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New Insights into the fourth-order Hankel determinant within a certain class of analytical functions

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Abstract This paper investigates the subclass $M(\beta,\gamma,\varepsilon)$ of analytic functions defined in the open unit disk by employing the principle of subordination as a key analytical tool. Coefficient bounds are obtained for the coefficients $|\lambda_\tau|$ of functions in this class for $\tau = 2,3,4,5,6,7$, providing insight into the coefficient structure of the proposed subclass. Furthermore, a general expression for the fourth Hankel determinant is derived for functions belonging to $M(\beta,\gamma,\varepsilon)$, which represents a new contribution to the study of analytic function classes. Several additional consequences are also established, further enriching the theory of analytic functions.

Keywords Analytical functions, fourth Hankel determinant, Chebyshev polynomials, Coefficient bounds, Subordination

AMS 2010 subject classifications 30C45, 30A36

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the family of functions $\mathtt{f}$ that are analytic in the open unit disk

$$\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\},$$

and normalized by the standard initial conditions

$$\mathtt{f}(0) = 0 \quad \text{and} \quad \mathtt{f}'(0) = 1.$$

Every function $\mathtt{f} \in \mathcal{A}$ admits the power series representation

$$\mathtt{f}(\zeta) = \zeta + \sum_{\tau=2}^{\infty} \lambda_\tau \zeta^\tau, \qquad \zeta \in \Delta, \tag{1}$$

where $\{\lambda_\tau\}_{\tau\geq 2}$ are complex coefficients.

We recall that for two analytic functions $\mathtt{f}$ and φ defined on Δ, the function $\mathtt{f}$ is said to be subordinate to φ, written as

$$\mathtt{f}(\zeta) \prec \varphi(\zeta), \quad \zeta \in \Delta,$$

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if there exists a Schwarz function $\mathcal{W}$ analytic in Δ satisfying

$$\mathcal{W}(0) = 0 \quad \text{and} \quad |\mathcal{W}(\zeta)| < 1 \text{ for all } \zeta \in \Delta,$$

such that

$$\mathtt{f}(\zeta) = \varphi(\mathcal{W}(\zeta)).$$

The concept of subordination plays a fundamental role in geometric function theory and provides a flexible framework for defining and investigating subclasses of analytic functions.

Recently, several subclasses generated by exponential-type dominant functions have attracted considerable attention. In particular, Ma and Minda [13] introduced a broad family of close-to-convex functions characterized by subordination. For an analytic function Φ satisfying suitable geometric conditions, they considered the class

$$\mathscr{C}^*(\Phi) = \{\mathtt{f} \in \mathcal{A} : \mathtt{f}'(\zeta) \prec \Phi(\zeta), \quad \zeta \in \Delta\}.$$

Motivated by this approach, and in order to emphasize exponential-type behavior, Mendiratta *et al.* [14] introduced the subclass

$$\mathscr{C}_e^* = \mathscr{C}^*(e^{\zeta}),$$

which can be written explicitly as

$$\mathscr{C}_e^* = \left\{\mathtt{f} \in \mathcal{A} : \mathtt{f}'(\zeta) \prec e^{\zeta}, \quad \zeta \in \Delta\right\}. \tag{2}$$

This class consists of normalized analytic functions whose derivatives are subordinate to the exponential function e^{ζ}, and hence it inherits geometric properties determined by the image domain of e^{ζ}.

In view of the preceding definitions, it follows that the class $\mathscr{C}_e^*$ introduced therein possesses symmetry with respect to the real axis, since the dominant function e^{ζ} itself enjoys this property.

Let $\sigma, \tau \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For a function $\mathtt{f} \in \mathcal{A}$ represented by (1), Pommerenke [17, 18] introduced the σ-th Hankel determinant $\mathscr{H}_{\sigma}(\tau)$ defined by

$$\mathscr{H}_{\sigma}(\tau) = \begin{vmatrix} \lambda_{\tau} & \lambda_{\tau+1} & \cdots & \lambda_{\tau+\sigma-1} \\ \lambda_{\tau+1} & \lambda_{\tau+2} & \cdots & \lambda_{\tau+\sigma} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\tau+\sigma-1} & \lambda_{\tau+\sigma} & \cdots & \lambda_{\tau+2\sigma-2} \end{vmatrix}, \qquad (\lambda_1 = 1). \tag{3}$$

The Hankel determinant plays a fundamental role in geometric function theory. It appears naturally in the investigation of singularities [7] and in the analysis of power series, particularly those with integer coefficients. For a comprehensive treatment, the reader may consult [7, 16]. Sharp bounds and sThe Hankel determinant plays a fundamental role in geometric function theory. It appears naturally in the investigation of singularities [7] and in the analysis of power series, particularly those with integer coefficients. For a comprehensive treatment, the reader may consult [7, 16]. Sharp bounds and structural properties of $\mathscr{H}_{\sigma}(\tau)$ have been extensively explored for various subclasses of univalent and bi-univalent functions. tructural properties of $\mathscr{H}_{\sigma}(\tau)$ have been extensively explored for various subclasses of univalent and bi-univalent functions.

In the particular case $\sigma = 2$, one obtains

$$\mathscr{H}_2(1) = \lambda_3 - \lambda_2^2,$$

which corresponds to the classical Fekete–Szegő functional for the special case $\mu = 1$. Similarly, the second Hankel determinant is given by

$$\mathscr{H}_2(2) = \lambda_2\lambda_4 - \lambda_3^2$$

has been studied for numerous families, including bi-starlike and bi-convex functions [1, 2, 4, 5, 8, 11, 12, 19, 20, 26, 27]. For the class of Bazilevič functions, Krishan *et al.* obtained sharp estimates of $\mathscr{H}_2(2)$. More recently, Srivastava and collaborators [24] derived bounds for the second Hankel determinant in the setting of bi-univalent functions associated with the symmetric σ-derivative operator. Related investigations concerning Hankel and

Toeplitz determinants for subclasses of σ-starlike functions linked to generalized conic domains were presented in [21]. Additional contributions may be found in [6, 15, 22, 23].

Higher-order Hankel determinants have also attracted considerable attention in geometric function theory. In particular, the fourth Hankel determinant for analytic functions has been investigated in [20, 25], where several new results were established.

For functions of the form (1), the third Hankel determinant takes the explicit form

$$\mathscr{H}_3(1) = \begin{vmatrix} 1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_3 & \lambda_4 & \lambda_5 \end{vmatrix} = -\lambda_5\lambda_2^2 + 2\lambda_2\lambda_3\lambda_4 - \lambda_3^3 + \lambda_3\lambda_5 - \lambda_4^2.$$

Notably, Babalola [3] initiated a systematic study of $\mathscr{H}_3(1)$ in 2010, and his results have since stimulated substantial further research activity. Subsequent work [9] analyzed this determinant for specific subclasses of starlike functions. Motivated by these developments, we introduce a new subclass of analytic functions defined via subordination and establish upper bounds for the fourth Hankel determinant within this framework. The proofs of our main results rely on the following auxiliary lemma.

Lemma 1.1
[10] Let $\mathcal{P}$ denote the family of analytic functions of the form

$$\flat(\zeta) = 1 + \sum_{\tau=1}^{\infty} c_\tau \zeta^\tau, \tag{4}$$

satisfying $\flat(0) = 1$ and $\mathrm{Re}\{\flat(\zeta)\} > 0$ for all $\zeta \in \Delta$. Then, for every $\tau \geq 1$, the sharp estimate

$$|c_\tau| \leq 2$$

holds.

Lemma 1.2
[27] Let $\mathtt{f} \in \mathcal{A}$ be represented by the series expansion (1). Then, the initial Taylor coefficients satisfy the sharp estimates

$$|\lambda_2| \leq 1, \qquad |\lambda_3| \leq \frac{3}{4}, \qquad |\lambda_4| \leq \frac{17}{36}, \qquad |\lambda_5| \leq 1.$$

Definition 1.3
[27] For a function $\mathtt{f} \in \mathcal{A}$ of the form (1), the fourth Hankel determinant corresponding to $\tau = 1$ is defined by

$$\mathscr{H}_4(1) = \begin{vmatrix} 1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \end{vmatrix}. \tag{5}$$

Expanding the determinant along the first row gives

$$\mathscr{H}_4(1) = -\lambda_4\,\delta_1 + \lambda_5\,\delta_2 - \lambda_6\,\delta_3 + \lambda_7\,\delta_4,$$

where δ_j, for $j = 1, 2, 3, 4$, denote the corresponding cofactors.

For later estimation, we introduce the auxiliary quantities

$$\begin{aligned}
\mathcal{S}_1 &= |\lambda_2|\,|\lambda_4\lambda_6 - \lambda_5^2| + |\lambda_3|\,|\lambda_3\lambda_6 - \lambda_4\lambda_5| - |\lambda_4|\,|\lambda_3\lambda_5 - \lambda_4^2|, \\
\mathcal{S}_2 &= |\lambda_4\lambda_6 - \lambda_5^2| - |\lambda_2|\,|\lambda_3\lambda_6 - \lambda_4\lambda_5| + |\lambda_3|\,|\lambda_3\lambda_5 - \lambda_4^2|, \\
\mathcal{S}_3 &= |\lambda_3\lambda_6 - \lambda_4\lambda_5| + |\lambda_2|\,|\lambda_2\lambda_6 - \lambda_3\lambda_5| - |\lambda_4|\,|\lambda_2\lambda_4 - \lambda_3^2|, \\
\mathcal{S}_4 &= |\lambda_3|\,|\lambda_2\lambda_4 - \lambda_3^2| - |\lambda_4|\,|\lambda_4 - \lambda_2\lambda_3| + |\lambda_5|\,|\lambda_3 - \lambda_2^2|.
\end{aligned} \tag{6}$$

These expressions will be instrumental in establishing upper bounds for $\mathscr{H}_4(1)$ in the forthcoming analysis.

Chebyshev polynomials constitute one of the most important and widely used families of orthogonal polynomials. Their significance has increased considerably in numerical analysis, approximation theory, spectral methods, and several branches of applied and theoretical mathematics. From both theoretical and computational points of view, Chebyshev polynomials provide powerful tools for approximation, interpolation, numerical integration, and the analysis of analytic functions. Classically, four kinds of Chebyshev polynomials are known; among them, the Chebyshev polynomials of the first kind $\mathcal{T}_\tau(\varkappa)$ and the second kind $\mathcal{U}_\tau(\varkappa)$ have received particular attention because of their rich structural properties and wide range of applications. Several authors have investigated Chebyshev polynomials and related orthogonal polynomial families in connection with analytic and univalent function theory; see, for example, [28, 29, 30].

Motivated by these developments, the present paper introduces and studies a new subclass of analytic functions, denoted by $\mathscr{M}(\beta,\gamma,\Im)$, by means of a subordination condition associated with the generating function of the Chebyshev polynomials of the second kind. The proposed class is significant for two main reasons. First, the parameters β and γ provide a flexible differential structure which includes, as particular cases, several previously studied subclasses defined through derivative-type and convexity-related operators. Secondly, the use of the Chebyshev generating function allows the coefficient structure of the class to be governed by the polynomials $\mathcal{U}_\tau(\Im)$, thereby connecting geometric function theory with a classical family of orthogonal polynomials. Consequently, this framework offers a unified setting for deriving coefficient estimates and related geometric inequalities for a broader class of analytic functions.

For a real variable $\varkappa \in (-1,1)$, the Chebyshev polynomials of the first and second kinds are defined, respectively, by

$$\mathcal{T}_\tau(\varkappa) = \cos\big(\tau \arccos \varkappa\big),$$

and

$$\mathcal{U}_\tau(\varkappa) = \frac{\sin\big((\tau+1)\arccos \varkappa\big)}{\sin(\arccos \varkappa)} = \frac{\sin\big((\tau+1)\arccos \varkappa\big)}{\sqrt{1-\varkappa^2}}.$$

In view of the generating function of the Chebyshev polynomials of the second kind, we consider the analytic function

$$\mathscr{H}(\Im,\zeta) = \frac{1}{1-2\Im\zeta+\zeta^2}, \qquad \Im \in \left(\frac{1}{2},1\right), \qquad \zeta \in \Delta.$$

It is well known that, if $\Im = \cos\alpha$, where

$$\alpha \in \left(0,\frac{\pi}{3}\right),$$

then the function $\mathscr{H}$ admits the series representation

$$\mathscr{H}(\Im,\zeta) = 1 + \sum_{\tau=1}^{\infty} \frac{\sin\big((\tau+1)\alpha\big)}{\sin\alpha}\,\zeta^\tau, \qquad \zeta \in \Delta.$$

In particular, the first few terms are given by

$$\mathscr{H}(\Im,\zeta) = 1 + 2\cos\alpha\,\zeta + \big(3\cos^2\alpha - \sin^2\alpha\big)\zeta^2 + \cdots.$$

Equivalently, since $\Im = \cos\alpha$, the above expansion can be written in terms of the Chebyshev polynomials of the second kind as

$$\mathscr{H}(\Im,\zeta) = 1 + \mathcal{U}_1(\Im)\zeta + \mathcal{U}_2(\Im)\zeta^2 + \mathcal{U}_3(\Im)\zeta^3 + \cdots, \qquad \Im \in \left(\frac{1}{2},1\right), \quad \zeta \in \Delta,$$

where

$$\mathcal{U}_\tau(\Im) = \frac{\sin\big((\tau+1)\arccos\Im\big)}{\sqrt{1-\Im^2}}, \qquad \tau \in \mathbb{N}.$$

The Chebyshev polynomials of the second kind satisfy the classical three-term recurrence relation

$$\mathcal{U}_{\tau+1}(\Im) = 2\Im\,\mathcal{U}_\tau(\Im) - \mathcal{U}_{\tau-1}(\Im), \qquad \tau \geq 1,$$

with the initial values

$$\mathcal{U}_0(\Im) = 1, \qquad \mathcal{U}_1(\Im) = 2\Im.$$

For convenience, the first few polynomials are listed in Table 1.

Table 1. First few Chebyshev polynomials of the second kind $\mathcal{U}_\tau(\Im)$.

Degree τ	**Polynomial** $\mathcal{U}_\tau(\Im)$
$\tau = 0$	1
$\tau = 1$	$2\Im$
$\tau = 2$	$4\Im^2 - 1$
$\tau = 3$	$8\Im^3 - 4\Im$

We now introduce the following subclass of analytic functions defined by subordination to the Chebyshev generating function.

Definition 1.4
Let $\mathtt{f} \in \mathcal{A}$ be of the form (1). Then $\mathtt{f}$ is said to belong to the class $\mathscr{M}(\beta,\gamma,\Im)$ if it satisfies the subordination condition

$$\beta\,\zeta\mathtt{f}''(\zeta) + \gamma\big(\mathtt{f}'(\zeta) - 1\big) + 1 \prec \mathscr{H}(\Im,\zeta), \tag{7}$$

where $\beta > 0$, $0 \leq \gamma < 1$, and

$$\mathscr{H}(\Im,\zeta) = 1 + \mathcal{U}_1(\Im)\zeta + \mathcal{U}_2(\Im)\zeta^2 + \mathcal{U}_3(\Im)\zeta^3 + \cdots, \qquad \Im \in \left(\frac{1}{2}, 1\right), \quad \zeta \in \Delta.$$

The defining subordination condition (7) indicates that the differential expression

$$\beta\,\zeta\mathtt{f}''(\zeta) + \gamma\big(\mathtt{f}'(\zeta) - 1\big) + 1 \tag{8}$$

is subordinate to the Chebyshev generating function $\mathscr{H}(\Im,\zeta)$. Hence, the analytic and geometric behaviour of the class $\mathscr{M}(\beta,\gamma,\Im)$ is governed jointly by the differential parameters β,γ and the Chebyshev parameter $\Im$. In particular, the image domain of $\mathscr{H}(\Im,\zeta)$ determines the admissible range of the expression in (8).

Figure 1 illustrates the images of selected concentric circles in the unit disk Δ under the mapping

$$\mathscr{H}(\Im,\zeta) = \frac{1}{1 - 2\Im\zeta + \zeta^2}, \qquad \Im \in \left(\frac{1}{2}, 1\right), \qquad \zeta \in \Delta.$$

This provides a visual description of the geometric structure induced by the subordinating function.

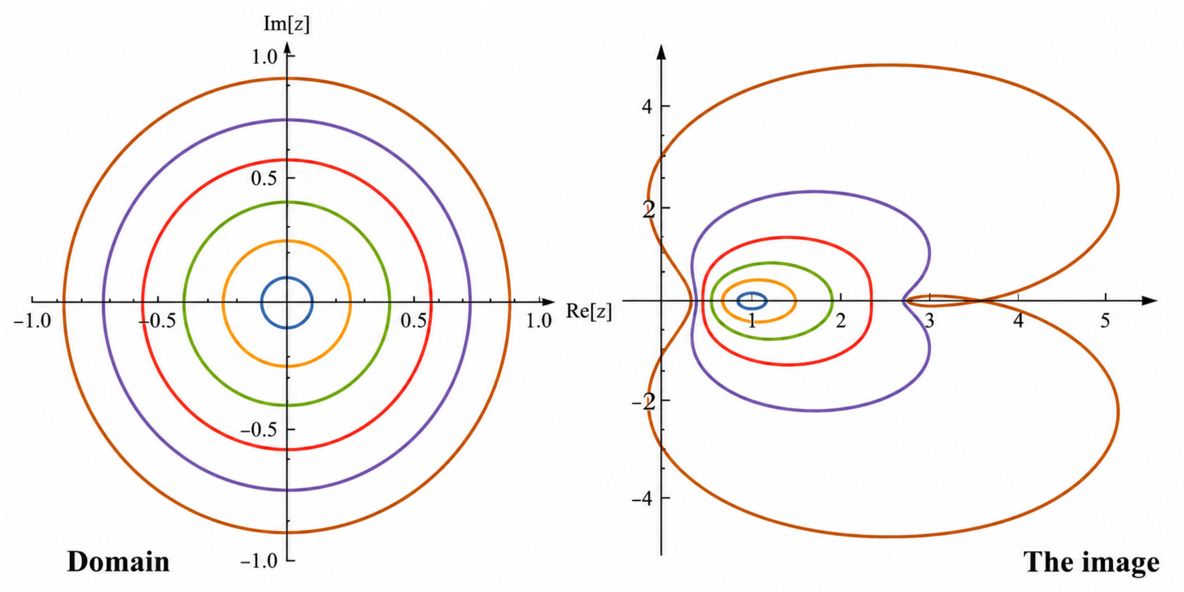

Figure 1. Images of selected concentric circles in the unit disk under the mapping $\mathscr{H}(\Im,\zeta)$.

Before deriving the main coefficient estimates, we first observe that the class $\mathscr{M}(\beta,\gamma,\Im)$ is non-empty.

Remark 1.5 (Non-emptiness of the class)
The class $\mathscr{M}(\beta,\gamma,\Im)$ is non-empty. Indeed, the identity function

$$\mathtt{f}_0(\zeta)=\zeta \tag{9}$$

belongs to $\mathscr{M}(\beta,\gamma,\Im)$. Since

$$\mathtt{f}_0'(\zeta)=1, \qquad \mathtt{f}_0''(\zeta)=0,$$

we obtain

$$\beta\zeta\mathtt{f}_0''(\zeta)+\gamma\big(\mathtt{f}_0'(\zeta)-1\big)+1=1=\mathscr{H}(\Im,0).$$

Thus, the subordination condition (7) is satisfied with the Schwarz function

$$\omega_0(\zeta)\equiv 0.$$

This proves that the class $\mathscr{M}(\beta,\gamma,\Im)$ is not void.

More generally, non-trivial admissible examples may be obtained by constructing a Schwarz function associated with the subordinating function $\mathscr{H}(\Im,\zeta)$. For instance, consider

$$\mathtt{f}_1(\zeta)=3\left(e^{\zeta/3}-1\right)=\zeta+\sum_{\tau=2}^{\infty}\frac{3^{1-\tau}}{\tau!}\zeta^{\tau}, \qquad \zeta\in\Delta. \tag{10}$$

Clearly, $\mathtt{f}_1\in\mathcal{A}$. Moreover,

$$\mathtt{f}_1'(\zeta)=e^{\zeta/3}, \qquad \mathtt{f}_1''(\zeta)=\frac{1}{3}e^{\zeta/3}.$$

Hence

$$\begin{aligned}P_1(\zeta)&:=\beta\zeta\mathtt{f}_1''(\zeta)+\gamma\big(\mathtt{f}_1'(\zeta)-1\big)+1\\&=1+\left(\gamma+\frac{\beta}{3}\zeta\right)e^{\zeta/3}-\gamma.\end{aligned} \tag{11}$$

Define

$$\omega_1(\zeta) = \Im - \sqrt{\Im^2 - 1 + \frac{1}{P_1(\zeta)}}, \tag{12}$$

where the branch of the square root is chosen so that

$$\sqrt{\Im^2} = \Im.$$

Since $P_1(0) = 1$, it follows that

$$\omega_1(0) = 0.$$

Furthermore, from (12), we have

$$\omega_1^2(\zeta) - 2\Im\omega_1(\zeta) = -1 + \frac{1}{P_1(\zeta)}.$$

Consequently,

$$1 - 2\Im\omega_1(\zeta) + \omega_1^2(\zeta) = \frac{1}{P_1(\zeta)}.$$

Therefore,

$$\mathscr{H}(\Im, \omega_1(\zeta)) = \frac{1}{1 - 2\Im\omega_1(\zeta) + \omega_1^2(\zeta)} = P_1(\zeta). \tag{13}$$

Thus, whenever ω_1 is a Schwarz function in Δ, that is,

$$\omega_1(0) = 0 \qquad \text{and} \qquad |\omega_1(\zeta)| < 1 \quad (\zeta \in \Delta),$$

we obtain

$$\beta\zeta \mathtt{f}_1''(\zeta) + \gamma\big(\mathtt{f}_1'(\zeta) - 1\big) + 1 = \mathscr{H}(\Im, \omega_1(\zeta)) \prec \mathscr{H}(\Im, \zeta).$$

Hence, under this admissibility condition, the function $\mathtt{f}_1$ defined in (10) belongs to $\mathscr{M}(\beta, \gamma, \Im)$.

Similarly, consider the function

$$\mathtt{f}_2(\zeta) = \frac{1}{2}\left((1-\zeta)^{-2} - 1\right) = \zeta + \sum_{\tau=2}^{\infty} \frac{\tau+1}{2}\zeta^\tau, \qquad \zeta \in \Delta. \tag{14}$$

Then $\mathtt{f}_2 \in \mathcal{A}$, and

$$\mathtt{f}_2'(\zeta) = \frac{1}{(1-\zeta)^3}, \qquad \mathtt{f}_2''(\zeta) = \frac{3}{(1-\zeta)^4}.$$

Therefore,

$$\begin{aligned} P_2(\zeta) &:= \beta\zeta \mathtt{f}_2''(\zeta) + \gamma\big(\mathtt{f}_2'(\zeta) - 1\big) + 1 \\ &= 1 + \frac{3\beta\zeta}{(1-\zeta)^4} + \gamma\left(\frac{1}{(1-\zeta)^3} - 1\right). \end{aligned} \tag{15}$$

Define

$$\omega_2(\zeta) = \Im - \sqrt{\Im^2 - 1 + \frac{1}{P_2(\zeta)}}, \tag{16}$$

where the branch of the square root is chosen so that

$$\sqrt{\Im^2} = \Im.$$

Since $P_2(0) = 1$, we have

$$\omega_2(0) = 0.$$

Moreover,

$$\omega_2^2(\zeta) - 2\Im\omega_2(\zeta) = -1 + \frac{1}{P_2(\zeta)},$$

and hence

$$1 - 2\Im\omega_2(\zeta) + \omega_2^2(\zeta) = \frac{1}{P_2(\zeta)}.$$

Consequently,

$$\mathscr{H}(\Im, \omega_2(\zeta)) = \frac{1}{1 - 2\Im\omega_2(\zeta) + \omega_2^2(\zeta)} = P_2(\zeta). \tag{17}$$

Thus, whenever ω_2 is a Schwarz function in Δ, we get

$$\beta\zeta \mathtt{f}_2''(\zeta) + \gamma\big(\mathtt{f}_2'(\zeta) - 1\big) + 1 = \mathscr{H}(\Im, \omega_2(\zeta)) \prec \mathscr{H}(\Im, \zeta).$$

Accordingly, under this admissibility condition, the function $\mathtt{f}_2$ defined in (14) also belongs to $\mathscr{M}(\beta, \gamma, \Im)$.

In the sequel, we establish the main coefficient estimates for functions belonging to the class $\mathscr{M}(\beta, \gamma, \Im)$. In particular, we obtain bounds for the coefficients $|\lambda_\tau|$, $2 \leq \tau \leq 7$, and discuss the consequences of these estimates for the geometric behaviour of the functions in this class.

2. Results

Theorem 2.1
Let $\mathtt{f}$ be a function given by (1) that belongs to the subclass $\mathscr{M}(\beta, \gamma, \Im)$. Then

$$|\lambda_2| \leq \frac{2}{(\beta+\gamma)}, \quad |\lambda_3| \leq \frac{4}{(2\beta+\gamma)}, \quad |\lambda_4| \leq \frac{6}{(3\beta+\gamma)}$$
$$|\lambda_5| \leq \frac{20.8}{(4\beta+\gamma)}, \quad |\lambda_6| \leq \frac{24.66}{(5\beta+\gamma)}, \quad |\lambda_7| \leq \frac{88.57}{(6\beta+\gamma)}.$$

Proof
If $\mathtt{f} \in \mathscr{M}(\beta, \gamma, \Im)$, then there exists an analytic function ϑ in $\mathscr{A}$ with $|\vartheta(\zeta)| \leq 1$ and we can write

$$\beta\zeta \mathtt{f}''(\zeta) + \gamma\left(\mathtt{f}'(\zeta) - 1\right) + 1 = \mathscr{H}(\Im, \vartheta(\zeta)).$$

Using the definition of subordination, there exists a Schwarz function $\mathcal{J}$, analytic in Δ, such that

$$\mathcal{J}(0) = 0, \qquad |\mathcal{J}(\zeta)| < 1 \quad (\zeta \in \Delta),$$

and

$$\mathcal{J}(\zeta) = \sum_{\tau=1}^{\infty} \flat_\tau \zeta^\tau, \qquad \zeta \in \Delta. \tag{18}$$

Associated with $\mathcal{J}$, we define the auxiliary function

$$\mathcal{L}(\zeta) = \frac{1 + \mathcal{J}(\zeta)}{1 - \mathcal{J}(\zeta)} = 1 + \sum_{\tau=1}^{\infty} L_\tau \zeta^\tau. \tag{19}$$

Since

$$\frac{1 + \mathcal{J}(\zeta)}{1 - \mathcal{J}(\zeta)} = 1 + 2\mathcal{J}(\zeta) + 2\mathcal{J}^2(\zeta) + 2\mathcal{J}^3(\zeta) + \cdots,$$

we obtain, up to the sixth order,

$$\begin{aligned}
\mathcal{L}(\zeta) = 1 &+ 2\flat_1\zeta + 2\left(\flat_2 + \flat_1^2\right)\zeta^2 + 2\left(\flat_3 + 2\flat_1\flat_2 + \flat_1^3\right)\zeta^3 \\
&+ 2\left(\flat_4 + 2\flat_1\flat_3 + \flat_2^2 + 3\flat_1^2\flat_2 + \flat_1^4\right)\zeta^4 \\
&+ 2\left(\flat_5 + 2\flat_1\flat_4 + 2\flat_2\flat_3 + 3\flat_1^2\flat_3 + 3\flat_1\flat_2^2 + 4\flat_1^3\flat_2 + \flat_1^5\right)\zeta^5 \\
&+ 2\left(\flat_6 + 2\flat_1\flat_5 + 2\flat_2\flat_4 + \flat_3^2 + 3\flat_1^2\flat_4 + 6\flat_1\flat_2\flat_3 + \flat_2^3 + 4\flat_1^3\flat_3 + 6\flat_1^2\flat_2^2 + 5\flat_1^4\flat_2 + \flat_1^6\right)\zeta^6 + \cdots.
\end{aligned}$$

Equivalently, the coefficients L_τ are given by

$$\begin{aligned}
L_1 &= 2\flat_1,\\
L_2 &= 2\left(\flat_2+\flat_1^2\right),\\
L_3 &= 2\left(\flat_3+2\flat_1\flat_2+\flat_1^3\right),\\
L_4 &= 2\left(\flat_4+2\flat_1\flat_3+\flat_2^2+3\flat_1^2\flat_2+\flat_1^4\right),\\
L_5 &= 2\left(\flat_5+2\flat_1\flat_4+2\flat_2\flat_3+3\flat_1^2\flat_3+3\flat_1\flat_2^2+4\flat_1^3\flat_2+\flat_1^5\right),\\
L_6 &= 2\left(\flat_6+2\flat_1\flat_5+2\flat_2\flat_4+\flat_3^2+3\flat_1^2\flat_4+6\flat_1\flat_2\flat_3+\flat_2^3+4\flat_1^3\flat_3+6\flat_1^2\flat_2^2+5\flat_1^4\flat_2+\flat_1^6\right).
\end{aligned}$$

Since $\mathtt{f}\in\mathscr{A}$ is written in the form given by (1), then

$$\begin{aligned}
\beta\zeta\mathtt{f}''(\zeta)+\gamma\left(\mathtt{f}'(\zeta)-1\right)+1 &= 1+2(\beta+\gamma)\lambda_2\zeta+3(2\beta+\gamma)\lambda_3\zeta^2\\
&\quad+4(3\beta+\gamma)\lambda_4\zeta^3+5(4\beta+\gamma)\lambda_5\zeta^4+6(5\beta+\gamma)\lambda_6\zeta^5+7(6\beta+\gamma)\lambda_7\zeta^6+\cdots.
\end{aligned} \tag{20}$$

Using (7) and (19) comparing the coefficients of ζ^τ for $\tau=,2,3,4,5,6$, we get

$$\begin{aligned}
2(\beta+\gamma)\lambda_2 &= 2\flat_1,\\
3(2\beta+\gamma)\lambda_3 &= 2\left(\flat_2+\flat_1^2\right),\\
4(3\beta+\gamma)\lambda_4 &= 2\left[\flat_3+\flat_1\left(2\flat_2+\flat_1^2\right)\right],\\
5(4\beta+\gamma)\lambda_5 &= 2\left[\flat_4+\flat_2^2+\flat_1^2\left(3\flat_2+\flat_1^2\right)+2\flat_1\flat_3\right],\\
6(5\beta+\gamma)\lambda_6 &= 2\left[\flat_5+2\flat_2\left(\flat_3+2\flat_1^3\right)+3\flat_1\left(\flat_1\flat_3+\flat_2^2\right)+\flat_1\left(2\flat_4+\flat_1^4\right)\right],\\
7(6\beta+\gamma)\lambda_7 &= 2\Big[\flat_6+\flat_1^3\left(4\flat_3+\flat_1^3\right)+\flat_1^2\left(3\flat_4+5\flat_2\flat_1^2\right)+2\flat_1\left(\flat_5+3\flat_2\flat_3\right)+\flat_3^2+\flat_2^2\left(\flat_2+6\flat_1^2\right)+2\flat_2\flat_4\Big].
\end{aligned} \tag{21}$$

by Lemma 1.1, we get

$$|\lambda_2|\le\frac{2}{\beta+\gamma}, \tag{22}$$

$$|\lambda_3|\le\frac{4}{2\beta+\gamma}, \tag{23}$$

$$|\lambda_4|\le\frac{6}{3\beta+\gamma}, \tag{24}$$

$$|\lambda_5|\le\frac{20.8}{4\beta+\gamma}, \tag{25}$$

$$|\lambda_6|\le\frac{24.66}{5\beta+\gamma}, \tag{26}$$

$$|\lambda_7|\le\frac{88.57}{6\beta+\gamma}. \tag{27}$$

The proof of this theorem is complete. □

In the following theorem, the estimates in $|\mathscr{H}_4(1)|$ are determined for the functions in the class $\mathscr{M}(\beta,\gamma,\Im)$.

Theorem 2.2
Let f be of the form (1) and suppose that $\mathtt{f} \in \mathscr{M}(\beta, \gamma, \Im)$, where $\beta > 0$ and $0 \leq \gamma < 1$. Define

$$\begin{aligned}
B_1 &= 6\beta^2 + 5\beta\gamma + \gamma^2,\\
B_2 &= 8\beta^2 + 6\beta\gamma + \gamma^2,\\
B_3 &= 24\beta^3 + 26\beta^2\gamma + 9\beta\gamma^2 + \gamma^3,\\
B_4 &= 125\beta^2 + 150\beta\gamma + 25\gamma^2,\\
B_5 &= 3000\beta^4 + 3850\beta^3\gamma + 1775\beta^2\gamma^2 + 350\beta\gamma^3 + 25\gamma^4.
\end{aligned}$$

Furthermore, let

$$\mathcal{R}(\beta, \gamma) = B_4 B_2^2 (3\beta + \gamma)^3 B_3^2 (2\beta + \gamma)^2 B_5 B_1^2 (\beta + \gamma)^3.$$

Set

$$\Phi_1(\beta, \gamma) = \frac{B_3^2 B_1^2 \big((\beta + \gamma)(2\beta + \gamma)\big)^3 B_5}{(3\beta + \gamma)\mathcal{R}(\beta, \gamma)},$$

$$\Phi_2(\beta, \gamma) = \frac{B_2^2 \big((\beta + \gamma)(2\beta + \gamma)(3\beta + \gamma)\big)^3 B_5}{(4\beta + \gamma)\mathcal{R}(\beta, \gamma)},$$

$$\Phi_3(\beta, \gamma) = \frac{B_4 B_2^2 \big((\beta + \gamma)(2\beta + \gamma)\big)^3 B_3^2}{(5\beta + \gamma)\mathcal{R}(\beta, \gamma)},$$

$$\Phi_4(\beta, \gamma) = \frac{B_4 B_2^2 B_5}{(6\beta + \gamma)\mathcal{R}(\beta, \gamma)}.$$

Then the fourth Hankel determinant satisfies

$$\begin{aligned}
|\mathscr{H}_4(1)| \leq{}& 6\,|\mathcal{K}_1(\beta, \gamma)|\,\Phi_1(\beta, \gamma) + 20.8\,|\mathcal{K}_2(\beta, \gamma)|\,\Phi_2(\beta, \gamma)\\
&+ 24.66\,|\mathcal{K}_3(\beta, \gamma)|\,\Phi_3(\beta, \gamma) + 88.57\,|\mathcal{K}_4(\beta, \gamma)|\,\Phi_4(\beta, \gamma),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\mathcal{K}_1(\beta, \gamma) ={}& 10416384\beta^6 + 24659712\beta^5\gamma + 23588688\beta^4\gamma^2 + 11630296\beta^3\gamma^3\\
&+ 347642\beta^2\gamma^4 + 430740\beta\gamma^5 + 23930\gamma^6,
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_2(\beta, \gamma) ={}& 998784\beta^5 + 2206080\beta^4\gamma + 1674872\beta^3\gamma^2 + 585788\beta^2\gamma^3\\
&+ 96448\beta\gamma^4 + 6028\gamma^5 - 1236672\beta^6 - 356800\beta^5\gamma\\
&- 4049900\beta^4\gamma^2 - 2333252\beta^3\gamma^3 - 722235\beta^2\gamma^4\\
&- 113872\beta\gamma^5 - 7117\gamma^6,
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_3(\beta, \gamma) ={}& \Big(1608\beta^4 - 7794\beta^3\gamma - 11418\beta^2\gamma^2 - 5886\beta\gamma^3 - 654\gamma^4\\
&- 3126\beta^3 - 31564\beta^2\gamma - 15246\beta\gamma^2 - 1694\gamma^3\Big) B_1^2\\
&- \left(96\beta\gamma + 24\gamma^2\right) B_5 (\beta + \gamma)^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_4(\beta, \gamma) ={}& \Big(576\beta^2\gamma + 224\beta\gamma^2 + 32\gamma^3 - 288\beta^4 - 720\beta^3\gamma\\
&- 648\beta^2\gamma^2 + 576\beta^3 - 252\beta\gamma^3 - 36\gamma^4\Big)\left(40\beta^2 + 30\beta\gamma + 5\gamma^2\right)(\beta + \gamma)^2\\
&+ (3\beta + \gamma)^2(\beta + \gamma)\left(416\beta^2 + 832\beta\gamma + 416(\beta^2 - \gamma) - 832\beta\right).
\end{aligned}$$

Proof
Let B_j, $j = 1, 2, \ldots, 5$, $\mathcal{R}(\beta,\gamma)$, and $\Phi_j(\beta,\gamma)$, $j = 1, 2, 3, 4$, be as defined in Theorem 2.2. The fourth Hankel determinant can be expanded as

$$\mathscr{H}_4(1) = |-\lambda_4\mathcal{S}_1 + \lambda_5\mathcal{S}_2 - \lambda_6\mathcal{S}_3 + \lambda_7\mathcal{S}_4|\,,$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{S}_4$ are the corresponding cofactors. Hence, by the triangle inequality, we obtain

$$|\mathscr{H}_4(1)| \le |\lambda_4|\,|\mathcal{S}_1| + |\lambda_5|\,|\mathcal{S}_2| + |\lambda_6|\,|\mathcal{S}_3| + |\lambda_7|\,|\mathcal{S}_4|. \tag{29}$$

The cofactors are estimated in terms of the coefficients as follows:

$$\begin{aligned}
|\mathcal{S}_1| &\le |\lambda_2|\,|\lambda_4\lambda_6 - \lambda_5^2| + |\lambda_3|\,|\lambda_3\lambda_6 - \lambda_4\lambda_5| + |\lambda_4|\,|\lambda_3\lambda_5 - \lambda_4^2|,\\
|\mathcal{S}_2| &\le |\lambda_4\lambda_6 - \lambda_5^2| + |\lambda_2|\,|\lambda_3\lambda_6 - \lambda_4\lambda_5| + |\lambda_3|\,|\lambda_3\lambda_5 - \lambda_4^2|,\\
|\mathcal{S}_3| &\le |\lambda_3\lambda_6 - \lambda_4\lambda_5| + |\lambda_2|\,|\lambda_2\lambda_6 - \lambda_3\lambda_5| + |\lambda_4|\,|\lambda_2\lambda_4 - \lambda_3^2|,\\
|\mathcal{S}_4| &\le |\lambda_3|\,|\lambda_2\lambda_4 - \lambda_3^2| + |\lambda_4|\,|\lambda_4 - \lambda_2\lambda_3| + |\lambda_5|\,|\lambda_3 - \lambda_2^2|.
\end{aligned}$$

Using the coefficient estimates obtained in (22)–(27), these inequalities give

$$\begin{aligned}
|\mathcal{S}_1| &\le \frac{|\mathcal{K}_1(\beta,\gamma)|}{\mathcal{D}_1(\beta,\gamma)},\\
|\mathcal{S}_2| &\le \frac{|\mathcal{K}_2(\beta,\gamma)|}{\mathcal{D}_2(\beta,\gamma)},\\
|\mathcal{S}_3| &\le \frac{|\mathcal{K}_3(\beta,\gamma)|}{\mathcal{D}_3(\beta,\gamma)},\\
|\mathcal{S}_4| &\le \frac{|\mathcal{K}_4(\beta,\gamma)|}{\mathcal{D}_4(\beta,\gamma)},
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
\mathcal{D}_1(\beta,\gamma) &= B_4B_2^2(3\beta+\gamma)^3,\\
\mathcal{D}_2(\beta,\gamma) &= B_4B_3^2,\\
\mathcal{D}_3(\beta,\gamma) &= B_5B_1^2,\\
\mathcal{D}_4(\beta,\gamma) &= \left(40\beta^2 + 30\beta\gamma + 5\gamma^2\right)\left((\beta+\gamma)(2\beta+\gamma)\right)^3(3\beta+\gamma)^2.
\end{aligned}$$

Moreover, using the estimates for the coefficients $\lambda_4, \lambda_5, \lambda_6$, and λ_7, and then reducing the resulting terms to the common denominator $\mathcal{R}(\beta,\gamma)$, we get

$$\begin{aligned}
|\lambda_4|\,|\mathcal{S}_1| &\le 6|\mathcal{K}_1(\beta,\gamma)|\Phi_1(\beta,\gamma),\\
|\lambda_5|\,|\mathcal{S}_2| &\le 20.8|\mathcal{K}_2(\beta,\gamma)|\Phi_2(\beta,\gamma),\\
|\lambda_6|\,|\mathcal{S}_3| &\le 24.66|\mathcal{K}_3(\beta,\gamma)|\Phi_3(\beta,\gamma),\\
|\lambda_7|\,|\mathcal{S}_4| &\le 88.57|\mathcal{K}_4(\beta,\gamma)|\Phi_4(\beta,\gamma).
\end{aligned}$$

Substituting these four estimates into (29), we obtain

$$\begin{aligned}
|\mathscr{H}_4(1)| \le 6|\mathcal{K}_1(\beta,\gamma)|\Phi_1(\beta,\gamma) &+ 20.8|\mathcal{K}_2(\beta,\gamma)|\Phi_2(\beta,\gamma)\\
&+ 24.66|\mathcal{K}_3(\beta,\gamma)|\Phi_3(\beta,\gamma) + 88.57|\mathcal{K}_4(\beta,\gamma)|\Phi_4(\beta,\gamma).
\end{aligned}$$

This is the required estimate in Theorem 2.2. Hence the proof is complete. □

Remark 2.3
The compact form (28) decomposes the lengthy rational bound into four explicit contributions. This presentation avoids a long single formula and makes the estimate easier to read, verify, and apply. Moreover, it clarifies the dependence of the upper bound on the parameters β and γ. Since

$$\beta + j\gamma > 0 \qquad (j = 1, 2, \dots, 6),$$

and since $B_j > 0$ for $j = 1, 2, \dots, 5$ under the assumptions $\beta > 0$ and $0 \le \gamma < 1$, the quantities $\Phi_j(\beta, \gamma)$, $j = 1, 2, 3, 4$, are positive. Hence the right-hand side of (28) gives an explicit non-negative upper bound for $|\mathscr{H}_4(1)|$.

To illustrate the applicability of the estimate, consider the special case $\gamma = 0$. Then

$$\beta + \gamma = \beta, \qquad 2\beta + \gamma = 2\beta, \qquad 3\beta + \gamma = 3\beta,$$

and similarly

$$4\beta + \gamma = 4\beta, \qquad 5\beta + \gamma = 5\beta, \qquad 6\beta + \gamma = 6\beta.$$

Thus the general bound is reduced to a simpler form. In particular, for $(\beta, \gamma) = (1, 0)$, we have

$$B_1 = 6, \qquad B_2 = 8, \qquad B_3 = 24, \qquad B_4 = 125, \qquad B_5 = 3000.$$

Consequently,

$$\Phi_1 = \frac{1}{324000}, \qquad \Phi_2 = \frac{1}{5184000}, \qquad \Phi_3 = \frac{1}{7290000}, \qquad \Phi_4 = \frac{1}{13436928}.$$

Also,

$$\mathcal{K}_1(1,0) = 10416384, \qquad \mathcal{K}_2(1,0) = -237888,$$
$$\mathcal{K}_3(1,0) = -54648, \qquad \mathcal{K}_4(1,0) = 11520.$$

Therefore, Theorem 2.2 yields

$$|\mathscr{H}_4(1)| \le 194.112.$$

This example shows how the general estimate can be evaluated explicitly for particular choices of the parameters and also demonstrates the advantage of the compact formulation in (28).

In case $\beta = 1$, we get the following Corollary.

Corollary 2.4
If a function f of the form (1) belongs to the subclass $\mathscr{M}(\gamma, \Im)$, then

$$\begin{aligned}
|\mathscr{H}_4(1)| \le\ & \frac{-6\,\mathcal{G}_1(\gamma)\left[\begin{array}{l}\left(24+26\gamma+9\gamma^2+\gamma^3\right)^2\left(6+5\gamma+\gamma^2\right)^2\left((1+\gamma)(2+\gamma)\right)^3\\ \times\left(3025+3850\gamma+1775\gamma^2+350\gamma^3\right)\end{array}\right]}{(3+\gamma)\,\mathcal{F}(\gamma)}\\
&+\frac{20.8\,\mathcal{G}_2(\gamma)\left[\left(8+6\gamma+\gamma^2\right)^2\left((1+\gamma)(2+\gamma)(3+\gamma)\right)^3\left(3025+3850\gamma+1775\gamma^2+350\gamma^3\right)\right]}{(4+\gamma)\,\mathcal{F}(\gamma)}\\
&-\frac{24.66\,\mathcal{G}_3(\gamma)\left[\begin{array}{l}\left(125+150\gamma+25\gamma^2\right)\left(8+6\gamma+\gamma^2\right)^2\left((1+\gamma)(2+\gamma)\right)^3\\ \times\left(24+26\gamma+9\gamma^2+\gamma^3\right)^2\end{array}\right]}{(5+\gamma)\,\mathcal{F}(\gamma)}\\
&+\frac{88.57\,\mathcal{G}_4(\gamma)\left[\left(125+150\gamma+25\gamma^2\right)\left(8+6\gamma+\gamma^2\right)^2\left(3025+3850\gamma+1775\gamma^2+350\gamma^3\right)\right]}{(6+\gamma)\,\mathcal{F}(\gamma)},
\end{aligned}$$

where

$$\mathcal{F}(\gamma) = \left(125 + 150\gamma + 25\gamma^2\right)\left(8 + 6\gamma + \gamma^2\right)^2 (3+\gamma)^3 \left(24 + 26\gamma + 9\gamma^2 + \gamma^3\right)^2 (2+\gamma)^2$$
$$\times \left(3025 + 3850\gamma + 1775\gamma^2 + 350\gamma^3\right)\left(6 + 5\gamma + \gamma^2\right)^2 (1+\gamma)^3,$$

$$\mathcal{G}_1(\gamma) = 10416384 + 24659712\gamma + 23588688\gamma^2 + 11630296\gamma^3 + 347642\gamma^4 + 430740\gamma^5 + 23930\gamma^6,$$

$$\mathcal{G}_2(\gamma) = 998784 + 2206080\gamma + 1674872\gamma^2 + 585788\gamma^3 + 96448\gamma^4 + 6028\gamma^5$$
$$- 1236672 - 4049900\gamma^2 - 2333252\gamma^3 - 113872\gamma^5 - 7117\gamma^6$$

$$\mathcal{G}_3(\gamma) = \Big(-1518 - 39358\gamma - 26664\gamma^2 - 7580\gamma^3 - 654\gamma^4\Big)\left(6 + 5\gamma + \gamma^2\right)^2$$
$$- \left(96\gamma + 24\gamma^2\right)\left(3025 + 3850\gamma + 1775\gamma^2 + 350\gamma^3\right)(1+\gamma)^2,$$

$$\mathcal{G}_4(\gamma) = \Big(288 - 144\gamma - 424\gamma^2 - 220\gamma^3 - 36\gamma^4\Big)\left(40 + 30\gamma + 5\gamma^2\right)(1+\gamma)^2$$
$$+ (3+\gamma)^2(1+\gamma)(832\gamma).$$

In case $\gamma = 0$, we get the following Corollary.

Corollary 2.5
If a function $\mathtt{f}$ of the form (1) belongs to the subclass $\mathscr{M}(\beta, t)$, then

$$|\mathscr{H}_4(1)| \leq -\frac{6\,\zeta_1(\beta)\,167270400\,\beta^{19}}{\mathcal{E}(\beta)} + \frac{20.8\,\zeta_2(\beta)\,10454400\,\beta^{16}}{\mathcal{E}(\beta)}$$
$$- \frac{24.66\,\zeta_3(\beta)\,7372800\,\beta^{17}}{\mathcal{E}(\beta)} + \frac{88.57\,\zeta_4(\beta,\gamma)\,12100000\,\beta^9}{3\,\mathcal{E}(\beta)},$$

where

$$\varepsilon(\beta) = 9032601600000\,\beta^{29},$$
$$\zeta_1(\beta) = 10416384\,\beta^6,$$
$$\zeta_2(\beta) = 576\,\beta^5\,(1734 - 2147\beta)\,,$$
$$\zeta_3(\beta) = 3\,\beta^7\,(19296\beta - 37548)\,,$$
$$\zeta_4(\beta) = 7488\,\beta^5 + 30240\,\beta^7 - 11520\,\beta^8 - 832\,\beta.$$

Example 2.6
If we put the function,

$$\mathtt{f}(\zeta) = 3\left(e^{\zeta/3} - 1\right) = z + \sum_{\tau=2}^{\infty} \frac{\zeta^\tau 3^{1-\tau}}{\tau!} \in \mathscr{M}(\beta, \gamma, \Im)$$

then, we have (5), therefore , we get

$$\begin{aligned}
|\mathcal{S}_1| &= |\lambda_2|\, |\lambda_4\lambda_6-\lambda_5^2| + |\lambda_3|\, |\lambda_3\lambda_6-\lambda_4\lambda_5| - |\lambda_4|\, |\lambda_3\lambda_5-\lambda_4^2| \\
&= \left|\frac{1}{6}\right| \left|\frac{1}{648}\cdot\frac{1}{174960}-\left(\frac{1}{9720}\right)^2\right| + \left|\frac{1}{54}\right| \left|\frac{1}{54}\cdot\frac{1}{174960}-\frac{1}{648}\cdot\frac{1}{9720}\right| \\
&\quad - \left|\frac{1}{648}\right| \left|\frac{1}{54}\cdot\frac{1}{9720}-\left(\frac{1}{648}\right)^2\right| = 5.39022\times 10^{-10},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{S}_2| &= |\lambda_4\lambda_6-\lambda_5^2| - |\lambda_2|\, |\lambda_3\lambda_6-\lambda_4\lambda_5| + |\lambda_3|\, |\lambda_3\lambda_5-\lambda_4^2| \\
&= \left|\frac{1}{648}\cdot\frac{1}{174960}-\left(\frac{1}{9720}\right)^2\right| - \left|\frac{1}{6}\right| \left|\frac{1}{54}\cdot\frac{1}{174960}-\frac{1}{648}\cdot\frac{1}{9720}\right| \\
&\quad + \left|\frac{1}{54}\right| \left|\frac{1}{54}\cdot\frac{1}{9720}-\left(\frac{1}{648}\right)^2\right| = 1.76407\times 10^{-9},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{S}_3| &= |\lambda_3\lambda_6-\lambda_4\lambda_5| + |\lambda_2|\, |\lambda_2\lambda_6-\lambda_3\lambda_5| - |\lambda_4|\, |\lambda_4\lambda_2-\lambda_3^2| \\
&= \left|\frac{1}{54}\cdot\frac{1}{174960}-\frac{1}{648}\cdot\frac{1}{9720}\right| + \left|\frac{1}{6}\right| \left|\frac{1}{6}\cdot\frac{1}{174960}-\frac{1}{54}\cdot\frac{1}{9720}\right| \\
&\quad - \left|\frac{1}{648}\right| \left|\frac{1}{648}\cdot\frac{1}{6}-\left(\frac{1}{54}\right)^2\right| = 7.93832\times 10^{-8},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{S}_4| &= |\lambda_3|\, |\lambda_4\lambda_2-\lambda_3^2| - |\lambda_4|\, |\lambda_4-\lambda_3\lambda_2| + |\lambda_5|\, |\lambda_3-\lambda_2^2| \\
&= \left|\frac{1}{54}\right| \left|\frac{1}{648}\cdot\frac{1}{6}-\left(\frac{1}{54}\right)^2\right| - \left|\frac{1}{648}\right| \left|\frac{1}{648}-\frac{1}{54}\cdot\frac{1}{6}\right| \\
&\quad + \left|\frac{1}{9720}\right| \left|\frac{1}{54}-\left(\frac{1}{6}\right)^2\right| = 1.58766\times 10^{-7}.
\end{aligned}$$

$$|\mathscr{H}_4(1)| = |-a_4\mathcal{S}_1+\lambda_5\mathcal{S}_2-\lambda_6\mathcal{S}_3+\lambda_7\mathcal{S}_4| = \frac{491}{462838344192000} \approx 1.0608455547\times 10^{-12}.$$

Example 2.7
If we choose the function

$$\mathtt{f}(\zeta) = \frac{1}{2}\left((1-\zeta)^{-2}-1\right) = \zeta + \sum_{\tau=2}^{\infty}\frac{\tau+1}{2}\zeta^{\tau} \in \mathscr{M}(\beta,\gamma,\Im),$$

then we have (5), therefore , we get

$$|\mathcal{S}_1| = |\lambda_2|\,|\lambda_4\lambda_6-\lambda_5^2|+|\lambda_3|\,|\lambda_3\lambda_6-\lambda_4\lambda_5|-|\lambda_4|\,|\lambda_3\lambda_5-\lambda_4^2|$$
$$=\left|\frac{3}{2}\right|\left|\frac{5}{2}\cdot\frac{7}{2}-3^2\right|+|2|\left|2\cdot\frac{7}{2}-\frac{5}{2}\cdot 3\right|-\left|\frac{5}{2}\right|\left|2\cdot 3-\left(\frac{5}{2}\right)^2\right|=0.75,$$

$$|\mathcal{S}_2| = |\lambda_4\lambda_6-\lambda_5^2|-|\lambda_2|\,|\lambda_3\lambda_6-\lambda_4\lambda_5|+|\lambda_3|\,|\lambda_3\lambda_5-\lambda_4^2|$$
$$=\left|\frac{5}{2}\cdot\frac{7}{2}-3^2\right|-\left|\frac{3}{2}\right|\left|2\cdot\frac{7}{2}-\frac{5}{2}\cdot 3\right|+|2|\left|2\cdot 3-\left(\frac{5}{2}\right)^2\right|=0,$$

$$|\mathcal{S}_3| = |\lambda_3\lambda_6-\lambda_4\lambda_5|+|\lambda_2|\,|\lambda_2\lambda_6-\lambda_3\lambda_5|-|\lambda_4|\,|\lambda_4\lambda_2-\lambda_3^2|$$
$$=\left|2\cdot\frac{7}{2}-\frac{5}{2}\cdot 3\right|+\left|\frac{3}{2}\right|\left|\frac{3}{2}\cdot\frac{7}{2}-2\cdot 3\right|-\left|\frac{5}{2}\right|\left|\frac{5}{2}\cdot\frac{3}{2}-2^2\right|=1,$$

$$|\mathcal{S}_4| = |\lambda_3|\,|\lambda_4\lambda_2-\lambda_3^2|-|\lambda_4|\,|\lambda_4-\lambda_3\lambda_2|+|\lambda_5|\,|\lambda_3-\lambda_2^2|$$
$$=|2|\left|\frac{5}{2}\cdot\frac{3}{2}-2^2\right|-\left|\frac{5}{2}\right|\left|\frac{5}{2}-2\cdot\frac{3}{2}\right|+|3|\left|2-\left(\frac{3}{2}\right)^2\right|=0,$$

$$|\mathscr{H}_4(1)| = |-\lambda_4\mathcal{S}_1+\lambda_5\mathcal{S}_2-\lambda_6\mathcal{S}_3+\lambda_7\mathcal{S}_4| = -5.375.$$

Theorem 2.8
If the function $\mathtt{f}\in\mathscr{M}(\beta,\gamma,\Im)$ and is of the form (1), then we have

$$|\mathscr{H}_4(1)| \le -\mathcal{O}_1(\beta,\gamma)+\mathcal{O}_2(\beta,\gamma)-\mathcal{O}_3(\beta,\gamma)+\mathcal{O}_4(\beta,\gamma),$$

where

$$\mathcal{O}_1(\beta,\gamma)=\frac{17}{36}\left(\frac{29762154-9348875\beta-1869775\gamma}{5832000\beta+1166400\gamma}\right),$$

$$\mathcal{O}_2(\beta,\gamma)=\frac{59184+5725\beta+1145\gamma}{43200\beta+8640\gamma},$$

$$\mathcal{O}_3(\beta,\gamma)=\frac{24.66}{5\beta+\gamma}\left(\frac{5592888-764375\beta-152875\gamma}{648000\beta+129600\gamma}\right),$$

$$\mathcal{O}_4(\beta,\gamma)=\frac{88.57}{6\beta+\gamma}\left(\frac{967}{5184}\right).$$

Proof
Let $\mathtt{f}\in\mathscr{M}(\beta,\gamma,\Im)$. Then, we can rewrite 4^{th} Hankel determinant as:

$$|\mathscr{H}_4(1)| = |-\lambda_4\mathcal{S}_1+\lambda_5\mathcal{S}_2-\lambda_6\mathcal{S}_3+\lambda_7\mathcal{S}_4|\,.$$

where

$$|\mathcal{S}_1| = |\lambda_2|\,|\lambda_4\lambda_6-\lambda_5^2|+|\lambda_3|\,|\lambda_3\lambda_6-\lambda_4\lambda_5|-|\lambda_4|\,|\lambda_3\lambda_5-\lambda_4^2|,$$

$$|\mathcal{S}_2| = |\lambda_4\lambda_6-\lambda_5^2|-|\lambda_2|\,|\lambda_3\lambda_6-\lambda_4\lambda_5|+|\lambda_3|\,|\lambda_3\lambda_5-\lambda_4^2|,$$

$$|\mathcal{S}_3| = |\lambda_3\lambda_6-\lambda_4\lambda_5|+|\lambda_2|\,|\lambda_2\lambda_6-\lambda_3\lambda_5|-|\lambda_4|\,|\lambda_4\lambda_2-\lambda_3^2|,$$

$$|\mathcal{S}_4| = |\lambda_3|\,|\lambda_4\lambda_2-\lambda_3^2|-|\lambda_4|\,|\lambda_4-\lambda_3\lambda_2|+|\lambda_5|\,|\lambda_3-\lambda_2^2|.$$

Applying Lemma 1.2 and using $|\lambda_6|, |\lambda_7|$ from Theorem 2.1, in the 4th Hankel determinant, we get

$$\begin{aligned}|\mathcal{S}_1| &= |\lambda_2|\,|\lambda_4\lambda_6-\lambda_5^2|+|\lambda_3|\,|\lambda_3\lambda_6-\lambda_4\lambda_5|-|\lambda_4|\,|\lambda_3\lambda_5-\lambda_4^2| \\ &= \frac{29762154-9348875\beta-1869775\gamma}{5832000\beta+1166400\gamma},\end{aligned} \tag{31}$$

$$|\mathcal{S}_2| = |\lambda_4\lambda_6-\lambda_5^2|-|\lambda_2|\,|\lambda_3\lambda_6-\lambda_4\lambda_5|+|\lambda_3|\,|\lambda_3\lambda_5-\lambda_4^2| = \frac{59184+5725\beta+1145\gamma}{43200\beta+8640\gamma}, \tag{32}$$

$$|\mathcal{S}_3| = |\lambda_3\lambda_6-\lambda_4\lambda_5|+|\lambda_2|\,|\lambda_2\lambda_6-\lambda_3\lambda_5|-|\lambda_4|\,|\lambda_4\lambda_2-\lambda_3^2| = \frac{5592888-764375\beta-152875\gamma}{648000\beta+129600\gamma}, \tag{33}$$

$$|\mathcal{S}_4| = |\lambda_3|\,|\lambda_4\lambda_2-\lambda_3^2|-|\lambda_4|\,|\lambda_4-\lambda_3\lambda_2|+|\lambda_5|\,|\lambda_3-\lambda_2^2| = \frac{967}{5184}. \tag{34}$$

Inserting values (31)-(34) in (5), we obtain

$$|\mathscr{H}_4(1)| \leq -\mathcal{O}_1(\beta,\gamma)+\mathcal{O}_2(\beta,\gamma)-\mathcal{O}_3(\beta,\gamma)+\mathcal{O}_4(\beta,\gamma),$$

where

$$\mathcal{O}_1(\beta,\gamma) = \frac{17}{36}\left(\frac{29762154-9348875\beta-1869775\gamma}{5832000\beta+1166400\gamma}\right),$$

$$\mathcal{O}_2(\beta,\gamma) = \frac{59184+5725\beta+1145\gamma}{43200\beta+8640\gamma},$$

$$\mathcal{O}_3(\beta,\gamma) = \frac{24.66}{5\beta+\gamma}\left(\frac{5592888-764375\beta-152875\gamma}{648000\beta+129600\gamma}\right),$$

$$\mathcal{O}_4(\beta,\gamma) = \frac{88.57}{6\beta+\gamma}\left(\frac{967}{5184}\right).$$

□

In case $\beta = 1$, we get the following Corollary.

Corollary 2.9
Let $\mathtt{f}(\zeta)$ given by (1) be in class $\mathscr{M}(\gamma,\Im)$. Then

$$|\mathscr{H}_4(1)| \leq -\delta_1(\gamma)+\delta_2(\gamma)-\delta_3(\gamma)+\delta_4(\gamma),$$

where

$$\delta_1(\gamma) = \frac{17}{36}\left(\frac{29762154-9348875-1869775\gamma}{5832000+1166400\gamma}\right),$$

$$\delta_2(\gamma) = \frac{59184+5725+1145\gamma}{43200+8640\gamma},$$

$$\delta_3(\gamma) = \frac{24.66}{5+\gamma}\left(\frac{5592888-764375-152875\gamma}{648000+129600\gamma}\right),$$

$$\delta_4(\gamma) = \frac{88.57}{6+\gamma}\left(\frac{967}{5184}\right).$$

In case $\gamma = 0$, we get the following Corollary.

Corollary 2.10
Let $\mathtt{f}(\zeta)$ given by (1) be in class $\mathscr{M}(\beta, \Im)$. Then

$$|\mathscr{H}_4(1)| \leq -\mathcal{W}_1(\beta) + \mathcal{W}_2(\beta) - \mathcal{W}_3(\beta) + \mathcal{W}_4(\beta),$$

where

$$\mathcal{W}_1(\beta) = \frac{17}{36}\left(\frac{29762154 - 9348875\beta}{5832000\beta}\right),$$

$$\mathcal{W}_2(\beta) = \frac{59184 + 5725\beta}{43200\beta},$$

$$\mathcal{W}_3(\beta) = \frac{24.66}{5\beta}\left(\frac{5592888 - 764375\beta}{648000\beta}\right),$$

$$\mathcal{W}_4(\beta) = \frac{88.57}{6\beta}\left(\frac{967}{5184}\right).$$

3. Conclusion

In this paper, we introduced and investigated the subclass $M(\beta, \gamma, \varepsilon)$ of analytic functions defined by means of subordination. By using the coefficient estimates associated with functions having positive real part, we obtained explicit upper bounds for the initial coefficients $|a_\tau|$, $\tau = 2, 3, 4, 5, 6, 7$, and derived an upper estimate for the fourth Hankel determinant for functions belonging to this class. These results contribute to the ongoing study of higher-order Hankel determinants in geometric function theory and provide further information on the coefficient structure of analytic functions defined through subordination.

The obtained estimates show how the parameters β, γ, and ε influence the size of the initial coefficients and the corresponding Hankel determinant. In particular, suitable choices of these parameters may lead to several known or related subclasses of analytic functions. Thus, the present results extend and complement earlier investigations on Hankel determinants for starlike, close-to-convex, and other subclasses of analytic functions studied in the literature. Moreover, the results may be compared with known bounds obtained for classes associated with Chebyshev polynomials and other dominant functions, showing that the present framework provides a flexible setting for deriving related coefficient inequalities.

It should be noted that the bounds obtained in this work are presented as explicit upper estimates. The question of sharpness, as well as the complete characterization of extremal functions for which equality is attained, requires additional investigation. Equality in the auxiliary coefficient estimates is usually connected with extremal functions in the Carathéodory class; however, simultaneous equality in the fourth Hankel determinant estimate is more delicate and depends on the interaction among the involved coefficients. Therefore, determining sharp bounds and identifying the corresponding extremal functions remain interesting problems for future research.

Finally, the method developed in this paper can be applied to other subclasses of analytic functions defined through different dominant functions, including those associated with Chebyshev polynomials, exponential functions, and other special functions. We expect that analogous estimates for higher-order Hankel and Toeplitz determinants can be obtained by adapting the present approach.

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