [25, 26, 37]. In this respect, MIVFS is related to interval-valued hesitant fuzzy sets. However, in the present framework, an interval family is equipped with explicit conjunctive core and disjunctive hull semantics. Hence the same family can support both conservative consensus decisions and exploratory possibility-based decisions.

Multi-source evidence intervals also record several interval estimates obtained from experts, sensors, models, or data sources. However, a flat multi-source representation does not by itself provide canonical core/hull operations, singleton reduction to ordinary IVFS, or extension to higher-order interval systems. By contrast, MIVFS treats the interval family as a structured membership value with mathematically defined aggregation semantics.

HIVFS extends this idea by treating an interval family itself as a hyper-level membership object. SHIVFS further allows nested families of intervals, so that evidence can be organized by layers such as sources within groups, groups within scenarios, or scenarios within decision contexts. This hierarchical organization is not captured by ordinary interval-valued hesitant fuzzy sets or by flat multi-source interval models.

Remark 1.1 (Additional capability)**.** The main contribution is not merely the storage of several interval-valued membership degrees, since this is already possible in interval-valued hesitant fuzzy sets and flat multi-source interval models. The added capability is the combination of core/hull semantics, singleton reduction to IVFS, hyper-level membership interpretation, and superhyper-level hierarchical organization of interval evidence. Thus, MIVFS/HIVFS/SHIVFS distinguish conservative consensus, exploratory possibility, flat multi-source evidence, and hierarchically structured evidence within a single framework.

2. Preliminaries

We collect the basic terminology and notation used in what follows. The definitions in this paper are assumed to be finite.

2.1. Interval set, HyperInterval set, and SuperHyperInterval set

An interval set collects subsets A with lower bound A_ℓ and upper bound A_u, requiring $A_\ell \subseteq A \subseteq A_u$ [16, 19, 20, 38, 39]. A HyperInterval set assigns each base interval $[A_\ell, A_u]$ a nonempty family of admissible interval sets, satisfying $A_\ell \subseteq B_\ell \subseteq B_u \subseteq A_u$ [40]. A SuperHyperInterval set of order n maps elements of $\mathcal{P}^n(U)$ to $(n-1)$-nested families of interval sets, supporting hierarchical interval evidence [40].

Definition 2.1 (Universe)**.** Let U be a nonempty finite set, called the *universe* or *base set*. All subsequent powerset constructions are formed relative to U.

Definition 2.2 (Powerset [41])**.** The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including both the empty set and S itself:

$$\mathcal{P}(S) \;=\; \{\, A \;\mid\; A \subseteq S \,\}.$$

Definition 2.3 (n-th Powerset [42–45])**.** For a nonempty set H and integer $n \geq 1$, the *n-th powerset* is defined recursively by

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) := \mathcal{P}\big(\mathcal{P}_n(H)\big).$$

Analogously, the *n-th nonempty powerset*, denoted $\mathcal{P}_n^*(H)$, is constructed by

$$\mathcal{P}_1^*(H) := \mathcal{P}^*(H), \qquad \mathcal{P}_{n+1}^*(H) := \mathcal{P}^*\big(\mathcal{P}_n^*(H)\big),$$

where $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\emptyset\}$.

Example 2.4 (n-th Powerset for Menu Planning Across Horizons)**.** Let the base set (“atomic choices”) be $H = \{\text{Salad}, \text{Soup}, \text{Pasta}\}$. *First powerset* $\mathcal{P}_1(H) = \mathcal{P}(H)$ lists all meal options as sets of dishes (e.g., $\varnothing$, $\{\text{Salad}\}$, $\{\text{Soup}, \text{Pasta}\}$, $\{\text{Salad}, \text{Soup}, \text{Pasta}\}$). A *Second powerset* element is a collection of such options; interpret each as a *daily plan with admissible alternatives*. For instance,

$$D = \big\{\ \{\text{Salad}\},\ \{\text{Soup}, \text{Pasta}\}\ \big\} \ \in\ \mathcal{P}_2(H)$$

means “one acceptable meal is just Salad, another acceptable choice is Soup together with Pasta.” A *Third powerset* element bundles multiple daily plans into a *weekly (or multi-day) template*; for example

$$W = \big\{\ D,\ \{\ \{\text{Soup}\},\ \{\text{Pasta}\}\ \}\ \big\} \ \in\ \mathcal{P}_3(H)$$

encodes a set of admissible daily-plan choices (e.g., day one uses D, day two allows either only Soup or only Pasta).

Thus, moving from $\mathcal{P}_1(H)$ to $\mathcal{P}_2(H)$ to $\mathcal{P}_3(H)$ adds planning layers: from *sets of dishes* (meal options), to *sets of meal options* (daily plans with alternatives), to *sets of daily plans* (weekly templates). In everyday terms, the n-th powerset models a hierarchy of “choices of choices” across n decision horizons (dish $\to$ day $\to$ week $\to \ldots$).

Definition 2.5 (Interval set)**.** (cf. [16]) Let U be a nonempty universe and let $\mathcal{P}(U)$ be its powerset, ordered by $\subseteq$. For any two subsets $A_\ell, A_u \subseteq U$ with $A_\ell \subseteq A_u$, the *interval set* determined by (A_ℓ, A_u) is

$$[A_\ell, A_u] \;:=\; \{\, A \subseteq U \mid A_\ell \subseteq A \subseteq A_u \,\}.$$

We call A_ℓ the *lower bound* and A_u the *upper bound*. The class of all (closed) interval sets on U is

$$\mathsf{I}\big(\mathcal{P}(U)\big) \;:=\; \{\, [A_\ell, A_u] \;:\; A_\ell, A_u \subseteq U,\ A_\ell \subseteq A_u \,\}.$$

Remark 2.6. Each $[A_\ell, A_u]$ is a complete sublattice of $(\mathcal{P}(U), \subseteq)$ with bottom A_ℓ and top A_u. Degenerate cases: $[A, A] = \{A\}$ (a crisp set) and $[\varnothing, U] = \mathcal{P}(U)$ (the largest interval set).

Example 2.7 (Interval set for a Shopping List Envelope)**.** Let the universe of items be

$$U = \{\text{milk}, \text{bread}, \text{eggs}, \text{apples}, \text{cheese}\}$$

. Suppose the *must–buy* core is $A_\ell = \{\text{milk}, \text{eggs}\}$ and the *may–buy* superset is $A_u = \{\text{milk}, \text{bread}, \text{eggs}, \text{apples}\}$ (cheese is excluded today). The interval set

$$[A_\ell, A_u] = \{\, A \subseteq U \mid A_\ell \subseteq A \subseteq A_u \,\}$$

collects all acceptable shopping lists between these bounds. Since $|A_u \setminus A_\ell| = 2$ (bread, apples), the members are exactly the $2^2 = 4$ sets

$$\{\text{milk}, \text{eggs}\}, \quad \{\text{milk}, \text{eggs}, \text{bread}\}, \quad \{\text{milk}, \text{eggs}, \text{apples}\}, \quad \{\text{milk}, \text{eggs}, \text{bread}, \text{apples}\}.$$

For instance, $\{\text{milk}, \text{eggs}, \text{bread}\} \in [A_\ell, A_u]$ while $\{\text{milk}, \text{eggs}, \text{cheese}\} \notin [A_\ell, A_u]$ (cheese $\notin A_u$).

Definition 2.8 (HyperInterval set)**.** Let $\mathsf{I}(\mathcal{P}(U))$ be as in Definition 2.5. A *HyperInterval set* on U is a map

$$\text{HI} \;:\; \mathsf{I}\big(\mathcal{P}(U)\big) \longrightarrow \mathcal{P}^{*}\big(\mathsf{I}(\mathcal{P}(U))\big),$$

assigning to each base interval set $I = [A_\ell, A_u]$ a nonempty family $\text{HI}(I) \subseteq \mathsf{I}(\mathcal{P}(U))$ of admissible interval sets. *(Refinement semantics, optional.)* If, moreover, every $[B_\ell, B_u] \in \text{HI}([A_\ell, A_u])$ satisfies $A_\ell \subseteq B_\ell \subseteq B_u \subseteq A_u$, we call HI a *refinement HyperInterval set*.

Remark 2.9. Via pointwise union and intersection, $(\text{HI}_1 \sqcup \text{HI}_2)(I) := \text{HI}_1(I) \cup \text{HI}_2(I)$ and $(\text{HI}_1 \sqcap \text{HI}_2)(I) := \text{HI}_1(I) \cap \text{HI}_2(I)$, the set of all HyperInterval sets on U inherits a natural (hyper)structure.

Example 2.10 (HyperInterval set refining an envelope by store policies)**.** Let $I_0 = [A_\ell, A_u]$ be as in Example 2.7. Define a HyperInterval set HI by assigning to I_0 two refinements coming from two store policies:

$$\text{HI}(I_0) = \{I_1, I_2\},$$

where

$$I_1 = \big[\{\text{milk}, \text{eggs}\}, \{\text{milk}, \text{eggs}, \text{bread}, \text{apples}\}\big],$$

and

$$I_2 = \big[\{\text{milk}, \text{eggs}, \text{bread}\}, \{\text{milk}, \text{eggs}, \text{bread}, \text{apples}\}\big].$$

For clarity, write

$$L_1 = \{\text{milk}, \text{eggs}\}, \qquad U_1 = \{\text{milk}, \text{eggs}, \text{bread}, \text{apples}\},$$

and

$$L_2 = \{\text{milk}, \text{eggs}, \text{bread}\}, \qquad U_2 = \{\text{milk}, \text{eggs}, \text{bread}, \text{apples}\}.$$

Then $I_1 = [L_1, U_1]$ and $I_2 = [L_2, U_2]$. Both intervals are refinements of I_0, since their lower and upper bounds lie between A_ℓ and A_u.

The conjunctive core is the intersection of the two interval sets:

$$I_1 \cap I_2 = [L_1, U_1] \cap [L_2, U_2].$$

Using the standard formula for the intersection of two interval sets,

$$[L_1, U_1] \cap [L_2, U_2] = [L_1 \cup L_2,\; U_1 \cap U_2],$$

provided that $L_1 \cup L_2 \subseteq U_1 \cap U_2$. In the present case,

$$L_1 \cup L_2 = \{\text{milk}, \text{eggs}\} \cup \{\text{milk}, \text{eggs}, \text{bread}\} = \{\text{milk}, \text{eggs}, \text{bread}\},$$

and

$$U_1 \cap U_2 = \{\text{milk, eggs, bread, apples}\} \cap \{\text{milk, eggs, bread, apples}\} = \{\text{milk, eggs, bread, apples}\}.$$

Since

$$\{\text{milk, eggs, bread}\} \subseteq \{\text{milk, eggs, bread, apples}\},$$

the core is nonempty and equals

$$I_1 \cap I_2 = \big[\{\text{milk, eggs, bread}\}, \{\text{milk, eggs, bread, apples}\}\big].$$

The disjunctive hull is the smallest interval set containing $I_1 \cup I_2$. It is given by

$$\text{Hull}(\{I_1, I_2\}) = [L_1 \cap L_2,\ U_1 \cup U_2].$$

Here,

$$L_1 \cap L_2 = \{\text{milk, eggs}\} \cap \{\text{milk, eggs, bread}\} = \{\text{milk, eggs}\},$$

and

$$U_1 \cup U_2 = \{\text{milk, eggs, bread, apples}\}.$$

Therefore,

$$\text{Hull}(\{I_1, I_2\}) = \big[\{\text{milk, eggs}\}, \{\text{milk, eggs, bread, apples}\}\big] = I_0.$$

Thus, the HyperInterval set records store-specific refinements. The conjunctive core gives the jointly required shopping-list envelope, while the disjunctive hull recovers the original global envelope.

Definition 2.11 (SuperHyperInterval set of order n)**.** For $n \geq 0$ define the iterated powersets by $\mathcal{P}^0(U) = U$ and $\mathcal{P}^{n+1}(U) = \mathcal{P}\big(\mathcal{P}^n(U)\big)$. Also write $\mathcal{P}^r\big(\mathsf{I}(\mathcal{P}(U))\big)$ for iterated powersets of the interval–set universe. A *SuperHyperInterval set of order* $n \geq 1$ on U is a map

$$\text{SHI}^{(n)}\ :\ \mathcal{P}^n(U) \longrightarrow \mathcal{P}^{\,n-1}\big(\,\mathsf{I}(\mathcal{P}(U))\,\big).$$

Thus, to each n–nested subset $A \in \mathcal{P}^n(U)$ the map assigns an $(n-1)$–nested family of interval sets. (When a refinement discipline is desired, one can require that the interval bounds appearing at the leaves are chosen compatibly with the subsets occurring in A.)

Example 2.12 (SuperHyperInterval set (order n=2) for Two Scenarios of a Dinner Plan)**.** Keep

$$U = \{\text{milk, bread, eggs, apples, cheese}\}$$

. Consider two scenario sets (elements of $\mathcal{P}(U)$):

$$S_{\text{family}} = \{\text{milk, eggs, bread}\},$$

$$S_{\text{guests}} = \{\text{bread, cheese}\}.$$

Form the nested input $A = \{S_{\text{family}}, S_{\text{guests}}\} \in \mathcal{P}^2(U)$. Define the SuperHyperInterval map of order 2 by

$$\text{SHI}^{(2)}(A)\ =\ \Big\{I_{\text{family}} = [\{\text{milk, eggs}\},\ \{\text{milk, eggs, bread}\}],$$

$$I_{\text{guests}} = [\{\text{bread}\},\ \{\text{milk, eggs, bread}\}]\Big\}\ \in\ \mathcal{P}\big(\mathsf{I}(\mathcal{P}(U))\big).$$

Here the family scenario insists on milk and eggs; the guest scenario insists on bread but allows the same upper bound to keep cooking logistics unified.

If one demands a list acceptable to *both* scenarios, the core is

$$I_{\text{family}} \cap I_{\text{guests}} = \Big[\{\text{milk}, \text{eggs}\} \cup \{\text{bread}\},\ \{\text{milk}, \text{eggs}, \text{bread}\} \cap \{\text{milk}, \text{eggs}, \text{bread}\}\Big]$$

$$= [\{\text{milk}, \text{eggs}, \text{bread}\},\ \{\text{milk}, \text{eggs}, \text{bread}\}],$$

a crisp recommendation $\{\text{milk}, \text{eggs}, \text{bread}\}$. If instead one allows either scenario, the hull is

$$\Big[\{\text{milk}, \text{eggs}\} \cap \{\text{bread}\},\ \{\text{milk}, \text{eggs}, \text{bread}\} \cup \{\text{milk}, \text{eggs}, \text{bread}\}\Big]$$

$$= [\varnothing,\ \{\text{milk}, \text{eggs}, \text{bread}\}],$$

recording every intermediate list up to the common upper bound. This illustrates how order-2 nesting organizes *scenario families* before interval selection.

2.2. *Interval-valued fuzzy set*

A fuzzy set assigns each element a membership degree between zero and one, modeling belonging and vagueness beyond crisp classification [1, 46]. An interval-valued fuzzy set assigns to each element u an interval $[\alpha, \beta] \subseteq [0, 1]$ of admissible membership degrees, modeling imprecision about exact values [21, 23, 47]. As related concepts, interval-valued intuitionistic fuzzy sets [48–50], interval-valued picture fuzzy sets [51, 52], and interval-valued hesitant fuzzy sets [25, 26, 37] have also been studied in the literature.

Definition 2.13 (Interval-valued fuzzy set)**.** [21, 22, 53] Let $U \neq \varnothing$ be a universe. Write

$$L([0, 1]) \ = \ \{\, [\alpha, \beta] \mid 0 \le \alpha \le \beta \le 1 \,\},$$

the set of all *closed* subintervals of $[0, 1]$. An *interval-valued fuzzy set (IVFS)* on U is a mapping

$$A : \ U \longrightarrow L([0, 1]),$$

so that each $u \in U$ is assigned an interval $A(u) = [\underline{A}(u), \overline{A}(u)] \in L([0, 1])$ of admissible membership degrees. We denote the class of all IVFSs on U by $\text{IVFS}(U)$.

Example 2.14 (IVFS for Fruit Ripeness in a Grocery Store)**.** Let $U = \{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3\}$ be three avocados on display. Consider the interval-valued fuzzy set $A : U \to L([0, 1])$ where $A(u)$ is the degree to which u is "ripe enough to eat tonight." Sensor readings (color, firmness) and staff judgment are summarized as

$$A(\mathrm{A}_1) = [0.65, 0.82], \qquad A(\mathrm{A}_2) = [0.30, 0.50], \qquad A(\mathrm{A}_3) = [0.75, 0.90].$$

A conservative customer requires membership at least $\alpha = 0.70$ under the *necessary* view (use lower bounds), yielding the α-cut

$$N_{0.70} = \{\, u \in U \mid \underline{A}(u) \ge 0.70 \,\} = \{\mathrm{A}_3\}.$$

An optimistic customer accepts if it is *possible* to meet $\alpha = 0.70$ (use upper bounds), giving

$$P_{0.70} = \{\, u \in U \mid \overline{A}(u) \ge 0.70 \,\} = \{\mathrm{A}_1, \mathrm{A}_3\}.$$

(Optionally) the IVFS complement $A^{\complement}(u) = [\, 1 - \overline{A}(u),\ 1 - \underline{A}(u) \,]$ quantifies "not ripe tonight": for A_1, $A^{\complement}(\mathrm{A}_1) = [0.18, 0.35]$.

2.3. Interval-valued neutrosophic set

A neutrosophic set assigns independent truth, indeterminacy, and falsity degrees to each element, capturing inconsistency and uncertainty beyond fuzzy membership [9, 10, 54]. An interval-valued neutrosophic set assigns each element intervals for truth, indeterminacy, and falsity, allowing independent bounded ranges for all three [30, 31, 55–57].

Definition 2.15 (Interval-valued neutrosophic set (IVNS))**.** Let U be a nonempty universe. An *interval-valued neutrosophic set* A on U is specified by three maps

$$T_A,\ I_A,\ F_A:\ U \longrightarrow \mathrm{Int}([0,1]),$$

assigning to each $u \in U$ closed intervals

$$T_A(u) = [T_A^-(u), T_A^+(u)], \quad I_A(u) = [I_A^-(u), I_A^+(u)], \quad F_A(u) = [F_A^-(u), F_A^+(u)]$$

interpreted respectively as the *truth*, *indeterminacy*, and *falsity* membership degrees of u. These components are independent; the only numeric bound required is

$$0 \ \le\ T_A^+(u) + I_A^+(u) + F_A^+(u) \ \le\ 3, \qquad \forall\, u \in U.$$

Equivalently, one writes

$$A \ =\ \big\{\ \langle T_A(u),\ I_A(u),\ F_A(u)\rangle\ /\ u \in U\ \big\}.$$

Example 2.16 (IVNS for Spam Detection of a Single Email)**.** Let $U = \{e\}$ where e is an incoming email. Define an IVNS $S = (T, I, F)$ for the statement "e is spam." From sender reputation, content filters, and user history we obtain

$$T(e) = [0.72, 0.86], \qquad I(e) = [0.10, 0.20], \qquad F(e) = [0.04, 0.12].$$

The numeric constraint holds:

$$T^+(e) + I^+(e) + F^+(e) = 0.86 + 0.20 + 0.12 = 1.18 \le 3.$$

A conservative rule declares spam when $T^-(e) \ge 0.70$ and $F^+(e) \le 0.20$. Here $T^-(e) = 0.72\ (\ge 0.70)$ and $F^+(e) = 0.12\ (\le 0.20)$, so e is classified as spam. The interval $T(e) = [0.72, 0.86]$ captures admissible truth, $I(e) = [0.10, 0.20]$ the uncertainty due to weak indicators, and $F(e) = [0.04, 0.12]$ the bounded counter-evidence (e.g., some benign content).

2.4. Multistructure

MultiStructure is a carrier set equipped with indexed multi-operations mapping tuples to sets of outcomes, enabling nondeterministic, multi-arity algebraic computation [58, 59].

Definition 2.17 (MultiOperation)**.** Fix an integer $m \ge 1$ and let H be a nonempty set. An *m-ary multi-operation* on H is a map

$$\#^{(m)}\colon H^m \ \longrightarrow\ \mathcal{M}(H), \quad (x_1, \ldots, x_m)\ \mapsto\ \#^{(m)}(x_1, \ldots, x_m),$$

assigning to each m-tuple $(x_1, \ldots, x_m)$ a finite multiset of elements of H rather than a single element.

Definition 2.18 (MultiStructure)**.** A *MultiStructure* is a pair

$$\mathcal{MS} \ =\ \big(H,\ \{\#^{(m)}\colon H^m \to \mathcal{M}(H)\}_{m\in\mathcal{I}}\big),$$

where H is a nonempty carrier set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ indexes a family of multi-operations of various arities. No further axioms are imposed unless explicitly stated.

Example 2.19 (MultiStructure on Interval Sets for Aggregating Requirements)**.** Let $H = \mathsf{I}(\mathcal{P}(U))$ with U as above, and take $\mathcal{M}(H) = \mathcal{P}(H)$. Define two multi-operations for any $m \geq 1$ and intervals $I_r = [L_r, U_r] \in H$:

$$\#_{\wedge}^{(m)}(I_1, \dots, I_m) = \begin{cases} \{[\bigcup_{r=1}^{m} L_r,\ \bigcap_{r=1}^{m} U_r\ ]\}, & \text{if } \bigcup_r L_r \subseteq \bigcap_r U_r, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\#_{\vee\text{hull}}^{(m)}(I_1, \dots, I_m) = \{[\ \bigcap_{r=1}^{m} L_r,\ \bigcup_{r=1}^{m} U_r\ ]\}.$$

Then $\mathcal{MS} = (H, \{\#_{\wedge}^{(m)}, \#_{\vee\text{hull}}^{(m)}\}_{m\geq 1})$ is a MultiStructure.

Concrete computation (two requirements). Let

$$I_1 = [\{\text{milk}\},\ \{\text{milk, bread, eggs}\}],$$

$$I_2 = [\{\text{bread}\},\ \{\text{milk, bread, eggs}\}].$$

Then

$$\#_{\wedge}^{(2)}(I_1, I_2) = \Big\{\ \big[\{\text{milk}\} \cup \{\text{bread}\},\ \{\text{milk, bread, eggs}\} \cap \{\text{milk, bread, eggs}\}\big]\ \Big\}$$
$$= \big\{\ [\{\text{milk, bread}\},\ \{\text{milk, bread, eggs}\}]\ \big\},$$

while

$$\#_{\vee\text{hull}}^{(2)}(I_1, I_2) = \Big\{\ \big[\{\text{milk}\} \cap \{\text{bread}\},\ \{\text{milk, bread, eggs}\} \cup \{\text{milk, bread, eggs}\}\big]\ \Big\}$$
$$= \big\{\ [\varnothing,\ \{\text{milk, bread, eggs}\}]\ \big\}.$$

Thus, the $\wedge$–operation returns the *jointly* acceptable envelope, and the $\vee$–hull returns the *least* envelope containing both requirements—illustrating multi-arity, set-valued outputs in a real aggregation workflow.

3. Main Results

In this section, we present the main results of this paper.

3.1. MultiInterval-valued Fuzzy Set

A MultiInterval-valued Fuzzy Set assigns each element a finite family of membership intervals, aggregating evidence using intersection cores and hulls.

Definition 3.1 (MultiInterval)**.** Let $U \neq \varnothing$ and let J be a finite, nonempty index set. A *MultiInterval* on U is a family

$$\mathbf{I}\ =\ \big\{\ [A_\ell^{(j)}, A_u^{(j)}]\ \big|\ j \in J,\ A_\ell^{(j)} \subseteq A_u^{(j)} \subseteq U\ \big\}\ \in\ \mathcal{P}^*\big(\mathsf{I}(\mathcal{P}(U))\big).$$

Its *conjunctive semantics* (feasible core) is the set

$$[\![\mathbf{I}]\!]_\wedge\ :=\ \bigcap_{j\in J} [A_\ell^{(j)}, A_u^{(j)}]\ =\ \Big\{\ A \subseteq U\ \Big|\ \forall j \in J:\ A_\ell^{(j)} \subseteq A \subseteq A_u^{(j)}\Big\}.$$

Its *disjunctive hull* is the smallest interval set containing $\bigcup_{j\in J}[A_\ell^{(j)}, A_u^{(j)}]$:

$$[\![\mathbf{I}]\!]_\vee^{\text{hull}}\ :=\ \Big[\ \bigcap_{j\in J} A_\ell^{(j)},\ \bigcup_{j\in J} A_u^{(j)}\ \Big].$$

Lemma 3.2 (Exact formula for the conjunctive core)
Let $\mathbf{I}$ be as in Definition 3.1 and set

$$L^{\wedge} := \bigcup_{j\in J} A_\ell^{(j)}, \qquad U^{\wedge} := \bigcap_{j\in J} A_u^{(j)}.$$

Then

$$\bigcap_{j\in J}[A_\ell^{(j)}, A_u^{(j)}] = \begin{cases} [L^{\wedge}, U^{\wedge}], & \text{if } L^{\wedge} \subseteq U^{\wedge}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof
($\subseteq$) Let $A \in \bigcap_j [A_\ell^{(j)}, A_u^{(j)}]$. Then $A_\ell^{(j)} \subseteq A \subseteq A_u^{(j)}$ for all j. Taking unions of the left bounds gives $L^{\wedge} = \bigcup_j A_\ell^{(j)} \subseteq A$, and taking intersections of the right bounds gives $A \subseteq \bigcap_j A_u^{(j)} = U^{\wedge}$. Hence $A \in [L^{\wedge}, U^{\wedge}]$, which already forces $L^{\wedge} \subseteq U^{\wedge}$ whenever the intersection is nonempty.
($\supseteq$) Conversely, if $L^{\wedge} \subseteq U^{\wedge}$ and $A \in [L^{\wedge}, U^{\wedge}]$, then $A_\ell^{(j)} \subseteq L^{\wedge} \subseteq A$ and $A \subseteq U^{\wedge} \subseteq A_u^{(j)}$ for each j, so $A \in [A_\ell^{(j)}, A_u^{(j)}]$. Therefore A lies in the intersection. □

Remark 3.3 (Order and feasibility)**.** Define a preorder $\preceq$ on MultiIntervals by $\mathbf{I}_1 \preceq \mathbf{I}_2$ iff $[\![\mathbf{I}_1]\!]_\wedge \subseteq [\![\mathbf{I}_2]\!]_\wedge$. By Lemma 3.2, feasibility of $\mathbf{I}$ is equivalent to the numeric inequality $\bigcup_j A_\ell^{(j)} \subseteq \bigcap_j A_u^{(j)}$.

Let $H := \mathsf{I}(\mathcal{P}(U))$. We define two canonical multi-operations on H.

Definition 3.4 (Meet and hull multi-operations on H)**.** For $m \geq 1$ and $(I_1, \ldots, I_m) \in H^m$ with $I_r = [L_r, U_r]$, set

$$\#_{\wedge}^{(m)}(I_1, \ldots, I_m) := \begin{cases} \{\, [\, \bigcup_{r=1}^m L_r \,,\, \bigcap_{r=1}^m U_r \,] \,\}, & \text{if } \bigcup L_r \subseteq \bigcap U_r, \\ \varnothing, & \text{otherwise,} \end{cases}$$

and

$$\#_{\vee\text{hull}}^{(m)}(I_1, \ldots, I_m) := \{\, [\, \bigcap_{r=1}^m L_r \,,\, \bigcup_{r=1}^m U_r \,] \,\}.$$

Then $\mathcal{MS}_U := (H, \{\#_{\wedge}^{(m)}, \#_{\vee\text{hull}}^{(m)}\}_{m\geq 1})$ is a MultiStructure in the sense of Definition 2.18.

Proposition 3.5 (MultiInterval as a MultiStructure computation)
Let $\mathbf{I} = \{I_j\}_{j\in J}$ be a MultiInterval on U with $I_j = [A_\ell^{(j)}, A_u^{(j)}]$. Then

$$[\![\mathbf{I}]\!]_\wedge = \bigcap_{j\in J} I_j = \begin{cases} I^\star, & \text{if } \#_{\wedge}^{(|J|)}\big((I_j)_{j\in J}\big) = \{I^\star\}, \\ \varnothing, & \text{otherwise,} \end{cases}$$

with $I^\star = \big[\bigcup_j A_\ell^{(j)}, \bigcap_j A_u^{(j)}\big]$. Moreover, the hull semantics satisfies

$$[\![\mathbf{I}]\!]_\vee^{\text{hull}} = \#_{\vee\text{hull}}^{(|J|)}\big((I_j)_{j\in J}\big).$$

Proof
Immediate from Definition 3.4 and Lemma 3.2, by expanding unions and intersections of bounds elementwise. □

Definition 3.6 (Singleton embedding)**.** Define $\iota : \mathsf{I}(\mathcal{P}(U)) \to \mathcal{P}^*(\mathsf{I}(\mathcal{P}(U)))$ by

$$\iota\big([A_\ell, A_u]\big) := \big\{[A_\ell, A_u]\big\}.$$

We identify $\iota([A_\ell, A_u])$ with a MultiInterval having index set $J = \{1\}$.

Theorem 3.7 (MultiInterval generalizes Interval)
For every interval set $I = [A_\ell, A_u] \in \mathsf{I}(\mathcal{P}(U))$,

$$[\![\, \iota(I) \,]\!]_\wedge \;=\; I \qquad \text{and} \qquad [\![\, \iota(I) \,]\!]_\vee^{\text{hull}} \;=\; I.$$

Hence the map ι is an order-embedding from $(\mathsf{I}(\mathcal{P}(U)), \subseteq)$ into the preorder of MultiIntervals under $\preceq$, and MultiIntervals strictly generalize intervals.

Proof
Let $J = \{1\}$ and $I = [A_\ell, A_u]$. By Lemma 3.2,

$$[\![\iota(I)]\!]_\wedge \;=\; \bigcap_{j\in J}[A_\ell^{(j)}, A_u^{(j)}] \;=\; \Big[\bigcup_{j\in J} A_\ell^{(j)},\; \bigcap_{j\in J} A_u^{(j)}\Big] \;=\; [A_\ell, A_u] \;=\; I,$$

since the union and intersection over a singleton index set return A_ℓ and A_u respectively. The hull equality is analogous:

$$[\![\iota(I)]\!]_\vee^{\text{hull}} \;=\; \Big[\bigcap_{j\in J} A_\ell^{(j)},\; \bigcup_{j\in J} A_u^{(j)}\Big] \;=\; [A_\ell, A_u] \;=\; I.$$

Monotonicity of ι with respect to $\subseteq$ and $\preceq$ follows directly from these equalities. □

Definition 3.8 (MultiInterval-valued fuzzy set (MIVFS))**.** Let $U \neq \varnothing$ and $L([0,1])$ as above. A *MIVFS* on U assigns to each $u \in U$ a finite nonempty family of numeric intervals

$$\mathcal{A}(u) \;=\; \big\{\, [\alpha_j(u), \beta_j(u)] \in L([0,1]) \;\big|\; j \in J(u) \text{ finite, nonempty} \,\big\}.$$

Its conjunctive (intersection) semantics and disjunctive (hull) semantics are

$$\Big[\!\!\Big[\mathcal{A}(u)\Big]\!\!\Big]_\wedge = \begin{cases} [\, \max_j \alpha_j(u)\,,\, \min_j \beta_j(u) \,], & \text{if } \max_j \alpha_j(u) \le \min_j \beta_j(u), \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\Big[\!\!\Big[\mathcal{A}(u)\Big]\!\!\Big]_\vee^{\text{hull}} = [\, \min_j \alpha_j(u)\,,\, \max_j \beta_j(u) \,].$$

Example 3.9 (MIVFS for Software Release Readiness Today)**.** Let $U = \{\nu\}$, where ν denotes the proposition

"Version v1.2 is ready for production today."

As a MultiInterval-valued fuzzy set, we assign to ν multiple membership intervals from independent engineering sources:

$$\mathcal{A}(\nu) = \{I_1, I_2, I_3, I_4\} \subseteq L([0,1]),$$

with

$I_1 = [0.78, 0.88]$ (unit + integration test trends),
$I_2 = [0.72, 0.85]$ (QA exploratory testing outcomes),
$I_3 = [0.80, 0.90]$ (SRE load/performance confidence),
$I_4 = [0.75, 0.83]$ (security/compliance quick audit).

By Definition (MIVFS), the conjunctive (intersection) semantics and the disjunctive (hull) semantics are

$$\Big[\!\!\Big[\mathcal{A}(\nu)\Big]\!\!\Big]_\wedge = \begin{cases} [\, \max_j \alpha_j,\, \min_j \beta_j \,], & \text{if } \max_j \alpha_j \le \min_j \beta_j, \\ \varnothing, & \text{otherwise,} \end{cases} \qquad \Big[\!\!\Big[\mathcal{A}(\nu)\Big]\!\!\Big]_\vee^{\text{hull}} = [\, \min_j \alpha_j,\, \max_j \beta_j \,],$$

where $I_j = [\alpha_j, \beta_j]$.

We compute the bounds explicitly.

Lower endpoints:

$$\alpha_1 = 0.78, \quad \alpha_2 = 0.72, \quad \alpha_3 = 0.80, \quad \alpha_4 = 0.75 \implies \max_j \alpha_j = \max\{0.78, 0.72, 0.80, 0.75\} = 0.80.$$

Upper endpoints:

$$\beta_1 = 0.88, \quad \beta_2 = 0.85, \quad \beta_3 = 0.90, \quad \beta_4 = 0.83 \implies \min_j \beta_j = \min\{0.88, 0.85, 0.90, 0.83\} = 0.83.$$

Since $0.80 \leq 0.83$, the intersection is feasible and hence

$$\left[\!\left[\mathcal{A}(\nu)\right]\!\right]_{\wedge} = [0.80,\ 0.83]\ .$$

Hull:

$$\min_j \alpha_j = \min\{0.78, 0.72, 0.80, 0.75\} = 0.72, \qquad \max_j \beta_j = \max\{0.88, 0.85, 0.90, 0.83\} = 0.90,$$

so

$$\left[\!\left[\mathcal{A}(\nu)\right]\!\right]_{\vee}^{\mathrm{hull}} = [0.72,\ 0.90]\ .$$

The interval $[0.80, 0.83]$ is the conservative consensus range endorsed by *all* engineering evidence streams, suited to a strict release gate. The hull $[0.72, 0.90]$ captures the full plausible readiness reported by *at least one* stream, useful for exploratory planning and risk negotiation.

Proposition 3.10 (Reduction to IVFS)
If each $J(u)$ is a singleton, say $J(u) = \{1\}$ with $\mathcal{A}(u) = \{[\alpha_1(u), \beta_1(u)]\}$, then

$$\left[\!\left[\mathcal{A}(u)\right]\!\right]_{\wedge} = \left[\!\left[\mathcal{A}(u)\right]\!\right]_{\vee}^{\mathrm{hull}} = [\,\alpha_1(u), \beta_1(u)\,],$$

so every IVFS is a special case of a MIVFS.

Proof
Compute $\max_j \alpha_j(u) = \min_j \beta_j(u)$ over a singleton index set. □

3.2. MultiInterval-valued Neutrosophic Set

A MultiInterval-valued Neutrosophic Set assigns each element interval families for truth, indeterminacy, and falsity, aggregated componentwise by cores and hulls.

Definition 3.11 (MIVNS)**.** A *MultiInterval-valued neutrosophic set (MIVNS)* on U is a triple

$$\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F}),$$

where for each $u \in U$, $\mathcal{T}(u), \mathcal{I}(u), \mathcal{F}(u)$ are finite nonempty families of numeric intervals in $L([0,1])$. Conjunctive semantics are computed componentwise by interval intersection:

$$[\![\mathcal{T}(u)]\!]_\wedge = [\, \max_j T_j^-(u),\ \min_j T_j^+(u)\,],\ [\![\mathcal{I}(u)]\!]_\wedge = [\, \max_j I_j^-(u),\ \min_j I_j^+(u)\,],\ [\![\mathcal{F}(u)]\!]_\wedge = [\, \max_j F_j^-(u),\ \min_j F_j^+(u)\,],$$

with the usual feasibility conditions $\max \le \min$ in each component; hull semantics take $[\min, \max]$ componentwise.

Example 3.12 (MIVNS for a Clinical Decision: "Pneumonia Present Today?")**.** Let $U = \{p\}$, where p denotes a particular patient. We assess the statement

"p has community–acquired pneumonia (CAP) today."

as a MultiInterval–valued neutrosophic set $\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F})$. Here $\mathcal{T}(p)$ collects *supporting* evidence intervals (truth), $\mathcal{I}(p)$ collects *uncertainty* intervals (indeterminacy), and $\mathcal{F}(p)$ collects *counter–evidence* intervals (falsity). Each interval $[\alpha, \beta] \subseteq [0,1]$ encodes an admissible range for the corresponding degree.

Concrete (illustrative) sources and intervals:

$$\mathcal{T}(p) = \big\{[0.72, 0.88] \text{ (chest X–ray report)},\ [0.65, 0.80] \text{ (clinical score)},\ [0.70, 0.90] \text{ (C–reactive protein model)}\big\},$$

$$\mathcal{I}(p) = \big\{[0.10, 0.25] \text{ (equivocal imaging)},\ [0.15, 0.30] \text{ (atypical symptom onset)},\ [0.05, 0.20] \text{ (comorbidity confounding)}\big\},$$

$$\mathcal{F}(p) = \big\{[0.08, 0.22] \text{ (normal WBC)},\ [0.12, 0.18] \text{ (viral/bacterial panel result)},\ [0.05, 0.15] \text{ (stable oxygenation)}\big\}.$$

By the MIVNS semantics, the *conjunctive core* and *disjunctive hull* are computed *componentwise* using interval intersection and least–containing–interval, respectively.

Truth component.

$$\begin{aligned}
\max \underline{\mathcal{T}}(p) &= \max\{0.72, 0.65, 0.70\} = 0.72,\\
\min \overline{\mathcal{T}}(p) &= \min\{0.88, 0.80, 0.90\} = 0.80\\
&\Longrightarrow \quad [\![\mathcal{T}(p)]\!]_\wedge = [0.72,\ 0.80] \text{ (feasible since } 0.72 \le 0.80),\\
\min \underline{\mathcal{T}}(p) &= \min\{0.72, 0.65, 0.70\} = 0.65,\\
\max \overline{\mathcal{T}}(p) &= \max\{0.88, 0.80, 0.90\} = 0.90\\
&\Rightarrow \quad [\![\mathcal{T}(p)]\!]_\vee^{\text{hull}} = [0.65,\ 0.90].
\end{aligned}$$

Indeterminacy component.

$$\begin{aligned}
&\max \underline{\mathcal{I}}(p) = \max\{0.10, 0.15, 0.05\} = 0.15,\\
&\min \overline{\mathcal{I}}(p) = \min\{0.25, 0.30, 0.20\} = 0.20 \quad \Longrightarrow \quad [\![\mathcal{I}(p)]\!]_\wedge = [0.15,\ 0.20],\\
&\min \underline{\mathcal{I}}(p) = \min\{0.10, 0.15, 0.05\} = 0.05, \qquad \max \overline{\mathcal{I}}(p) = \max\{0.25, 0.30, 0.20\} = 0.30\\
&\qquad \Rightarrow \quad [\![\mathcal{I}(p)]\!]_\vee^{\text{hull}} = [0.05,\ 0.30].
\end{aligned}$$

Falsity component.

$$\begin{aligned}
&\max \underline{\mathcal{F}}(p) = \max\{0.08, 0.12, 0.05\} = 0.12,\\
&\min \overline{\mathcal{F}}(p) = \min\{0.22, 0.18, 0.15\} = 0.15 \quad \Longrightarrow \quad [\![\mathcal{F}(p)]\!]_\wedge = [0.12,\ 0.15],\\
&\min \underline{\mathcal{F}}(p) = \min\{0.08, 0.12, 0.05\} = 0.05, \qquad \max \overline{\mathcal{F}}(p) = \max\{0.22, 0.18, 0.15\} = 0.22\\
&\qquad \Rightarrow \quad [\![\mathcal{F}(p)]\!]_\vee^{\text{hull}} = [0.05,\ 0.22].
\end{aligned}$$

Vector–valued result.

$$\mathrm{core}_\wedge\big(\mathcal{A}\big)(p) = \big([0.72, 0.80],\ [0.15, 0.20],\ [0.12, 0.15]\big),$$

$$\mathrm{hull}_\vee\big(\mathcal{A}\big)(p) = \big([0.65, 0.90],\ [0.05, 0.30],\ [0.05, 0.22]\big)\ .$$

The conjunctive core gives the consensus ranges simultaneously supported by all sources in each component (truth, indeterminacy, falsity). The hulls summarize the full plausible spans reported by at least one source. A conservative decision maker would consult the core; exploratory or safety–margin assessments can reference the hulls.

Theorem 3.13 (MIVNS generalizes IVNS)
Define the embedding

$$\iota:\ \big(L([0,1])\big)^3 \longrightarrow \Big(\mathcal{P}^*\big(L([0,1])\big)\Big)^3, \qquad \iota\big([a,b],[c,d],[e,f]\big) := \big(\{[a,b]\},\{[c,d]\},\{[e,f]\}\big).$$

If $A = (T, I, F)$ is an IVNS on U, then $\mathcal{A} := \iota \circ A$ is a MIVNS and, for every $u \in U$,

$$\mathrm{core}_\wedge(\mathcal{A})(u) = (T(u), I(u), F(u)) \qquad \text{and} \qquad \mathrm{hull}_\vee(\mathcal{A})(u) = (T(u), I(u), F(u)).$$

Hence every IVNS is a (canonically embedded) special case of a MIVNS.

Proof
Fix $u \in U$ and write $T(u) = [t^-(u), t^+(u)]$, $I(u) = [i^-(u), i^+(u)]$, $F(u) = [f^-(u), f^+(u)]$. By definition of ι,

$$\mathcal{T}(u) = \{[t^-(u), t^+(u)]\}, \quad \mathcal{I}(u) = \{[i^-(u), i^+(u)]\}, \quad \mathcal{F}(u) = \{[f^-(u), f^+(u)]\}.$$

Evaluating maxima/minima over a singleton gives, componentwise,

$$\max_{[\alpha,\beta]\in\mathcal{T}(u)} \alpha = t^-(u), \qquad \min_{[\alpha,\beta]\in\mathcal{T}(u)} \beta = t^+(u) \implies \mathrm{core}_\wedge(\mathcal{T}(u)) = [t^-(u), t^+(u)] = T(u),$$

and similarly $\mathrm{core}_\wedge(\mathcal{I}(u)) = I(u)$, $\mathrm{core}_\wedge(\mathcal{F}(u)) = F(u)$. The hull equalities follow analogously: $\mathrm{hull}_\vee(\mathcal{T}(u)) = [t^-(u), t^+(u)] = T(u)$, etc. Thus $\mathrm{core}_\wedge(\mathcal{A})(u) = \mathrm{hull}_\vee(\mathcal{A})(u) = (T(u), I(u), F(u))$. □

Theorem 3.14 (MIVNS generalizes MIVFS)
Let $\mathcal{A}_\mathrm{F}$ be a MIVFS on U. Define $\Phi(\mathcal{A}_\mathrm{F})$ to be the MIVNS $\big(\mathcal{T}, \mathcal{I}, \mathcal{F}\big)$ given by

$$\mathcal{T}(u) := \mathcal{A}_\mathrm{F}(u), \qquad \mathcal{I}(u) := \{\,[0,0]\,\}, \qquad \mathcal{F}(u) := \{\,[0,0]\,\}, \quad \text{for each } u \in U.$$

Then, for every $u \in U$,

$$\mathrm{core}_\wedge\big(\mathcal{T}(u)\big) = \mathrm{core}_\wedge\big(\mathcal{A}_\mathrm{F}(u)\big), \quad \mathrm{hull}_\vee\big(\mathcal{T}(u)\big) = \mathrm{hull}_\vee\big(\mathcal{A}_\mathrm{F}(u)\big),$$

and

$$\mathrm{core}_\wedge\big(\mathcal{I}(u)\big) = \mathrm{hull}_\vee\big(\mathcal{I}(u)\big) = [0,0], \qquad \mathrm{core}_\wedge\big(\mathcal{F}(u)\big) = \mathrm{hull}_\vee\big(\mathcal{F}(u)\big) = [0,0].$$

Consequently, projecting the MIVNS $\Phi(\mathcal{A}_\mathrm{F})$ onto its truth component recovers exactly the original MIVFS semantics (both core and hull), so MIVNS strictly extends MIVFS.

Proof
Fix $u \in U$ and write $\mathcal{A}_\mathrm{F}(u) = \{[\alpha_j(u), \beta_j(u)]\}_{j\in J(u)}$ with $J(u)$ finite nonempty. By definition of MIVNS

core/hull, applied to $\mathcal{T}(u) = \mathcal{A}_{\mathrm{F}}(u)$,

$$\mathrm{core}_{\wedge}\big(\mathcal{T}(u)\big) = \begin{cases} [\max_{j\in J(u)} \alpha_j(u),\ \min_{j\in J(u)} \beta_j(u)], & \text{if } \max \alpha_j \le \min \beta_j, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\mathrm{hull}_{\vee}\big(\mathcal{T}(u)\big) = [\min_{j\in J(u)} \alpha_j(u),\ \max_{j\in J(u)} \beta_j(u)],$$

which are exactly the MIVFS core/hull for $\mathcal{A}_{\mathrm{F}}(u)$. For the neutrosophic indeterminacy and falsity components, each is the singleton family $\{[0,0]\}$; hence

$$\max_{[\alpha,\beta]\in\{[0,0]\}} \alpha = 0, \qquad \min_{[\alpha,\beta]\in\{[0,0]\}} \beta = 0 \implies \mathrm{core}_{\wedge}(\{[0,0]\}) = [0,0],$$

and trivially $\mathrm{hull}_{\vee}(\{[0,0]\}) = [0,0]$. Therefore the truth-component of $\Phi(\mathcal{A}_{\mathrm{F}})$ reproduces the MIVFS, while the neutrosophic extras are neutralized at $[0,0]$. □

3.3. HyperInterval-Valued Fuzzy Set

A HyperInterval-Valued Fuzzy Set permits arbitrary sets of membership intervals per element, deriving intersection cores and minimal containing hulls afterward.

Definition 3.15 (HyperInterval-valued fuzzy set (HIVFS))**.** A *HIVFS* on U assigns to each $u \in U$ a nonempty family $\mathrm{H}(u) \subseteq L([0,1])$. Its conjunctive core and hull are, respectively,

$$[\![\mathrm{H}(u)]\!]_{\wedge} = \bigcap_{I\in\mathrm{H}(u)} I = \begin{cases} [\,\sup_{I\in\mathrm{H}(u)} \inf I,\ \inf_{I\in\mathrm{H}(u)} \sup I\,], & \text{if feasible,} \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$[\![\mathrm{H}(u)]\!]_{\vee}^{\mathrm{hull}} = [\,\inf_{I\in\mathrm{H}(u)} \inf I,\ \sup_{I\in\mathrm{H}(u)} \sup I\,].$$

Remark 3.16 (Relation with interval-valued hesitant fuzzy sets)**.** At the raw membership-data level, a HyperInterval-valued fuzzy set $H : U \to \mathcal{P}^*(L([0,1]))$ is mathematically equivalent to an interval-valued hesitant fuzzy set with the same value space. Therefore, the terminology "HyperInterval-valued" is not intended to claim a new carrier-level structure at this flat level. The distinction in this paper lies in the added core/hull semantics, which interpret each interval family through conservative consensus and exploratory possibility, and in the extension to SuperHyperInterval-valued models, where nested interval families are handled by non-flattening hierarchical semantics.

Remark 3.17 (Information not preserved by IVNS or HFS)**.** A HyperInterval-valued fuzzy set can preserve source-wise interval information that is lost in a standard IVNS or HFS representation. Let $U = \{x\}$ and define

$$\mathrm{H}(x) = \{[0.20, 0.40], [0.60, 0.80]\}.$$

This means that two evidence sources support two separated admissible membership ranges for x. Its conjunctive core and disjunctive hull are

$$[\![\mathrm{H}(x)]\!]_{\wedge} = [0.20, 0.40] \cap [0.60, 0.80] = \varnothing,$$

and

$$[\![\mathrm{H}(x)]\!]_{\vee}^{\mathrm{hull}} = [0.20, 0.80].$$

A standard interval-valued neutrosophic set could record only one truth interval, for example $T(x) = [0.20, 0.80]$, but this loses the fact that no source supports the intermediate region $(0.40, 0.60)$. A hesitant fuzzy set could record finitely many point values, for example $\{0.20, 0.40, 0.60, 0.80\}$, but this loses the interval nature of each source estimate. Hence $\mathrm{H}(x)$ preserves both the source-wise interval uncertainty and the gap between incompatible evidence ranges, which are not retained by a single IVNS interval or by an ordinary HFS.

Example 3.18 (HIVFS in a Morning Commute Decision)**.** Let U be the set of candidate routes for today's commute. For the proposition "Arrive on time via route r_1", define the HyperInterval-valued fuzzy set

$$\mathrm{H}(r_1) = \big\{I_1, I_2, I_3\big\} \subseteq L([0,1]),$$

where each interval encodes an admissible membership range (probability-like confidence) from an independent source:

$$I_1 = [0.62, 0.78] \text{ (live traffic app)},$$
$$I_2 = [0.55, 0.72] \text{ (weather-adjusted model)},$$
$$I_3 = [0.68, 0.80] \text{ (historical punctuality)}.$$

By Definition (HIVFS), the *conjunctive core* and *disjunctive hull* at r_1 are

$$[\![\mathrm{H}(r_1)]\!]_{\wedge} = \bigcap_{I\in\mathrm{H}(r_1)} I = \begin{cases} \big[\, \sup_{I\in\mathrm{H}(r_1)} \inf I,\ \inf_{I\in\mathrm{H}(r_1)} \sup I \,\big], & \text{if feasible,} \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$[\![\mathrm{H}(r_1)]\!]_{\vee}^{\mathrm{hull}} = \big[\, \inf_{I\in\mathrm{H}(r_1)} \inf I,\ \sup_{I\in\mathrm{H}(r_1)} \sup I \,\big].$$

We compute these bounds explicitly. First the lower endpoints:

$$\inf I_1 = 0.62, \quad \inf I_2 = 0.55, \quad \inf I_3 = 0.68 \implies \sup_{I\in\mathrm{H}(r_1)} \inf I = \max\{0.62, 0.55, 0.68\} = 0.68.$$

Then the upper endpoints:

$$\sup I_1 = 0.78, \quad \sup I_2 = 0.72, \quad \sup I_3 = 0.80 \implies \inf_{I\in\mathrm{H}(r_1)} \sup I = \min\{0.78, 0.72, 0.80\} = 0.72.$$

Therefore the conjunctive core is feasible and equals

$$[\![\mathrm{H}(r_1)]\!]_{\wedge} = [0.68,\ 0.72].$$

The disjunctive hull aggregates the most permissive range:

$$\inf_{I\in\mathrm{H}(r_1)} \inf I = \min\{0.62, 0.55, 0.68\} = 0.55, \qquad \sup_{I\in\mathrm{H}(r_1)} \sup I = \max\{0.78, 0.72, 0.80\} = 0.80,$$

so

$$[\![\mathrm{H}(r_1)]\!]_{\vee}^{\mathrm{hull}} = [0.55,\ 0.80].$$

The interval $[0.68, 0.72]$ is the consensus range supported *simultaneously* by all sources for "arrive on time via r_1". The hull $[0.55, 0.80]$ captures the full plausible spectrum reported by *at least one* source. Decision makers can require the core (conservative) or consult the hull (exploratory) depending on risk tolerance.

3.4. SuperHyperInterval-Valued Fuzzy Set

A SuperHyperInterval-Valued Fuzzy Set assigns nested families of membership intervals via iterated powersets. In contrast to a flattened HyperInterval interpretation, the nesting may itself carry semantic information, such as evidence groups, source hierarchies, or trust layers. Therefore, besides the flat core/hull interpretation, we introduce a non-flattening hierarchical semantics in which aggregation is performed level by level.

Definition 3.19 (SuperHyperInterval-valued fuzzy set (order n))**.** Fix $n \geq 1$. A *SuperHyperInterval-valued fuzzy set of order* n on U is a map

$$\mathrm{SH}^{(n)} : U \longrightarrow \mathcal{P}^{n-1}\big(L([0,1])\big),$$

assigning to each u an $(n-1)$–nested family of numeric intervals. Conjunctive cores and hulls are computed by iterated application of intersection and hull at the leaves.

Definition 3.20 (Hierarchical trust-weighted semantics)**.** Let $X = L([0,1])$, and let $\mathcal{S} \in \mathcal{P}^r(X)$ be a finite nonempty r-level nested family of intervals, where $r \geq 0$. A *hierarchical trust system* on $\mathcal{S}$ assigns, to every non-leaf nested family $\mathcal{G}$, a weight function

$$\omega_{\mathcal{G}} : \mathcal{G} \longrightarrow [0,1]$$

satisfying

$$\sum_{G \in \mathcal{G}} \omega_{\mathcal{G}}(G) = 1.$$

The *hierarchical trust-weighted value* $\mathsf{HW}_\omega(\mathcal{S}) \in L([0,1])$ is defined recursively as follows.

If $r = 0$, then $\mathcal{S} \in X = L([0,1])$ is a single interval and

$$\mathsf{HW}_\omega(\mathcal{S}) = \mathcal{S}.$$

If $r \geq 1$, then each $G \in \mathcal{S}$ is an $(r-1)$-level nested family. Write

$$\mathsf{HW}_\omega(G) = [\ell_G, u_G].$$

Then

$$\mathsf{HW}_\omega(\mathcal{S}) := \left[\sum_{G \in \mathcal{S}} \omega_{\mathcal{S}}(G)\ell_G,\ \sum_{G \in \mathcal{S}} \omega_{\mathcal{S}}(G)u_G\right].$$

Proposition 3.21 (Well-definedness of hierarchical trust-weighted semantics)
Let $\mathcal{S} \in \mathcal{P}^r(L([0,1]))$ be finite and nonempty, and let ω be a hierarchical trust system on $\mathcal{S}$. Then

$$\mathsf{HW}_\omega(\mathcal{S}) \in L([0,1]).$$

In particular, the hierarchical trust-weighted semantics always returns a closed subinterval of $[0,1]$.

Proof
The proof is by induction on r. For $r = 0$, the assertion is immediate because $\mathcal{S} \in L([0,1])$. Assume the assertion holds for all $(r-1)$-level nested interval families. Let $\mathcal{S} \in \mathcal{P}^r(L([0,1]))$. For each $G \in \mathcal{S}$, the induction hypothesis gives

$$\mathsf{HW}_\omega(G) = [\ell_G, u_G] \in L([0,1]).$$

Hence $0 \leq \ell_G \leq u_G \leq 1$ for every $G \in \mathcal{S}$. Since the weights are nonnegative and sum to 1, we have

$$0 \leq \sum_{G \in \mathcal{S}} \omega_{\mathcal{S}}(G)\ell_G \leq \sum_{G \in \mathcal{S}} \omega_{\mathcal{S}}(G)u_G \leq 1.$$

Therefore

$$\mathsf{HW}_\omega(\mathcal{S}) = \left[\sum_{G \in \mathcal{S}} \omega_{\mathcal{S}}(G)\ell_G,\ \sum_{G \in \mathcal{S}} \omega_{\mathcal{S}}(G)u_G\right] \in L([0,1]).$$

□

Remark 3.22 (Why the SuperHyper level is not merely a flattened HyperInterval level)**.** Flattening a nested family $\mathcal{S} \in \mathcal{P}^r(L([0,1]))$ ignores all intermediate groupings and retains only the leaf intervals. The hierarchical trust-weighted semantics in Definition 3.20 does not do this. It aggregates intervals level by level, and therefore the final interval may depend on how the same leaf intervals are grouped.

For example, let

$$I_1 = [0,0], \qquad I_2 = [1,1], \qquad I_3 = [1,1].$$

Consider two nested families with the same leaves:

$$\mathcal{S}_1 = \big\{\{I_1, I_2\}, \{I_3\}\big\}, \qquad \mathcal{S}_2 = \big\{\{I_1\}, \{I_2, I_3\}\big\}.$$

Both flatten to the same set $\{I_1, I_2, I_3\}$. However, if all local weights are uniform, then

$$\mathsf{HW}_\omega(\mathcal{S}_1) = \left[\frac{3}{4}, \frac{3}{4}\right], \qquad \mathsf{HW}_\omega(\mathcal{S}_2) = \left[\frac{1}{2}, \frac{1}{2}\right].$$

Thus the nesting itself affects the result. This is the main reason why the SuperHyperInterval-valued model is not reducible, in general, to a flat HyperInterval-valued model.

Example 3.23 (SHIVFS (order n=3) for a Same-Day Delivery Promise)**.** Let $U = \{\tau\}$, where τ denotes today's task "Deliver the parcel by 6 pm". Fix $n = 3$. A SuperHyperInterval-valued fuzzy set of order 3 maps τ to a two-level nested family of numeric intervals (elements of $L([0,1])$), grouped by evidence sources:

$$\mathrm{SH}^{(3)}(\tau) = \big\{S_{\text{logistics}},\ S_{\text{conditions}},\ S_{\text{recipient}}\big\} \in \mathcal{P}^2\big(L([0,1])\big),$$

where each $S_\bullet$ is a finite set of closed intervals $[\alpha, \beta] \subseteq [0,1]$.

We instantiate concretely (all numbers are unit-free confidence levels):

$$\begin{aligned}
S_{\text{logistics}} &= \big\{I_1 = [0.78, 0.90] \text{ (Carrier A dispatch+capacity)},\ I_2 = [0.80, 0.88] \text{ (Carrier B historical on-time)}\big\},\\
S_{\text{conditions}} &= \big\{I_3 = [0.75, 0.92] \text{ (traffic nowcast)},\ I_4 = [0.77, 0.89] \text{ (weather-adjusted travel time)}\big\},\\
S_{\text{recipient}} &= \big\{I_5 = [0.76, 0.93] \text{ (recipient availability window)},\ I_6 = [0.82, 0.87] \text{ (building access constraints)}\big\}.
\end{aligned}$$

By definition, conjunctive cores and hulls are computed at the leaves. Flatten the nesting by taking the union of groups:

$$\mathrm{Leaves}(\tau) \ := \ S_{\text{logistics}} \cup S_{\text{conditions}} \cup S_{\text{recipient}} = \{I_1, I_2, I_3, I_4, I_5, I_6\} \subseteq L([0,1]).$$

Conjunctive core (intersection of all leaf intervals) is

$$\begin{aligned}
[\![\mathrm{SH}^{(3)}(\tau)]\!]_\wedge &= \bigcap_{k=1}^{6} I_k\\
&= \Big[\sup_{1\le k\le 6} \inf I_k,\ \inf_{1\le k\le 6} \sup I_k\Big],
\end{aligned}$$

provided feasibility $\sup\inf \le \inf\sup$ holds. We compute explicitly:

$$\begin{aligned}
&\inf I_1 = 0.78,\ \inf I_2 = 0.80,\ \inf I_3 = 0.75,\ \inf I_4 = 0.77,\ \inf I_5 = 0.76,\ \inf I_6 = 0.82\\
&\implies \sup_k \inf I_k = \max\{0.78, 0.80, 0.75, 0.77, 0.76, 0.82\} = 0.82,
\end{aligned}$$

$$\sup I_1 = 0.90,\ \sup I_2 = 0.88,\ \sup I_3 = 0.92,\ \sup I_4 = 0.89,\ \sup I_5 = 0.93,\ \sup I_6 = 0.87$$
$$\Longrightarrow\ \inf_k \sup I_k = \min\{0.90, 0.88, 0.92, 0.89, 0.93, 0.87\} = 0.87.$$

Since $0.82 \le 0.87$, the intersection is feasible and

$$[\![\mathrm{SH}^{(3)}(\tau)]\!]_\wedge = [0.82,\ 0.87]\ .$$

Disjunctive hull (least interval containing all leaf intervals) is

$$[\![\mathrm{SH}^{(3)}(\tau)]\!]_\vee^{\mathrm{hull}} = \Big[\ \inf_k \inf I_k,\ \sup_k \sup I_k\ \Big]$$

$$= \Big[\ \min\{0.78, 0.80, 0.75, 0.77, 0.76, 0.82\},\ \max\{0.90, 0.88, 0.92, 0.89, 0.93, 0.87\}\ \Big] = [0.75,\ 0.93].$$

The nested structure records *hierarchical* evidence: logistics providers, external conditions, and recipient constraints. Flattening aggregates every leaf interval; the core $[0.82, 0.87]$ captures the consensus range simultaneously supported by *all* groups and sources, while the hull $[0.75, 0.93]$ captures the full plausible spectrum reported by *at least one* leaf source. A planner requiring high reliability would use the core; exploratory planning or contingency analysis may reference the hull.

3.5. HyperInterval-Valued Neutrosophic Set

A HyperInterval-Valued Neutrosophic Set uses sets of intervals for truth, indeterminacy, and falsity, yielding componentwise intersection cores and hulls semantics.

Definition 3.24 (HyperInterval-Valued Neutrosophic Set (HIVNS))**.** Let $U \neq \varnothing$ be a universe, and let

$$L([0,1]) \ :=\ \big\{[\alpha,\beta] \subseteq [0,1]\ \big|\ 0 \le \alpha \le \beta \le 1\big\}$$

be the set of all closed numeric intervals in $[0, 1]$. A *HyperInterval-Valued Neutrosophic Set (HIVNS)* on U is a triple of maps

$$\mathcal{A}\ =\ (\mathcal{T}, \mathcal{I}, \mathcal{F}),$$
$$\mathcal{T}, \mathcal{I}, \mathcal{F} : U \longrightarrow \mathcal{P}^*\big(L([0,1])\big),$$

such that for each $u \in U$ the sets $\mathcal{T}(u)$, $\mathcal{I}(u)$, and $\mathcal{F}(u)$ are *finite, nonempty* families of intervals in $L([0, 1])$.

For a finite family $\mathcal{S} = \{[\alpha_j, \beta_j]\}_{j\in J} \subseteq L([0, 1])$, define the *conjunctive core* and the *disjunctive hull* by

$$\mathrm{core}_\wedge(\mathcal{S}) = \begin{cases} \big[\max_{j\in J}\alpha_j,\ \min_{j\in J}\beta_j\big], & \text{if } \max_j \alpha_j \le \min_j \beta_j,\\ \varnothing, & \text{otherwise,}\end{cases}$$

$$\mathrm{hull}_\vee(\mathcal{S}) = \big[\min_{j\in J}\alpha_j,\ \max_{j\in J}\beta_j\big].$$

We interpret $\mathrm{core}_\wedge(\mathcal{S})$ as the precise admissible range under *all* hyper-interval declarations, and $\mathrm{hull}_\vee(\mathcal{S})$ as the least interval containing *some* declaration.

The *conjunctive semantics* and *hull semantics* of $\mathcal{A}$ at $u \in U$ are then given componentwise by

$$\mathrm{core}_\wedge(\mathcal{A})(u)\ :=\ \big(\mathrm{core}_\wedge(\mathcal{T}(u)),\ \mathrm{core}_\wedge(\mathcal{I}(u)),\ \mathrm{core}_\wedge(\mathcal{F}(u))\big),$$

$$\mathrm{hull}_{\vee}(\mathcal{A})(u) \;:=\; \big(\mathrm{hull}_{\vee}(\mathcal{T}(u)),\ \mathrm{hull}_{\vee}(\mathcal{I}(u)),\ \mathrm{hull}_{\vee}(\mathcal{F}(u))\big),$$

with feasibility of the conjunctive semantics requiring, for each component,

$$\max_{[\alpha,\beta]\in\mathcal{T}(u)} \alpha \;\le\; \min_{[\alpha,\beta]\in\mathcal{T}(u)} \beta,$$
$$\max_{[\alpha,\beta]\in\mathcal{I}(u)} \alpha \;\le\; \min_{[\alpha,\beta]\in\mathcal{I}(u)} \beta,$$
$$\max_{[\alpha,\beta]\in\mathcal{F}(u)} \alpha \;\le\; \min_{[\alpha,\beta]\in\mathcal{F}(u)} \beta.$$

Theorem 3.25 (HIVNS generalizes IVNS)
Let an *Interval-Valued Neutrosophic Set (IVNS)* be given by three maps

$$T, I, F : U \longrightarrow L([0,1]), \qquad T(u) = [t^-(u), t^+(u)],\ I(u) = [i^-(u), i^+(u)],\ F(u) = [f^-(u), f^+(u)].$$

Define the embedding

$$\iota:\ \big(L([0,1])\big)^3 \;\longrightarrow\; \Big(\mathcal{P}^{*}\big(L([0,1])\big)\Big)^3,$$
$$\iota\big([a,b],[c,d],[e,f]\big) \;:=\; \big(\{[a,b]\},\{[c,d]\},\{[e,f]\}\big).$$

Then $\mathcal{A} := \iota(T, I, F)$ is a HIVNS and, for every $u \in U$,

$$\mathrm{core}_{\wedge}(\mathcal{A})(u) = \big(T(u), I(u), F(u)\big) = \mathrm{hull}_{\vee}(\mathcal{A})(u).$$

Consequently, every IVNS is (canonically) a special case of a HIVNS.

Proof
Fix $u \in U$. By construction,

$$\mathcal{T}(u) = \{[t^-(u), t^+(u)]\},$$
$$\mathcal{I}(u) = \{[i^-(u), i^+(u)]\},$$
$$\mathcal{F}(u) = \{[f^-(u), f^+(u)]\}.$$

Evaluating the maxima/minima over a singleton set gives

$$\max_{[\alpha,\beta]\in\mathcal{T}(u)} \alpha = t^-(u),$$
$$\min_{[\alpha,\beta]\in\mathcal{T}(u)} \beta = t^+(u),$$

and analogously for the I and F components. Hence

$$\mathrm{core}_{\wedge}(\mathcal{T}(u)) = [t^-(u), t^+(u)],$$
$$\mathrm{hull}_{\vee}(\mathcal{T}(u)) = [t^-(u), t^+(u)],$$

with identical equalities for $\mathcal{I}(u)$ and $\mathcal{F}(u)$. Therefore

$$\mathrm{core}_{\wedge}(\mathcal{A})(u) = \mathrm{hull}_{\vee}(\mathcal{A})(u) = (T(u), I(u), F(u)).$$

□

Example 3.26 (Concrete computation at a point)**.** Let $\mathcal{T}(u) = \{[0.6, 0.9], [0.7, 0.8]\}$, $\mathcal{I}(u) = \{[0.1, 0.3]\}$, $\mathcal{F}(u) = \{[0.05, 0.2], [0.0, 0.15]\}$. Then

$$\mathrm{core}_{\wedge}(\mathcal{T}(u)) = [\max\{0.6, 0.7\}, \min\{0.9, 0.8\}] = [0.7, 0.8],$$
$$\mathrm{hull}_{\vee}(\mathcal{T}(u)) = [\min\{0.6, 0.7\}, \max\{0.9, 0.8\}] = [0.6, 0.9],$$
$$\mathrm{core}_{\wedge}(\mathcal{I}(u)) = \mathrm{hull}_{\vee}(\mathcal{I}(u)) = [0.1, 0.3],$$
$$\mathrm{core}_{\wedge}(\mathcal{F}(u)) = [\max\{0.05, 0.0\}, \min\{0.2, 0.15\}] = [0.05, 0.15],$$
$$\mathrm{hull}_{\vee}(\mathcal{F}(u)) = [\min\{0.05, 0.0\}, \max\{0.2, 0.15\}] = [0.0, 0.2].$$

Thus $\mathrm{core}_{\wedge}(\mathcal{A})(u) = \big([0.7, 0.8], [0.1, 0.3], [0.05, 0.15]\big)$ and $\mathrm{hull}_{\vee}(\mathcal{A})(u) = \big([0.6, 0.9], [0.1, 0.3], [0.0, 0.2]\big)$.

3.6. SuperHyperInterval-Valued Neutrosophic Set

A SuperHyperInterval-Valued Neutrosophic Set organizes nested interval families for truth, indeterminacy, and falsity. When the nested levels represent evidence groups, expert hierarchies, or trust layers, flattening may erase relevant information. Therefore, the non-flattening hierarchical semantics introduced above is applied componentwise to the truth, indeterminacy, and falsity components.

Definition 3.27 (SuperHyperInterval-Valued Neutrosophic Set (order n))**.** Fix $n \geq 1$. A *SuperHyperInterval-Valued Neutrosophic Set (SHIVNS) of order* n on U is a triple

$$\mathcal{S}^{(n)} = \big(\mathsf{T}^{(n)}, \mathsf{I}^{(n)}, \mathsf{F}^{(n)}\big), \qquad \mathsf{T}^{(n)}, \mathsf{I}^{(n)}, \mathsf{F}^{(n)} : U \longrightarrow \mathcal{P}^{\,n-1}\big(L([0,1])\big),$$

with the requirement that for each $u \in U$ the sets $\mathsf{T}^{(n)}(u)$, $\mathsf{I}^{(n)}(u)$, $\mathsf{F}^{(n)}(u)$ are finite at every nesting level.

Write the *leaf families* at u as

$$\mathcal{T}_{\mathrm{leaf}}(u) := \mathrm{Flat}_{n-1}\big(\mathsf{T}^{(n)}(u)\big) \subseteq L([0,1]), \quad \mathcal{I}_{\mathrm{leaf}}(u) := \mathrm{Flat}_{n-1}\big(\mathsf{I}^{(n)}(u)\big), \quad \mathcal{F}_{\mathrm{leaf}}(u) := \mathrm{Flat}_{n-1}\big(\mathsf{F}^{(n)}(u)\big).$$

The *conjunctive semantics* and *hull semantics* of $\mathcal{S}^{(n)}$ are defined by applying Definition 3.24 to these leaf families:

$$\mathrm{core}_{\wedge}\big(\mathcal{S}^{(n)}\big)(u) := \Big(\mathrm{core}_{\wedge}\big(\mathcal{T}_{\mathrm{leaf}}(u)\big),\ \mathrm{core}_{\wedge}\big(\mathcal{I}_{\mathrm{leaf}}(u)\big),\ \mathrm{core}_{\wedge}\big(\mathcal{F}_{\mathrm{leaf}}(u)\big)\Big),$$

$$\mathrm{hull}_{\vee}\big(\mathcal{S}^{(n)}\big)(u) := \Big(\mathrm{hull}_{\vee}\big(\mathcal{T}_{\mathrm{leaf}}(u)\big),\ \mathrm{hull}_{\vee}\big(\mathcal{I}_{\mathrm{leaf}}(u)\big),\ \mathrm{hull}_{\vee}\big(\mathcal{F}_{\mathrm{leaf}}(u)\big)\Big).$$

Definition 3.28 (Non-flattening semantics for SHIVNS)**.** Let

$$\mathcal{A}^{(n)} = (\mathcal{T}^{(n)}, \mathcal{I}^{(n)}, \mathcal{F}^{(n)})$$

be a SuperHyperInterval-valued neutrosophic set of order n on U. For each $u \in U$, suppose that

$$\mathcal{T}^{(n)}(u), \quad \mathcal{I}^{(n)}(u), \quad \mathcal{F}^{(n)}(u)$$

are finite nonempty nested interval families in $\mathcal{P}^{n-1}(L([0,1]))$. Given hierarchical trust systems $\omega_T, \omega_I, \omega_F$ on these three nested families, define

$$\mathsf{HW}_{\omega}(\mathcal{A}^{(n)})(u) := \Big(\mathsf{HW}_{\omega_T}(\mathcal{T}^{(n)}(u)), \mathsf{HW}_{\omega_I}(\mathcal{I}^{(n)}(u)), \mathsf{HW}_{\omega_F}(\mathcal{F}^{(n)}(u))\Big).$$

That is, truth, indeterminacy, and falsity are aggregated by the same recursive non-flattening semantics, but the three components may use different hierarchical trust systems.

Proposition 3.29 (Well-definedness of non-flattening SHIVNS semantics)
For every $u \in U$,

$$\mathsf{HW}_{\omega}(\mathcal{A}^{(n)})(u) = \big([T^-(u), T^+(u)], [I^-(u), I^+(u)], [F^-(u), F^+(u)]\big)$$

is an interval-valued neutrosophic membership triple. In particular,

$$[T^-(u), T^+(u)], \quad [I^-(u), I^+(u)], \quad [F^-(u), F^+(u)] \in L([0,1]),$$

and hence

$$0 \leq T^+(u) + I^+(u) + F^+(u) \leq 3.$$

Proof
By Proposition 3.21, the hierarchical trust-weighted value of each nested component is an element of $L([0,1])$. Therefore the truth, indeterminacy, and falsity components are closed subintervals of $[0,1]$. Since each upper endpoint is at most 1, we obtain

$$0 \le T^+(u) + I^+(u) + F^+(u) \le 1 + 1 + 1 = 3.$$

Thus the resulting triple is a well-defined interval-valued neutrosophic membership triple. □

Definition 3.30 (Canonical nesting)**.** For any set S and $k \ge 0$, define $\mathrm{Nest}_0(S) := S$ and $\mathrm{Nest}_{k+1}(S) := \{\mathrm{Nest}_k(S)\}$. Then $\mathrm{Nest}_k(S) \in \mathcal{P}^k(S)$ and, crucially,

$$\mathrm{Flat}_k\big(\mathrm{Nest}_k(S)\big) \;=\; S \qquad \text{for all } k \ge 0. \tag{1}$$

Lemma 3.31 (Flattening a canonical nest)
Let S be a finite nonempty family of intervals. For $k \ge 0$, define the canonical k-fold nest recursively by

$$\mathrm{Nest}_0(S) := S, \qquad \mathrm{Nest}_{k+1}(S) := \{\mathrm{Nest}_k(S)\}.$$

Let the flattening maps be defined by

$$\mathrm{Flat}_0(S) := S,$$

and, for $k \ge 0$,

$$\mathrm{Flat}_{k+1}(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathrm{Flat}_k(A).$$

Then, for every $k \ge 0$,

$$\mathrm{Flat}_k\big(\mathrm{Nest}_k(S)\big) = S.$$

Proof
We prove the statement by induction on k.

For $k = 0$, we have

$$\mathrm{Flat}_0\big(\mathrm{Nest}_0(S)\big) = \mathrm{Flat}_0(S) = S,$$

by the definitions of Nest_0 and Flat_0. Hence the claim holds for $k = 0$.

Assume now that the claim holds for some $k \ge 0$, that is,

$$\mathrm{Flat}_k\big(\mathrm{Nest}_k(S)\big) = S.$$

We show that it also holds for $k + 1$. By the definition of the canonical nest,

$$\mathrm{Nest}_{k+1}(S) = \{\mathrm{Nest}_k(S)\}.$$

Therefore,

$$\begin{aligned}
\mathrm{Flat}_{k+1}\big(\mathrm{Nest}_{k+1}(S)\big) &= \mathrm{Flat}_{k+1}\big(\{\mathrm{Nest}_k(S)\}\big) \\
&= \bigcup_{A \in \{\mathrm{Nest}_k(S)\}} \mathrm{Flat}_k(A) \\
&= \mathrm{Flat}_k\big(\mathrm{Nest}_k(S)\big) \\
&= S,
\end{aligned}$$

where the last equality follows from the induction hypothesis. Thus the claim holds for $k + 1$. By induction, the equality

$$\mathrm{Flat}_k\big(\mathrm{Nest}_k(S)\big) = S$$

holds for all $k \ge 0$. □

Theorem 3.32 (SHIVNS generalizes HIVNS)
Let $\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F})$ be a HIVNS on U (Definition 3.24). For any $n \geq 1$, define the embedding

$$\jmath_n : \left(\mathcal{P}^*(L([0,1]))\right)^3 \longrightarrow \left(\mathcal{P}^{n-1}(L([0,1]))\right)^3$$

by

$$\left(\mathcal{T}(u), \mathcal{I}(u), \mathcal{F}(u)\right) \longmapsto \left(\mathrm{Nest}_{n-1}(\mathcal{T}(u)),\ \mathrm{Nest}_{n-1}(\mathcal{I}(u)),\ \mathrm{Nest}_{n-1}(\mathcal{F}(u))\right).$$

Then $\mathcal{S}^{(n)} := \jmath_n(\mathcal{A})$ is a SHIVNS of order n and, for every $u \in U$,

$$\mathrm{core}_\wedge\left(\mathcal{S}^{(n)}\right)(u) \ = \ \mathrm{core}_\wedge(\mathcal{A})(u), \qquad \mathrm{hull}_\vee\left(\mathcal{S}^{(n)}\right)(u) \ = \ \mathrm{hull}_\vee(\mathcal{A})(u).$$

Hence every HIVNS is (canonically) a special case of a SHIVNS (for any $n \geq 1$).

Proof
Fix $u \in U$. By Lemma 3.31 with $S = \mathcal{T}(u)$ (and similarly for $\mathcal{I}, \mathcal{F}$),

$$\mathcal{T}_{\mathrm{leaf}}(u) = \mathrm{Flat}_{n-1}\left(\mathrm{Nest}_{n-1}(\mathcal{T}(u))\right) = \mathcal{T}(u).$$

Therefore the leaf families of $\mathcal{S}^{(n)}$ coincide with the original hyper-interval families of $\mathcal{A}$:

$$\mathcal{T}_{\mathrm{leaf}}(u) = \mathcal{T}(u), \quad \mathcal{I}_{\mathrm{leaf}}(u) = \mathcal{I}(u), \quad \mathcal{F}_{\mathrm{leaf}}(u) = \mathcal{F}(u).$$

Since both $\mathrm{core}_\wedge$ and $\mathrm{hull}_\vee$ depend only on the sets of leaf intervals via $\max / \min$ and $\min / \max$, it follows immediately that

$$\mathrm{core}_\wedge\left(\mathcal{S}^{(n)}\right)(u) = \mathrm{core}_\wedge(\mathcal{A})(u), \qquad \mathrm{hull}_\vee\left(\mathcal{S}^{(n)}\right)(u) = \mathrm{hull}_\vee(\mathcal{A})(u).$$

□

3.7. *Algebraic Properties of Core and Hull Semantics*

In this subsection, we record several elementary algebraic properties of the core and hull semantics. We state the results for finite families of closed intervals in $L([0,1])$. The corresponding statements for MultiInterval-valued neutrosophic sets hold componentwise for the truth, indeterminacy, and falsity components.

Let

$$\mathcal{A} = \{[\alpha_j, \beta_j] \mid j \in J\} \subseteq L([0,1])$$

be a finite nonempty family of intervals. Its core and hull are denoted by

$$\mathrm{Core}(\mathcal{A}) = \begin{cases} [\max_j \alpha_j, \min_j \beta_j], & \text{if } \max_j \alpha_j \leq \min_j \beta_j, \\ \varnothing, & \text{otherwise}, \end{cases}$$

and

$$\mathrm{Hull}(\mathcal{A}) = [\min_j \alpha_j, \max_j \beta_j].$$

Proposition 3.33 (Basic monotonicity)
If $\mathcal{A} \subseteq \mathcal{B}$, then

$$\mathrm{Core}(\mathcal{B}) \subseteq \mathrm{Core}(\mathcal{A})$$

whenever both cores are nonempty, and

$$\mathrm{Hull}(\mathcal{A}) \subseteq \mathrm{Hull}(\mathcal{B}).$$

Thus adding more interval evidence can only shrink the conjunctive core and enlarge the disjunctive hull.

Proposition 3.34
The lower endpointive core and enlarge the disjunctive hull.

Proof
The lower of the core is a maximum of lower endpoints, while its upper endpoint is a minimum of upper endpoints. Adding more intervals can only increase the former and decrease the latter. Hence the core becomes smaller. Conversely, the hull uses the minimum of the lower endpoints and the maximum of the upper endpoints, so adding more intervals can only enlarge the hull. □

Remark 3.35 (Distributivity is conditional)**.** Let $X, Y, Z \in L([0,1])$. Define

$$X \wedge Y := X \cap Y, \qquad X \vee Y := \mathrm{Hull}(\{X, Y\}).$$

In general,

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$$

does not always hold. For example, take

$$X = [0.45, 0.55], \qquad Y = [0, 0.2], \qquad Z = [0.8, 1].$$

Then

$$Y \vee Z = [0, 1],$$

so

$$X \wedge (Y \vee Z) = [0.45, 0.55].$$

However,

$$X \wedge Y = \varnothing, \qquad X \wedge Z = \varnothing,$$

and hence the right-hand side is empty. Therefore distributivity fails without additional assumptions.

A sufficient condition for distributivity is that $Y \cap Z \neq \varnothing$. In that case, $Y \cup Z$ is already an interval and

$$X \cap \mathrm{Hull}(\{Y, Z\}) = \mathrm{Hull}\big(\{X \cap Y,\ X \cap Z\}\big),$$

with the convention that empty intersections are ignored whenever the remaining side is nonempty.

Definition 3.36 (Complement of an interval family)**.** For an interval $I = [a, b] \in L([0,1])$, define its complement by

$$I^c := [1 - b, 1 - a].$$

For a finite interval family

$$\mathcal{A} = \{I_j \mid j \in J\},$$

define

$$\mathcal{A}^c := \{I_j^c \mid j \in J\}.$$

Proposition 3.37 (Complement dualities)
Let $\mathcal{A} \subseteq L([0,1])$ be finite and nonempty. If $\mathrm{Core}(\mathcal{A}) \neq \varnothing$, then

$$\mathrm{Core}(\mathcal{A}^c) = \mathrm{Core}(\mathcal{A})^c.$$

Moreover,

$$\mathrm{Hull}(\mathcal{A}^c) = \mathrm{Hull}(\mathcal{A})^c.$$

Proof
Write $I_j = [\alpha_j, \beta_j]$. Then
$$I_j^c = [1-\beta_j, 1-\alpha_j].$$
Therefore
$$\mathrm{Core}(\mathcal{A}^c) = [\max_j(1-\beta_j), \min_j(1-\alpha_j)] = [1-\min_j \beta_j, 1-\max_j \alpha_j] = \mathrm{Core}(\mathcal{A})^c.$$
Similarly,
$$\mathrm{Hull}(\mathcal{A}^c) = [\min_j(1-\beta_j), \max_j(1-\alpha_j)] = [1-\max_j \beta_j, 1-\min_j \alpha_j] = \mathrm{Hull}(\mathcal{A})^c.$$
□

Remark 3.38 (Galois-type interpretation)**.** A direct Galois connection between core and hull is not canonical under ordinary inclusion alone, because both maps send interval families to intervals. However, a standard Galois-type interpretation is obtained after choosing natural orders on interval families.

Let $\mathfrak{F}$ be the set of all finite nonempty families of intervals in $L([0,1])$, and let
$$\eta : L([0,1]) \to \mathfrak{F}, \qquad \eta(K) = \{K\}$$
be the singleton embedding. Define the Hoare-type order
$$\mathcal{A} \preceq_H \mathcal{B} \iff \forall I \in \mathcal{A},\ \exists J \in \mathcal{B} \text{ such that } I \subseteq J.$$
Then
$$\mathrm{Hull}(\mathcal{A}) \subseteq K \iff \mathcal{A} \preceq_H \eta(K).$$
Thus the hull is the least interval containing all intervals in $\mathcal{A}$.

Similarly, define the Smyth-type order
$$\mathcal{A} \preceq_S \mathcal{B} \iff \forall J \in \mathcal{B},\ \exists I \in \mathcal{A} \text{ such that } I \subseteq J.$$
Then, whenever $\mathrm{Core}(\mathcal{A}) \neq \varnothing$,
$$K \subseteq \mathrm{Core}(\mathcal{A}) \iff \eta(K) \preceq_S \mathcal{A}.$$
Thus the core is the greatest interval contained in all intervals of $\mathcal{A}$. Under these Hoare- and Smyth-type orders, hull and core serve respectively as lower and upper adjoint-type constructions relative to the singleton embedding.

4. Conclusion

In this paper, we introduced HyperInterval-valued and SuperHyperInterval-valued fuzzy/neutrosophic sets. We also defined conjunctive core semantics, based on intersection, and disjunctive hull semantics, based on the least interval containing the relevant interval families. Furthermore, we proved embedding theorems showing that classical interval, fuzzy, and neutrosophic models arise as singleton or degenerate cases of the proposed framework.

In addition, we introduced a non-flattening hierarchical trust-weighted semantics for SuperHyperInterval-valued fuzzy/neutrosophic sets. This semantics aggregates interval evidence level by level, thereby preserving the information encoded by nested evidence structures. Consequently, the SuperHyperInterval construction is not, in general, reducible to the HyperInterval case obtained by simply flattening all leaf intervals.

For future work, we plan to investigate further extensions based on HyperFuzzy Sets [33], HyperNeutrosophic Sets [60, 61], and Plithogenic Sets [62, 63]. We also intend to examine possible structural applications and generalizations in the settings of Graphs [64], HyperGraphs [65], and SuperHyperGraphs [66, 67]. Future work will include small numerical simulations comparing core and hull decisions across Multi, Hyper, and SuperHyper models. Also, as a direction for future work, we would like to investigate computational feasibility. In particular, we hope to examine time complexity and space complexity. We also intend to consider possible approximation algorithms.

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Author's Contributions

Conceptualization, Takaaki Fujita; Investigation, Takaaki Fujita; Methodology, Takaaki Fujita; Writing – original draft, Takaaki Fujita; Writing – review & editing, All authors.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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