STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, July 2026, pp 777–789.
Published online in International Academic Press (www.IAPress.org)

Grundy Chromatic Number of Shuriken Graphs and Applications to Regulatory Network Modeling

M. Kamalnath, T. Muthukani Vairavel*

Department of Mathematics, School of Advanced Sciences, Kalasalingam Academy of Research and Education, Krishnankovil, Virudhunagar–626126, Tamil Nadu, India
Email:rmvkamalnath@gmail.com, muthukanivairavel@gmail.com

Abstract The concept of Grundy coloring plays an important role in the study of graph coloring and its applications in sequential allocation problems. In this paper, we study the Grundy (first-fit) chromatic number of the Shuriken graph Sh_n and some of its associated derived graphs obtained through standard graph operations. In particular, we determine the Grundy chromatic numbers of the line graph $L(Sh_n)$, the middle graph $M(Sh_n)$, and the total graph $T(Sh_n)$. The results are obtained by utilizing the structural properties of the Shuriken graph together with the principles of Grundy coloring and suitable upper bound techniques. In addition, we discuss the relevance of these graphs in the framework of Gene Regulatory Networks (GRNs), where the Grundy chromatic number may represent the maximum level of hierarchical activations arising in sequential regulatory interactions. Thus, the Shuriken graph and its related derived graphs provide meaningful mathematical models for studying interaction complexity and stability in biological systems.

Keywords Grundy chromatic number, Shuriken graph, Line graph, Middle graph, Total graph, Gene regulatory networks, Graph coloring algorithms.

AMS 2010 subject classifications 05C15;05C75; 05C76; 05C85; 05C90; 05C99.

DOI: 10.19139/soic-2310-5070-3731

1. Introduction

Graph coloring is an important area in graph theory that has been widely studied because of its many practical applications, such as scheduling, frequency assignment in communication systems, register allocation in compilers, and network design. Among the different forms of graph coloring, vertex coloring is one of the most fundamental. In vertex coloring, colors are assigned to the vertices of a graph in such a way that no two adjacent vertices receive the same color. The minimum number of colors required to achieve such a coloring is known as the chromatic number of the graph. In contrast to the chromatic number, which focuses on minimizing the number of colors used, the concept of the Grundy (First-Fit) chromatic number arises from greedy coloring procedures. This parameter reflects the behavior of the First-Fit coloring algorithm when vertices are processed in different orders. The Grundy chromatic number was originally motivated by ideas from game theory and was later incorporated into graph theory. Christen and Selkow [1] extended this concept to graphs. Subsequently, Hedetniemi et al. [2] introduced Grundy functions and developed theoretical foundations related to greedy vertex coloring. For a graph G, the maximum number of colors that may appear when vertices are colored using a greedy (First-Fit) algorithm is called the *Grundy chromatic number* and is denoted by $\Gamma(G)$. This parameter captures the worst-case behavior of greedy coloring algorithms when vertices are considered in unfavorable orders. Considerable research has been

*Correspondence to: T. Muthukani Vairavel (Email: muthukanivairavel@gmail.com). Department of Mathematics, School of Advanced Sciences, Kalasalingam Academy of Research and Education, Krishnankovil, Virudhunagar–626126, Tamil Nadu, India.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

devoted to understanding the theoretical and algorithmic aspects of the Grundy chromatic number. Zaker [3] studied several properties of the Grundy chromatic number and proved that determining this parameter is NP-complete in general. Effantin and Kheddouci [4] investigated structural properties and bounds related to the Grundy number. In particular, it satisfies the well-known inequality

$$\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1,$$

where $\chi(G)$ denotes the chromatic number and $\Delta(G)$ represents the maximum degree of the graph G.

The relationship between the Grundy chromatic number and other coloring parameters has also received significant attention. Erdős et al. [5] studied the equality of the partial Grundy number and the upper ochromatic number of graphs. Füredi et al. [6] derived several inequalities concerning the first-fit chromatic number. Furthermore, a comprehensive survey on Nordhaus-Gaddum relations for various graph invariants, especially those closely related to the Grundy number, was presented by Aouchiche and Hansen in 2013 [7]. Subsequent studies were carried out on the calculation of the Grundy chromatic number for certain graph families. Germain and Kheddouci [8] studied the calculation of graph power for the Grundy number, while Asté et al. [9] studied the calculation of the Grundy chromatic number for graph products. Araujo and Linhares Sales [10] studied graphs with a restricted number of induced P_4 subgraphs, while Gastineau et al. [11] studied certain families of r-regular graphs with a Grundy number equal to $r + 1$. Recently, several researchers studied the calculation of the Grundy chromatic number for certain graph families, especially for structured graph families such as prism graphs and comb product graphs.[12, 13] Graph coloring parameters were studied for certain special graph families. For example, Kaliraj et al. [14] studied the calculation of equitable coloring for sunlet graphs, while Sundaram et al. [15] studied domination parameters for Shuriken graphs. In addition, George et al. [16] studied the calculation of equitable dominator coloring for certain graph families related to helm graphs. Graph transformation and derived graphs are recent developments in graph theory, especially in graph coloring, with increased attention from researchers in recent times. Muralidharan et al. [17] studied splitting graphs and middle graphs of standard graph classes, while Djuang et al. [18] studied structural properties of Shuriken graphs derived from clean graphs of rings. These papers demonstrate the increasing interest in the study of structural properties of graphs through various classical operators of graph theory, including line graphs, middle graphs, and total graphs. Apart from classical graph theory, various derived structures of graphs have also been used in modern computational paradigms. For example, line graph neural networks have been proposed for link prediction problems [19]. Similarly, total variation graph neural networks have also been proposed, where various transformations of graphs are used for improving representation learning [20]. These approaches are based on machine learning paradigms, but we have also seen that various versions of greedy coloring algorithms have been used for solving various optimization problems in real-world networks, including distributed peer-to-peer networks based on blockchain technology [21]. In spite of the vast literature on the study of the Grundy chromatic number of various classes of graphs, this area of study is still not well explored, including the study of the Grundy chromatic number of Shuriken graphs and their line graphs, middle graphs, and total graphs.

Apart from its theoretical significance, the concept of the Grundy chromatic number may provide useful insights into the structural organization of certain real-world networks. In particular, Gene Regulatory Networks (GRNs), where vertices represent genes and edges denote regulatory interactions, have been widely modeled using graph-theoretic frameworks [22, 23]. These networks often exhibit hierarchical organization, modularity, and layered regulatory mechanisms [24].

The Grundy chromatic number, derived from a greedy (First-Fit) coloring process, can be interpreted as capturing a form of sequential dependency or layered activation within a graph. While biological systems are not governed by greedy algorithms in a strict sense, sequential activation and regulatory cascades are well-documented phenomena in GRNs, particularly in transcriptional regulation and signaling pathways [23]. In this context, the Grundy number may serve as an abstract measure of the depth of such dependency structures, rather than a direct mechanistic model.

Furthermore, different graph transformations may offer perspectives on various levels of biological interaction. For example, line graphs emphasize interactions between regulatory relationships (edge-based interactions), while total graphs incorporate both gene-level and interaction-level dependencies. Such representations are conceptually aligned with multilevel modeling approaches in systems biology, where both node dynamics and interaction dynamics are considered [22]. Similarly, structured graph families such as Shuriken graphs can be viewed as idealized models capturing repeated regulatory motifs, which are known to play a crucial role in the functional organization of GRNs [25].

However, it is important to emphasize that these interpretations remain theoretical. The Grundy chromatic number represents a worst-case outcome of a greedy coloring process, whereas biological systems are shaped by evolutionary optimization and robustness constraints. Therefore, while the proposed framework may offer qualitative insights into hierarchical and dependency structures, its direct applicability to biological systems requires further empirical validation. Future work may focus on integrating these combinatorial models with experimental data and established dynamical models of GRNs.

The rest of the paper is organized as follows. In Section 2, we give the required definitions and notation. This includes the definition of the Shuriken graph as well as the line graphs, middle graphs, and total graphs. The following sections will show how we find the Grundy chromatic number of Sh_n, $L(Sh_n)$, $M(Sh_n)$, and $T(Sh_n)$. In Section 4, we will show a possible application of our results in the context of Gene Regulatory Networks. The Grundy chromatic number is a measure of the worst-case hierarchical activation depth.

2. Preliminaries

This section introduces the fundamental definitions and notation that will be used throughout the paper. Unless otherwise specified, all graphs considered in this study are finite, simple, and undirected.

Definition 2.1
[15] The Shuriken graph Sh_n is obtained from the gear graph G_n as follows. Let $v_1, v_2, \ldots, v_{2n}$ denote the vertices on the outer cycle of G_n, and let u_0 represent the central vertex. Consider the subset of vertices $\{v_i : i = 1, 3, 5, \ldots, 2n-1\}$. For each vertex in this set, create a corresponding copy denoted by $u_1, u_2, \ldots, u_n$. The central vertex of each copy is then identified with the central vertex of G_n, resulting in the common vertex u_0. For every $i = 1, 2, \ldots, n$, the vertex u_i is connected to all neighbors of the corresponding vertex v_{2i-1} in G_n. The graph obtained through this construction is referred to as the Shuriken graph and is denoted by Sh_n.

Definition 2.2
[14] Let G be a graph. The line graph of a graph G, denoted by $L(G)$, is a graph whose vertices represent the edges of the graph G. Hence, the vertex set of the line graph of G is equal to the edge set of G, that is, $V(L(G)) = E(G)$. Two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a common endpoint.

Definition 2.3
[14] The middle graph of a graph G, denoted by $M(G)$, is a graph whose vertex set is given by $V(M(G)) = V(G) \cup E(G)$. Two vertices in $M(G)$ are adjacent if and only if one of the following conditions holds:

$\Longrightarrow$ the vertices represent two adjacent edges of G, or
$\Longrightarrow$ one vertex represents a vertex of G and the other represents an edge incident with that vertex in G.

Thus, the middle graph captures both edge–edge adjacency and vertex–edge incidence relationships present in the original graph.

Definition 2.4
[14] The total graph of a graph G, denoted by $T(G)$, is a graph whose vertex set is given by $V(T(G)) = V(G) \cup E(G)$. Two vertices of $T(G)$ are adjacent whenever one of the following conditions holds:

$\Longrightarrow$ the vertices correspond to two adjacent vertices in G, or
$\Longrightarrow$ the vertices correspond to two adjacent edges in G, or
$\Longrightarrow$ one vertex corresponds to a vertex of G and the other corresponds to an edge incident with that vertex in G.

Hence, the total graph represents vertex–vertex, edge–edge, and vertex–edge adjacency relations of the original graph.

The above definitions establish the framework required for studying the Grundy (first-fit) chromatic numbers of the Shuriken graph and its associated derived graphs.

3. Main Results

Theorem 3.1
Let $n \geq 3$ be an integer. Then the Grundy (First–Fit) chromatic number of the Shuriken graph Sh_n equals 4.

Proof
Let Sh_n be the Shuriken graph with vertex set $V(Sh_n) = \{u_0\} \cup \{u_1, u_2, \ldots, u_{2n}\} \cup \{v_1, v_2, \ldots, v_n\}$. The edge set is defined by $E(Sh_n) = \{u_0u_{2i-1}, u_0v_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_{2n}u_1\} \cup \{v_iu_{2i} : 1 \leq i \leq n\} \cup \{v_nu_{2n}\} \cup \{u_{2i}v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{2n}v_1\}$.

Thus Sh_n consists of a cycle on the vertices $u_1, u_2, \ldots, u_{2n}$ together with a central vertex u_0 and pendant vertices v_i attached to u_{2i} and u_0.

Define a coloring function $c : V(Sh_n) \to \{1, 2, 3, 4\}$ as follows:

$$\begin{aligned} c(u_0) &= 4, \\ c(u_{2i-1}) &= 3, 2, 1, \quad 1 \leq i \leq n, \\ c(u_{2i}) &= 1, 3, 2, \quad 1 \leq i \leq n, \\ c(v_i) &= 3, 2, 1, \quad 1 \leq i \leq n. \end{aligned}$$

Hence the color partitions are

$$\begin{aligned} V_1 &= \{u_{2i-1} : 1 \leq i \leq n\}, \quad V_2 = \{u_{2i} : 1 \leq i \leq n\}, \\ V_3 &= \{v_i : 1 \leq i \leq n\}, \quad V_4 = \{u_0\}. \end{aligned}$$

Thus every vertex colored k has neighbors colored with all colors $1, 2, \ldots, k-1$. Hence this is a Grundy coloring using 4 colors and therefore $\Gamma(Sh_n) \geq 4$.

The central vertex u_0 is adjacent to $N(u_0) = \{u_{2i-1}, v_i : 1 \leq i \leq n\}$, hence $d(u_0) = 2n$. All remaining vertices have degree 3. Therefore, $\Delta(Sh_n) = 2n$ and $\delta(Sh_n) = 3$. Although the general bound $\Gamma(G) \leq \Delta(G) + 1$ yields $\Gamma(Sh_n) \leq 2n + 1$, this bound is not tight for the Shuriken graph. We now show that Sh_n does not admit a Grundy coloring with five colors.

Assume, toward a contradiction, that there exists a Grundy coloring of Sh_n using five distinct colors. In such a case, there must be a vertex x colored 5 whose neighbors include vertices with colors $1, 2, 3$, and 4.

If $x = u_0$, then its neighbors belong to the set $\{u_{2i-1}, v_i : 1 \leq i \leq n\}$. However, every vertex u_{2i-1} is adjacent only to u_{2i-2}, u_{2i} and u_0, and each v_i is adjacent only to u_{2i} and u_0. Thus the neighbors of u_0 can realize at most three distinct Grundy colors in any greedy coloring process, since no vertex in $N(u_0)$ can be forced to take color 4 while simultaneously seeing colors $1, 2$, and 3 in its own neighborhood.

If $x \neq u_0$, then x has degree 3, and hence it can be adjacent to at most three differently colored vertices. Therefore, x cannot have neighbors colored with all four colors $1, 2, 3, 4$.

In both cases, no vertex in Sh_n can satisfy the Grundy condition required for color 5. Hence, $\Gamma(Sh_n) \leq 4$. Therefore, $\Gamma(Sh_n) = 4, \quad \forall n \geq 3$. □

Algorithm 1 Construction of a Worst-Case Grundy Ordering for Sh_n

---

Input: An integer $n \geq 3$.
Output: A vertex ordering ϕ of Sh_n such that First–Fit coloring with respect to ϕ uses $\Gamma(Sh_n) = 4$ colors.

1: Initialize ϕ as an empty ordering.
2: Place all odd rim vertices u_{2i-1}, $1 \leq i \leq n$, at the beginning of ϕ.
3: Append all even rim vertices u_{2i}, $1 \leq i \leq n$, to ϕ.
4: Insert all pendant vertices v_i, $1 \leq i \leq n$, next in ϕ.
5: Place the central vertex u_0 last in the ordering.
6: **return** ϕ.

This algorithm constructs a vertex ordering of Sh_n that forces the First–Fit coloring to use exactly four colors, achieving $\Gamma(Sh_n) = 4$.

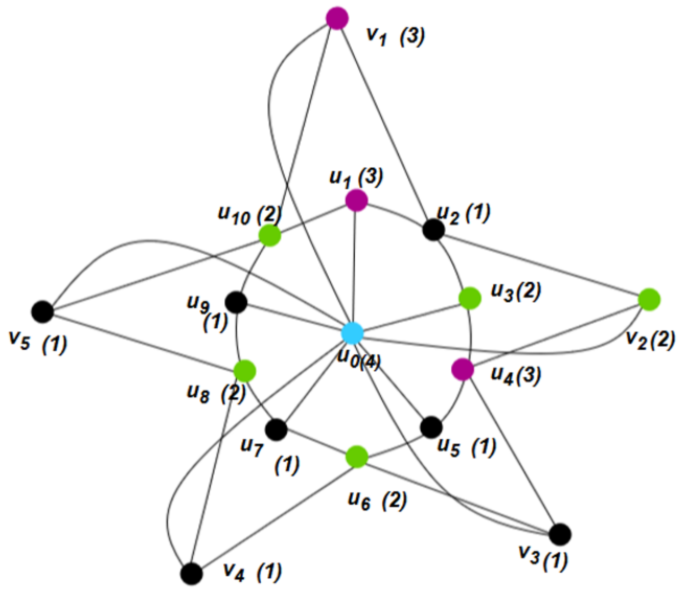

Figure 1. An example of a Grundy coloring of the Shuriken graph Sh_5 using four colors.

Theorem 3.2
If $n \geq 5$, then the Grundy (first - fit) chromatic number of $L(Sh_n)$, the line graph of the Shuriken graph Sh_n, is equal to $2n$.

Proof
Let Sh_n be the Shuriken graph with vertex set $V(Sh_n) = \{u_0\} \cup \{u_1, u_2, \ldots, u_{2n}\} \cup \{v_1, v_2, \ldots, v_n\}$, and edge set $E(Sh_n) = \{u_0u_{2i-1}, u_0v_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_{2n}u_1\} \cup \{v_iu_{2i} : 1 \leq i \leq n\} \cup \{v_nu_{2n}\} \cup \{u_{2i}v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{2n}v_1\}$. The line graph $L(Sh_n)$ is defined by $V(L(Sh_n)) = E(Sh_n)$, and two vertices of $L(Sh_n)$ are adjacent whenever the corresponding edges in Sh_n share a common endpoint. The vertex u_0 in Sh_n is incident with $2n$ edges. Hence, in $L(Sh_n)$, these $2n$ vertices induce a clique K_{2n}.

Define a coloring function $c : V(L(Sh_n)) \to \{1, 2, \ldots, 2n\}$ by assigning distinct colors to the vertices corresponding to the edges incident with u_0 as follows:

$$\begin{aligned}
c(u_0u_{2i-1}) &= i, \qquad c(u_0v_i) = n+i, \quad 1 \leq i \leq n; \\
c(u_{2i}u_{2i+1}) &= 1, 2 \leq i \leq n; \qquad c(u_{2i+1}u_{2i+2}) = 2, 2 \leq i \leq n; \qquad c(u_{2n}u_1) = 3; \\
c(u_{2i}v_{i+1}) &= 4, 2 \leq i \leq n-1; \qquad c(u_{2n}v_1) = 4, 2 \leq i \leq n; \qquad c(v_iu_{2i}) = 3, 3 \leq i \leq n; \\
c(u_1u_2) &= 2; \qquad c(u_2u_3) = 1; \qquad c(u_3u_4) = 3; \\
c(v_1u_2) &= 3; \qquad c(v_2u_4) = 1;
\end{aligned}$$

The remaining vertices of $L(Sh_n)$ are assigned colors greedily by choosing the smallest available color. The vertices corresponding to the $2n$ edges incident with u_0 induce a clique in $L(Sh_n)$; therefore, each of these vertices must receive a distinct color from the set $1, 2, \ldots, 2n$. Consequently, we obtain $\Gamma(L(Sh_n)) \geq 2n$. Assume, for the sake of contradiction, that $L(Sh_n)$ admits a Grundy coloring using $2n+1$ colors. Then there must exist a vertex x in $L(Sh_n)$ assigned the color $2n+1$ such that its neighbors include vertices colored with all colors $1, 2, \ldots, 2n$.

If x corresponds to an edge incident with u_0, then x belongs to the clique K_{2n} and hence can be adjacent to at most $2n-1$ vertices in this clique. Therefore, it cannot have neighbors with all $2n$ colors $1, 2, \ldots, 2n$.
If x corresponds to an edge not incident with u_0, then this edge has at most two endpoints in Sh_n, each of which has degree at most 3. Hence, in $L(Sh_n)$ the vertex x has degree at most 4, and therefore it cannot be adjacent to $2n$ differently colored vertices.
In both cases, no vertex of $L(Sh_n)$ can satisfy the Grundy condition required for color $2n+1$. Thus, $\Gamma(L(Sh_n)) \leq 2n$. Therefore,

$$\Gamma(L(Sh_n)) = 2n \quad \text{for all } n \geq 5.$$

□

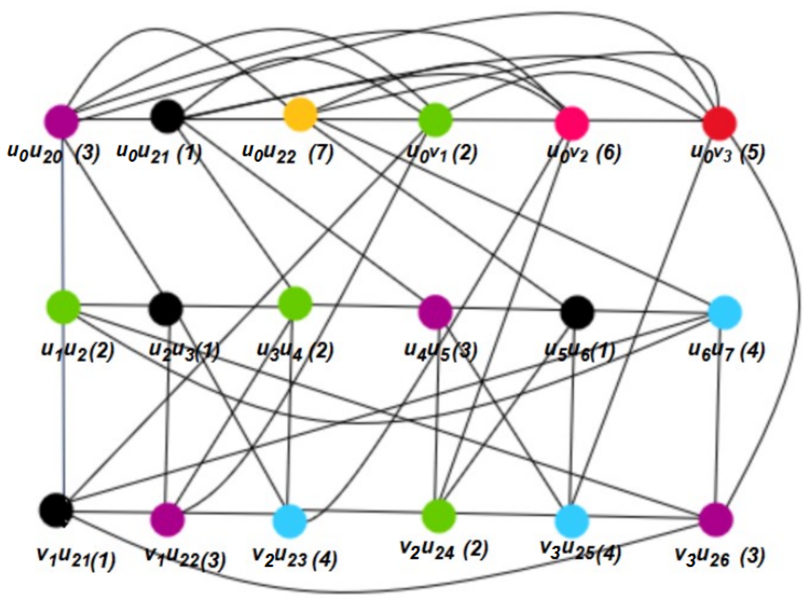

Figure 2. $L(SH_3)$

Remark 3.1
For small values n = 3,4, computational verification shows that the Grundy chromatic number of the line graph satisfies$\Gamma(L(Sh_n)) = 2n+1$. Thus the formula $\Gamma(L(Sh_n)) = 2n$ holds for all $n \geq 5$, while n = 3,4 exhibit one additional color.

Algorithm 2 Construction of a Worst-Case Grundy Ordering for $L(Sh_n)$

Input: An integer $n \geq 5$.
Output: A vertex ordering ϕ of $L(Sh_n)$ such that First–Fit coloring with respect to ϕ uses $\Gamma(L(Sh_n)) = 2n$ colors.

Initialize ϕ as an empty ordering.
Insert vertices corresponding to the edges u_0u_{2i-1}, $1 \leq i \leq n$, into ϕ.
Append vertices corresponding to the edges u_0v_i, $1 \leq i \leq n$, into ϕ.
Append all remaining vertices of $L(Sh_n)$ in arbitrary order.
return ϕ.

This procedure orders the vertices of $L(Sh_n)$ so that the clique formed by edges incident to u_0 generates $2n$ distinct colors under First–Fit coloring.

Theorem 3.3
For every integer $n \geq 5$, the middle graph of the Shuriken graph Sh_n has Grundy (First–Fit) chromatic number $2n+1$, that is,

$$\Gamma(M(Sh_n)) = 2n+1.$$

Proof
Let Sh_n be the Shuriken graph with vertex set $V(Sh_n) = \{u_0\} \cup \{u_1, u_2, \ldots, u_{2n}\} \cup \{v_1, v_2, \ldots, v_n\}$, and edge set

$$\begin{aligned} E(Sh_n) = \{&u_0u_{2i-1},\, u_0v_i : 1 \leq i \leq n\} \\ &\cup \{u_iu_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_{2n}u_1\} \\ &\cup \{v_iu_{2i} : 1 \leq i \leq n\} \\ &\cup \{u_{2i}v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{2n}v_1\}. \end{aligned}$$

The middle graph $M(Sh_n)$ has vertex set

$$\begin{aligned} V(M(Sh_n)) &= V(Sh_n) \cup E(Sh_n) \\ &= \{u_0\} \cup \{u_1, u_2, \ldots, u_{2n}\} \cup \{v_1, v_2, \ldots, v_n\} \\ &\quad \cup \{e_{0,2i-1},\, e_{0,i} : 1 \leq i \leq n\} \\ &\quad \cup \{e_{i,i+1} : 1 \leq i \leq 2n-1\} \cup \{e_{2n,1}\} \\ &\quad \cup \{f_{i,2i} : 1 \leq i \leq n\} \\ &\quad \cup \{g_{2i,i+1} : 1 \leq i \leq n-1\} \cup \{g_{2n,1}\}, \end{aligned}$$

where $e_{0,2i-1}$ represents the edge u_0u_{2i-1}, $e_{0,i}$ represents the edge u_0v_i, $e_{i,i+1}$ represents the edge u_iu_{i+1}, $f_{i,2i}$ represents the edge v_iu_{2i} and $g_{2i,i+1}, g_{2n,1}$ represents the edge $u_{2i}v_{i+1}$ in Sh_n.

Define a coloring function $c : V(M(Sh_n)) \to \{1, 2, \ldots, 2n+1\}$ as follows:

$$\begin{aligned} c(e_{0,2i-1}) &= i, \qquad 1 \leq i \leq n, \\ c(e_{0,i}) &= n+i, \qquad 1 \leq i \leq n, \\ c(u_0) &= 2n+1. \end{aligned}$$

All remaining vertices in $\{u_1, u_2, \ldots, u_{2n}\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{e_{i,i+1} : 1 \leq i \leq 2n-1\} \cup \{e_{2n,1}\} \cup \{f_{i,2i} : 1 \leq i \leq n\} \cup \{g_{2i,i+1} : 1 \leq i \leq n-1\} \cup \{g_{2n,1}\}$, are colored greedily with the smallest available color. Hence, the induced color partitions are $V_k = \{x \in V(M(Sh_n)) : c(x) = k\}, \quad 1 \leq k \leq 2n+1$. By construction, the vertex u_0 is adjacent in $M(Sh_n)$ to the vertices $\{e_{0,2i-1}, e_{0,i} : 1 \leq i \leq n\}$, which are colored with all colors $1, 2, \ldots, 2n$. Therefore, u_0 receives color $2n+1$ and the coloring is a Grundy coloring.

Assume, to the contrary, that $M(Sh_n)$ admits a Grundy coloring with $2n+2$ colors. Then there exists a vertex x colored $2n+2$ such that x has neighbors colored with all colors $1, 2, \ldots, 2n+1$.

If $x = u_0$, this is impossible since u_0 has exactly $2n$ neighbors corresponding to the incident edges and cannot be adjacent to $2n+1$ distinct colors.

If x corresponds to an edge of Sh_n, then this vertex is adjacent in $M(Sh_n)$ only to its two end vertices and to edges sharing a common endpoint with it. Hence its degree is bounded by a constant independent of n, and it cannot be adjacent to $2n+1$ differently colored vertices.

If x corresponds to a vertex of Sh_n different from u_0, then this vertex has degree at most 3 in Sh_n and therefore at most 3 neighbors corresponding to incident edges in $M(Sh_n)$, which is insufficient to realize $2n+1$ distinct colors.

In all cases, no vertex of $M(Sh_n)$ can satisfy the Grundy condition required for color $2n+2$. Thus, $\Gamma(M(Sh_n)) \leq 2n+1$. Therefore,

$$\Gamma(M(Sh_n)) = 2n+1 \quad \text{for all } n \geq 5.$$

□

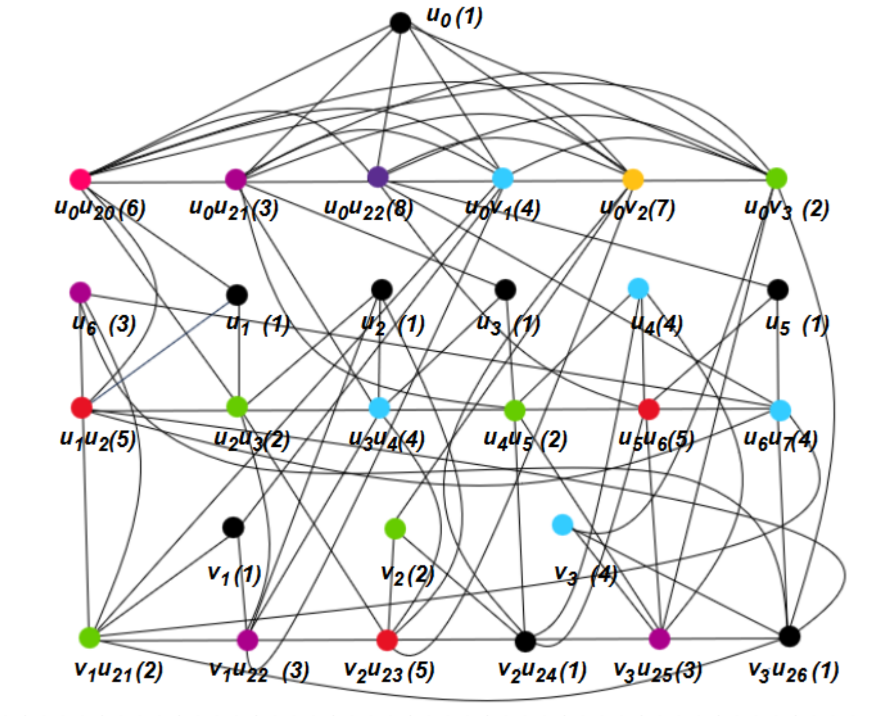

Figure 3. $M(SH_3)$

Remark 3.2
For n = 3,4, the middle graph admits one additional Grundy color, and hence $\Gamma(M(Sh_n)) = 2n + 2$. For $n \geq 5$, the general result $\Gamma(M(Sh_n)) = 2n + 1$ holds.

Algorithm 3 Construction of a Worst-Case Grundy Ordering for $M(Sh_n)$

Input: An integer $n \geq 5$.
Output: A vertex ordering ϕ of $M(Sh_n)$ such that First–Fit coloring with respect to ϕ uses $\Gamma(M(Sh_n)) = 2n + 1$ colors.

1: Initialize ϕ as an empty ordering.
2: Insert vertices corresponding to the edges u_0u_{2i-1}, $1 \leq i \leq n$, into ϕ.
3: Append vertices corresponding to the edges u_0v_i, $1 \leq i \leq n$, into ϕ.
4: Place the central vertex u_0 next in ϕ.
5: Append all remaining vertices of $M(Sh_n)$ (original rim vertices, pendant vertices, and remaining edge-vertices) in arbitrary order.
6: **return** ϕ.

This algorithm produces a worst-case ordering of $M(Sh_n)$ ensuring that the central vertex u_0 receives color $2n + 1$, establishing $\Gamma(M(Sh_n)) = 2n + 1$.

Theorem 3.4
For the Shuriken graph Sh_n, the Grundy chromatic number of its total graph satisfies

$$\Gamma(T(Sh_n)) = \begin{cases} 2n+3, & n = 3,4,5, \\ 2n+2, & 6 \le n \le 15, \\ 2n+1, & n \ge 16. \end{cases}$$

Proof
Let Sh_n be the Shuriken graph with vertex set $V(Sh_n) = \{u_0\} \cup \{u_1, u_2, \dots, u_{2n}\} \cup \{v_1, v_2, \dots, v_n\}$, and edge set

$$\begin{aligned} E(Sh_n) = \{&u_0u_{2i-1},\, u_0v_i : 1 \le i \le n\} \\ &\cup \{u_iu_{i+1} : 1 \le i \le 2n-1\} \cup \{u_{2n}u_1\} \\ &\cup \{v_iu_{2i} : 1 \le i \le n\} \\ &\cup \{u_{2i}v_{i+1} : 1 \le i \le n-1\} \cup \{u_{2n}v_1\}. \end{aligned}$$

The total graph $T(Sh_n)$ has vertex set

$$\begin{aligned} V(T(Sh_n)) &= V(Sh_n) \cup E(Sh_n) \\ &= \{u_0\} \cup \{u_1, u_2, \dots, u_{2n}\} \cup \{v_1, v_2, \dots, v_n\} \\ &\quad \cup \{e_{0,2i-1},\, e_{0,i} : 1 \le i \le n\} \\ &\quad \cup \{e_{i,i+1} : 1 \le i \le 2n-1\} \cup \{e_{2n,1}\} \\ &\quad \cup \{f_{i,2i} : 1 \le i \le n\} \\ &\quad \cup \{g_{2i,i+1} : 1 \le i \le n-1\} \cup \{g_{2n,1}\}, \end{aligned}$$

where $e_{0,2i-1}$ represents the edge u_0u_{2i-1}, $e_{0,i}$ represents the edge u_0v_i, $e_{i,i+1}$ represents the edge u_iu_{i+1}, $f_{i,2i}$ represents the edge v_iu_{2i} and $g_{2i,i+1}, g_{2n,1}$ represents the edge $u_{2i}v_{i+1}$ in Sh_n.
Define a coloring function $c : V(T(Sh_n)) \to \mathbb{N}$ by

$$c(u_0u_{2i-1}) = i, \qquad c(u_0v_i) = n+i, \quad 1 \le i \le n, \text{ and } c(u_0) = 2n+1.$$

All remaining vertices of $T(Sh_n)$ are colored greedily with the smallest available color. Hence, the induced color partition of $V(T(Sh_n))$ is given by

$$V(T(Sh_n)) = \bigcup_{k=1}^{2n+1} C_k,$$

where the color classes are $C_i = \{u_0u_{2i-1}\}, \quad 1 \le i \le n$, $C_{n+i} = \{u_0v_i\}, \quad 1 \le i \le n$, $C_{2n+1} = \{u_0\}$, and for the remaining vertices $x \in V(T(Sh_n)) \setminus \Big(\{u_0\} \cup \{u_0u_{2i-1}, u_0v_i : 1 \le i \le n\}\Big)$,

$$c(x) = \min\{k \in \mathbb{N} : x \text{ is not adjacent to any vertex in } C_k\}.$$

Thus the color partition explicitly consists of $\mathcal{C} = \{C_1, C_2, \dots, C_{2n+1}\}$, which defines a valid Grundy coloring of $T(Sh_n)$.

Case 1: For $n = 3, 4, 5$.
For these values, there exist vertices in $T(Sh_n)$ corresponding to edges not incident with u_0 whose neighborhoods contain vertices colored with all colors $1, 2, \dots, 2n+2$. Hence such a vertex must receive color $2n+3$. Therefore, $\Gamma(T(Sh_n)) \ge 2n+3$. Moreover, no vertex can be adjacent to vertices colored with $1, 2, \dots, 2n+3$, so $\Gamma(T(Sh_n)) \le 2n+3$. Thus,

$$\Gamma(T(Sh_n)) = 2n+3, \quad n = 3, 4, 5.$$

Case 2: For $6 \leq n \leq 15$**.**

In this range, there exists a vertex in $T(Sh_n)$ adjacent to vertices colored with $1, 2, \ldots, 2n+1$, forcing the use of color $2n+2$. Hence, $\Gamma(T(Sh_n)) \geq 2n+2$. However, no vertex in $T(Sh_n)$ can be adjacent to vertices colored with $1, 2, \ldots, 2n+2$, and therefore, $\Gamma(T(Sh_n)) \leq 2n+2$. Thus,

$$\Gamma(T(Sh_n)) = 2n+2, \quad 6 \leq n \leq 15.$$

Case 3: For $n \geq 16$**.**

The vertex u_0 is adjacent in $T(Sh_n)$ to exactly $2n$ vertices corresponding to the incident edges, which realize colors $1, 2, \ldots, 2n$. Hence u_0 receives color $2n+1$, and $\Gamma(T(Sh_n)) \geq 2n+1$. Assume there exists a vertex with color $2n+2$. Then it must be adjacent to vertices colored with all colors $1, 2, \ldots, 2n+1$, which is impossible since every vertex other than u_0 has bounded degree independent of n. Therefore, $\Gamma(T(Sh_n)) \leq 2n+1$. Thus,

$$\Gamma(T(Sh_n)) = 2n+1, \quad n \geq 16.$$

Combining all three cases, we conclude that

$$\Gamma(T(Sh_n)) = \begin{cases} 2n+3, & n = 3, 4, 5, \\ 2n+2, & 6 \leq n \leq 15, \\ 2n+1, & n \geq 16. \end{cases}$$

□

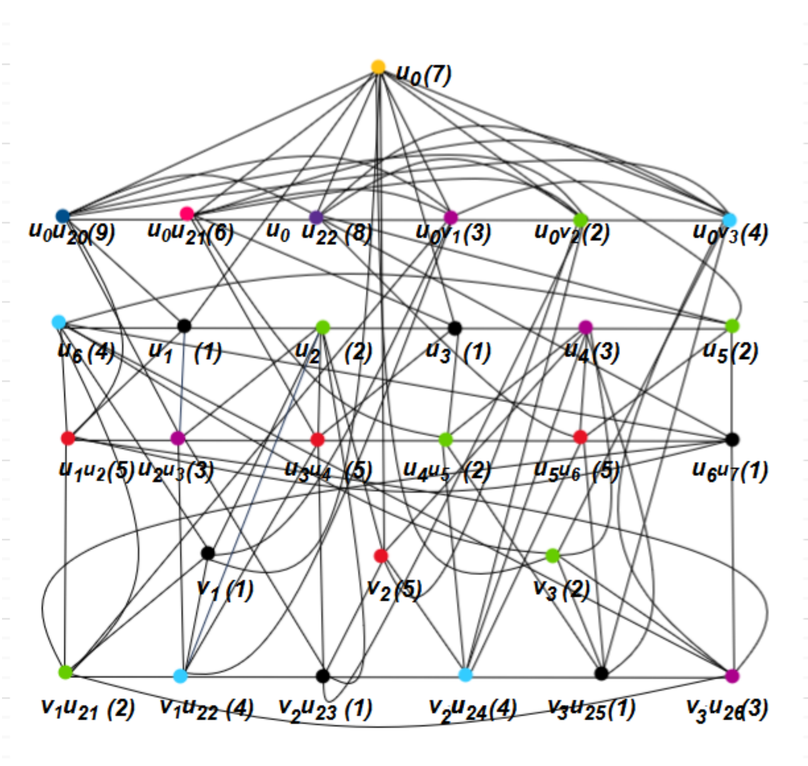

Figure 4. $(T(SH_3)$

Algorithm 4 Construction of a Worst-Case Grundy Ordering for $T(Sh_n)$

Input: An integer $n \geq 3$.
Output: A vertex ordering ϕ of $T(Sh_n)$ such that First–Fit coloring with respect to ϕ uses $\Gamma(T(Sh_n))$ colors.

```
Initialize φ as an empty ordering.
                                   ▷ Step 1: Generate 2n colors from edges incident with u_0
Insert vertices corresponding to the edges u_0 u_{2i-1}, 1 ≤ i ≤ n, into φ.
Append vertices corresponding to the edges u_0 v_i, 1 ≤ i ≤ n, into φ.
                                   ▷ Step 2: Force color 2n + 1
Place the central vertex u_0 next in φ.
if n ≥ 16 then
    Append all remaining vertices in arbitrary order.
end if
if 6 ≤ n ≤ 15 then
    Insert a vertex corresponding to a carefully chosen edge not incident with u_0 whose neighborhood contains
    colors 1, 2, ..., 2n + 1.
    Append all remaining vertices in arbitrary order.
end if
if n = 3, 4, 5 then
    Insert two designated trigger vertices (corresponding to suitable non-central edges or rim vertices) whose
    neighborhoods successively realize colors 1, 2, ..., 2n + 2.
    Append all remaining vertices in arbitrary order.
end if
return φ.
```

With this vertex ordering in $T(Sh_n)$, the First–Fit coloring process is forced to use the largest possible number of colors for each value of n, which results in the piecewise expression for $\Gamma(T(Sh_n))$..

4. Application to Gene Regulatory Networks

The results obtained for Shuriken graphs and their derived graphs highlight how hierarchical depth varies across different structural representations. The base graph Sh_n satisfies $\Gamma(Sh_n) = 4$, which indicates that the hierarchical depth remains bounded, independent of n. In contrast, the derived graphs exhibit increasing complexity: the line graph $L(Sh_n)$ and middle graph $M(Sh_n)$ show linear growth, while the total graph $T(Sh_n)$ captures the highest level of interaction complexity with controlled asymptotic behavior.

Comparative Perspective: Unlike related graph families such as gear graphs, where the Grundy number typically scales with structural expansion, the Shuriken graph exhibits a bounded Grundy number. This highlights a key structural distinction: hub-dominated architectures impose constraints on hierarchical growth, whereas interaction-driven structures allow greater expansion.

Theoretical Context: The obtained bounds are consistent with known properties of Grundy coloring, where $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$. In particular, the result $\Gamma(Sh_n) = 4$ reflects the influence of bounded degree and structural symmetry, distinguishing this family from trees and other graph classes where the Grundy number may grow with size.

Remark: The interpretation of Grundy numbers in GRNs should be viewed as a theoretical abstraction. While it provides insight into worst-case hierarchical activation patterns, it does not directly model biological processes and requires further validation in systems biology contexts.

5. Conclusion

In this paper, we have examined the Grundy chromatic number of the Shuriken graph and its associated line, middle, and total graphs. By constructing explicit Grundy colorings and establishing sharp upper bounds, we analyzed how the Grundy chromatic number behaves under classical graph operators. The results presented in this work extend existing studies on Grundy colorings of derived graphs and highlight the influence of graph structure on greedy coloring parameters. These findings provide a foundation for further investigations on Grundy chromatic numbers of other graph families and related graph transformations. In addition to their theoretical relevance, the results obtained in this work also suggest useful applications in network modeling. In particular, viewing the Grundy chromatic number as an indicator of worst-case hierarchical activation depth provides a meaningful perspective for analyzing Gene Regulatory Networks (GRNs). The structural patterns observed in Sh_n, $L(Sh_n)$, $M(Sh_n)$, and $T(Sh_n)$ demonstrate how interaction-level and integrated regulatory mechanisms can increase hierarchical complexity while still remaining structurally limited. Therefore, this study contributes not only to the theory of graph coloring but also to the understanding of hierarchical activation processes in complex biological and networked systems.

REFERENCES

1. C. A. Christen, and S. M. Selkow, *Some perfect coloring properties of graphs*, Journal of Combinatorial Theory, Series B, vol. 27, pp. 49–59, 1979.
2. S. M. Hedetniemi, S. T. Hedetniemi, and A. Beyer, *A linear algorithm for the Grundy (coloring) number of a tree*, Congressus Numerantium, vol. 36, pp. 351–363, 1982.
3. M. Zaker, *Results on the Grundy chromatic number of graphs*, Discrete Mathematics, vol. 306, pp. 3166–3173, 2006.
4. B. Effantin, and H. Kheddouci, *Grundy number of graphs*, Discussiones Mathematicae Graph Theory, vol. 27, no. 1, pp. 5–18, 2007.
5. P. Erdős, S. T. Hedetniemi, R. C. Laskar, and G. C. E. Prins, *On the equality of the partial Grundy and upper ochromatic numbers of a graph*, Discrete Mathematics, vol. 272, no. 1, pp. 53–64, 2003.
6. Z. Füredi, A. Gyárfás, G. Sárközy, and S. Selkow, *Inequalities for the First-Fit chromatic number*, Journal of Graph Theory, vol. 59, no. 1, pp. 75–88, 2008.
7. M. Aouchiche, and P. Hansen, *A survey of Nordhaus–Gaddum type relations*, Discrete Applied Mathematics, vol. 161, pp. 466–546, 2013.
8. C. Germain, and H. Kheddouci, *Grundy numbers of powers of graphs*, Electronic Notes in Discrete Mathematics, vol. 15, pp. 65–67, 2003.
9. M. Asté, F. Havet, and C. Linhares-Sales, *Grundy number and products of graphs*, Discrete Mathematics, vol. 310, pp. 1482–1490, 2010.
10. J. Araujo, and C. Linhares Sales, *On the Grundy number of graphs with few P_4's*, Discrete Applied Mathematics, vol. 160, pp. 2514–2522, 2012.
11. N. Gastineau, H. Kheddouci, and O. Togni, *On the family of r-regular graphs with Grundy number $r+1$*, Discrete Mathematics, vol. 328, pp. 5–15, 2014.
12. K. Annathurai, P. Periasamy, and V. Sankar Raj, *Results on Grundy chromatic number of prism graphs*, Statistics, Optimization & Information Computing, vol. 15, no. 3, pp. 1810–1820, 2026.
13. M. Kamalnath, and T. Muthukani Vairavel, *Results on Grundy chromatic number of comb product of some graphs*, Journal of Contemporary Applied Mathematics, vol. 16, no. 1, 2026.
14. K. Kaliraj, J. Vernold Vivin, and M. M. Akbar Ali, *On equitable coloring of sunlet graph families*, Ars Combinatoria, vol. 103, pp. 497–504, 2012.
15. M. Sundaram, R. Ponraj, and P. Somasundaram, *On the domination number of Shuriken graphs*, International Journal of Mathematical Combinatorics, vol. 3, pp. 45–52, 2013.
16. P. S. George, S. Madhumitha, and S. Naduvath, *Equitable dominator coloring of helm-related graphs*, Journal of Interconnection Networks, vol. 26, no. 02, Article 2550004, 2026.
17. D. Muralidharan, M. S. Paulraj, and D. Yokesh, *Differential value of splitting and middle graph of some standard graphs*, Journal of Mathematics and Computer Science, vol. 12, 2022.
18. F. Servina Djuang, I. E. Wijayanti, and Y. Susanti, *Shuriken graphs arising from clean graphs of rings and their properties relative to base graphs*, arXiv e-prints, 2026.
19. L. Cai, J. Wang, et al., *Line graph neural networks for link prediction*, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 44, no. 9, pp. 5103–5113, 2021.
20. J. B. Hansen, and F. M. Bianchi, *Total variation graph neural networks*, in Proceedings of the International Conference on Machine Learning (ICML), PMLR, 2023.
21. M. Švarcmajer et al., *Application of graph theory and variants of greedy graph coloring algorithms for optimization of distributed peer-to-peer blockchain networks*, Technologies, vol. 13, no. 1, Article 33, 2025.
22. U. Alon, *An Introduction to Systems Biology: Design Principles of Biological Circuits*, Chapman & Hall/CRC, 2006.

23. G. Karlebach, and R. Shamir, *Modelling and analysis of gene regulatory networks*, Nature Reviews Molecular Cell Biology, vol. 9, pp. 770–780, 2008.
24. A.-L. Barabási, and Z. N. Oltvai, *Network biology: understanding the cell's functional organization*, Nature Reviews Genetics, vol. 5, pp. 101–113, 2004.
25. R. Milo et al., *Network motifs: simple building blocks of complex networks*, Science, vol. 298, pp. 824–827, 2002.